Nuclear Structure

In this lecture we explain the ideas behind the most popular nuclear models: the liquid drop model, the shell model and the collective model. We begin by explaining the general properties of the nuclei that we expect to describe with these models.

I. GENERAL PROPERTIES OF NUCLEI

Before introducing the nuclear models, it is useful to explain what nuclear properties they are expected to explain. These usually include the binding energy, the radius of the nuclei, the angular momentum and the electromagnetic moments.

A. Binding Energy

The most obvious property of a nucleus is probably its binding energy: nuclei are bound states after all. For the deuteron the binding energy is given by

$$B_d = (m_p + m_n) - m_d \simeq 2.2\,\mathrm{MeV}\,, \tag{1}$$

where m_p, m_n are the masses of the proton and the neutron and m_d the mass of the deuteron. If we consider a nucleus with Z protons and N neutrons, the binding energy is determined by the analogous formula

$$B(Z,N) = (Zm_p + Nm_n) - m(Z,N)\,, \tag{2}$$

where $m(Z,N)$ is the mass of the nucleus we are interested in. If we define $A = N + Z$, a general observation is that for $A \geq 30$ the binding energy per nucleon is roughtly constant

$$\frac{B}{A} \sim 8\,\mathrm{MeV}\,. \tag{3}$$

This observation is indeed the basis for the liquid drop model, which we will study later.

Other useful quantity is the separation energy, which represents a difference in binding energies. For example, the neutron and proton separation energies are

$$S_p(Z,N) = B(Z,N) - B(Z-1,N)\,, \tag{4}$$

$$S_n(Z,N) = B(Z,N) - B(Z,N-1)\,. \tag{5}$$

A sudden increase in the separation energy indicates a nucleus that is particularly stable: it is much more difficult to extract a nucleon from these nuclei. These sudden increases happen at N or $Z = 2, 8, 20, 28, 50, 82, \ldots$, which are called the magic numbers. This series of numbers is explained with another nuclear model: the shell model. We can also define the separation energy of an α particle i.e. a ^{4}He nucleus, as

$$S_\alpha(Z,N) = B(Z,N) - B(Z-2,N-2) - B(2,2)\,, \tag{6}$$

which is useful if we consider α decay. For nuclei with $A > 150$ it often happens that $S_\alpha < 0$, which indicates that these heavy nuclei will be unstable with respect to α decay.

Other interesting quantity is the pairing energy, which is defined as

$$\delta_n = S_n(Z,N) - S_n(Z,N-1)\,, \tag{7}$$

which is defined for N even and $N-1$ odd. This energy happens to be about 2 MeV. It is also related with the magic numbers and the shell model.

B. Size

It is not easy to define the size of a nucleon, but typical numbers are

$$\sqrt{\langle r^2 \rangle_N} \sim 0.5 - 1.0\,\mathrm{fm}\,. \tag{8}$$

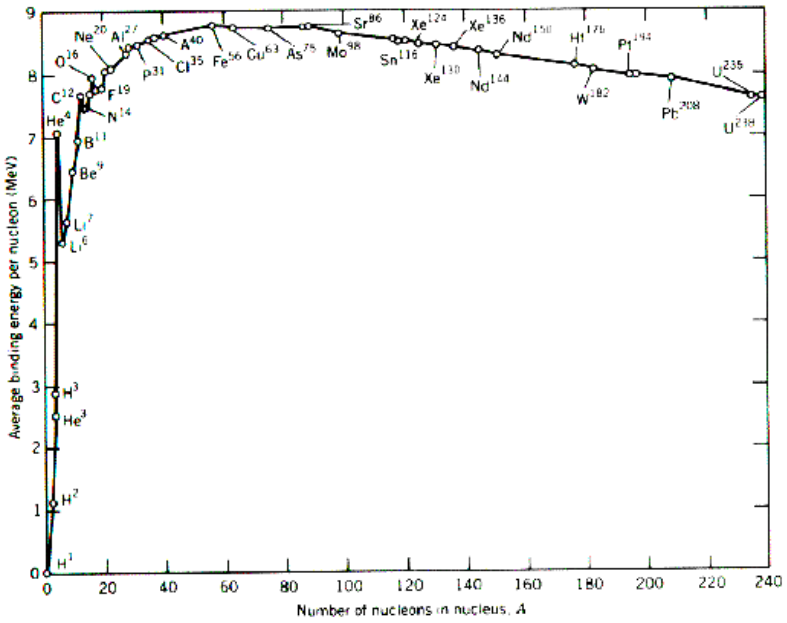


FIG. 1. Binding energy per nucleon for a series of stable nuclei as a function of A.

If we consider a nucleus of radius R and nucleons of radius r_0, we expect the number of nucleons A be

$$A \sim \frac{R^3}{r_0^3} \,. \tag{9}$$

Nuclei are however not extremely packed, i.e. the radius r_0 that the nucleons will occuppy will be larger than their size $r_0 \geq \sqrt{\langle r^2 \rangle_N}$. Actually $r_0 \sim 1.2 - 1.3\,\text{fm}$.

The density of a nucleus can be determined by electron-nucleus scattering experiments. This was done for the first time by Robert Hofstadter [1], an experimentalist. In that case we find a Fermi distribution of the type:

$$\rho(r) = \frac{\rho_0}{1 + e^{(r-R_0)/a}} \,. \tag{10}$$

where the parameters are $\rho_0 = 0.17\,\text{fm}^{-3}$, $a = 0.54\,\text{fm}$, $R_0 = 1.128\,A^{1/3}$.

C. Angular Momentum and Parity

Nuclei have well-defined angular momentum and parity:

$$J^P \,. \tag{11}$$

The J^P of the nucleon and the lightest nuclei is

$$J^P(N = n, p) = \frac{1}{2}^+ \quad , \quad J^P(^2H) = 1^+ \,,$$
$$J^P(^3H, ^3\,He) = \frac{1}{2}^+ \quad , \quad J^P(^4He) = 0^+ \,.$$

The case of ^4He (the α particle) is indeed interesting: it happens that the fundamental state of all known even-even nuclei is 0^+, where even-even refers to the fact that N and Z are both even.

Nuclei also have excited states with their own quantum numbers. For example, if we consider ^4He the fundamental state has $J^P = 0^+$ and $B \simeq 28\,\text{MeV}$, but it has also a series of excited states. The first excited state has $J^P = 0^+$ and $B \simeq 8\,\text{MeV}$. Explaining the J^P (and the binding energy, if possible) of these excited states is also part of the job of nuclear models.

