Nuclear Forces: the Nucleon-Nucleon Interaction

In this lecture we will consider the two-nucleon problem. We will cover the following things: (i) the partial wave projection of the Schrödinger equation and its solutions, (ii) scattering of two particles, partial wave decomposition and the phase shifts, (iii) the effective range expansion, (iv) formal scattering theory and the T-matrix, (v) the one boson exchange model of the nuclear forces, (vi) the effective field theory approach to the nuclear forces.

I. TWO-BODY SYSTEM

This section is a reminder of the physics of two-body systems, in particular of bound states and scattering.

A. Schrödinger Equation

We begin with the Schödinger equation for two particles

$$\left[\frac{p_1^2}{2m_1}+\frac{p_2^2}{2m_2}+V(\vec{r}_1-\vec{r}_2)\right]\,\Psi(\vec{r}_1,\vec{r}_2)=E_T\,\Psi(\vec{r}_1,\vec{r}_2)\,. \tag{1}$$

This can be simplified by considering the center of mass system, which implies the following changes of variables

$$\begin{aligned}\vec{p}_1&=\frac{m_2}{m_1+m_2}\,\vec{P}+\vec{p} \qquad & \vec{p}_2&=\frac{m_1}{m_1+m_2}\,\vec{P}-\vec{p}\\ \vec{r}_1&=\vec{R}+\frac{m_2}{m_1+m_2}\,\vec{r} \qquad & \vec{r}_2&=\vec{R}-\frac{m_1}{m_1+m_2}\,\vec{r},\end{aligned} \tag{2}$$

which lead to

$$\left[\frac{p^2}{2\mu}+\frac{P^2}{2M}+V(\vec{r})\right]\,\Psi(\vec{r},\vec{R})=E_T\,\Psi(\vec{r},\vec{R})\,, \tag{3}$$

where

$$\mu=\frac{m_1m_2}{m_1+m_2}\quad\text{and}\quad M=m_1+m_2\,. \tag{4}$$

Here we can make a clean separation between the relative wave function of particles 1 and 2 and the center of mass movement

$$\Psi(\vec{r},\vec{R})=\Psi(\vec{r})\,\Psi_{CM}(\vec{R})\,, \tag{5}$$

$$\frac{P^2}{2M}\,\Psi(\vec{R})=E_{\rm CM}\,\Psi_{\rm CM}(\vec{R})\,, \tag{6}$$

$$\left[\frac{p^2}{2\mu}+V(\vec{r})\right]\,\Psi(\vec{r})=E\,\Psi(\vec{r})\,, \tag{7}$$

with $E_T=E_{\rm CM}+E$. The equation for the total center of mass movement is straightforward to solve

$$\Psi_{\rm CM}(\vec{R})=e^{i\vec{K}\cdot\vec{R}}\,,\quad\text{with}\quad E_{\rm CM}=\frac{K^2}{2M}\,. \tag{8}$$

For central forces, the relative wave function can be separater into an angular and a radial piece

$$\Psi(\vec{r})=\frac{u_l(r)}{r}\,Y_{lm}(\hat{r})\,, \tag{9}$$

where Y_{lm} are the spherical harmonics and we are assuming a solution with good angular momentum l. In addition the Schrödinger equation for the reduced wave function is

$$-u_l''(r)+\left[2\,\mu V(r)+\frac{l(l+1)}{r^2}\right]\,u_l(r)=k^2\,u_l(r)\,, \tag{10}$$

where $E = k^2/2\mu$. For bound states we have that $E < 0$ and sometimes we define

$$E = -B \quad \text{with} \quad B = \frac{\gamma^2}{2\mu}\,, \tag{11}$$

for which the Schrödinger equation reads

$$-u_l''(r) + \left[2\,\mu V(r) + \frac{l(l+1)}{r^2}\right]\,u_l(r) = -\gamma^2\,u_l(r)\,. \tag{12}$$

We call γ the wave number.

B. Asymptotic Solutions of the Schrödinger Equation ($r \to 0$)

1. *Bound States ($E < 0$)*

Now we will take a look at the solutions for $r \to \infty$. For potentials with a finite range, i.e. potentials that fulfill the following property

$$\lim_{r\to\infty} r^n V(r) \to 0\,, \tag{13}$$

where $n \geq 0$, we have that the Schrödinger equation at long distances can be written as

$$-u_l''(r) + \frac{l(l+1)}{r^2}\,u_l(r) = -\gamma^2\,u_l(r)\,, \tag{14}$$

$$-u_l''(r) + \frac{l(l+1)}{r^2}\,u_l(r) = +k^2\,u_l(r)\,, \tag{15}$$

depending on whether we are considering bound states ($E < 0$) or scattering states ($E > 0$). For bound states the wave functions is

$$u_l(r) \to A_l\,e^{-\gamma r}\left(1 + \mathcal{O}(\frac{1}{\gamma r})\right)\,, \tag{16}$$

i.e. a decaying exponential multiplied by a polynomial in $1/\gamma r$. The factor A_l is just a number which goes by the name of *asymptotic normalization*. For s-wave ($l = 0$) it is easy to show that

$$u_0(r) \to A_S\,e^{-\gamma r}\,, \tag{17}$$

and here I can propose a really easy **exercise**: if we assume that the potential is

$$V(r) = 0 \quad \text{for} \quad r \neq 0\,, \tag{18}$$

then show that $A_S = \sqrt{2\gamma}$ and that the mean square radius of the bound state is $\langle r^2\rangle = 1/(2\gamma^2)$. Other simple **exercise** one can try here is to find the exact asymptotic form of the $l = 1$ and $l = 2$ solutions

$$u_1(r) \to A_P\,e^{-\gamma r}\,(1 + \dots)\,, \tag{19}$$

$$u_2(r) \to A_D\,e^{-\gamma r}\,(1 + \dots)\,, \tag{20}$$

that is, to find the complete expression that should go inside the brackets.

2. *Scattering States ($E \geq 0$)*

For scattering states we have two possible solutions for the partial wave l

$$u_l(r) \to +\sin\,(kr - l\frac{\pi}{2})\,, \tag{21}$$

$$u_l(r) \to -\cos\,(kr - l\frac{\pi}{2})\,, \tag{22}$$

which is really easy to check and where the signs are merely a convention. Actually the exact solution is

$$u_l(r) = a_l(k)\,(kr)\,j_l(kr) + b_l(k)\,(kr)\,y_l(kr)\,, \tag{23}$$

where $j_l(x)$ and $y_l(x)$ are the spherical bessel functions, which can be defined as

$$j_l(x) = +(-x)^l \left(\frac{1}{x}\frac{d}{dx}\right)^l \frac{\sin(x)}{x}\,, \tag{24}$$

$$y_l(x) = -(-x)^l \left(\frac{1}{x}\frac{d}{dx}\right)^l \frac{\cos(x)}{x}\,, \tag{25}$$

and which for $r \to \infty$ behave as

$$j_l(x) \to +\frac{1}{x}\,\sin(x - l\frac{\pi}{2})\,, \tag{26}$$

$$y_l(x) \to -\frac{1}{x}\,\cos(x - l\frac{\pi}{2})\,. \tag{27}$$

Usually these solutions are presented in the form of a linear combination as follows

$$\begin{aligned} u_l(r) &= \cot\delta_l\,(kr)\,j_l(kr) - (kr)\,j_l(kr) \\ &\to \frac{\sin\left(kr - l\frac{\pi}{2} + \delta_l\right)}{\sin\delta_l} \end{aligned} \tag{28}$$

where $\delta_l = \delta_l(k)$ is the phase shift. For s-waves ($l = 0$), which is usually the most important partial wave in all systems, we have the following

$$u_0(r) \to \sin(kr + \delta_0)\,, \tag{29}$$

which is relatively easy.

3. Zero Energy and The Scattering Length

The phase shift is actually important because its connection with scattering theory, which we will explain below. In addition, for $k \to 0$, the phase shift exhibits the following behaviour

$$\delta_l(k) \to -\alpha_l\, k^{2l+1} + \mathcal{O}(k^{2l+1})\,. \tag{30}$$

This in turn implies that we can take the previous solution for k and check that the limit for $k \to 0$ is

$$\begin{aligned} \lim_{k\to 0} u_l(r) &= \lim_{k\to 0} \left(\cot\delta_l(kr)\,j_l(kr) - (kr)\,y_l(kr)\right) \\ &\to \frac{(2l-1)!!}{r^l} - \frac{r^{l+1}}{(2l+1)!!} r^{l+1}\,. \end{aligned} \tag{31}$$

Again the most important case is $l = 0$ for which we have

$$u_0(r) = 1 - \frac{r}{\alpha_0}\,. \tag{32}$$

C. The Schrödinger Equation near the Origin ($r \to 0$)

Now we study what happens with the wave function when the radius is very small. In general we want the reduced wave function $u_l(r)$ to be zero at the origin

$$u_l(0) = 0\,, \tag{33}$$

because in such a way the full wave function Ψ_{lm} will go to a constant value. In addition, we will be able to properly normalize the wave function

$$\langle Psi|Psi\rangle = \int d^3\vec{r}\,|\Psi_{lm}(\vec{r})|^2 = \int dr u_l(r)^2 = 1\,. \tag{34}$$

However we mention in passing that for s-waves, the solution $u_0 \neq 0$ also leads to a wave function that can be normalized.

1. Regular Potential

Now the most common case is that of a regular potential, by which we mean a potential that fulfills the following condition

$$\lim_{r\to 0} 2\mu r^2 V(r) = 0\,. \tag{35}$$

In such a case we have two type of solutions at short distances

$$u_l(r) \propto r^l \quad \text{and} \quad u_l(r) \propto \frac{1}{r^{l+1}}\,. \tag{36}$$

The regularity condition implies that we choose $u_l(r) \propto r^{l+1}$.

This leads us to a simple **exercise**, solving the square well for s-wave

$$2\mu\, V(r) = U(R)\theta(R-r)\,, \tag{37}$$

which is a very well-known potential. In particular we can try to solve it for $k = 0$ and obtain the scattering length from the formula

$$u_0(r) = 1 - \frac{r}{\alpha_0} \quad \text{for } r > R\,. \tag{38}$$

To make the calculations a bit more simple we will assume that

$$\frac{\alpha_0}{R} \ll 1\,, \tag{39}$$

which means that we can do a few Taylor expansions that will simplify the results. Now imagine that you don't know the potencial $U(R)$ or R, but you know the value of the scattering length. For instance, in the two-nucleon system we have for the singlet channel ($S = 0$) that the scattering length is $\alpha_0 = -23.7\,\text{fm}$ and for the triplet channel $\alpha_0 = 5.4\,\text{fm}$. From this try to find $U = U(R;\alpha_0)$, i.e. U as a function of R and the scattering length. If you find this relation, you would have solved a renormalization group equation for the two-nucleon system.

2. Singular Potential

Now we might find ourselves with the situation that

$$\lim_{r\to 0} 2\mu r^2 V(r) \to \pm\infty\,. \tag{40}$$

This situation seems terrible, but actually it is sort of okay. The previous type of potential is called a singular potential. For simplicity let us consider a singular potential of the type

$$2\mu V(r) = \pm\frac{a^{n-2}}{r^n}\,. \tag{41}$$

We have two possibilities, that the potential is repulsive and that the potential is attractive.

If the potential is repulsive, the solution for $r \ll a$ takes the form

$$u_l(r) = c_+ \left(\frac{r}{a}\right)^{1/4} \exp\left[+\frac{2}{n-1}\left(\frac{a}{r}\right)^{(n-2)/2}\right] + c_- \left(\frac{r}{a}\right)^{1/4} \exp\left[-\frac{2}{n-1}\left(\frac{a}{r}\right)^{(n-2)/2}\right] \tag{42}$$

which looks bad, except that by a close inspection we discover that

$$\lim_{r\to 0} \exp\left[+\frac{2}{n-1}\left(\frac{a}{r}\right)^{(n-2)/2}\right] \to \infty\,, \tag{43}$$

$$\lim_{r\to 0} \exp\left[-\frac{2}{n-1}\left(\frac{a}{r}\right)^{(n-2)/2}\right] \to 0\,. \tag{44}$$

This means that there is a simple regular solution

$$u_l(r) = c_- \left(\frac{r}{a}\right)^{1/4} \exp\left[-\frac{2}{n-1}\left(\frac{a}{r}\right)^{(n-2)/2}\right]. \tag{45}$$

Now if the potential is attractive, things are different. The most general solution is

$$u_l(r) = c \left(\frac{r}{a}\right)^{1/4} \sin\left[\frac{2}{n-1}\left(\frac{a}{r}\right)^{(n-2)/2} + \varphi\right], \tag{46}$$

and it happens that this solution is always regular, independently of the value of φ. As a consequence we cannot determine which solution is the physical one.

