Assignments: Problems

This is the problem set for the nuclear physics course (Spring 2023). The course contain two types of assignments: theory and problems. Theory includes easy theoretical questions, while the problems require more thinking, with a few of the questions being really difficult.

Regarding this problem set it is enough to choose a unique problem for this course. It is not necessary to arrive at a correct or complete answer (as they can be hard to solve): attemps at arriving at an answer will be evaluated positively, particularly if the idea is good.

(1) Explain why vector meson exchange generates a negative quadrupole moment in the deuteron and why pseudoscalar meson exchange (e.g. the pion) give us the correct sign for the quadrupole moment (check the slides of lesson 6 for context).

(2) Explain how nuclear physics will change if chiral symmetry was broken differently: that is, instead of $G = SU(3)_L \otimes SU(3)_R \otimes U(1)_{L+R}$ breaking into the subgroup $F = SU(3)_{L+R} \otimes U(1)_{L+R}$, how do you think nuclear physics will change if the conserved subgroup was $F = SU(3)_{L-R} \otimes U(1)_{L+R}$ instead (check the slides of lesson 12 for context).

(3) A comparison between the electromagnetic, the nuclear and the strong force: when we consider a bound state in quantum mechanics, for instance the hydrogen atom, we see that the total mass of the bound state is a bit smaller than their components. For example, in the hydrogen atom we will have

$$m(H) = (m_e + m_p) - B_2^H < (m_e + m_p)\,, \tag{1}$$

where B_2 is the binding energy. The same happens with the deuteron

$$m(d) = (m_p + m_n) - B_2^d < (m_n + m_p)\,, \tag{2}$$

and to all other nuclei

$$m(A,Z) = (Zm_p + (A-Z)m_n) - B < (Zm_p + (A-Z)m_n)\,. \tag{3}$$

However, when we consider hadrons it happens exactly the contrary as here. For the pion we have

$$m_\pi > (m_u + m_d)\,, \tag{4}$$

where $m_\pi \simeq 140\,\mathrm{MeV}$ and $(m_u + m_d) \sim 8\,\mathrm{MeV}$, while for the proton we have instead

$$m_p > (2m_u + m_d)\,, \tag{5}$$

with $m_p \simeq 940\,\mathrm{MeV}$ and $(2m_u + m_d) \sim 11\,\mathrm{MeV}$. Explain why this is happening.

(4) Imagine a two-body system interacting via an attractive central potential $V(r) < 0$ that is short-ranged. Using the Wronskian identity trick from lesson 15, try to argue how the phase shift for this potential will change if we change the reduced mass of the system.

(5) Imagine a two-body system interacting via a power-law potential of the type $V(r) = C/r^n$, with n a positive integer. This is not a short-ranged potential and as a consequence the effective range expansion that we studied in lesson 15 does not apply, at least completely, though in a few cases the first few parameters of this expansion will be well defined. Find an argument explaining why for $n = 1, 2, 3$ the scattering length is not well defined, while for $n \geq 4$ it is. Find also the minimum value of n required for the effective range to be well-defined.

(6) In lesson 15 we commented that for a short-ranged potential showing an exponential decay at long distances of the type $V(r) = f(r)\,e^{-mr}$ (with $f(r)$ non-exponential), the effective range expansion only converges for $k < m/2$. Try to find an argument of why this is the case.

(7) Imagine a two-body potential that is zero for $r > R$, that is

$$V(r) = V(r)\,\theta(R - r)\,. \tag{6}$$

Find what is the maximum value of the effective range r_0 for such type of potential (check lesson 15).

(8) As we briefly commented in lesson 15, the effective range expansion (ERE) is not valid for infinite range potentials such as Coulomb. Yet, there exists a Coulomb ERE for proton-proton scattering that looks like

$$\mathcal{C}^2(\eta)\, k\, \cot\delta_C(k) + \frac{1}{a_B}\, h(\eta) = -\frac{1}{a_C} + \frac{1}{2}\, r_C\, k^2 + \sum_{n=2}^{\infty} v_{n,C} k^{2n}\,, \tag{7}$$

where a_B is the Bohr radius (which we define as $1/(m_p\alpha)$, and thus a bit different from how we defined it in lesson 2 for the electron-proton system; check also the expressions in the next problem), $\eta = 1/(2ka_B)$ and the functions $\mathcal{C}^2(\eta)$ and $h(\eta)$ are defined as:

$$\mathcal{C}^2(\eta) = \frac{2\pi\eta}{e^{2\pi\eta}-1} \quad \text{and} \quad h(\eta) = \eta^2 \sum_{n=1}^{\infty} \frac{1}{n(n^2+\eta^2)} - \log\eta - \gamma_E \tag{8}$$

with γ_E the Euler-Mascheroni constant. In the Coulomb ERE, δ_C is the Coulomb-modified phase shift and a_C, r_C and $v_{n,C}$ are the Coulomb scattering length, effective range and shape parameters. In turn the Coulomb phase shift is derived from the asymptotic form ($r \to \infty$) of the wave functions in the presence of a repulsive Coulomb potential:

$$u(r) \to \cot\delta_C(k)\, F_0(r) - G_0(r) \tag{9}$$

where the functions F_0 and G_0 have the asymptotic behavior

$$F_0 \to \sin\left(kr - \eta\log(2kr) + \sigma_0\right), \tag{10}$$

$$G_0 \to -\cos\left(kr - \eta\log(2kr) + \sigma_0\right), \tag{11}$$

with σ_0 defined as

$$e^{2i\sigma_0} = \frac{\Gamma(1+i\eta)}{\Gamma(1-i\eta)}\,. \tag{12}$$

In addition, the behavior of F_0 and G_0 for $kr \to 0$ is given by

$$F_0 \to k\, C(\eta) \left[r + \frac{r^2}{2a_B} + O(r^3) \right],$$

$$G_0 \to -\frac{1}{C(\eta)} \left[1 + \frac{r}{a_B} \left(\log\frac{r}{a_B} + 2\,\gamma_E - 1 + h(\eta) \right) + O(r^2) \right]. \tag{13}$$

Try to derive the formula for the Coulomb ERE by following the steps from lesson 15 for the usual ERE for finite range interactions.

(9) Consider proton-proton scattering at low energies, where the Coulomb scattering length is defined as

$$\lim_{k\to 0} \mathcal{C}^2(\eta)\, k\, \cot\delta_C(k) = -\frac{1}{a_C}\,. \tag{14}$$

Imagine that you want to describe proton-proton scattering with a contact-range potential (plus Coulomb), where for simplicity we regulate the potential as

$$V_{pp}(r; R_c) = C_0(R_c)\, \frac{\delta(r-R_c)}{4\pi R_c^2} + \frac{1}{2\mu}\, \frac{1}{a_B r}\, \theta(r-R_c)\,, \tag{15}$$

with $a_B = 1/(m_p\alpha)$ and R_c the cutoff. By using the expressions from the previous problem and following the same steps as in lesson 13 (but adapted to the zero energy case), find the running of the coupling $C_0(R_c)$ as a function of the cutoff, the proton Bohr radius a_B and the Coulomb scattering length a_C (check also lesson 17 for comparison purposes). Then, using the expresion you obtain, try to deduce the value of the strong scattering length a_S (that is, the scattering length in case that Coulomb was turned off) and comment on the result. In particular, is it cutoff independent? What interpretation do you give to this fact?

(10) In lesson 15 we explained that the Lippmann-Schwinger equation had analytic solutions for separable potentials of the type $\langle p'|V|p\rangle = \lambda\, f(p')\, f(p)$. There are also analytic solutions of the T-matrix for potentials that are a sum of separable pieces (which we call a and b), for instance

$$\langle p'|V|p\rangle = \langle p'|V_a|p\rangle + \langle p'|V_b|p\rangle = \lambda_a\, f_a(p')\, f_a(p) + \lambda_b\, f_b(p')\, f_b(p)\,. \tag{16}$$

Try to find the analytic T-matrix corresponding to this particular potential.

(11) Try to solve the Lippmann-Schwinger equation for a contact-range potential with a term proportional to $(p'^2 + p^2)$, that is, for

$$\langle p'|V|p\rangle = \left[C_0(\Lambda) + C_2(\Lambda)\,(p'^2 + p^2)\right]\, f(\frac{p'}{\Lambda})\, f(\frac{p}{\Lambda})\,. \tag{17}$$

For this the best strategy is to begin by proposing an ansatz for the T-matrix with this potential (for instance, by playing with the first few iterations of the $V + VG_0V + \dots$ and trying to identify a pattern... or by any other method you devise).

(12) A resonance is a quasi-bound state, i.e. a bound state at $E \geq 0$ that decays after some time. The way to check for the existence of a resonance is to look for solutions of the Schrödinger equation at complex energy

$$\left[-\frac{\nabla^2}{2\mu} + V(r)\right]\Psi(r) = E^*\Psi(r)\,, \tag{18}$$

where E^* is the energy of the resonance

$$E^* = E_R - i\frac{\Gamma_R}{2}\,, \tag{19}$$

where Γ_R is called the width of the resonance. The wave function of a resonant state behaves as

$$\Psi(r) \to \frac{e^{ik^* r}}{r}\, Y_{lm}(\hat{r})\,, \tag{20}$$

with $k^* = \sqrt{2\mu E^*}$, which is the complex energy analog of the asymptotic behaviour of the bound state solution. Equivalently, if we consider the reduced wave function $u_l(r)$ instead of the full wave function, a resonance is a solution that behaves as

$$u_l(r) \to e^{ik^* r}\,, \tag{21}$$

with k^* a complex momentum (alternatively, check lesson 16 for a more lightweight introduction to resonances).