D. Electric Quadrupole Moment

As seen in a previous lecture, the potential energy of a charge distribution in an external magnetic field can be written as a multipolar expansion

$$V = q\,\Phi + d_i\,\partial_i\Phi + \frac{1}{6}Q_{ij}\partial_{ij}\Phi + \dots \tag{12}$$

where the coefficients q, d_i, Q_{ij} are the charge, electric dipole and electric quadrupole moments respectively. If the object we are dealing with has a known charge distribution, we can write these coefficients as

$$q = \int d^3\vec{r}\,\rho(\vec{r})\,, \tag{13}$$

$$d_i = \int d^3\vec{r}\,r_i\rho(\vec{r})\,, \tag{14}$$

$$Q_{ij} = \int d^3\vec{r}\,\left(3r_ir_j - r^2\delta_{ij}\right)\,\rho(\vec{r})\,, \tag{15}$$

where $\rho(\vec{r})$ is the charge distribution of the object we have put in the external electric field. It happens that nuclei have well-defined parity:

$$\Psi(r_1,\dots,r_A) = \pm\Psi(-r_1,\dots,-r_A)\,. \tag{16}$$

As a consequence the electric dipole moment is zero, as can be checked by taking into account that $\rho = |\Psi|^2$. The first non-zero multipole moment is the electric quadrupole moment, as we saw in the deuteron case. The quadrupole moment of the deuteron is $Q_d = 0.256\,\mathrm{fm}^2 > 0$, from which we deduce that this is a *prolate* nucleus (i.e. it is like a *youtiao*). For most nuclei the quadrupole moment is negative however, $Q_d < 0$, indicating that they are *oblate*.

E. Magnetic Dipole Moment

Nucleons cannot have an electric dipole moment, but can have a magnetic dipole moment. From classical mechanics, the potential energy between a magnetic dipole and a magnetic field is given by

$$H = -\vec{\mu}\cdot\vec{B}\,. \tag{17}$$

The energy is minimized when the magnetic dipole is paralell to the magnetic field. The typical example of a magnetic dipole is a compass needle, which by virtue of the hamiltonian above will point north.

In classical mechanics a point particle of charge e that has orbital angular momentum will have a magnetic moment of

$$\vec{\mu}_L = \frac{e}{2m}\,\vec{L}\,, \tag{18}$$

where $\vec{L}$ is the orbital angular momentum. In quantum mechanics, a point particle also has spin and the total angular momentum is $\vec{J} = \vec{L} + \vec{S}$. As a consequence we have to modify the previous formula for the magnetic dipole moment to take into account the spin. We usually write

$$\vec{\mu} = \vec{\mu}_L + \vec{\mu}_S\,, \tag{19}$$

where $\vec{\mu}_L$ is the orbital part of the magnetic moment, which we have written above, and $\vec{\mu}_S$ is the intrinsic part of the magnetic moment. One might be tempted to write $\vec{\mu}_S$ as

$$\vec{\mu}_S = \frac{e}{2m}\,\vec{S}\,, \tag{20}$$

in analogy with the classical expression for the orbital magnetic moment, but this would be incorrect. After the formulation of the Dirac equation we know that the correct expression for the intrinsic magnetic moment is

$$\vec{\mu}_S = g_S\frac{|e|}{2m}\,\vec{S}\,, \tag{21}$$

where $g_S \simeq -2$ for an electron (and for any other fundamental spin 1/2 particle), where the minus sign is there to take into account that electrons have negative charge. This g_S is called the gyromagnetic factor of the electron. Then after

the discovery of quantum electrodynamics and renormalization, we found out that the previous value is subjected to quantum corrections

$$g_s = -2\left(1 + \frac{\alpha}{2\pi} + \mathcal{O}(\alpha^2)\right). \tag{22}$$

Ignoring the quantum corrections, if we consider the proton and the neutron as point-like particles we would expect the following

$$g_p = 2 \quad , \quad g_n = 0\,, \tag{23}$$

where we have taken into account that the proton has positive electric charge and the neutron is neutral. It actually happens that the gyromagnetic factors of the proton and neutron are nothing like the previous values. Instead we have

$$g_p = 5.586 \quad , \quad g_n = -3.826\,. \tag{24}$$

This indicates that the proton and the neutron have indeed internal structure. These numbers indicate the internal distribution of quarks and gluons inside the nucleons. Yet it is interesting to make a stop here and study the ideas that existed to explain this discrepancy before it was known that the nucleons were composite.

1. Vector Meson Dominance

In a forgotten time before the dawn of QCD, physicists though that the nucleons were fundamental particles without internal parts. In this picture the previous two values of the gyromagnetic ratios were really mysterious, to say the least, and required an explanation. This explanation was vector meson dominance. The idea is really simple: the neutral vector mesons, ρ^0 and ω^0, interact strongly with the nucleons. Besides the neutral vector mesons have exactly the same quantum numbers as the photon. In particular it can happen that a vector meson turns into a photon in flight.

This idea looks complicated, but it can be summarized in a really simple formula for the gyromagnetic factor of the nucleon

$$g_N = 2\,e_N + (\kappa_\omega + \tau_3\,\kappa_\rho)\,, \tag{25}$$

where κ_ρ and κ_ω represent the contribution of each of the vector mesons to the gyromagnetic factor and τ_3 is the third component of the isospin vector $\vec{\tau}$. If we particularize for protons and neutrons we find

$$g_p = 2 + \kappa_\omega + \kappa_\rho\,, \tag{26}$$

$$g_n = \kappa_\omega - \kappa_\rho\,, \tag{27}$$

which translates into $\kappa_\rho = 3.7$ and $\kappa_\omega = -0.13$. A more canonical version of vector meson dominance indicates that the nucleon electric charge is also a consequence of the vector mesons (while the direct coupling of the nucleon to the photon is negligible). We will not expose this idea here, though it is mathematically very simple, but simply indicate that the outcome is that κ_ρ and κ_ω are the ratio of the electric and magnetic type couplings of the vector mesons with the nucleons

$$\kappa_\rho = \frac{f_\rho}{g_\rho} \quad , \quad \kappa_\omega = \frac{f_\omega}{g_\omega}\,. \tag{28}$$

2. Conventions and Definitions

Previously we have written the magnetic moment operator as

$$\vec{\mu} = \vec{\mu}_L + \vec{\mu}_S = \frac{e}{2m}\,\vec{L} + g_S\frac{e}{2m}\,\vec{S}\,.$$

It is customary however to simplify the previous expression with the definiton of the *nuclear magneton*

$$\mu_N = \frac{e}{2M_N}\,, \tag{29}$$

where $m = M_N$, the nucleon mass. With this we write

$$\vec{\mu} = \mu_N\left(\vec{L} + g_S\,\vec{S}\right), \tag{30}$$

but sometimes we will simply remove the μ_N altogether (for the simple reason that μ_N is just a unit). The intrinsic magnetic moment is also written sometimes in terms of $\mu_s = g_s/2$, in which case we write

$$\vec{\mu}_S = \mu_N \left(g_S \vec{S} \right) = \mu_N \left(\mu_S \vec{\sigma} \right) , \tag{31}$$

where $\vec{\sigma}$ refer to the Pauli matrices. Equivalently, if we remove μ_N (because we already know that the nuclear magnetic moments are measured in nuclear magnetons)

$$\vec{\mu}_S = \mu_S \vec{\sigma} \, , \tag{32}$$

which is definitely a very convenient expression.