3. The Borderline Case

There are two potentials that lies in between these two cases, and which we can analyze here as **exercises**. The first **exercise** (two points) is the potential

$$2\mu V(r) = \frac{g}{r^2}\,, \tag{47}$$

which is singular, but not too much. For $l = 0$ and $k = 0$ we can indeed check a solution of the type

$$u_0(r) = c_+\, r^{\frac{1}{2}+\nu} + c_-\, r^{\frac{1}{2}-\nu}\,. \tag{48}$$

Then by plugging this solution into the Schrödinger equation, it is possible to calculate the equation

$$\nu = \nu(g)\,, \tag{49}$$

which has real solution for $g > g_{\rm crit}$. There is a critical value of the coupling $g_{\rm crit}$ for which we have

$$\nu(g_{\rm crit}) = 0\,, \tag{50}$$

find this value and also that for $g < g_{\rm crit}$ the solutions are complex. Now for complex values of ν, one can use the following identities

$$\frac{r^{i\alpha} + r^{-i\alpha}}{2} = \frac{e^{i\alpha\log r} + e^{-i\alpha\log r}}{2} = \cos\left(\alpha\log r\right), \tag{51}$$

$$\frac{r^{i\alpha} - r^{-i\alpha}}{2i} = \frac{e^{i\alpha\log r} - e^{-i\alpha\log r}}{2} = \sin\left(\alpha\log r\right), \tag{52}$$

to find the general solution for $u_0(r)$. The situation is analogous to those of the attractive singular potentials: there are an infinite number of solutions. Yet there is a very interesting property for this solution. If we look at the original Schrödinger equation

$$-\frac{d}{dr^2}\,u_0(r) + \frac{g}{r^2}\,u_0(r) = 0\,, \tag{53}$$

it happens that it has a very interesting symmetry, which is that if we do the transformation

$$r \to r' = \lambda r\,, \tag{54}$$

then the Schrödinger equation remains the same:

$$-\frac{d^2}{dr'^2}\,u_0(r') + \frac{g}{r'^2}\,u_0(r') = 0\,. \tag{55}$$

The transformation $r \to \lambda r$ is called a *dilatation* and systems that do not change with dilatations are calles *scale invariant*. Now the thing is that if $g > g_{\rm crit}$, we have that the solutions scale as

$$r^{\frac{1}{2}\pm\nu} \to \lambda^{\frac{1}{2}\pm\nu}\, r^{\frac{1}{2}\pm\nu}\,, \tag{56}$$

and if we ignore the normalization of the wave function (which is only important for bound states), we see that the wave function is scale invariant. However for $g < g_{\rm crit}$ we see that

$$u_0(\lambda r) = \lambda^a u_0(r) \quad \text{only if} \quad \lambda = \lambda_0\,. \tag{57}$$

Find the value of $\lambda_0 = \lambda_0(g)$. In this case we have something called an *anomaly*: a classical symmetry of a system, which is broken in quantum mechanics. The classical two-body system is invariant under $r \to \lambda r$ for any λ, but the quantum mechanical two body system is not, except if $\lambda = \lambda_0$. The residual symmetry of the quantum system is usually referred to as *discrete scale invariance* ($\lambda = \lambda_0$) to distinguish it from *continuous scale invariance* (λ arbitrary).

The second **exercise** (two points) has to do with the delta-shell potential

$$2\mu\, V(r; R_c) = \frac{C_0(R_c)}{4\pi R_c^2}\,\delta(r - R_c)\,, \tag{58}$$

and the problem of how to solve the s-wave reduced Schrödinger equation with this potential

$$-u_0''(r) + 2\mu\, V(r; R_c) u_0(r) = k^2\, u_0(r)\,. \tag{59}$$

For $r \neq R_c$ we have two free Schrödinger equations

$$\begin{aligned} -u_0''(r) &= k^2\, u_0(r) \quad \text{for } r < R_c, \text{ and} \\ -u_0''(r) &= k^2\, u_0(r) \quad \text{for } r > R_c. \end{aligned} \tag{60}$$

The problem here is how to treat the singular point $r = R_c$. The trick will be to integrate around $r = R_c \pm \epsilon$ with $\epsilon > 0$, that is

$$-\int_{R_c-\epsilon}^{R_c+\epsilon} u_0''(r)dr + 2\mu \int_{R_c-\epsilon}^{R_c+\epsilon} V(r;R_c)u_0(r)dr = k^2 \int_{R_c-\epsilon}^{R_c+\epsilon} u_0(r)dr\,, \tag{61}$$

and then take the limit $\epsilon \to 0^+$, where the plus "+" indicates that the limit is done with the condition $\epsilon > 0$. In additon we have to add the continuity condition

$$\lim_{\epsilon\to 0^+} u_0(R_c - \epsilon) = \lim_{\epsilon\to 0^+} u_0(R_c + \epsilon)\,. \tag{62}$$

With that find the equation for the phase shift of this potential.

D. Scattering Theory

1. *Definition of the Cross Section*

When two particles collides, the quantity we measure in the laboratory is the cross section. If we have a beam of particle species A hitting a target of particle species B, the cross section is proportional to the number of particles of the type A that are scattered by the target. The definition of the cross section is then

$$\sigma = \frac{N_S}{N_A N_B} S\,, \tag{63}$$

where σ is the cross section, N_S the number of particles that have been scattered, N_A the number of particles of type A, N_B the number of particles of type B and S the area of the target (which are the B-type particles) or the area of the beam (whichever is bigger). Equivalently we can re-order the terms as

$$\sigma = \frac{N_S}{N_A} \frac{1}{(N_B/S)}\,, \tag{64}$$

where N_S/N_A is now the fraction of particles of type A that have been scattered and N_B/S the density of the target.

This is not the only definition of the cross section and other equivalent definitions exist. Now we consider one definition in which the flux of particle type A is taken into account. In this case the cross section is normalized in such a way that it does not depend on the incoming flux of the beam A or the density of the target B. This is done as follows:

$$\sigma = \frac{N_S}{T_A\,(n_A v_A) N_B}\,, \tag{65}$$

where σ is the cross section, N_S is the number of particles that are scattered, T_A the time during which the beam is hitting the target, $(n_A v_A)$ is the flux of the beam A and N_B is the number of particles in the target. The flux $(n_A v_A)$ refers to the number of particles of type A that are traversing the target per unit area and per unit time, that is $n_A = N_A/S/L$ with N_A the number of particles in the beam, S the area of the beam and L the length of the beam in the direction of v_A. Then when we multiply by v_A we obtain the particles of type A reaching the target per unit of area and per unit of time. Both definitions are equivalent, we just have to take into account that $T_A(n_A v_A) = N_A N_B/S_A$, but the second one is more common in textbooks when they cover scattering.

We can better understand the cross section with an example. Let us assume a target of rigid spheres of radius R and a beam of tiny particles (much much smaller than R) that is hitting the target. A particle in the beam is only scattered if it hits a sphere. Thus the probability of an individual particle in the beam to be scattered is

$$P_S = \frac{N_B S_B}{S_A}\,, \tag{66}$$

where N_B is the number of spheres in the region where the beam is hitting the target, S_B is the area of each sphere from the point of view of the beam A (that is, the cross section of the sphere) and S_A is the surface of the beam of particles A. From this we can compute the number of particles that are scattered as

$$N_S = P_S N_A\,, \tag{67}$$

which is pretty simple. For doing the connection with the cross section we do the following

$$\sigma = \frac{S_A N_S}{N_A N_B} = \frac{S_A P_S}{N_B} = S_B \,. \tag{68}$$

Thus the cross section turns out to be

$$\sigma = S_B = 4\pi R^2 \,, \tag{69}$$

which is indeed the cross section of the rigid spheres.

2. *Quantum Mechanical Cross Section*

Now to the quantum mechanical problem. In this case we begin by considering the scattering (positive energy) wave function of two particles, which we can write as

$$\Psi_k(\vec{r}) = e^{i\vec{k}\cdot\vec{r}} + f(\hat{k}\cdot\hat{r})\,\frac{e^{ikr}}{r} \,, \tag{70}$$

where we express the wave function as a plane wave (the incoming beam) plus a spherical wave (representing the scattered particles). Obviously in the absence of a potential, we will simply get $f(\hat{k}\cdot\hat{r}) = 0$ and the cross section will be zero. This also indicates that the cross section will have something to do with the amplitude f.

The first thing we do is to separate the wave function into an incoming and an outgoing piece

$$\Psi_k(\vec{r}) = \Psi_{\rm in}(\vec{r}) + \Psi_{\rm out}(\vec{r}) \,, \tag{71}$$

where

$$\Psi_{\rm in}(\vec{r}) = e^{i\vec{k}\cdot\vec{r}} \,, \tag{72}$$

$$\Psi_{\rm out}(\vec{r}) = f(\hat{k}\cdot\hat{r})\,\frac{e^{ikr}}{r} \,. \tag{73}$$

From here it is trival to compute is the incoming flux, which is given by

$$\vec{j}_{\rm in} = -\frac{i}{2m}\left[\Psi^*_{\rm in}(\vec{r})\left(\nabla\Psi_{\rm in}(\vec{r})\right) - \left(\nabla\Psi^*_{\rm in}(\vec{r})\right)\Psi_{\rm in}(\vec{r})\right] = \frac{1}{m}{\rm Im}\left[\Psi^*_{\rm in}(\vec{r})\left(\nabla\Psi_{\rm in}(\vec{r})\right)\right] = \frac{\vec{k}}{m} \tag{74}$$

which is the quantum mechanical equivalent of N_A/T, the number of beam particles A per unit time. This is also the equivalent of $(n_A v_A)$ if we use the second definition of the cross section. Now the scattering probability per unit time is proportional to the outgoing flux multiplied by the area where the flux is escaping (just look at the equations below). First, the outgoing flux is

$$\vec{j}_{\rm out} = \frac{1}{m}{\rm Im}\left[\Psi^*_{\rm out}(\vec{r})\nabla\Psi_{\rm out}(\vec{r})\right] = \frac{k}{m}\left[\left|f(\hat{k}\cdot\hat{r})\right|^2\frac{\hat{e}_R}{R^2} + \mathcal{O}(\frac{1}{R^3})\right] \,, \tag{75}$$

which means that the probability of the scattered particles to escape per unit time is

$$P_S = \frac{k}{m}\int\left|f(\hat{k}\cdot\hat{r})\right|^2 d^2\hat{k} \,, \tag{76}$$

where we have already taken the limit $R\to\infty$. This is the quantum mechanical equivalent of $N_S/T/N_B$. Finally, putting the pieces together

$$\sigma = \frac{P_S}{|j_{\rm in}|} = \int\left|f(\hat{k}\cdot\hat{r})\right|^2 d^2\hat{k} \,. \tag{77}$$

Equivalently we can write the differential cross section as

$$\frac{d\sigma}{d^2\hat{k}} = \left|f(\hat{k}\cdot\hat{r})\right|^2 \quad \text{or simply} \quad \frac{d\sigma}{d\Omega} = |f(\Omega)|^2 \,, \tag{78}$$

where in the second expression Ω is the solid angle. This is a pretty simple result. Now we can fiddle with these expressions a bit more to find a few more interesting results.

3. Partial Wave Expansion

We can learn a great deal about scattering by performing the partial wave expansion of the wave function. We begin with

$$e^{i\vec{k}\cdot\vec{r}} = 4\pi \sum_{lm} i^l j_l(kr) Y^*_{lm}(\hat{k}) Y_{lm}(\hat{r}) = \sum_l (2l+1) i^l j_l(kr) P_l(\hat{k}\cdot\hat{r})\,. \tag{79}$$

Actually the scattering amplitude f can be expanded in the same way

$$f(\hat{k}\cdot\hat{r}) = \sum_l (2l+1) f_l(k) P_l(\hat{k}\cdot\hat{r})\,. \tag{80}$$

We can also do exactly the same with the full wave function

$$\Psi_k(\vec{r}) = 4\pi \sum_{lm} i^l \frac{u_l(r;k)}{r} Y^*_{lm}(\hat{k}) Y_{lm}(\hat{r}) = \sum_l (2l+1) i^l \frac{u_l(r;k)}{r} P_l(\hat{k}\cdot\hat{r})\,, \tag{81}$$

which coincides with the plane wave once we turn off the potential, in which case $u_l(r,k) \to r j(kr)$. We also know that at large distances the functions $u_l(r,k)$ take the form [1]

$$\frac{u_l(k;r)}{r} \to e^{i\delta_l} \left[\cos\delta_l(k)\, j_l(kr) - \sin\delta_l(k) y_l(kr)\right]\,. \tag{82}$$

We can manipulate the expressions to obtain (**exercise**)

$$f_l(k) = \frac{e^{i\delta_l}\sin\delta_l}{k} = \frac{1}{k\,\cot\delta_l(k) - ik}\,, \tag{83}$$

which is pretty simple. This expression also allows us to write the cross section as

$$\sigma = \frac{4\pi}{k^2} \sum_l \sin\delta_l{}^2\,. \tag{84}$$

Other interesting limit is low energy scattering. In this case we have to use the relation

$$\delta_l(k) = -\alpha_l k^{2l+1} + \mathcal{O}(k^{2l+3})\,, \tag{85}$$

from which we end up with

$$\sigma = 4\pi\alpha_0^2 + \mathcal{O}(k^2)\,, \tag{86}$$

which is reminiscent of the rigid sphere cross section. In this sense the scattering length can be interpreted as sort of the effective rigid sphere radius that a particle spans at low energy in quantum mechanics.

