Assignments: Theory

This is the theory set for the nuclear physics course (Spring 2023). The course contain two types of assignments: theory and problems. Theory includes easy theoretical questions, while the problems require more thinking, with a few of the questions being really difficult.

Regarding this theory set, you might choose whatever questions that you like or find easier to solve, with the only condition that they sum three points.

(1) As explained in the lectures, one of the reasons why we know that the nuclear force has a finite-range is saturation, i.e. the idea that the binding energy of a nucleus is proportional to the number of nucleons

$$B(\text{nucleus}) \propto A \tag{1}$$

with A the number of nucleons. Had the nuclear force be infinite-ranged, as the Coulomb force or gravitation, the binding energy would have been proportional to A^2 instead of A.

To understand this idea a bit more directly let's consider the gravitational binding energy of a spherical distribution of mass, such as a planet. This binding energy can be calculated as follows

$$V_G = -\frac{G}{2} \int d^3\vec{r}_1 d^3\vec{r}_2 \, \frac{\rho(\vec{r}_1)\, \rho(\vec{r}_2)}{|\vec{r}_1 - \vec{r}_2|} \tag{2}$$

with ρ the mass density of the object and G Newton's constant. For a spherical mass distribution of constant density ($\rho(\vec{r}) = \rho$), show that the gravitational binding energy is in fact proportional to the mass squared (M^2) of the object (**1 point**).

Now, if you want to go a step further, assume that the graviton has a finite mass ϵ, in which case Newton's law of gravitation changes to

$$V = -G\frac{m_1 m_2}{r}\, e^{-\epsilon r} \, . \tag{3}$$

This will naturally change the expression for the gravitational binding energy that we had written before. Compute the gravitational binding energy in this case and show that if the product of the graviton mass with the radius of the planet is large ($R\epsilon \gg 1$) then the gravitational binding energy will be proportional to the mass of the planet (M), instead of the square of the mass as in the previous case (**2 points**).

(2) Explain how a nucleon-nucleon potential that is strongly repulsive at short distances while attractive at larger distances will lead to a constant nucleon density in the nucleus (**1 point**).

(3) Fine tuning and the deuteron: the deuteron is a bound state of a neutron and a proton, with a binding energy of $E_B = -2.2\,\text{MeV}$. Consider the following idea: try to describe the deuteron with a square well with a range of $1\,\text{fm}$:

$$V(r) = V_0 \theta(r - a) \, , \tag{4}$$

with $a = 1\,\text{fm}$. Reproducing the location of a bound state will give you a particular value of V_0 (the depth of the potential). Now consider that the binding energy E_B is the sum of two contributions:

$$E_B = \langle T \rangle + \langle V \rangle \, , \tag{5}$$

that is, the sum of the kinetic and potential energy. Find the ratio $R_{FT} = |E_B|/(|\langle T \rangle| + |\langle V \rangle|)$, which describes the level of fine tuning (**1 point**).

(4) As already explained, the standard Dirac-delta in three dimensions:

$$\delta^{(3)}(\vec{r}) \, ,$$

is rather inconvenient to use. For this reason we usually *smear* the delta, i.e. we make it a bit broader by including a cutoff

$$\delta^{(3)}(\vec{r}) \to \delta^{(3)}(\vec{r}; R_c) \, .$$

One example is

$$\delta^{(3)}(\vec{r}; R_c) = \frac{e^{-(r/R_c)^2}}{\pi^{3/2} R_c^3} \,. \tag{6}$$

Show that the normalization of this smeared delta is indeed the correct one, that is,

$$\int d^3\vec{r}\,\delta^{(3)}(\vec{r}; R_c) = 1 \,. \tag{7}$$

(1 point)

(5) Let's assume a two-body system in S-wave, which follows the reduced Schrödinger equation

$$-u_0''(r) + 2\mu V(r) u_0(r) = -\gamma^2 u_0(r) \,, \tag{8}$$

with $\gamma^2 = -2\mu E$ and $E < 0$ (i.e. we have a bound state). Assume a potential of the type

$$V(r) = 0 \quad \text{for} \quad r \neq 0 \,, \tag{9}$$

which generates a bound state (despite being a contact-range potential). Show that the wave function is

$$u_0(r) = A_S\, e^{-\gamma r} \,, \tag{10}$$

and compute the value of A_S. Show also that the mean square radius of such a bound state is $\langle r^2 \rangle = 1/(2\gamma^2)$. **(1 point)**