3. Magnetic Moment of the Deuteron and the D-wave Probability

Now we apply the previous ideas to the magnetic moment of the deuteron, which is experimentally known to be

$$\mu_d = 0.8573 \, \mu_N \, . \tag{33}$$

Here an explanation is in order: previously we have indicated that the magnetic moment is a vector, but we have shown a pure number for the magnetic moment of the deuteron. This is simply a matter of definition. Usually for a system with angular momentum J, we define the magnetic moment simply as

$$\mu = \langle JJ | \vec{\mu} | JJ \rangle \, , \tag{34}$$

that is, we define it as the matrix element of the magnetic moment operator for the state with maximum third component of the spin operator.

With this definition in mind, we begin the theoretical calculation of this magnetic moment with the magnetic moment operator for the deuteron

$$\vec{\mu}_d = \vec{\mu}_p + \vec{\mu}_n = \mu_N \left(\vec{L}_p + g_p \vec{S}_p + g_n \vec{S}_n \right) = \mu_N \left(\vec{L}_p + \mu_p \vec{\sigma}_p + \mu_n \vec{\sigma}_n \right) , \tag{35}$$

where $\mu_p = 2.793$ and $\mu_n = -1.913$. The deuteron has angular momentum $J = 1$, which means that

$$\mu = \langle 11 | \vec{\mu} | 11 \rangle \, , \tag{36}$$

where $|11\rangle$ refers to the $J = +1$, $M = +1$ total angular momentum wave function If the deuteron is indeed $J^P = 1^+$ and the spin of the deuteron is $S = 1$, then we can construct the deuteron as a sum of a S-wave and a D-wave state

$$|\Psi_d\rangle = a_s |^3S_1\rangle + a_d |^3D_1\rangle \, , \tag{37}$$

where $^3S_1$ and $^3D_1$ refer to the S- and D-wave (in spectroscopic notation). It turns out that if the deuteron was a pure S-wave or a pure D-wave state, the calculation of the magnetic moment is easy (**exercise**, 2 points)

$$\mu(^3S_1) = \mu_p + \mu_n = 0.88 \tag{38}$$

$$\mu(^3D_1) = \frac{3}{4} - \frac{1}{2} \left(\mu_p + \mu_n \right) = 0.31 \, , \tag{39}$$

where everything is in units of nuclear magnetons. Then for a real world deuteron that contains a mixture of S- and D-wave

$$\mu_d = |a_s|^2 \, \mu(^3S_1) + |a_d|^2 \, \mu(^3D_1) = P_S \, \mu(^3S_1) + P_D \, \mu(^3D_1) \, , \tag{40}$$

where we have taken into account that $P_S = |a_s|^2$ and $P_D = |a_d|^2$ are the S- and D-wave probabilities respectively. From this and $\mu_d = 0.8573$, we get $P_D = 0.04$.

4. Meson Exchange Currents

Actually the previous estimation of the D-wave probability of the deuteron is incomplete: it only takes into account the existence of the nucleons. However the deuteron also contains virtual pions. It can happen that an external photon hits a pion that is being exchanged between the neutron and the proton. This type of process, though not so probable as the photon hitting a nucleon, will indeed change the predictions for the deuteron electromagnetic properties. The

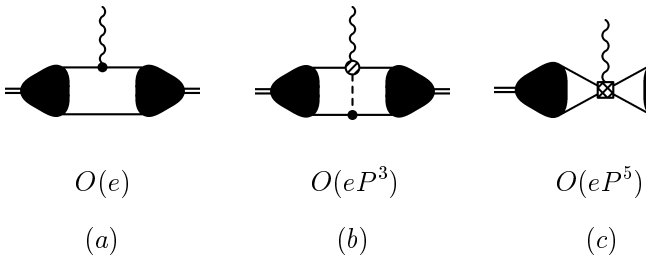


FIG. 2. Different contributions to the electromagnetic current for the deuteron. Figure (a) shows the contribution from the one body current, i.e. when a photon simply hits a nucleon, which is the contribution that we consider in the main text when we discuss the magnetic moments. Figures (b) and (c) discuss other type of contributions, basically when the photon does something different than hittin a nucleon. It could be hitting a pion-in-flight, though this contribution does not appear in the figure above, hitting a pion while emitted (b) or probing the short-range structure of the deuteron (c). The image is taken from Ref. [2], which discusses the impact of two-body currents on the description of electromagnetic properties of the deuteron.

change is small indeed, yet the accuracy of theoretical calculations and of experiments tells us that this effect can indeed be observed. The name of these processes is "two-body currents". By the way, the two-body currents are also important for another theoretical reason: the D-wave probability is not an observable quantity. This indicates that in fact the calculation that we did in the previous section has to be incomplete, and the two-body currents are the answer.

In the case of the deuteron the impact of two-body currents on the magnetic moment is relatively small, maybe of the order of 1%, though the exact number depends on the details of the deuteron description. For the deuteron quadrupole moment, which is experimentally known to be $Q_d = 0.2859(3)\,\mathrm{fm}^2$, the usual two-body contribution is $Q_{d,2B} \sim 0.01\,\mathrm{fm}^2$, about $3-4\%$. Yet the process in which the two-body currents were firmly stablished as a relevant physical effect is *radiative neutron capture by a proton*

$$n + p \to d + \gamma\,, \tag{41}$$

that is, a neutron hitting a proton to become a deuteron, with the emission of a photon. This process is important for a few reasons: (i) the cross section is really big owing to the very large singlet scattering length, (ii) it is the reason why slow neutrons are really bad for human health (they love to combine with the free protons in all the water molecules of our bodies) and (iii) meson exchange contributions acccount for more than 10% of the capture cross section, as shown for the first time in Ref. [3].

5. Magnetic Moments of Even-Even and Even-Odd Nuclei

The magnetic moment of even-even and even-odd nuclei are amenable to easy theoretical estimations. In particular for even-even nuclei they are trivial:

$$\mu(0^+) = 0\,. \tag{42}$$

which is a consequence of all the pairs of neutrons and protons coupling to angular momentum zero (this also entails that their magnetic moments cancel out).