4. Effective Range Expansion

The partial wave expansion of the scattering amplitude indicates the importance of the phase shifts. Besides at very low energy only the S-wave ($l = 0$) contributes and scattering can be described in terms of the scattering length. In particular we have seen that

$$\delta_0(k) = -\alpha_0 k^1 + \mathcal{O}(k^3)\,, \tag{87}$$

which implies the following

$$k\,\cot\delta_0(k) = -\frac{1}{\alpha_0} + \mathcal{O}(k^2)\,. \tag{88}$$

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[1] We have previously written

$$\frac{u_l(k,r)}{kr} = \cot\delta_l(k) j_l(kr) - y_l(kr)\,,$$

which is just the same result in a different normalization. The reason for the new normalization is that we want $u_l \to r\, j_l(kr)$ when there is no interaction (i.e. when $\delta_l \to 0$). Then we have $\Psi_k(\vec{r}) \to e^{i\vec{k}\cdot\vec{r}}$ in that limit.

Now it would be very interesting to extend this formula to higher powers of the momentum. The way to do it is the effective range expansion, which states that $k\,\cot\delta$ is an expansion of the type

$$k\,\cot\delta_0(k) = -\frac{1}{\alpha_0} + \frac{1}{2}\,r_0^2\,k^2 + \sum_{n=2}^{\infty} v_n k^{2n}\,, \tag{89}$$

where the radius of convergence of the expansion is $k < m/2$ for a scattering problem involving a potential that behaves as $f(r)e^{-mr}$ at long distances.

The derivation of this formula involves a really interesting trick called a *Wronskian identity*, which is worth mastering. The starting point is the Schrödinger equation for S-wave scattering

$$-u_k''(r) + 2\mu\,V(r)\,u_k(r) = k^2 u_k(r)\,, \tag{90}$$

for which we use the asymptotic normalization

$$u_k(r) \to \frac{\sin(kr+\delta)}{\sin\delta}\,, \tag{91}$$

for $r \to \infty$. Now we compare the equation above with the Schrödinger equation for a problem with no potential

$$-v_k''(r) = k^2\,v_k(r) \quad \text{with} \quad v_k(r) = \frac{\sin(kr+\delta)}{\sin\delta}\,. \tag{92}$$

And now we add the zero energy solutions to the mix, i.e. we consider the zero energy solution with a potential

$$-u_0''(r) + 2\mu\,V(r)\,u_0(r) = 0 \quad \text{with} \quad u_0(r) \to 1 - \frac{r}{\alpha_0}\,, \tag{93}$$

and without a potential

$$-v_0''(r) = 0 \quad \text{with} \quad v_0(r) = 1 - \frac{r}{\alpha_0}\,. \tag{94}$$

Now the trick is as follows: we multiply the Schrödinger equation for u_k (v_k) by u_0 (v_0) and vice versa and then compute the difference:

$$\begin{aligned}\left[-u_k'' + 2\mu V(r)u_k = k^2 u_k\right] \times u_0(r) - \left[-u_0'' + 2\mu V(r)u_0 = 0\right] \times u_k(r) = \\ -(u_k''u_0 - u_k u_0'') = k^2 u_k u_0\,,\end{aligned} \tag{95}$$

eplus the analogous equation for v_k and v_0

$$-(v_k''v_0 - v_k v_0'') = k^2 v_k v_0\,. \tag{96}$$

We can notice that there is an exact derivative hidden in the previous expressions

$$-(u_k''u_0 - u_k u_0'') = -(u_k'u_0 - u_k u_0')'\,, \tag{97}$$

$$-(v_k''v_0 - v_k v_0'') = -(v_k'v_0 - v_k v_0')'\,, \tag{98}$$

which allows us to perform the following integration

$$-(u_k'u_0 - u_k u_0')\Big|_0^R = k^2 \int_0^R dr\,u_k(r)u_0(r)\,, \tag{99}$$

$$-(v_k'v_0 - v_k v_0')\Big|_0^R = k^2 \int_0^R dr\,v_k(r)v_0(r)\,. \tag{100}$$

Then by calculating the different and taking the $R \to infty$ limit, we arrive (**exercise**)

$$k\,\cot\delta = -\frac{1}{\alpha_0} + k^2 \int_0^\infty dr\,\left(v_k(r)v_0(r) - u_k(r)u_0(r)\right)\,. \tag{101}$$

If the potential has a finite range (for instance, it contains an exponential falloff at long distance), the integral above is well defined. Besides, the wave functions can be expanded in powers of k^2

$$u_k(r) = u_0(r) + k^2\,u_2(r) + k^4\,u_4(r) + \dots\,, \tag{102}$$

$$v_k(r) = v_0(r) + k^2\,v_2(r) + k^4\,v_4(r) + \dots\,, \tag{103}$$

from which we arrive at the expression

$$k\,\cot\delta_0(k) = -\frac{1}{\alpha_0} + \frac{1}{2}\,r_0^2\,k^2 + \sum_{n=2}^{\infty} v_n k^{2n}\,, \tag{104}$$

that is, the effective range expansion. The parameter r_0 is the effective range and can be defined as the integral

$$r_0 = 2\int_0^\infty (v_0^2(r) - u_0^2(r))\,dr\,. \tag{105}$$

It is relatively easy to see that if we have a potential with a finite range a, i.e. for $r > a$ the potential quickly goes to zero, then for $r > a$ it will also happen that $u_0 \to v_0$. As a consequence the integral above receive most of its contributions from the region where the potential is non-zero

$$\int_0^\infty (v_0^2(r) - u_0^2(r))\,dr \sim \int_0^a (v_0^2(r) - u_0^2(r))\,dr\,, \tag{106}$$

from which we are led to the conclusion that r_0 is proportional to the range of the potential a. This is why r_0 is called the effective range. The other parameters, v_2, v_3 and so on, are called the shape parameters and are defined as

$$v_n = 2\int_0^\infty (v_0(r)v_{2n-2}(r) - u_0(r)u_{2n-2}(r))\,dr\,. \tag{107}$$

E. Formal Scattering Theory and the T-matrix

Now we will introduce a different scattering formalism: the T-matrix. We begin with the Schrödinger equation, but now we write it as

$$H\,|\phi\rangle = E\,|\phi\rangle\,, \tag{108}$$

where $H = H_0 + V$ with H_0 the kinetic term and V the potential. We can rewrite the Schrödinger equation in the form

$$(E - H_0)|\phi\rangle = V|\phi\rangle\,, \tag{109}$$

which accepts a general solution that is iterative

$$\phi(\vec{r}) = e^{i\vec{k}\cdot\vec{r}} + \int d^3\vec{r}\,'\,G_0(\vec{r} - \vec{r}\,')\,V(r')\phi(\vec{r}\,')\,. \tag{110}$$

It is easy to show that the Green function G fulfills

$$(E - H_0)\,G_0(\vec{r} - \vec{r}\,') = \delta^{(3)}(\vec{r} - \vec{r}\,')\,. \tag{111}$$

At first sight this equation looks rather mysterious, but if we write it in momentum space it will look much less menacing

$$(E - H_0)\,G_0(\vec{l}) = 1 \quad \text{with} \quad H_0 = \frac{\vec{l}^2}{2\mu}\,, \tag{112}$$

with $\vec{l}$ the momentum and which admits the solution

$$G_0(\vec{l}) = \frac{1}{E - H_0} = \frac{1}{E - \frac{l^2}{2\mu}}\,. \tag{113}$$

Henceforth we can obtain the coordinate space expression from Fourier-transforming the previous solution

$$G(\vec{r}) = \int \frac{d^3l}{(2\pi)^3}\,\frac{e^{i\vec{l}\cdot\vec{r}}}{E - H_0(\vec{l})}\,. \tag{114}$$

Now you are welcome to enter the magical realm of the complex plane! The standard way of solving the previous equation is by doing first the angular momentum integral

$$\begin{aligned} G(\vec{r}) &= \frac{1}{2\pi^2 r}\int_0^\infty dl\,\frac{l\sin(lr)}{E-\frac{l^2}{2\mu}} \\ &= \frac{1}{4\pi^2 r}\,\mathrm{Im}\left[\int_{-\infty}^{\infty} dl\,\frac{le^{ilr}}{E-\frac{l^2}{2\mu}}\right]. \end{aligned} \tag{115}$$

And now it is when the complex plane enters. We are going to solve the previous integral by residues, which means that we have first to take a look at the poles of the integral:

$$l = \pm\sqrt{2\mu E} = \pm k\,. \tag{116}$$

But the problem is whether they are inside or outside the countour of integration. For defining the countour we will make a little change to the energy of the system:

$$E \to E + i\epsilon\,, \tag{117}$$

with ϵ a tiny positive number. With this choice we end up with

$$G_0(r) = -\frac{\mu}{2\pi}\frac{e^{ikr}}{kr}\,. \tag{118}$$

while if we had choosen $E \to E - i\epsilon$ we would have obtained

$$G_0(r) = -\frac{\mu}{2\pi}\frac{e^{-ikr}}{kr}\,. \tag{119}$$

instead. From this we can understand that $E + i\epsilon$ give us an outgoing wave, which is compatible with the idea that we indeed have an outgoing wave for the scattering solution

$$\phi^+(\vec{r}) \to e^{i\vec{k}\cdot\vec{r}} + f(\Omega)\,\frac{e^{ikr}}{r}\,. \tag{120}$$

By the way, from now on we will add a $\pm$ superscript to ϕ to indicate whether it is the solution with $E+i\epsilon$ or $E-i\epsilon$. For ϕ^+ the scattering equation now looks like

$$\phi^+(\vec{r}) = e^{i\vec{k}\cdot\vec{r}} - \frac{\mu}{2\pi}\int d^3\vec{r}\,'\,\frac{e^{ik|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|}\,V(r')\phi^+(\vec{r}\,')\,. \tag{121}$$

If we take the limit $|\vec{r}| \to \infty$

$$\begin{aligned} \phi^+(\vec{r}) &= e^{i\vec{k}\cdot\vec{r}} - \frac{\mu}{2\pi}\frac{e^{ikr}}{r}\int d^3\vec{r}\,'\,e^{-ikr'\hat{r}\cdot\hat{r}\,'}\,V(r')\phi^+(\vec{r}\,') \\ &= e^{i\vec{k}\cdot\vec{r}} - \frac{\mu}{2\pi}\frac{e^{ikr}}{r}\int d^3\vec{r}\,'\,e^{-i\vec{k}\cdot\vec{r}\,'}\,V(r')\phi^+(\vec{r}\,')\,, \end{aligned} \tag{122}$$

where in the second line we have played with the conventions for the angles ($kr'\hat{r}\cdot\hat{r}\,' = \vec{k}\cdot\vec{r}\,'$). From this we find that

$$\begin{aligned} f(\Omega) &= -\frac{\mu}{2\pi}\int d^3\vec{r}\,'\,e^{-i\vec{k}\cdot\vec{r}\,'}\,V(r')\phi^+(\vec{r}\,') \\ &= -\frac{\mu}{2\pi}\,\langle\vec{k}|V|\phi^+\rangle\,, \end{aligned} \tag{123}$$

where $|\phi^+\rangle$ follows the equation

$$|\phi^+\rangle = |\vec{k}\rangle + G_0(E+i\epsilon)V|\phi^+\rangle \tag{124}$$

$$= |\vec{k}\rangle + G_0V|\vec{k}\rangle + G_0VG_0V|\vec{k}\rangle + G_0VG_0VG_0V|\vec{k}\rangle + \dots\,. \tag{125}$$

Now there is a trick here: instead of using the matrix element

$$\langle\vec{k}\,'|V|\phi^+\rangle\,, \tag{126}$$

which is a bit akward and requires the calculation of ϕ^+, we can define

$$\langle \vec{k}\,'|V|\phi^+\rangle = \langle \vec{k}\,'|T|\vec{k}\rangle\,, \tag{127}$$

where T is the T-matrix, a new operator that we have included to make things a bit more convenient. If we combine now the equation for ϕ^+

$$|\phi^+\rangle = |\vec{k}\rangle + G_0 V|\phi^+\rangle\,, \tag{128}$$

then we can arrive at the equation for T, also known as the Lippmann-Schwinger equation

$$T = V + VG_0T\,, \tag{129}$$

which looks really mysterious, but it's actually not such a difficult equation.