To show how to calculate a resonant state solution, let us consider the repulsive delta-shell potential of the previous exercise

$$2\mu V(r) = \lambda\delta(r - a)\,, \tag{22}$$

where $\lambda > 0$ and a is the range of the delta-shell. For $\lambda \to \infty$ this potential has bound state solutions for $r < a$, where the binding energy is

$$E_B = \frac{1}{2\mu}\left(n\frac{\pi}{a}\right)^2\,, \tag{23}$$

with n a non-zero positive integer. If λ is not infinite, it happens that these bound state solutions can *escape* the potential barrier provided by the repulsive delta-shell potential. Hence they adquire a width

$$E_B \to E^* = E_R - i\frac{\Gamma_R}{2}\,, \tag{24}$$

where the half-life of these bound states (or, more properly, resonances) is

$$\tau = \frac{1}{\Gamma_R} \,. \tag{25}$$

Show that for λ very big (but not infinite), we have that

$$E_R = E_B + \mathcal{O}(\frac{1}{\lambda^2}) \quad , \quad \Gamma_R = \mathcal{O}(\frac{1}{\lambda}) \,. \tag{26}$$

Compute the explicit expression of Γ_R at order $1/\lambda$.

(13) Consider an attractive square well with range a and strength V_0, such that it has a bound state near threshold. Now, suppose that we add a second square well that is repulsive and that acts between $r = a$ and $r = 2a$. That is, we have

$$V(r) = V_0\, \theta(a - r) + W_0\, \theta(2a - r)\, \theta(r - a) \,, \tag{27}$$

with $V_0 < 0$ and $W_0 > 0$. How does the location of the shallow bound state change with W_0? As W_0 increases, does the bound state becomes a virtual state or a resonance?

(14) Consider the on-shell T-matrix

$$\langle k|T(k)|k\rangle = -\frac{2\pi}{\mu}\, \frac{1}{-\frac{1}{a_0} + \frac{1}{2}\, r_0\, k^2 - ik} \,, \tag{28}$$

that is, a T-matrix that contains a scattering length and an effective range (originally shown in lesson 16). Discuss the types of poles of the previous T-matrix depending on the values of a_0 and r_0: under what circumstances will we have a resonance? For the particular case in which this T-matrix has a bound state, use the technique of calculating the residue of the T-matrix to obtain the bound state wave function. In particular find the asymptotic normalization of the bound state wave function, that is, the A_S factor in the asymptotic behavior ($r \to \infty$) of the wave function

$$u(r) \to A_S\, e^{-\gamma r} \,. \tag{29}$$

In which sense is it different from the case when $r_0 = 0$ (that is, the case we calculated in lesson 16) ?

(15) The quadrupole moment of the deuteron can be calculated by using the well-known formula

$$Q_d = \frac{1}{20} \int_0^\infty dr\, r^2 w(r)\, (2\sqrt{2}u(r) - w(r)) \,, \tag{30}$$

with u, w the S- and D-wave components of the wave function. However, we have not studied the derivation of this formula, which actually makes for a good exercise. For understanding where this formula comes from, we begin with the definition of the quadrupole moment of the deuteron

$$Q_d = \langle \Psi_d(11)|\hat{Q}_{33}|\Psi_d(11)\rangle \,, \tag{31}$$

where $\Psi_d(Sm_S)$ is the wave function for a deuteron state with spin S and third component of the spin m_S (for which we take $S = m_S = 1$). In turn the quadrupole operator is simply

$$\begin{aligned} \hat{Q}_{33} &= e_p(3z_p^2 - r_p^2) + e_n(3z_n^2 - r_p^2) \\ &= \frac{e_p}{4}(3z^2 - r^2) = \frac{1}{4}(3z^2 - r^2) \,, \end{aligned} \tag{32}$$

where in the first line we have written it in terms of the proton and neutron coordinates, while in the second we have use the relative coordinate $\vec{r} = \vec{r}_p - \vec{r}_n$ (with z the third component of $\vec{r}$); e_p and e_n refer to the proton and neutron electric charge, which we take to be $e_p = 1$ and $e_n = 0$. Notice that for simplicity we have omited the Dirac-delta factors and thus the matrix elements of $\hat{Q}_{33}$ are simply given by

$$\langle \Psi_d | \hat{Q}_{33} | \Psi_d \rangle = \int d^3\,\vec{r}\,\Psi_d^\dagger(\vec{r})\frac{1}{4}(3z^2 - r^2)\Psi_d(\vec{r})\,. \tag{33}$$

The full deuteron wave function (including the D-wave) is

$$\Psi_d(\vec{r}) = \frac{u(r)}{r}\,\mathcal{Y}^{01}_{1m_d}(\hat{r}) + \frac{w(r)}{r}\,\mathcal{Y}^{21}_{1m_d}(\hat{r})\,, \tag{34}$$

where $\mathcal{Y}^{ls}_{jm}(\hat{r})$ are generalized spherical harmonics that combine the spin and angular momentum wave functions of the deuteron, which are define as

$$\mathcal{Y}^{ls}_{jm}(\hat{r}) = \sum_{m_l m_s} Y_{lm_l}(\hat{r})|sm_s\rangle\,\langle lm_l 1m_s|jm\rangle\,, \tag{35}$$

with Y_{lm_l} the standard spherical harmonics, $|sm_s\rangle$ the spin wave function and $\langle lm_l 1m_s|jm\rangle$ a Clebsch-Gordan coefficient. Putting all the previous pieces together, deduce the formula for the deuteron quadrupole form factor that we wrote at the beginning of this exercise.