For even-odd nuclei with J^P, we can consider these nuclei as an even-even core with $J^P_{\rm core} = 0^+$ plus one *unpaired* nucleon that carries the angular momentum and parity of the nucleus. If this nucleon has angular momentum L, then the total angular momentum J is $J = L \pm \frac{1}{2}$. In terms of the magnetic moment we can write

$$\mu(A) = \mu_{\rm core} + \mu_N = \mu_N\,, \tag{43}$$

because the core is 0^+ and its magnetic moment is zero. In this case we can compute the magnetic moment for each of the configurations (**exercise**, 2 points)

$$\mu_N(J = L + \frac{1}{2}) = g_L(J - \frac{1}{2}) + \frac{1}{2}g_S\,, \tag{44}$$

$$\mu_N(J = L - \frac{1}{2}) = g_L\frac{J(J+\frac{3}{2})}{J+1} - \frac{J}{2(J+1)}g_S\,, \tag{45}$$

which are called the Schmidt values. Actually most magnetic moments of even-odd nuclei do not really correspond with the Schmidt values, which indicates that the previous picture of an unpaired nucleon and an inert core is not that correct. Yet most magnetic moments of even-odd nuclei fall within the two Schmidt values, which indicates that the previous picture is not such a bad approximation.

F. Nuclear Stability

Nuclei are only stable if their decays are energetically forbidden (or if there is a symmetry protecting them to decay). A very simple type of decay is α decay, which consists on the emission of an alpha particle by a nucleus

$$\alpha\colon \quad {}^{A}_{Z}X_N \quad \rightarrow \quad {}^{A-4}_{Z-2}X_{N-2} + \alpha\,, \tag{46}$$

which happens when the α separation energy is negative

$$S_\alpha(Z,N) = B(Z,N) - B(Z-2,N-2) - B(2,2)\,. \tag{47}$$

As previously mentioned, most heavy nuclei are α-unstable. However α decay times can be rather long.

Other two types of decays are β^- and β^+, which correspond to the emission of an electron and a positron respectively and can be written as

$$\beta^-\colon \quad {}^{A}_{Z}X_N \quad \rightarrow \quad {}^{A}_{Z+1}X_{N-1} + e^- + \bar{\nu}_e\,, \tag{48}$$

$$\beta^+\colon \quad {}^{A}_{Z}X_N \quad \rightarrow \quad {}^{A}_{Z-1}X_{N+1} + e^+ + \nu_e\,. \tag{49}$$

They involve an electroweak transition in which a proton/neutron is turned into a neutron/proton. Taking into account that the weak interaction is involved, the half-lives of these two processes are also rather long. The classical example of β^- decay is ${}^3H$

$${}^3H \quad \rightarrow \quad {}^3He + e^- + \bar{\nu}_e\,, \tag{50}$$

which is energetically possible because

$$Q = -B({}^3H) + (m_n - m_p) - B({}^3He) - m_e \geq 0\,. \tag{51}$$

In particular if we take into account that $B({}^3H) = 8.48\,\text{MeV}$ and $B({}^3He) = 7.72\,\text{MeV}$, we find that $Q = 0.0186\,\text{MeV}$ (pretty close to zero). The half-life of the triton is about 12.3 years.

There are other forms of decay. For instance, γ decay is an electromagnetic type of decay which happens between an excited state and the fundamental (or a less excited) state of a nucleus

$$\alpha\colon \quad {}^{A}_{Z}X^*_N \quad \rightarrow \quad {}^{A}_{Z}X_N + \gamma\,. \tag{52}$$

Being electromagnetic it happens really fast. Other instability is nuclear fission, which however does not happen that fast as it requires intermediate states with more energy than the inital state. Yet eventually it could happen through quantum tunneling, only that the times required are long (much longer than, for instance, the evaporation of a supermassive black hole).

II. THE LIQUID DROP MODEL

Nuclei have a lot of properties, of which the most obvious one is probably its binding energy. As previously mentioned, a really interesting regularity is that nuclear binding energy *saturates*. This means that the average binding energy per nucleon is roughly independent of the number of nucleons, with this number of the order of

$$\frac{B}{A} \simeq 8\,\text{MeV}\,. \tag{53}$$

Actually saturation is a consequences of the properties of the nuclear force:

(i) nuclear forces have finite range of about $1/m_\pi \simeq 1.4\,\text{fm}$

(ii) nuclear forces display a strong medium range attraction at about $1 - 2\,\text{fm}$

(iii) nuclear forces display a strong short range repulsion for $r < 1\,\text{fm}$.

From (ii) and (iii) we infer that if a system of A nucleons binds, its density should be roughly constant. This is because the potential has a minimum and the nucleons will have the tendency to be at that minimum, while the short-range repulsion prevents the nucleons to be squeezed together more tightly. Finally from (iii) we infer that the binding energy scales as the number of nucleons A. Had the nuclear force been long range, then every nucleon would have interacted with all the other nucleons and the binding energy would have been proportional to the number of pairs of A nucleons: $A(A-1)/2$. However this is not the case and each nucleon can only interact with the nucleons in its vicinity, i.e. with a given number of nucleons. As a consequence the binding energy scales as A.

Actually this situation is not unique to nucleons. The atom-atom interaction is indeed similar in a few aspects to the nucleon-nucleon one. For instance, the Lenard-Johnes potential

$$V_{AA}(r) = \frac{1}{2\mu_{AA}} \left[-\frac{R_6^4}{r^6} + \frac{R_{12}^{10}}{r^{12}} \right] , \tag{54}$$

also displays finite range, intermediate range attraction and short range repulsion. The comparison with atom-atom systems also leads to another interesting idea: systems of many atoms form drops of liquid. Thus by analogy we can expect systems of many nucleons to also form drops of liquid. This is the basis of the liquid drop model.

In the liquid drop model the nuclei are not considered as many body systems composed of A nucleons, but rather as small droplets of nuclear matter (which happens to be liquid). The energy of the liquid drop is then the consequence of different contributions. The most obvious contributions are volume and surface:

$$B(Z,A) = a_V A - a_S A^{2/3} + \dots , \tag{55}$$

which reflect that the binding of a droplet should be proportional to its volume, but includes a correction from the surface tension. As for a standard liquid the sign of the surface contribution is the contrary to the volume one. This ensures for instance that droplets are form: the way to minimize the surface energy is to enclose the liquid within the minimum surface possible. Typical sizes of the volume and surface term are $a_v \sim 16\,\mathrm{MeV}$ and $a_S \sim 18\,\mathrm{MeV}$. In addition nuclei are positively charged, as they contain protons. This indicates that we have to include another term that takes into account the Coulomb repulsion among the protons

$$B(Z,A) = a_V A - a_S A^{2/3} - a_C \frac{Z(Z-1)}{A^{1/3}} + \dots . \tag{56}$$

where the factors involved are $Z(Z-1)$ (the number of interacting proton pairs) and $1/A^{1/3}$ for the average separation between the protons. The size of $a_V \sim 0.7\,\mathrm{MeV}$, smaller than the other two terms owing to its electromagnetic origin. But as the number of protons grows, it eventually implies that nuclei with a very large number of protons will become unstable. The next term that we have to include is the asymmetry term, which takes into account that a nucleus will tend to have the same number of protons and neutrons (while they dislike having a really different number of protons and neutrons)

$$B(Z,A) = a_V A - a_S A^{2/3} - a_C \frac{Z(Z-1)}{A^{1/3}} - a_A \frac{(Z-\frac{A}{2})^2}{A} + \dots . \tag{57}$$

This symmetry is broken owing to the Coulomb term, which implies that large nuclei tend to have more neutrons than protons. The size of $a_A \sim 23\,\mathrm{MeV}$. Last there is a paring term, which tries to explain while even-even nuclei are the most common. The reason for this term is that neutron and protons tend to pair to total angular momentum zero, as this configuration has minimum energy. The addition of this term completes the standard liquid drop formula (or semi-empirical mass formula)

$$B(Z,A) = a_V A - a_S A^{2/3} - a_C \frac{Z(Z-1)}{A^{1/3}} - a_A \frac{(Z-\frac{A}{2})^2}{A} + \frac{((-1)^Z + (-1)^N)}{2} a_p A^{-1/2} . \tag{58}$$

The size of the pairing coefficient is usually $a_p \sim 11\,\mathrm{MeV}$.