F. Solving the Lippmann-Schwinger Equation

1. The Contact-Range Potential

The usual way to handle the Lippmann-Schwinger equation is to first include the initial and final momenta explicitly, in which case we have

$$\langle \vec{k}\,'|T(E)|\vec{k}\rangle = \langle \vec{k}\,'|V|\vec{k}\rangle + \int \frac{d^3\vec{q}}{(2\pi)^3}\,\langle \vec{k}\,'|V|\vec{q}\rangle\,\frac{1}{E-\frac{q^2}{2\mu}}\,\langle \vec{q}|T|\vec{k}\rangle\,. \tag{130}$$

At first sight this equation loops unfathomable, there seems to be no way to solve it. Yet we will use an example to show how this is done. For that we will consider a contact interaction, which in coordinate space reads:

$$V(\vec{r}) = C_0\,\delta^{(3)}(\vec{r})\,. \tag{131}$$

If we Fourier-transform it into momentum space, we will find

$$\langle \vec{p}\,'|V|\vec{p}\rangle = \int d^3\vec{r}\,e^{-i\vec{p}\,'\cdot r}\,V(\vec{r})\,e^{i\vec{p}\cdot r} = C_0\,, \tag{132}$$

which is surprisingly simple. However, as usual, a pure delta-type interaction is an approximation to something we cannot really distinguish in detail. According to the EFT ideas we have been studying, the good way to do things is to include a cut-off. A very simple example is the following one

$$\langle \vec{p}\,'|V|\vec{p}\rangle = C_0\,\theta(\Lambda-|\vec{p}\,'|)\,\theta(\Lambda-|\vec{p}|)\,, \tag{133}$$

which is called a sharp cut-off regulator. Now if we include this potential in the Lippmann-Schwinger equation, we arrive to

$$\langle \vec{k}\,'|T(E)|\vec{k}\rangle = C_0\,\theta(\Lambda-|\vec{k}\,'|)\,\theta(\Lambda-|\vec{k}|) + C_0\,\theta(\Lambda-|\vec{k}\,'|)\int \frac{d^3\vec{q}}{(2\pi)^3}\,\theta(\Lambda-|\vec{q}|)\,\frac{1}{E-\frac{q^2}{2\mu}}\,\langle \vec{q}|T|\vec{k}\rangle\,. \tag{134}$$

This still has a mysterious halo around it, but it is easy to see that the general structure of the T-matrix should be

$$\langle \vec{k}\,'|T(E)|\vec{k}\rangle = \tau_0(E)\,\theta(\Lambda-|\vec{k}\,'|)\,\theta(\Lambda-|\vec{k}|)\,. \tag{135}$$

If we use the previous form and plug it into the Lippmann-Schwinger equation, we obtain

$$\tau_0(E) = C_0\left(1+\tau_0(E)\int \frac{d^3\vec{q}}{(2\pi)^3}\,\frac{\theta(\Lambda-|\vec{q}|)}{E-\frac{q^2}{2\mu}}\right)\,, \tag{136}$$

finally an expression at the grasp of human understanding! Now we define

$$I_0(E;\Lambda) = \int \frac{d^3\vec{q}}{(2\pi)^3}\,\frac{\theta(\Lambda-|\vec{q}|)}{E-\frac{q^2}{2\mu}}\,, \tag{137}$$

we end up with the really simple expression

$$\tau_0(E) = \frac{1}{\frac{1}{C_0}-I_0(E;\Lambda)}\,. \tag{138}$$

The only problem standing in our way is the integral $I_0(E;\Lambda)$, which for $E>0$ is a bit tricky because this guy:

$$\frac{1}{E-\frac{q^2}{2\mu}}\,. \tag{139}$$

The way to overcome this obstacle is exactly as in the previous section: include a small imaginary piece in the energy, in which case we have

$$I_0(E+i\epsilon;\Lambda)=\int\frac{d^3\vec{q}}{(2\pi)^3}\,\frac{\theta(\Lambda-|\vec{q}|)}{E+i\epsilon-\frac{q^2}{2\mu}}\,. \tag{140}$$

Now we define

$$E=\frac{k^2}{2\mu}\,, \tag{141}$$

and do the angular piece of the integral

$$I_0(E+i\epsilon;\Lambda)=\frac{\mu}{\pi^2}\int_0^\Lambda\frac{q^2\,dq}{k^2+2\mu i\epsilon-q^2}\,. \tag{142}$$

Now the problem is that unlike the integral for $G_0(\vec{r})$ that we computed in the previous section, this integral does not have a e^{iqr} exponential that vanishes for very large q (after including a little positive imaginary piece to it). The trick to do it is something called the principal value of an integral with a pole in it. Let's see this with an example:

$$I(a,b)=\int_a^b\frac{dx}{x}\,, \tag{143}$$

where $a<0$. Though this integral has a pole, if we define a way to deal with this pole the integral will have a defined value. This way is the following

$$I(a,b)=\int_a^{-\epsilon}\frac{dx}{x}+\int_{-\epsilon}^{+\epsilon}\frac{dx}{x}+\int_\epsilon^b\frac{dx}{x}\,, \tag{144}$$

i.e. to separate the integral in pieces. The only piece that is problematic is

$$\int_{-\epsilon}^{+\epsilon}\frac{dx}{x}\,. \tag{145}$$

The principal value prescription means that we are really performing the integral in a symmetric way, which means that the infinities cancel out exactly

$$\mathcal{P}\int_{-\epsilon}^{+\epsilon}\frac{dx}{x}=0\,, \tag{146}$$

and we reach the result

$$I(a,b)=\mathcal{P}\int_a^b\frac{dx}{x}=\log\left|\frac{b}{a}\right|. \tag{147}$$

But things can get more complicated because of the $i\epsilon$ trick. What we usually have is

$$\frac{1}{x\pm i\epsilon}\,, \tag{148}$$

and we have to handle the $i\epsilon$ somehow, but we have another ace under the sleeve:

$$\lim_{\epsilon\to0}\left(\frac{1}{x-i\epsilon}-\frac{1}{x+i\epsilon}\right)=\lim_{\epsilon\to0}\left(\frac{2i\epsilon}{x^2+\epsilon^2}\right)=2\pi i\delta(x)\,. \tag{149}$$

Now we now that

$$\mathrm{Re}\left[\frac{1}{x-i\epsilon}\right]=+\mathrm{Re}\left[\frac{1}{x+i\epsilon}\right]\quad\text{and}\quad\mathrm{Im}\left[\frac{1}{x-i\epsilon}\right]=-\mathrm{Im}\left[\frac{1}{x+i\epsilon}\right], \tag{150}$$

and we arrive to

$$\frac{1}{x \pm i\epsilon} = \mathcal{P}\left(\frac{1}{x}\right) \mp i\pi\delta(x)\,. \tag{151}$$

If we now apply this idea to the loop integral

$$I_0(E;\Lambda) = \frac{\mu}{\pi^2}\,\mathcal{P}\left[\int_0^\Lambda \frac{q^2 dq}{k^2 - q^2}\right] - \frac{\mu}{2\pi}\,ik\,, \tag{152}$$

with

$$\mathcal{P}\left[\int_0^\Lambda \frac{q^2 dq}{k^2 - q^2}\right] = -\Lambda + \frac{k}{2}\log\frac{\Lambda + k}{\Lambda - k}\,. \tag{153}$$

After this long derivation it is time to put all the pieces together. We were solving the Lippmann-Schwinger equation

$$T(E) = V + V\,G_0(E)\,T(E)\,, \tag{154}$$

with a regularized contact-range potential of the type

$$\langle \vec{k}\,'|V|\vec{k}\rangle = C_0\,\theta(\Lambda - |\vec{k}\,'|)\,\theta(\Lambda - |\vec{k}|)\,, \tag{155}$$

which admits a solution of the type

$$\langle \vec{k}\,'|T(E)|\vec{k}\rangle = \tau_0(E)\,\theta(\Lambda - |\vec{k}\,'|)\,\theta(\Lambda - |\vec{k}|)\,, \tag{156}$$

where $\tau_0(E)$ follows the equation

$$\tau_0(E) = \frac{1}{\frac{1}{C_0} - I_0(E;\Lambda)}\,. \tag{157}$$

For scattering we take $E \to E + i\epsilon$, in which case we end up with

$$I_0(E;\Lambda) = \frac{\mu}{\pi^2}\,\left[-\Lambda + \frac{k}{2}\log\frac{\Lambda + k}{\Lambda - k}\right] - \frac{\mu}{2\pi}\,ik\,. \tag{158}$$

Now, to put theory into practice, I propose the following **exercise** (two points). If you have not already noticed, the solution of the Lippmann-Schwinger equation for the contact-range interaction is spherically symmetric. This means in particular that we only have s-wave. As a consequence the T-matrix can be expressed as ($|\vec{k}\,'| = |\vec{k}| = k$)

$$\frac{\theta(\Lambda - |\vec{k}\,'|)\theta(\Lambda - |\vec{k}|)}{\frac{1}{C_0} - I_0(E + i\epsilon;\Lambda)} = \langle \vec{k}\,'|T(E = \frac{k^2}{2\mu} + i\epsilon)|\vec{k}\rangle = -\frac{2\pi}{\mu}\,f(\hat{k}\,'\cdot\hat{k}) = -\frac{2\pi}{\mu}\,\frac{1}{k\,\cot\delta_0(k) - ik}\,. \tag{159}$$

where in the last equality we have used the idea that there is only s-wave scattering. If we use $\delta_0(k) \to -\alpha_0\,k$, we find that

$$\frac{1}{\frac{1}{C_0} - I_0(0;\Lambda)} = \frac{2\pi}{\mu}\,\alpha_0\,. \tag{160}$$

Now we are going to *renormalize* the previous equation, i.e. we will assume that C_0 depends on Λ

$$C_0 = C_0(\Lambda)\,, \tag{161}$$

and assume that we already know the value of α_0. What you need to do is the following:

(i) Find the equation relating C_0, α_0 and Λ.

(ii) With the value of $k\,\cot\delta_0$ that corresponds to the $C_0(\Lambda)$ found in the previous step.

(iii) Calculate the value of $k\,\cot\delta_0$ when $\Lambda \to \infty$.

2. The Partial Wave Expansion for the T-matrix

Just like the scattering amplitude f, the T-matrix accepts a partial wave expansion

$$\langle \vec{k}\,'|T(E)|\vec{k}\rangle = \sum_{l=0}^{\infty}(2l+1)\,\langle k\,'|T_l(E)|k\rangle\,P_l(\hat{k}\,'\cdot\hat{k})\,. \tag{162}$$

where

$$\begin{aligned}\langle k\,'|T_l(E)|k\rangle &= \int\frac{d^2\hat{k}'}{4\pi}\int\frac{d^2\hat{k}}{4\pi}\langle\vec{k}\,'|T(E)|\vec{k}\rangle\,P_l(\hat{k}\,'\cdot\hat{k})\\ &= \int_{-1}^{+1}\frac{du}{2}\langle\vec{k}\,'|T(E)|\vec{k}\rangle\,P_l(u)\,,\end{aligned} \tag{163}$$

with $u=\cos\theta$, with θ the angle between $\hat{k}\,'$ and $\hat{k}$. i.e. $\hat{k}\,'\cdot\hat{k}=u$. We can use the same expression for defining the potential V_l

$$\langle k\,'|V|k\rangle = \int\frac{d^2\hat{k}'}{4\pi}\int\frac{d^2\hat{k}}{4\pi}\langle\vec{k}\,'|V|\vec{k}\rangle\,P_l(\hat{k}\,'\cdot\hat{k})\,. \tag{164}$$

This expression can be manipulated in different ways. For instance, if we consider the following formula

$$P_l(\hat{k}\,'\cdot\hat{k}) = \frac{4\pi}{2l+1}\sum_{m=-l}^{+l}Y^*_{lm}(\hat{k}\,')\,Y_{lm}(\hat{k})\,, \tag{165}$$

we arrive to

$$\begin{aligned}\langle k\,'|V|k\rangle &= \frac{1}{2l+1}\sum_m\int\frac{d^2\hat{k}'}{4\pi}\int\frac{d^2\hat{k}}{4\pi}Y^*_{lm}(\hat{k}\,')\,\langle\vec{k}\,'|V|\vec{k}\rangle\,Y_{lm}(\hat{k})\\ &= \int\frac{d^2\hat{k}'}{4\pi}\int\frac{d^2\hat{k}}{4\pi}Y^*_{lm}(\hat{k}\,')\,\langle\vec{k}\,'|V|\vec{k}\rangle\,Y_{lm}(\hat{k})\,,\end{aligned} \tag{166}$$

where in the second formula we have used that the matrix element of the potential do not depend on m (because it is a central potential). We can also use the expression for the Fourier-transform of the potential

$$\langle\vec{k}\,'|V|\vec{k}\rangle = \int d^3\vec{r}\,e^{-i\vec{k}\,'\cdot r}\,V(r)\,e^{i\vec{k}\cdot r}\,, \tag{167}$$

to arrive to this other expression (**exercise**)

$$\langle k\,'|V|k\rangle = 4\pi\int_0^\infty r^2dr\,j_l(k'r)V(r)j_l(kr)\,. \tag{168}$$