(6) Consider the l-wave reduced Schrödinger equation

$$-u_l''(r) + \left[2\mu V(r) + \frac{l(l+1)}{r^2} \right] u_l(r) = -\gamma^2 u_l(r) \,, \tag{11}$$

with $\gamma^2 = -2\mu E$, $E < 0$ and a contact-range potential

$$V(r) = 0 \quad \text{for} \quad r \neq 0 \,. \tag{12}$$

Find the complete form of the bound state solution for P- and D-waves (i.e. for $l = 1$ and $l = 2$)

$$u_1(r) \to A_P\, e^{-\gamma r} \left(1 + \dots \right) \,, \tag{13}$$

$$u_2(r) \to A_D\, e^{-\gamma r} \left(1 + \dots \right) \,, \tag{14}$$

that is, found the complete expression that goes inside the brackets. **(2 points)**

(7) Knowing that the $l = 0$, $k = 0$ solution of the wave function behaves asymptotically as

$$u_0(r) \to 1 - \frac{r}{a_0} \,, \tag{15}$$

with a_0 the scattering length, find the scattering length for the well-known square-well potential

$$2\mu\, V(r) = U(R) \theta(R - r) \,. \tag{16}$$

(1 point)

(8) Assuming that the deuteron was a pure S-wave or a pure D-wave state — a ${}^3S_1$ or ${}^3D_1$ partial wave in the spectroscopic notation — show that the magnetic moment in units of the nuclear magneton should be

$$\mu({}^3S_1) = \mu_p + \mu_n = 0.88 \tag{17}$$

$$\mu({}^3D_1) = \frac{3}{4} - \frac{1}{2} \left(\mu_p + \mu_n \right) = 0.31 \,. \tag{18}$$

(2 points)

(9) Even-odd and odd-even nuclei can be understood as an even-even core with spin-parity $J^P = 0^+$ plus an impaired nucleon which generates the spin-parity and the magnetic moment of the nucleus. In particular for the magnetic moment we can write:

$$\mu(A) = \mu_{\rm core} + \mu_N = \mu_N \, , \tag{19}$$

because the core is 0^+ and its magnetic moment is zero. The unpaired nucleon has intrinsic spin $S = \frac{1}{2}$ and orbital angular momentum L, which can coupled to total angular momentum $J = L \pm \frac{1}{2}$. Show that the magnetic moment for the $J = L \pm \frac{1}{2}$ configurations is

$$\mu_N(J = L + \frac{1}{2}) = g_L (J - \frac{1}{2}) + \frac{1}{2} g_S \, , \tag{20}$$

$$\mu_N(J = L - \frac{1}{2}) = g_L \frac{J(J+\frac{3}{2})}{J+1} - \frac{J}{2(J+1)} g_S \, , \tag{21}$$

which are called the Schmidt values **(2 points)**.

(10) In the Hartree-Fock method, the mean field potential is defined as

$$U(\vec{r}, \vec{r}') = \delta^3(\vec{r} - \vec{r}') \sum_{j=1}^{A} \int d^3 \vec{r}' \, \phi_j^*(\vec{r}') \, V_{2B}(\vec{r}, \vec{r}') \, \phi_j(\vec{r}') - \sum_{j=1}^{A} V_{2B}(\vec{r}, \vec{r}') \, \phi_j^*(\vec{r}) \, \phi_j(\vec{r}') \, . \tag{22}$$

Show that for a pure contact-range potential

$$V_{2B}(\vec{r}, \vec{r}') = C \, \delta(\vec{r} - \vec{r}') \, , \tag{23}$$

the mean field potential vanishes (i.e. $U(\vec{r}, \vec{r}') = 0$). Why does this happens? How does the Skyrme force avoid this problem? **(2 points)**