III. THE SHELL MODEL

The shell model exploits the fact that nucleons are fermions that live in a sort of mean field potential to explain several features of nuclear structure. The starting point for the shell model is the A-body hamiltonian

$$H = \frac{1}{2M_N} \sum p_i^2 + \sum V_{2B}(r_i, r_j) + \sum V_{3B}(r_i, r_j, r_k) + \dots \tag{59}$$

which includes the kinetic energy, the two-body potential, three-body potential and where the dots include higher body forces. Now we can include a mean-field potential V_{MF} that acts on all the particles. The way to do it is the following trick

$$H = \frac{1}{2M_N}\sum p_i^2 + \sum V_{MF}(r_i) + \Delta V\,, \tag{60}$$

where ΔV, the residual interaction, is defined as

$$\Delta V = \sum V_{2B}(r_i, r_j) + \sum V_{3B}(r_i, r_j, r_k) + \cdots - \sum V_{MF}(r_i)\,, \tag{61}$$

that is, we have simply added and removed the mean field potential. The expectation is that for a good choice of the mean field potential the residual interaction will be small and hence perturbative. Thus we will be left with a very simple hamiltonian which accepts a solution of the type

$$\Psi_A = \Pi_{i=1}^A \phi_i(r_i) \quad \text{such that} \quad H_i\phi_i(r) = \epsilon_i\phi_i\,, \tag{62}$$

where Π represents the product of states and H_i is given by

$$H_i = \frac{p_i^2}{2M_N} + V_{MF}(r_i)\,. \tag{63}$$

This is indeed a pretty simple solution, but we have not yet taken into account the fact that nucleons are fermions and that the total wave function must be antisymmetric. This is easy to amend though with

$$\Psi_A = \frac{1}{A!}\sum_\sigma \Pi_{i=1}^A (-)^\sigma \phi_{\sigma(i)}(r_i)\,, \tag{64}$$

where σ is a permutation and $(-1)^\sigma$ refers to the sign of the permutation [1]. The total energy of such a state is

$$E = \sum \epsilon_i\,. \tag{65}$$

If our choice of a mean field has been good enough, this might represent a good approximation for both the wave function and the energy of the system.

Now we can try to explore the consequences of the shell-model with a few choices of mean interactions. A really simple choice is the harmonic oscillator potential

$$V_{\rm MF}(r) = \frac{1}{2}M_N\,\omega^2\,r^2\,, \tag{66}$$

for which the spectrum of the single-particle states is given by

$$\epsilon_i(n,l) = \omega(n + l + 3/2) \tag{67}$$

Where n, l are integers. Now we explore what happens when we start to include nucleons. In this potential the lowest energy shell is the one defined by $n = 0$ and $l = 0$, with energy $\frac{3}{2}\,\omega$. Taking into account that nucleons have spin, in this shell we can have a total of two neutrons or two protons, giving us the *magic number* $N = 2$. The next shell has energy $\frac{5}{2}\,\omega$, which corresponds to $l = 1$: if there was no spin, this would imply a total of $(2l + 1) = 3$ fermions, but after taking into account spin there should be 6 neutrons or protons in this shell. Together with the 2 fermions in the previous shell, we find the new magic number $N = 2 + 6 = 8$. If we repeat this process, we see that the magic numbers we predict are

$$N = 2, 8, 20, 40, 70, \ldots. \tag{68}$$

This series does not match exactly with the known magic numbers ($N = 2, 8, 20, 28, \ldots$), but is able to reproduce the first three of them, which is not that bad for such a simple model. However it clearly indicates that there is a missing ingredient, which has to do with the fact that the nuclear force contain more operators besides a central force.

The missing ingredient is in fact the spin-orbit force, that is, a component of the potential that behaves as

$$V_{LS} = \vec{l}\cdot\vec{s}\,v_{ls}(r)\,. \tag{69}$$

---

[1] The sign of a permutation is (-1) for every pair that is exchange. For example, if we have $A = 3$ particles, the permutation $(123) \to (123)$ is even ($(-1)^\sigma = +1$), the permutation $(123) \to (213)$ is odd ($(-1)^\sigma = -1$), the permutation $(123) \to (231)$ is even ($(-1)^\sigma = +1$), etc. Cyclic permutations $(123\ldots A) \to (23\ldots A1)$, $(34\ldots A12)$, etc., are always even.

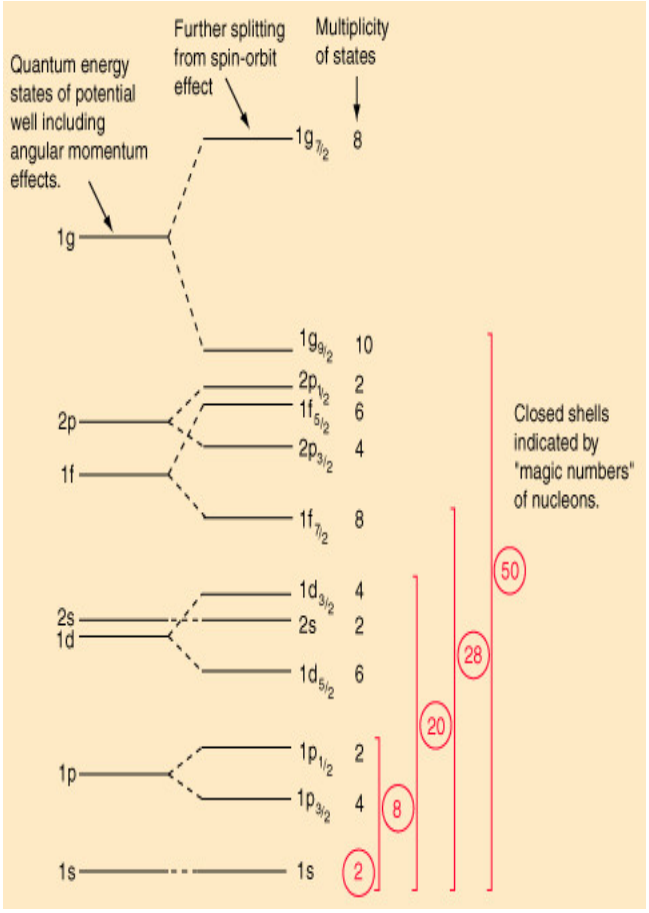


FIG. 3. Ordering of the orbitals in the shell model. Image taken from hyperphysics.