Be it as it may, the final Lippmann-Schwinger equation at which we arrive is

$$\langle k\,'|T_l(E)|k\rangle = \langle k\,'|V_l|k\rangle + \frac{\mu}{\pi^2}\int_0^\infty\frac{q^2dq}{k^2-q^2}\,\langle k\,'|V_l|q\rangle\,\langle q|T_l(E)|k\rangle\,, \tag{169}$$

where for $E=\frac{k^2}{2\mu}+i\epsilon$ we have

$$\langle k|T_l(E)|k\rangle = -\frac{2\pi}{\mu}\,\frac{1}{k\,\cot\delta_l(k)-ik}\,. \tag{170}$$

3. Numerical Solutions of the T-matrix for a General Potential

Here we will explain how to solve numerically the T-matrix for a general potential V. Usually it is much easier to use a matrix called the R-matrix or reactance matrix, that is defined as

$$\frac{1}{R} = \mathrm{Re}\left[\frac{1}{T}\right]\,. \tag{171}$$

If we project into the partical wave l, we have for the R-matrix that

$$\langle k|R|k\rangle = -\frac{2\pi}{\mu}\,\frac{1}{k\,\cot\delta_l}\,, \tag{172}$$

which will be useful for calculating the phase shifts. The R-matrix follows the real version of the Lippmann-Schwinger equation

$$\langle k\,'|R_l(E)|k\rangle = \langle k\,'|V_l|k\rangle + \frac{\mu}{\pi^2}\,\mathcal{P}\int_0^\infty \frac{q^2dq}{k^2-q^2}\,\langle k\,'|V_l|q\rangle\,\langle q|R_l(E)|k\rangle\,. \tag{173}$$

The problem is how to compute numerically the principal value of the integral. The normal way to perform a numerical integral of the type above is to use Gauss points, a series of points and weights that are able to give a very good approximation of an integral without having to use too many points:

$$I = \int dx f(x) \simeq \sum_{i=1}^{N} f(x_i)w_i\,, \tag{174}$$

where x_i with $i = 1,\ldots,N$ are the Gauss points and w_i the weights. However if we have an integral of the type

$$\mathcal{P}\int dq\frac{f(q)}{k-q}\,, \tag{175}$$

it happens that it is not practical to find a set of Gauss points that are symmetrically distribited around the pole $q = k$. Then we have to perform the integral in a different way. A possible method is to modify the integral as follows

$$\mathcal{P}\int dq\frac{f(q)}{k-q} = \int dq\frac{f(q)-f(k)}{k-q}\,, \tag{176}$$

where the evaluation of the right hand side of the equation does not contain any pole and gives the principal value. Other possibility for the case at hand is

$$\mathcal{P}\int dq\frac{f(q)}{k^2-q^2} = \int dq\frac{f(q)-f(k)}{k^2-q^2}\,, \tag{177}$$

which also removes the pole and it is far easier to implement for the Lippmann-Schwinger equation, which we now modify to

$$\langle k\,'|R_l(E)|k\rangle = \langle k\,'|V_l|k\rangle + \frac{\mu}{\pi^2}\int_0^\infty \frac{dq}{k^2-q^2}\,\left[q^2\langle k\,'|V_l|q\rangle\,\langle q|R_l(E)|k\rangle - k^2\langle k\,'|V_l|k\rangle\,\langle k|R_l(E)|k\rangle\right]\,. \tag{178}$$

Actually there is a little trick behind this expression, which we leave as an **exercise**. A careful inspection would indicate that the modifications we have done might be imcomplete once we consider the integration range. For instance

$$\begin{aligned}\mathcal{P}\int_0^\infty dq\frac{f(q)}{k-q} &= \int_0^{2k} dq\frac{f(q)-f(k)}{k-q} + \int_{2k}^\infty dq\frac{f(q)}{k-q}\\ &= \int_0^\infty dq\frac{f(q)-f(k)}{k-q} - f(k)\int_{2k}^\infty \frac{dq}{k-q}\,,\end{aligned} \tag{179}$$

However it happens that

$$\mathcal{P}\int_0^\infty dq\frac{f(q)}{k^2-q^2} = \int_0^\infty dq\frac{f(q)-f(k)}{k^2-q^2}\,. \tag{180}$$

Prove it.

Now we continue with how to numerically solve the Lippmann-Schwinger equation. The next step is to introduce a series of Gauss points for the integration range $q = 0$ to $q = \infty$:

$$q = \{q_1,\ldots,q_N\} \quad \text{with weights} \quad w = \{w_1,\ldots,w_N\}\,. \tag{181}$$

Yet there is a problem here, because the point $k = \sqrt{2\mu E}$ is very likely not to be included within the Gauss points. The solution is to include it by hand as an additional Gauss point with zero weight:

$$q = \{q_1,\ldots,q_N,k\} \quad \text{with weights} \quad w = \{w_1,\ldots,w_N,0\}\,. \tag{182}$$

We also use the extended Gauss points for the momenta at which we evaluate the R-matrix and the potential:

$$\langle k'|R_l(E)|k\rangle \to R_{ij} = \langle q_i|R_l(E)|q_j\rangle\,, \tag{183}$$

$$\langle k'|V_l|k\rangle \to V_{ij} = \langle q_i|V_l|q_j\rangle\,, \tag{184}$$

where R_{ij} and V_{ij} are (N+1) x (N+1) matrices. It is straigthforward to see that

$$R_{ij} = V_{ij} + \sum_{k=1}^{N+1} V_{ik} G_k R_{kj}\,, \tag{185}$$

with

$$G_k = \frac{\mu}{\pi^2}\,\frac{q_k^2\,w_k}{q_k^2 - q_{N+1}^2}\,, \tag{186}$$

$$G_{N+1} = -\frac{\mu}{\pi^2}\sum_{k=1}^{N}\frac{q_N^2\,w_k}{q_k^2 - q_{N+1}^2}\,. \tag{187}$$

Now we rearrange the equation as

$$\sum_{k=1}^{N+1}\left(\delta_{ik} - V_{ik}G_k\right)\,R_{kj} = V_{ij}\,, \tag{188}$$

which is simply a matrix equation

$$B\,R = V \quad \text{with} \quad B_{ij} = \delta_{ij} - V_{ij}G_j\,, \tag{189}$$

with B, R and V (N+1) x (N+1) matrices. Then we can solve this by inverting the matrix B and computing R:

$$R = B^{-1}V\,. \tag{190}$$

Finally we find that

$$R_{N+1,N+1} = -\frac{2\pi}{\mu}\,\frac{1}{k\,\cot\delta_l(k)}\,, \tag{191}$$

which is what we were looking for.

G. Bound States and the Lippmann-Schwinger Equation

The Lippmann-Schwinger equation can also be applied to bound states. The relation between the T-matrix and the bound states is as follows. We can rewrite the Lippmann-Schwinger equation as follows

$$\begin{aligned} T &= V + VG_0T = V + VG_0V + VG_0VG_0V + \ldots \\ &= V + VGV\,, \end{aligned} \tag{192}$$

where G is defined as

$$G(E) = G_0(E) + G_0(E)VG(E) = \frac{1}{E - H}\,. \tag{193}$$

Now it is time to remember an interesting result in quantum mechanics, or rather a mathematical result about operators in a Hilbert space. If we have a hermitian matrix A that can be inverted, we can write this matrix as follows

$$A = \sum_i |\lambda_i\rangle\lambda_i\langle\lambda_i|\,, \tag{194}$$

where λ_i and $|\lambda_i\rangle$ are the eigenvalues and eigenvectors of A respectively, and where the eigenvectors are normalized to one: $\langle\lambda_i|\lambda_i\rangle = 1$. This is sometimes calles the *spectral decomposition* of A. We also have that we can write the identity as

$$1 = \sum_i |\lambda_i\rangle\langle\lambda_i|\,. \tag{195}$$

This idea extends to infinite dimensional matrices, i.e. operators in a Hilbert space. For instance, if we consider H_0 we can write

$$H_0 = \int \frac{d^3\vec{p}}{(2\pi)^3} |\vec{p}\rangle \, \frac{p^2}{2\mu} \langle \vec{p}| \,, \tag{196}$$

and the identity as

$$1 = \int \frac{d^3\vec{p}}{(2\pi)^3} |\vec{p}\rangle \langle \vec{p}| \,. \tag{197}$$

Now we can extend this idea to $H = H_0 + V$. The presence of the potential means that there can be bound states and that the positive energy eigenvalues are no longer $|\vec{p}\rangle$ but $|\phi^+\rangle$:

$$H = \sum_n |B_n\rangle \, E_n \langle B_n| + \int \frac{d^3\vec{p}}{(2\pi)^3} |\phi^+\rangle \, \frac{p^2}{2\mu} \langle \phi^+| \,, \tag{198}$$

where $|B_n\rangle$ are the bound states. In addition the indentity reads

$$1 = \sum_n |B_n\rangle \langle B_n| + \int \frac{d^3\vec{p}}{(2\pi)^3} |\phi^+\rangle \langle \phi^+| \,. \tag{199}$$

If we apply this identity to the resolvent operator G, we find

$$G = \sum_n |B_n\rangle \frac{1}{E - E_n} \langle B_n| + \int \frac{d^3\vec{p}}{(2\pi)^3} |\phi^+\rangle \frac{1}{E - \frac{p^2}{2\mu}} \langle \phi^+| \,. \tag{200}$$

Now if we go back to the one of the equations for the T matrix

$$T(E) = V + VG(E)V \,, \tag{201}$$

and this is very interesting because it tell us that the T-matrix has poles at the energies corresponding to bound states

$$T(E \to E_n) \to V|B_n\rangle \frac{1}{E - E_n} \langle B_n|V \,. \tag{202}$$

In addition, if we take the limit of $E \to E_n$ in the Lippmann-Schwinger equation

$$\lim_{E \to E_n} T(E) = \lim_{E \to E_n} \left[V + VG_0T(E) \right] \,, \tag{203}$$

we arrive to the Lippmann-Schwinger equation for the bound states

$$|B\rangle = G_0(E)V|B\rangle \,. \tag{204}$$

The way to solve this equation is actually analogous to what we have already seen for the standard Lippmann-Schwinger equation, i.e. we write the matrix elements explicitly and check the resulting equation

$$\langle \vec{p}|B\rangle = -\frac{2\mu}{p^2 + \gamma^2} \int \frac{d^3\vec{q}}{(2\pi)^2} \langle \vec{p}|V|\vec{q}\rangle \langle \vec{q}|B\rangle \,. \tag{205}$$

However as we can see, the G_0 factorizes. For this reason sometines one can also write a different equation

$$|B\rangle = G_0(E) \, |\phi\rangle \,, \tag{206}$$

where $|\phi\rangle$ is called the vertex function, for which the equation reads

$$|\phi\rangle = VG_0(E)|\phi\rangle \,. \tag{207}$$

After taking explicit matrix elements

$$\langle \vec{p}|\phi\rangle = -\int \frac{d^3\vec{q}}{(2\pi)^2} \frac{2\mu}{q^2 + \gamma^2} \, \langle \vec{p}|V|\vec{q}\rangle \, \langle \vec{q}|\phi\rangle \,. \tag{208}$$

With this we can try an **exercise** (two points): solve the bound state equation for the vertex function $|\phi\rangle$ with a contact-range potential

$$\langle \vec{p}\,'|V|\vec{p}\rangle = C_0 \, \theta(\Lambda - |\vec{p}\,'|) \, \theta(\Lambda - |\vec{p}|) \,, \tag{209}$$

with $C_0 < 0$. Find the relation between Λ, C_0 and γ. Also find the form of the wave function in the $\Lambda \to \infty$ (keeping γ fixed) in coordinate space by Fourier-transforming the momentum space wave function

$$\Psi_B(\vec{r}) = \int \frac{d^3\vec{p}}{(2\pi)^3} \, \frac{2\mu\phi(\vec{p})}{p^2 + \gamma^2} \,. \tag{210}$$

II. FINE TUNING AND NUCLEAR FORCES

Now we consider the actual case of nucleon-nucleon scattering. The most important difference between nucleon-nucleon scattering and the type of scattering we have been studying so far is that nucleons have spin. This will generate all type of complications once we consider one pion exchange, but for the moment we will concentrate on low momenta $k < m_\pi$ for which pions are not important. For these low momenta we determined that S-wave scattering can be well-described with the effective range expansion

$$k\cot\delta(k) = -\frac{1}{\alpha_0} + \frac{1}{2}\, r_0\, k^2 + \dots \,. \tag{211}$$

Now we are considering nucleons that have spin: for the two-nucleon system this entails that the total spin can be $S = 0$ (singlet) or $S = 1$ (triplet). At this moment the only complication that spin brings into the problem of scattering is that: there are different scattering quantities for the singlet and triplet cases. If we consider the scattering length, there is a singlet and triplet one

$$\alpha_s = -23.7\,\text{fm} \quad , \quad \alpha_t = 5.4\,\text{fm}\,. \tag{212}$$

Are these quantities big or small? For answering this question we have to compare with a length scale that can be consider natural in the two-nucleon system. The most obvious answer has to do with pion exchanges, which behave as

$$V_\pi \sim \frac{e^{-m_\pi r}}{r}\,, \tag{213}$$

that is, the pion exchange potential has a range $R_\pi \sim 1/m_\pi \simeq 1.4\,\text{fm}$. From this we can express the scattering lengths in units of R_π

$$\frac{\alpha_s}{R_\pi} = -16.9 \quad , \quad \frac{\alpha_t}{R_\pi} = 3.9\,. \tag{214}$$

Conclusion: they are big, in particular the singlet scattering length.

To further illustrate the necessary sense of amazement that we should be feeling at this point, let us consider this. Assume we are theoretical physicist and we know about quantum chromodynamics, chiral symmetry breaking, the nucleon and the pion. But for whatever reasons there have never been nucleon-nucleon scattering experiments, particularly at low energy. A group of experimental physicist plan to measure for the first time the nucleon-nucleon cross section at very low momenta, $k \ll m_\pi$. Before doing the experiment they ask us theoreticians what is our expectation for the cross section to be for the singlet and triplet. Probably we will simply give the following answer

$$\sigma_{s,\text{th}} \sim 4\pi R_\pi^2 \quad , \quad \sigma_{t,\text{th}} \sim 4\pi R_\pi^2\,, \tag{215}$$

which should be about right, modulo numerical factors that we expect neither to be too big or too small. Then the experiments do their thing and come us with the following numbers

$$\frac{\sigma_{s,\text{exp}}}{\sigma_{s,\text{th}}} \sim 290 \quad , \quad \frac{\sigma_{t,\text{exp}}}{\sigma_{t,\text{th}}} \sim 15\,. \tag{216}$$

In the case of the singlet the natural theoretical expectations were off by a factor of a few hundreds! (Luckily for the triplet they were off only by a factor of fifteen.) This is why nuclear physics is interesting.

Fine tuning is the name we give to this phenomenon. The reason for that name is that unnaturally large scattering lengths require careful cancellations happening at shorter distances. This is more easy to understand if we consider binding energies instead of scattering lengths, because they are the sum of a kinetic and potential energy term

$$B = -\langle\Psi|H|\Psi\rangle = \langle\Psi|V|\Psi\rangle - \langle\Psi|T|\Psi\rangle\,. \tag{217}$$

From that it can be appreciated that $B \to 0$ requires a precise cancellation of the potential and kinetic terms.

A different aspect of fine tuning can be appreaciated if we consider what would be the change in the scattering length if we do a small change in the potential. For this we consider two scattering problems at zero energy, the first with a potential V_0 and scattering length α_0 and the second with potential V_1 and scattering length α_1. Their Schrödinger equations read

$$-u_0'' + 2\mu V_0(r) u_0 = 0 \qquad \text{with } u_0 \to 1 - \frac{r}{\alpha_0} \quad \text{for } r \to \infty\,, \tag{218}$$

$$-u_1'' + 2\mu V_1(r) u_1 = 0 \qquad \text{with } u_1 \to 1 - \frac{r}{\alpha_1} \quad \text{for } r \to \infty\,. \tag{219}$$

If we build a Wronskian identity for the wave functions u_0 and u_1

$$-(u_0' u_1 - u_0 u_1')\Big|_{r_c}^{R} = \int_{r_c}^{R} dr\,[V_0(r) - V_1(r)]\,u_0(r)u_1(r)\,, \tag{220}$$

which after taking the limits $R \to \infty$ and $r_c \to 0$ reduces to

$$-\left(\frac{1}{\alpha_0} - \frac{1}{\alpha_1}\right) = \int_0^{\infty} dr\,[V_0(r) - V_1(r)]\,u_0(r)u_1(r)\,. \tag{221}$$

Now if we consider that $V_1 = V_0 + \Delta V$, i.e. V_1 is merely a small variation of V_0, we find that

$$\Delta\alpha_0 = \alpha_0^2 \int_0^{\infty} dr \Delta V(r) u_0^2(r) + \mathcal{O}((\Delta V)^2)\,, \tag{222}$$

and we see that for $\alpha_0 \to \infty$, even a small change in the potential will trigger a big change in the scattering length. That is, the scattering length is fined-tuned.

III. ONE PION EXCHANGE

Here we will consider the problem of scattering for particles with spin when there is a tensor force. It happens that one of the defining features of the one pion exchange (OPE) potential is the existence of a strong tensor force. This tensor force generates a few complications regarding scattering.

We begin with the OPE potential

$$V_{\rm OPE}(r) = \frac{g_A^2}{4f_\pi^2}\,\vec{\tau}_1\cdot\vec{\tau}_2\,\left[\vec{\sigma}_1\cdot\vec{\sigma}_2\,W_C(r) + S_{12}(\hat{r})\,W_T(r)\right]\,. \tag{223}$$

where $g_A = 1.26$ is the axial coupling and $f_\pi = 92.4\,\text{MeV}$ the pion weak decay constant. The functions W_C and W_T are defined as

$$W_c(r) = \frac{m^3}{12\pi}\,\frac{e^{-mr}}{mr}\,, \tag{224}$$

$$W_T(r) = \frac{m^3}{12\pi}\,\frac{e^{-mr}}{mr}\,\left(1 + \frac{3}{mr} + \frac{3}{m^2r^2}\right)\,. \tag{225}$$

The OPE potential contains a piece proportional to $\vec{\sigma}_1\cdot\vec{\sigma}_2$ and another piece proportional to $S_{12}(\hat{r})$. The first piece is really easy, it gives us

$$\vec{\sigma}_1\cdot\vec{\sigma}_2 = -3 \quad \text{for the singlet, and} \quad \vec{\sigma}_1\cdot\vec{\sigma}_2 = +1 \quad \text{for the triplet.} \tag{226}$$

The second piece, the tensor piece, will be the problematic one. The reason is that it mixes partial waves of different angular momentum. This is because the properties of the tensor force, which can be summarized as

$$[\vec{S}^2, S_{12}] = 0 \quad , \quad [\vec{L}^2, S_{12}] \neq 0 \quad , \quad [\vec{J}^2, S_{12}] = 0\,, \tag{227}$$

that is, the tensor force conserves spin and total angular momentum but not orbital angular momentum. Yet the fact that it converses spin is already a relief because (**exercise**)

$$S_{12} = 0 \quad \text{for the singlet } (S = 0). \tag{228}$$

As a consequence singlet scattering is exactly as spinless scattering: we can directly use all the things we have learned in the previous sections.

The first step to trying to understand the effect of the tensor force is to obtain its matrix elements between partial waves. If we use the spectroscopic notation, ${}^{2S+1}L_J$, then we can consider the basis

$${}^3C_J = \{{}^3(J-1)_J, {}^3J_J, {}^3(J+1)_J\}\,. \tag{229}$$

In this basis the matrix elements of the tensor force are given by

$$S_J = \begin{pmatrix} -\frac{2(J-1)}{2J+1} & 0 & \frac{6\sqrt{J(J+1)}}{2J+1} \\ 0 & 2 & 0 \\ \frac{6\sqrt{J(J+1)}}{2J+1} & 0 & -\frac{2(J+2)}{2J+1} \end{pmatrix}\,. \tag{230}$$

From this we see that the triplet partial waves with $L = J \pm 1$ mix together.

A. The Schrödinger Equation for the Deuteron

Before considering the modifications that the tensor force entails to scattering, we can consider the much more simple problem of what happens with the description of the deuteron. The wave function of the deuteron contains a S- and D-wave component, like this

$$|\Psi_d\rangle = \frac{u(r)}{r}|^3S_1\rangle + \frac{w(r)}{r}|^3D_1\rangle\,, \tag{231}$$

where the ket s$|^3S_1\rangle$ and $|^3D_1\rangle$ indicates the S- and D-wave pieces. These S- and D-wave piece are a mixture of spin and angular components, and can be written as follows

$$|^3S_1(1m_d)\rangle = Y_{00}(\hat{r})\,|1m_d\rangle\,, \tag{232}$$

$$|^3D_1(1m_d)\rangle = \sum_{m_l,m_s} Y_{2m_l}(\hat{r})\,|1m_s\rangle\langle 2m_l 1m_s|1m_d\rangle\,, \tag{233}$$

where $(1m_d)$ indicates that the deuteron has $J=1$ and $M=m_d$, $|1m_s\rangle$ refers to the spin wave function of the deuteron and $Y_{lm_l}(\hat{r})$ is a spherical harmonic to represent the angular dependence of the wave function. If we take into account that the OPE potential for the deuteron can be written as

$$V_d = V_C + S_{12}(\hat{r})\,V_T\,, \tag{234}$$

where we are ignoring the spin-spin structure (because it will not affect the outcome), and also take into account that the matrix elements of the tensor force between the $|^3S_1\rangle$ and $|^3D_1\rangle$ partial waves is given by

$$S = \begin{pmatrix} 0 & 2\sqrt{2} \\ 2\sqrt{2} & -2 \end{pmatrix}\,, \tag{235}$$

we end up with the following reduced Schrödinger equation

$$-u''(r) + 2\mu\,\Big[V_C\,u(r) + 2\sqrt{2}V_T\,w(r)\Big] = -\gamma^2\,u(r)\,, \tag{236}$$

$$-w''(r) + 2\mu\,\Big[2\sqrt{2}V_T\,u(r) + (V_C - 2V_T)\,w(r)\Big] = -\gamma^2\,w(r)\,, \tag{237}$$

which is a coupled-channel equation. The solution of this equation behaves for $r\to\infty$ as

$$u(r) = A_S\,e^{-\gamma r}\,, \tag{238}$$

$$w(r) = A_D\,e^{-\gamma r}\,\left(1 + \frac{3}{\gamma r} + \frac{3}{\gamma^2 r^2}\right)\,. \tag{239}$$

B. Scattering with Tensor Forces

In this section we want to find out how to define the phase shifts for the case of triplet coupled channels. For that we have to study the scattering of particles with spin. The starting point for the scattering of spinless particles is the wave function

$$\Psi_k(\vec{r}) \to e^{i\vec{k}\cdot\vec{r}} + f(\hat{k}\cdot\hat{r})\,\frac{e^{ikr}}{r}\,. \tag{240}$$

The difference if we include spin is the following one

$$\Psi_k(\vec{r}) \to e^{i\vec{k}\cdot\vec{r}}|00\rangle + f(\hat{k}\cdot\hat{r})\,\frac{e^{ikr}}{r}|00\rangle \quad \text{for the singlets}\,, \tag{241}$$

$$\Psi_k(\vec{r}) \to e^{i\vec{k}\cdot\vec{r}}|1m_s\rangle + f_{m_s,m_s'}(\hat{k}\cdot\hat{r})\,\frac{e^{ikr}}{r}|1m_s'\rangle \quad \text{for the triplets}\,, \tag{242}$$

that is, if we include the spin we have to take into account that the scattering amplitude f can flip the spins. For the case of the singlet there is no difference with the spinless case, as previously mentioned, and we will not consider it further. For the triplet case, however, the addition of the spin wave function will create a series of complications that we will consider below.