In fact if we try the following model

$$V_{\rm MF}(r) = \frac{1}{2}M_N\omega^2\, r^2 - \kappa\vec{l}^2 - \zeta\vec{l}\cdot\vec{s}\,, \tag{70}$$

the new energy levels will be

$$\epsilon_i(n,l,j=l+\frac{1}{2}) = \omega(2n+l+3/2) - \kappa l(l+1) - \zeta\frac{l}{2} \tag{71}$$

$$\epsilon_i(n,l,j=l-\frac{1}{2}) = \omega(2n+l+3/2) - \kappa l(l+1) + \zeta\frac{1}{2}(l+1) \tag{72}$$

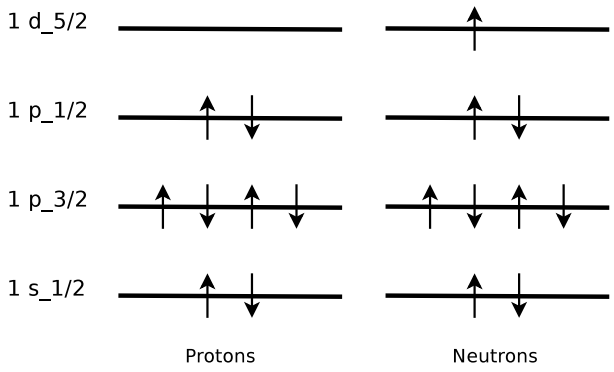
By tuning the parameters we can get in fact the magic number series $N = 2, 8, 20, 28, 50, 82, 126, 184$. The ordering of the orbitals is the one given in Fig. 3.

A. Basic Examples of the Shell-Model: filling shells

Now we will consider a few examples from which we can understand how the shell-model is used in practical applications. Probably the most simple case is the ^4He nucleus, which contains two protons and two neutrons and it is thus doubly magical. The way in which we fill the shells for the two protons and two neutrons is:

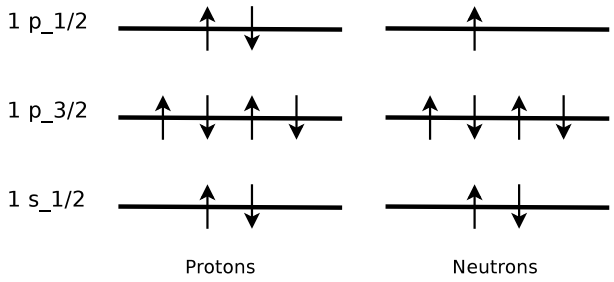
in which the arrows pointing in opposite directions indicate that spin of the two protons and the two neutrons add up to zero. This is a general feature of the shell model called *pairing*: two protons or two neutrons in the same shell will always add their total angular momentum to zero. The parity is $P = +1$, because the protons and neutrons also couple to orbital angular momentum zero. That is, the spin-parity of ^4He is $J^P = 0^+$, a prediction which is correct. This also means that if we want to determine the angular momentum of some configurarion of neutrons and protons in the shell-model we only have to take a look at the unpaired protons/neutrons.

The next example are the ^{17}O and ^{17}F nuclei, which have 8 proton + 9 neutrons and 9 protons + 8 neutrons respectively (they are *mirror nuclei*). In both cases what we have to do is to fill the shells in the order shown in Fig. 3, which leads to



for the case of ^{17}O, plus a very similar diagram for ^{17}F in which we only have to exchange protons by nucleons. Owing to pairing, the only neutron/proton that contributes to the angular momentum and parity of the $^{17}O/^{17}F$ nucleus is the one in the $^1d_{5/2}$ shell, which has $J = \frac{5}{2}$ and parity $P = (-1)^2 = +1$ (as it is a d-wave with $L = 2$). Thus for ^{17}O and ^{17}F the ground state is $J^P = \frac{5}{2}^+$.

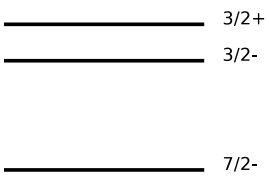
Now we can check what happens if a shell is missing a nucleon. The examples will be ^{15}O and ^{15}N, with 8 protons + 7 neutrons and 7 protons and 8 neutrons respectively. For the case of ^{15}O the filling of the shells happens as follows



while for ^{15}N we only have to exchange the neutrons and protons of this diagram. In both case the outcome is that we have an unpaired nucleon in the $1p_{1/2}$ shell, which means $J^P = \frac{1}{2}^-$.

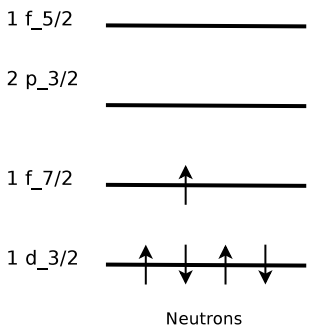
B. Excited states in the Shell-Model

The next example calculation we will tackle is that of the groung and excited states of ^{41}Ca, a nucleus with 20 protons and 21 neutrons with the following energy levels:



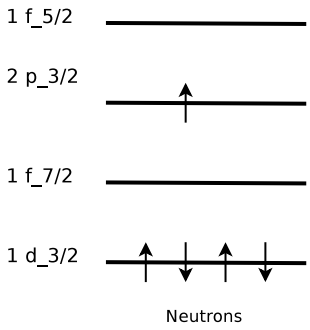
where the $\frac{7}{2}^-$ is the ground state, the $\frac{3}{2}^-$ the first excited state and the $\frac{3}{2}^+$ the second excited state.

The first thing to notice is that the number of protons is $Z = 20$, which is magical. As a consequence, the quantum numbers of the ground and first few excited states will probably only depend on the configuration of the neutrons. Thus from now on we will only consider the neutrons for this nuclei. For the ground state we simply have to fill the shells, where we end up with



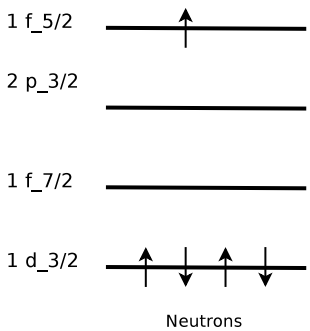
in which we do not show the $1s_{1/2}$, $1p_{3/2}$, $1p_{1/2}$, $1d_{5/2}$ and $2s_{1/2}$ shells, because they will be simply full and will play no role in the discussion to come. From the diagram above it is clear that the ground state of ^{41}Ca is $J^P = \frac{7}{2}^-$.

The first excited state is actually really easy to explain, we simply move the unpair neutron from the $1f_{7/2}$ shell to the $2p_{3/2}$ shell

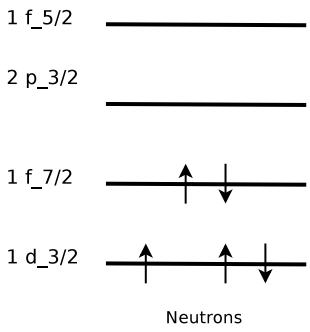


from which we reproduce the $J^P = \frac{3}{2}^-$ for the first excited state.

For the second excited state, the situation is more compex: there are two different possibilities of constructing the excited state, which we will call configuration A and B. In configuration A the neutron jumps from the $2p_{3/2}$ orbital to the $1f_{5/2}$ orbital:

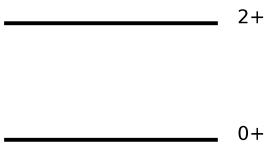


which results in the prediction $J^P = \frac{5}{2}^-$, which is not the correct one. In configuration B what we do is a bit different, we move a neutron from the $1d_{3/2}$ orbital (which was originally full) to the $1f_{7/2}$ one:

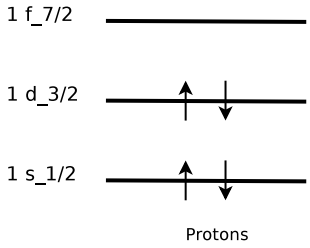


where the two neutrons in the $1f_{7/2}$ shell couple their total angular momentum to zero. Thus the angular momentum and parity of configuration B is determined from the unpaired neutron in the $1d_{3/2}$ shell, leading to $J^P = \frac{3}{2}^+$. This second configuration is the one that happens for the ^{41}Ca nucleus.

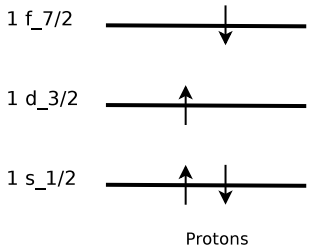
Other nucleus in which a similar situation happens is ^{38}Ar, which has 18 protons and 20 neutrons and for which the ground and first excited levels are:



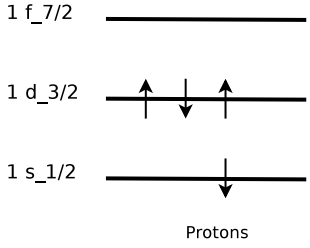
where the ground state is 0^+ and the first excited state is 2^+. Actually this type of ground/excited state combination is almost universal for even-even nuclei (nuclei with an even number of protons and neutrons). Owing to pairing, it is trivial to explain why the fundamental state is $J^P = 0^+$, where the configuration of the last and next-to-last shells is:



For the first excited state we have two possible configurations, A and B. Configuration A is as follows



while configuration B is



where each of the configurations yield different predictions. For configuration A the J^P of the first excited state can be 2^-, 3^-, 4^- and 5^-, while for configuration B we have $J^P = 1^+$ or 2^+. It happens that configuration B is again the preferred one.

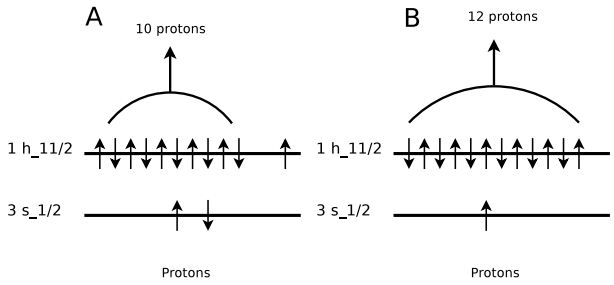
The reason why configuration B is chosen in both the second excited state of ^{41}Ca and the first excited state of ^{38}Ar is because of the residual interactions of the shell model. The type of residual interaction involved in this case is called *pairing force* and we will explain it in more detail in the following lines.

C. Residual pairing interaction in the Shell-Model

The pairing force is the most simple type of residual interaction and it can be written as

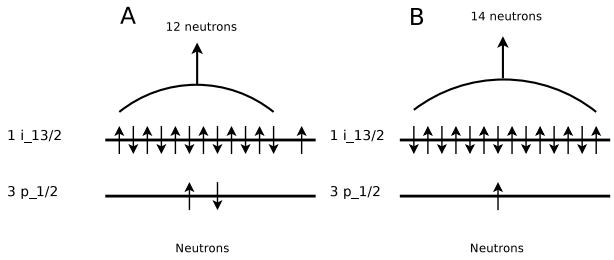
$$\langle jj(JM)|V_{\text{pairing}}|jj(JM)\rangle = -\frac{1}{2}\, g(j+1)\, \delta_{J0}\delta_{M0}\,, \tag{73}$$

where jj indicates two protons/neutrons in the same shell with angular momentum j (they couple to $J = 0$). That is, configurations in which two protons/neutrons with high angular momentum couple to total angular momentum zero are energetically more favorable. This is particular important when we are near shells with high j, in which case nucleons will tend to go from lower shells with low angular momentum to higher shells with high angular momentum if they can pair in the later. A classical example is the ground state of two isotopes of Thallium, ^{203}Tl and ^{205}Tl, both with 81 protons and 122 and 124 neutrons depending on the case. The angular momentum and parity of the fundamental state are determined from the protons, beacuse the neutrons will couple to angular momentum zero. Two possible configurations for the protons in Thallium are:



where A is the standard configuration that we will expect in the shell model when there are no residual interactions, and B is the preferred configuration if the pairing force is strong enough. Configuration A predicts $J^P = \frac{11}{2}^-$, while configuration B predicts $J^P = \frac{1}{2}^+$. Owing to the pairing force, the ground states of ^{203}Tl and ^{205}Tl are always in configuration B, with $J^P = \frac{1}{2}^+$.

Another example is ^{207}Pb (Lead), which has 82 protons and 125 neutrons. In this case the spin-parity of the ground state is determined by the neutrons, where two possible configurations are:



where configuration A predicts $J^P = \frac{13}{2}^+$ and configuration B predicts $J^P = \frac{1}{2}^-$. Owing to the pairing force, configuration B represents a lower energy state and it is the configuration of the ground state of ^{207}Pb.