First, the expansion of the plane wave when we include spin in principle is the same

$$e^{i\vec{k}\cdot\vec{r}}\,|1m_s\rangle = 4\pi \sum_{lm_l} i^l\, j_l(kr)\, Y^*_{lm_l}(\hat{k})\, Y_{lm_l}(\hat{r})\,|1m_s\rangle\,, \tag{243}$$

but this expansion is in orbital angular momentum, which is not a good quantum number if we have a tensor force. We have to change it to an expansion in j. For this we define the *generalized* spherical harmonics as

$$\mathcal{Y}^l_{jm}(\hat{r}) = \sum_{m_l m_s} \langle lm_l 1m_s|jm\rangle\, Y_{lm}(\hat{r})\,|1m_s\rangle\,, \tag{244}$$

but here we rather need the standard spherical harmonics as a sum of generalized spherical harmonics

$$Y_{lm_l}(\hat{r})\,|1m_s\rangle = \sum_{j=l-1}^{l+1} \mathcal{Y}^l_{jm}(\hat{r})\,\langle jm|lm_l 1m_s\rangle\,. \tag{245}$$

We can include this relation in the plane wave expansion

$$e^{i\vec{k}\cdot\vec{r}}\,|1m_s\rangle = 4\pi \sum_{lm_l} i^l\, j_l(kr)\, Y^*_{lm_l}(\hat{k}) \sum_{j=l-1}^{l+1} \mathcal{Y}^l_{jm}(\hat{r})\,\langle jm|lm_l 1m_s\rangle\,, \tag{246}$$

and rearrange the previous expansion as

$$e^{i\vec{k}\cdot\vec{r}}\,|1m_s\rangle = 4\pi \sum_{jm} \sum_{l=j-1}^{l=j+1} i^l\, j_l(kr)\,\langle jm|lm_l 1m_s\rangle\, Y^*_{lm_l}(\hat{k})\, \mathcal{Y}^l_{jm}(\hat{r}) \tag{247}$$

where $m_l = m - m_s$. We can further simplify the previous expression by defining

$$\mathcal{Z}^{lsm_s}_{jm}(\hat{k}) = \langle lm_l 1m_s|jm\rangle\, Y_{lm_l}(\hat{k})\,. \tag{248}$$

where the sum on the m_l coefficient is cosmetical. So we get

$$\begin{aligned} e^{i\vec{k}\cdot\vec{r}}\,|1m_s\rangle &= 4\pi \sum_{jm} \sum_{l=j-1}^{l=j+1} i^l\, j_l(kr)\, \mathcal{Z}^{lsm_s\,*}_{jm}(\hat{k})\, \mathcal{Y}^l_{jm}(\hat{r}) && (249)\\ &= 4\pi \sum_{jm} \left\{ i^{j-1}\, j_{j-1}(kr)\, \mathcal{Z}^{j-1sm_s\,*}_{jm}(\hat{k})\, \mathcal{Y}^{j-1}_{jm}(\hat{r}) + i^{j+1}\, j_{j+1}(kr)\, \mathcal{Z}^{j+1sm_s\,*}_{jm}(\hat{k})\, \mathcal{Y}^{j+1}_{jm}(\hat{r}) \right\} + \\ &\quad 4\pi \sum_{jm} i^j\, j_j(kr)\, \mathcal{Z}^{jsm_s\,*}_{jm}(\hat{k})\, \mathcal{Y}^{j}_{jm}(\hat{r}) && (250) \end{aligned}$$

So the first piece of the sum is over coupled triplet states, while the second part is over uncoupled triplet states.

Now we can do exactly the same expansion for the full scattering wave function

$$\begin{aligned} \Psi_k(\vec{r}) &= 4\pi \sum_{jm} \sum_{l=j-1}^{l=j+1} i^l\, \frac{u_{jl}(r;k)}{r}\, \mathcal{Z}^{lsm_s\,*}_{jm}(\hat{k})\, \mathcal{Y}^l_{jm}(\hat{r}) && (251)\\ &= 4\pi \sum_{jm} \left\{ i^{j-1}\, \frac{u_j(r;k)}{r}\, \mathcal{Z}^{j-1sm_s\,*}_{jm}(\hat{k})\, \mathcal{Y}^{j-1}_{jm}(\hat{r}) + i^{j+1}\, \frac{w_j(r;k)}{r}\, \mathcal{Z}^{j+1sm_s\,*}_{jm}(\hat{k})\, \mathcal{Y}^{j+1}_{jm}(\hat{r}) \right\} + \\ &\quad 4\pi \sum_{jm} i^j\, \frac{v_j(r;k)}{r}\, \mathcal{Z}^{jsm_s\,*}_{jm}(\hat{k})\, \mathcal{Y}^{j}_{jm}(\hat{r}) && (252) \end{aligned}$$

where in the second and third lines we are using the notation u_j, w_j for the $l = j \pm 1$ wave functions of the coupled channels and v_j for the $l = j$ wave function. Now in analogy with the spinless case, we should include the asymptotic solution of the wave functions and then we would be done. But here there is a complication though: there are multiple definitions for the phase shifts. For this reason we are going back to the single channel, where we originally used

$$\frac{u_l(k;r)}{r} \to e^{i\delta_l}\,\left[\cos\delta_l(k)\, j_l(kr) - \sin\delta_l(k) y_l(kr)\right]\,. \tag{253}$$

There we could have used instead this other definition

$$\frac{u_l(k;r)}{r} \to h_l^-(kr) - h_l^+(kr) S_l(k)\,, \tag{254}$$

where the $h_l^\pm$ are defined as

$$h_l^\pm(x) = x^n \left(-\frac{1}{x}\frac{d}{dx}\right)^n \frac{e^{\pm ix}}{x}\,. \tag{255}$$

With this definition it happens that $S_l(k) = e^{2i\delta_l(k)}$, but it also happens that S_l is unitary

$$S_l^\dagger S_l = 1\,. \tag{256}$$

The extension for the coupled channel case is done as follows

$$\frac{u_j(k;r)}{r} \to a_{j-1}(k)\, h_{j-1}^-(kr) - b_{j-1}(k)\, h_{j-1}^+(kr)\,, \tag{257}$$

$$\frac{w_j(k;r)}{r} \to a_{j+1}(k)\, h_{j+1}^-(kr) - b_{j+1}(k)\, h_{j+1}^+(kr)\,, \tag{258}$$

where the relation between the a_l and b_l is

$$\begin{pmatrix} b_{j-1} \\ b_{j+1} \end{pmatrix} = S_j(k) \begin{pmatrix} a_{j-1} \\ a_{j+1} \end{pmatrix}\,. \tag{259}$$

In the equation above, the S-matrix S_j is unitary

$$S_j^\dagger S_j = 1\,, \tag{260}$$

from which we can propose several parametrizations that will include phase shifts. The first parametrization is the eigen phase shifts (also known as Blatt-Biedenharn paramterization)

$$S_j(k) = \begin{pmatrix} \cos\epsilon_j & -\sin\epsilon_j \\ \sin\epsilon_j & \cos\epsilon_j \end{pmatrix} \begin{pmatrix} e^{2\mathrm{i}\delta_{1j}} & 0 \\ 0 & e^{2\mathrm{i}\delta_{2j}} \end{pmatrix} \begin{pmatrix} \cos\epsilon_j & \sin\epsilon_j \\ -\sin\epsilon_j & \cos\epsilon_j \end{pmatrix} \tag{261}$$

The second are the nuclear bar phase shifts (or Stapp-Ypsilantis-Metropolis parametrization)

$$S_j(k) = \begin{pmatrix} \cos(2\bar{\epsilon}_j) e^{2i\bar{\delta}_{1j}} & i\sin(2\bar{\epsilon}_j) e^{i(\bar{\delta}_{1j}+\bar{\delta}_{2j})} \\ i\sin(2\bar{\epsilon}_j) e^{i(\bar{\delta}_{1j}+\bar{\delta}_{2j})} & \cos(2\bar{\epsilon}_j) e^{2i\bar{\delta}_{2j}} \end{pmatrix}\,. \tag{262}$$

The most popular one is the nuclear bar or SYM parametrization. They are equivalent however and the relation among them is given by

$$\bar{\delta}_{1j} + \bar{\delta}_{2j} = \delta_{1j} + \delta_{2j} \tag{263}$$

$$\sin(\bar{\delta}_{1j} - \bar{\delta}_{2j}) = \frac{\tan(2\bar{\epsilon}_j)}{\tan(2\epsilon_j)}\,. \tag{264}$$

It is interesting to consider the type of asymptotic wave function solutions available for a triplet coupled channel and their relation with the eigen phase shift, for instance. We begin with the uncoupled channel case

$$\frac{u_l(k;r)}{r} \to \left[\cos\delta_l(k)\, j_l(kr) - \sin\delta_l(k) y_l(kr)\right]\,, \tag{265}$$

where we have chosen a real normalization. If we solve the wave function from the Schrödinger equation and match the solution for $r \to \infty$ with the previous form, we can extract the phase shifts. For the coupled channel case things are however not so easy because there will be two linearly independent asymptotic solutions. They are called the α and β scattering states and behave as

$$\frac{u_{\alpha,j}(r)}{r} \to \cos\epsilon_j \left(\cos\delta_j^{(\alpha)} j_{j-1}(kr) - \sin\delta_j^{(\alpha)} y_{j-1}(kr)\right) \tag{266}$$

$$\frac{w_{\alpha,j}(r)}{r} \to \sin\epsilon_j \left(\cos\delta_j^{(\alpha)} j_{j+1}(kr) - \sin\delta_j^{(\alpha)} y_{j+1}(kr)\right) \tag{267}$$

$$\frac{u_{\beta,j}(r)}{r} \to -\sin\epsilon_j \left(\cos\delta_j^{(\beta)} j_{j-1}(kr) - \sin\delta_j^{(\beta)} y_{j-1}(kr)\right) \tag{268}$$

$$\frac{w_{\beta,j}(r)}{r} \to \cos\epsilon_j \left(\cos\delta_j^{(\beta)} j_{j+1}(kr) - \sin\delta_j^{(\beta)} y_{j+1}(kr)\right) \tag{269}$$

That is, they correspond to the two eigen phase shifts conveniently mixed by the mixing angle ϵ_j

IV. THE ONE BOSON EXCHANGE MODEL

The idea of Yukawa — nuclear forces are the consequence of the exchange of a boson — led to the OPE potential, which explains indeed a lot of features of the nucleon-nucleon interaction. This idea can be easily extended to the exchange of other bosons, which in turn led to the first precision description of the nuclear force. In addition, this description is relatively simple and intuitive. The most important bosons within this one boson exchange (OBE) model are the σ, the ρ and the ω (plus the pion, of course). The interaction lagrangian of these mesons with the nucleon are given by

$$\mathcal{L}_{\sigma NN} = g_\sigma \, \bar{\Psi}_N \sigma \Psi_N \,, \tag{270}$$

$$\mathcal{L}_{\rho NN} = g_\rho \, \bar{\Psi}_N \gamma^\mu \, \vec{\tau} \cdot \vec{\rho}_\mu \Psi_N + \frac{f_\rho}{4M_N} \bar{\Psi}_N \sigma_{\mu\nu} \vec{\tau} \Psi_N \, \cdot \, (\partial_\mu \vec{\rho}_\nu - \partial_\nu \vec{\tau} \vec{\rho}_\mu) \,, \tag{271}$$

$$\mathcal{L}_{\omega NN} = g_\omega \, \bar{\Psi}_N \gamma^\mu \, \omega_\mu \Psi_N + \frac{f_\omega}{4M_N} \bar{\Psi}_N \sigma_{\mu\nu} \Psi_N \, (\partial_\mu \omega_\nu - \partial_\nu \omega_\mu) \,, \tag{272}$$

and the corresponding momentum space potentials in the non-relativistic limit are (**exercise**, three points: one for each of the mesons)

$$V_\sigma(\vec{q}) = -\frac{g_\sigma^2}{q^2 + m_\sigma^2} \tag{273}$$

$$V_\rho(\vec{q}) = \vec{\tau}_1 \cdot \vec{\tau}_2 \left[\frac{g_\rho^2}{q^2 + m_\rho^2} - \frac{(f_\rho + g_\rho)^2}{4M_N^2} \, \frac{(\vec{\sigma}_1 \times \vec{q}) \cdot (\vec{\sigma}_2 \times \vec{q})}{q^2 + m_\rho^2} \right] , \tag{274}$$

$$V_\omega(\vec{q}) = \frac{g_\omega^2}{q^2 + m_\omega^2} - \frac{(f_\omega + g_\rho)^2}{4M_N^2} \, \frac{(\vec{\sigma}_1 \times \vec{q}) \cdot (\vec{\sigma}_2 \times \vec{q})}{q^2 + m_\omega^2} \,. \tag{275}$$

Equivalently, we can write them in coordinate space by means of a Fourier transform (**exersice**: also three points, or one point per meson)

$$V_\sigma(\vec{r}) = -g_\sigma^2 \, m_\sigma \, W_Y(m_\sigma r) \,, \tag{276}$$

$$\begin{aligned} V_\rho(\vec{r}) = \vec{\tau}_1 \cdot \vec{\tau}_2 \Big[& g_\rho^2 \, m_\rho \, W_Y(m_\rho r) + \frac{(f_\rho + g_\rho)^2}{4M_N^2} \Big(- \frac{2}{3} \, \vec{\sigma}_1 \cdot \vec{\sigma}_2 \, \delta(\vec{r}) \\ & + \frac{2}{3} \, \vec{\sigma}_1 \cdot \vec{\sigma}_2 \, m_\rho^3 \, W_Y(m_\rho r) - \frac{1}{3} \, S_{12}(\hat{r}) \, m_\rho^3 \, W_T(m_\rho r) \Big) \Big] \,, \end{aligned} \tag{277}$$

$$\begin{aligned} V_\omega(\vec{r}) = & \, g_\omega^2 \, m_\omega \, W_Y(m_\omega r) + \frac{(f_\omega + g_\omega)^2}{4M_N^2} \Big(- \frac{2}{3} \, \vec{\sigma}_1 \cdot \vec{\sigma}_2 \, \delta(\vec{r}) \\ & + \frac{2}{3} \, \vec{\sigma}_1 \cdot \vec{\sigma}_2 \, m_\omega^3 \, W_Y(m_\omega r) - \frac{1}{3} \, S_{12}(\hat{r}) \, m_\omega^3 \, W_T(m_\omega r) \Big) \,, \end{aligned} \tag{278}$$

where the functions $W_Y(x)$ and $W_T(x)$ are now defined as

$$W_Y(x) = \frac{e^{-x}}{4\pi x} \,, \tag{279}$$

$$W_T(x) = \left(1 + \frac{3}{x} + \frac{3}{x^2} \right) \frac{e^{-x}}{4\pi x} \,. \tag{280}$$

(Notice that we have used a different definition as in the OPE case in the previous section). Now if we include the OPE potential $V_\pi(\vec{r})$, we get the OBE potential in its simplest encarnation

$$V_{\rm OBE}(\vec{r}) = V_\pi(\vec{r}) + V_\sigma(\vec{r}) + V_\rho(\vec{r}) + V_\omega(\vec{r}) \,. \tag{281}$$

A very compact and neat expression, indeed.

However the OBE model usually entails a few modifications over the previous version. The first, most obvious one, is the different roles of ρ and ω. It happens that

$$f_\rho \gg g_\rho \quad \text{and} \quad f_\omega \ll g_\omega \,, \tag{282}$$

which usually translates into the following simplifications of the ρ and ω potentials

$$V_\rho(\vec{q}) = \vec{\tau}_1 \cdot \vec{\tau}_2 \, \frac{f_\rho^2}{4M_N^2} \, \frac{(\vec{\sigma}_1 \times \vec{q}) \cdot (\vec{\sigma}_2 \times \vec{q})}{q^2 + m_\rho^2} \,, \tag{283}$$

$$V_\omega(\vec{q}) = \frac{g_\omega^2}{q^2 + m_\omega^2} \,. \tag{284}$$

As a result, the ρ and ω end up fulfilling very different jobs in the OBE potential. Other modification is the inclusion of the first relativistic corrections for the σ and ω exchange potentials, which in turn entails the appearance of a spin-orbit force (the $\vec{L}\cdot\vec{S}$ term below)

$$V_\sigma(\vec{q}) = \frac{g_\sigma^2}{q^2+m_\sigma^2}\left[-1+\frac{k^2}{2M_N^2}-\frac{q^2}{8M_N^2}-\frac{\vec{L}\cdot\vec{S}}{2M_N^2}\right], \tag{285}$$

$$V_\omega(\vec{q}) = \frac{g_\omega^2}{q^2+m_\omega^2}\left[1-3\frac{\vec{L}\cdot\vec{S}}{M_N^2}\right], \tag{286}$$

where in the expression above $\vec{k}=\frac{1}{2}(\vec{p}+\vec{p}')$, with $\vec{p}$ and $\vec{p}'$ the initial and final two-nucleon momenta in the center-of-mass system. Notice also that in the case of the ρ meson the potential is already expanded up to $1/M_N^2$. The spin-orbit term above is indeed very important for the description of the P-waves in nucleon-nucleon scattering. Besides the simplifications in the ρ and ω potential and the inclusion of relativistic effects to obtain the spin-orbit term, other essential ingredient of the OBE model are the form factors. The form factors are modifications on the meson-nucleon-nucleon vertex that take into account the finite size of these systems. In addition their inclusion makes the potentials softer at high momenta, which means that they become much more tractable mathematically. The typical way to include a form factor is

$$V_M(\vec{q},\Lambda) = V_M(\vec{q})\,F_{M,\Lambda}^2(\vec{q}), \tag{287}$$

where M refers to the meson and $F_M(x)$ to its form factor. A typical choice of form factor is the multipolar form factor

$$F_{M,\Lambda}(\vec{q}) = \left(\frac{\Lambda^2-m_M^2}{\Lambda^2+\vec{q}^2}\right)^n, \tag{288}$$

with m_M the mass of the meson and n some exponent: for $n=1$ we have a monopolar form factor, for $n=2$ dipolar and so on. The form factor can be different for each of the mesons, with pions in general having the larger values of $\Lambda\sim 1.3\,\mathrm{GeV}$ while for the other mesons $\Lambda\sim 0.8-1.0\,\mathrm{GeV}$ is enough.

Let us review the OBE model with a list of the mesons and their jobs

(π) The pion provides a weak spin-spin term and a strong tensor term, both of them long-range

(σ) The sigma provides a strong medium-range attraction and a moderate spin-orbit force.

(ρ) The rho provides a weak spin-spin term of the same sign as the pion, and a moderate tensor term that cancels the one of the pion at short-distances.

(ω) The omega provides a strong short-range repulsion and a strong spin-orbit term of the same sign as the sigma.

The OBE model also have a few deficiencies and limitations. One of the most conspicuous is the problem of the very large g_ω. According to SU(3)-flavour symmetry, which is an extension of isospin symmetry to include the strange quark, and the OZI rule (a phenomenological rule that says that annihilation of strange and anti-strange quarks is suppressed relative to the u and d quarks) it should happen that

$$g_\omega = 3\,g_\rho, \tag{289}$$

where $g_\rho = m_\rho/2f_\pi \simeq 2.9$ aproximately. However the short-range repulsion required in the OBE model requires a considerably larger g_ω.

V. EFFECTIVE FIELD THEORIES

Nowadays the effective field theory (EFT) formulation of the nuclear forces is slowly becoming the standard in nuclear physics (a very pedagogical introduction can found in Ref. [1], while more complete reviews in Refs. [2, 3]). The idea behind EFT is that physics at long distances should not depend on the short distance details. We have seen a few examples of this idea in the exercises, in particular the constant repetition of exercises relative with a contact-range potential that is regularized with a cut-off function

$$\langle\vec{p}'|V_C^{(-1)}|\vec{p}\rangle = C_0(\Lambda)\,g(\frac{p'}{\Lambda})g(\frac{p}{\Lambda}), \tag{290}$$

which is then included in the Schrödinger or Lippmann-Schwinger equation to obtain scattering amplitudes, phase shifts and the like

$$k\,\cot\delta\,, \tag{291}$$

as a function of C_0 and Λ. Then we include a *renormalization condition*, for example

$$\lim_{k\to 0} k\,\cot\delta = -\frac{1}{\alpha_0}\,. \tag{292}$$

From this condition we can determine the relation among C_0, Λ and the scattering length. In addition we can determine all the observables in the theory with this simple theory: the observables will still depend on the cut-off Λ, but the dependence is under control. Why? Because is proportional to $1/\Lambda$. For instance, if we compute again $k\,\cot\delta$ we will obtain

$$k\,\cot\delta = -\frac{1}{\alpha_0} + c_2\frac{k^2}{\Lambda} + \dots\,, \tag{293}$$

which means that the cut-off is harmless. We can reduce its influence by simply increasing the value of the cut-off, and in the $\Lambda\to\infty$ limit we simply obtain

$$k\,\cot\delta = -\frac{1}{\alpha_0}\,. \tag{294}$$

This example is not particularly impressive because we are using a really simple theory: the only prediction that we are making is that the scattering for $k \ll m_\pi$ is very similar to zero energy scattering. Yet this is still a prediction. When we combine this idea with a more complete version of the long range physics, the outcome can be much mode impressive. A very simple example is provided by the following theory

$$V^{(-1)} = V_C^{(-1)} + V_F^{(-1)}\,, \tag{295}$$

a potential composed of a contact- and finite-range piece, where they are given by

$$\langle\vec{p}\,'|V_C^{(0)}|\vec{p}\rangle = C_0(\Lambda)\,g(\frac{p'}{\Lambda})g(\frac{p}{\Lambda})\,, \tag{296}$$

$$\langle\vec{p}\,'|V_F^{(0)}|\vec{p}\rangle = -\frac{g_A^2}{4f_\pi^2}\,\frac{\sigma_1\cdot\vec{q}\,\sigma_2\cdot\vec{q}}{q^2+m_\pi^2}\,g(\frac{p'}{\Lambda})g(\frac{p}{\Lambda})\,, \tag{297}$$

that is, the basic contact-range potential that we already know so well plus the OPE potential. In this case if we determine the value of the coupling C_0 from the condition of reproducing the deuteron binding energy, we will be able to reproduce all the deuteron properties really well, as can be seen in Table I. And the interesting thing is that the predictions do not significantly depend on the cut-off provided that $\Lambda \gg m_\pi$ (with $\Lambda\sim 0.5\,\text{GeV}$ usually enough). This is very different from the OBE model, where the cut-off Λ in the dipolar form factor cannot be taken too large and has an ideal size of about 1 GeV. Besides within the EFT framework there are clearly defined rules as to how to improve the calculation. The previous example, with a contact-range interaction and the OPE potential, is what is called a *leading order* calculation. But the EFT tell us that there are more orders beyong leading order and that we can indeed do the following

$$V = V^{(-1)} + V^{(0)} + V^{(1)} + V^{(2)} + \dots\,, \tag{298}$$

until we reach the desired level of accuracy in the calculation. That is, the physics of any physical problem we might think of are translated into a power series. Here we provide the nucleon-nucleon phase shifts as an example in Fig. 1, which corresponds to the Q^4 potential of Ref. [4].

The current problem however with EFT is that the renormalization process, that we have illustrated a few times with the contact-range theory, is in general not that easy. When we consider the higher orders of the theory it becomes exceedingly complicated indeed, a problem that has led to several incompatible proposals. In this regard, the EFT formulation of nuclear physics is a very active research field in which new findinds are being made routinely (but also in which new setbacks are met routinely).

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Set	γ (fm^{-1})	η	A_S (fm$^{-1/2}$)	r_m (fm)	Q_d(fm^2)	P_D
$V^{(-1)}$	Input	0.02633	0.8681(1)	1.9351(5)	0.2762(1)	7.31(1) %
$V^{(3)}$	Input	Input	0.884(4)	1.967(6)	0.276(3)	8(1) %
Exp.	0.231605	0.0256(4)	0.8846(9)	1.971(6)	0.2859(3)	-

TABLE I. Properties of the deuteron within pionful EFT: the $V^{(-1)}$ calculation refers to the leading order calculation, as explained in the main text, while the $V^{(3)}$ calculation refers to a higher order calculation and hence will yield more accurate predictions. The observables are the deuteron binding momentum $\gamma = \sqrt{M_N B_d}$, the asymtotic D/S ratio $\eta = A_D/A_S$, the mean square radius r_m, the quadrupole moment Q_d and the D-wave probability P_D (though this latter one is not really an observable). Input means that this observable has been used as a *renormalization condition*. Results are taken from Ref. [5].

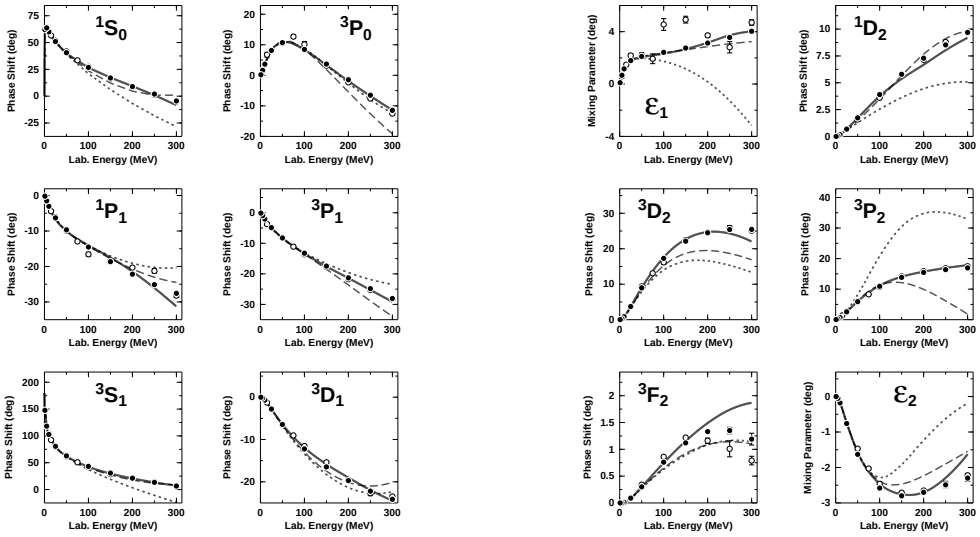

FIG. 1. Phase shifts for nucleon-nucleon scattering for the chiral potential of Ref. [4], computed up to order Q^4 in the chiral expansion.