IV. THE COLLECTIVE MODEL

The shell model is in general able to explain pretty well the properties of nuclei that are not far from a closed shell (i.e. such that N and Z are close to a magic number). However explaining the quantum number of the excited states

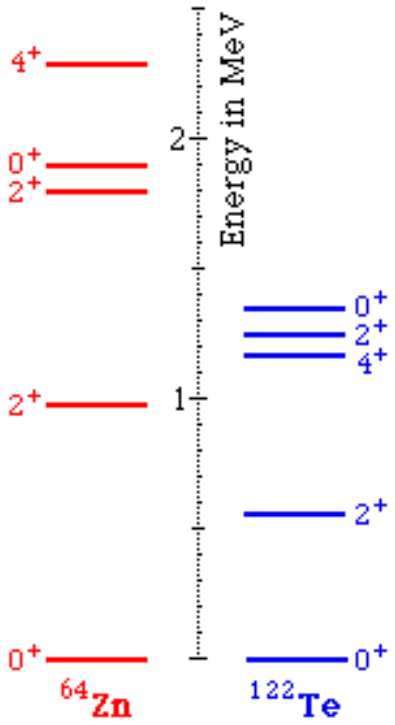


FIG. 4. Vibration spectrum in ^{64}Zn and ^{122}Te, where it can be seen that the first excited state is a 2^+ and that the following 0^+, 2^+, 4^+ excited states have an energy that is roughly twice that of the 0^+ state.

of nuclei far from the closed shells requires other types of explanations. Particularly interesting are a few properties of nuclei that are *collective*, i.e. they do not require at all to consider the nuclei as a compound object but rather as a unit (as happened for instance in the liquid drop model).

There are two examples of these type of collective behaviour: one are the vibrational modes and the other the rotational modes. The vibrational modes are natural to expect if we consider nuclear matter as a liquid: in this case it would be perfectly possible to have vibrations of this nuclear liquid, and these vibrations generate the excited states of many nuclei, The rotational modes happen if we consider a deformed nuclei as a rigid solid that can rotate about an axis. This will generate a different series of excited states.

A. Vibrational Modes

If nuclear matter is indeed a liquid, vibrational modes simply are vibrations of the nuclear fluid. If a heavy nucleus far from a closed shell is very much like a liquid drop of nuclear matter then there the surface of the drop can vibrate, much like a drop of a classical fluid. The obvious difference with a classical fluid is that these vibrations will be quantized. First, we begin by describing the form of the surface of a liquid drop as a sum of spherical harmonics

$$R(\theta,\phi;t) = R\sum \alpha^*_{\lambda\mu}(t)Y_{\lambda\mu}(\theta,\phi)\,, \tag{74}$$

where R is the radius of the liquid drop, $Y_{\lambda\mu}$ the spherical harmonics and $\alpha_{\lambda\mu}$ the coefficients in the expansion, where we assume that the coefficients depend on time because we are after all describing vibrations,

Now we can distinguish between several types of vibrations. We will begin with $\lambda = 0$, for which we have

$$R(\theta,\phi;t) = R\,\frac{\alpha_{00}(t)}{\sqrt{4\pi}}\,, \tag{75}$$

where it is useful to remember that $Y_{00} = 1/\sqrt{4\pi}$. This type of vibrations correspond to the expansion and contraction of a liquid drop and they are called the *breathing mode*. If in a first approximation we assume the nuclear liquid to be incompressible (i.e. the density of the liquid is constant), then this vibrations are impossible. As a consequence we can ignore this type of surface vibration. As a matter of fact nuclear matter can be compressed just like any

other liquid. Yet this is not particularly easy and requires a lot of energy. This means that when one quantizes the breathing mode of the nucleus one will get states that are very energetic. A possible example of a collective breathing mode might be the first excited state of the ^{4}He nucleus, which is a 0^+ state located about 20 MeV above the ground state. This 20 MeV energy for the breathing excitation is indeed much larger than the typical excitation energy for higher multipolar excitations in the collective model.

If we continue with $\lambda = 1$, this corresponds to

$$R(\theta, \phi; t) = R \sum \alpha^*_{1\mu}(t) Y_{1\mu} = R\, \vec{\alpha}(t) \cdot \hat{r}\,. \tag{76}$$

Now if we reinterpret carefully the previous equation, it actually corresponds to a translation of the drop by a distance $|\alpha(t)|$ in the direction $\hat{\alpha}(t)$. In other words, there is no deformation whatsoever of the surface of the liquid drop. This type of vibration mode is actually an spurious translation, i.e. an artifact of the description we have employed for the surface of the liquid drop. We can simply ignore it.

Finally for $\lambda \geq 2$ we have the actual deformations of the liquid drop. For $\lambda = 2$ we have the quadrupole modes, for $\lambda = 3$ the octupole modes, for $\lambda = 4$ the hexadecupole modes and so on. In general the larger the λ the less important these modes will be for the description of the excited states of the nucleus. For that reason here we will mostly consider the quadrupole and octupole modes only.

The vibrations of the surface of the nuclear fluid are quantized in a very characteristic way (similar to the excitations of a harmonic oscillator, i.e. we apply the idea of *second quantization* on the vibrational modes). Mathematically it is convenient to think of these quantized vibrations as a type of particle, which we will call *phonon*. Their origin is indeed similar to that of the photon, which are excitations of the electromagnetic field. They are also similar to the photon in the sense that the phonons are bosons. The phonon for the quadrupole excitation has quantum numbers 2^+ and a fixed energy $\omega_2 \sim 0.5 - 1.0$ MeV, where the number depends on the nucleus. For two quadrupole phonons the energy will be $2\omega_2$ and the quantum numbers will be 0^+, 2^+ and 4^+, where 1^+ and 3^+ do not happen because they require an antisymmetric wave function and phonons are bosons. The octupole phonon has quantum numbers 3^- and energy $\omega_3 \sim 2\omega_2$, though this relation is not exact and can vary from one nucleus to another. Examples of vibrational spectra are ^{64}Zn and ^{122}Te, which can be seen in Fig. 4

B. Rotational Modes

Now we can check the spectrum of ^{164}Er, which is quite different from the one we would expect in the case of a vibrational spectrum. What is happening here? One thing is that the quadrupole moment of ^{164}W is relatively big in comparison with other nuclei. In particular nuclei in the region $A = 150 - 190$ tend to have very large quadrupole moments. This indicates that they are deformed nuclei and can have energy levels owing to rotations. A very simple explanation of these rotational levels begins with the energy of a simple rotor

$$H = \frac{1}{2} I \omega^2\,, \tag{77}$$

where I is the momentum of inertia and ω the angular momentum. We can rewrite this hamiltonian by taking into account that the angular momentum of a rigid solid can be defined as

$$\vec{L} = I\, \vec{\omega}\,, \tag{78}$$

from which we can rewrite the hamiltonian as

$$H = \frac{\vec{L}^2}{2I}\,. \tag{79}$$

From this we can obtain the quantum levels of a rigid rotor, which happen to be given by the quantization of angular momentum, i.e. $\vec{L}^2 = L(L+1)$ with $L = 0, 1, 2, 3, 4, \ldots$. Now it happens that the levels with odd angular momentum $L = 1, 3, 5, \ldots$ imply an antisymmetric wave function which is imcomparible with the symmetric wave function of the rotor. From this we are left with $L = 0, 2, 4, \ldots$, yielding the series 0^+, 2^+, 4^+, 6^+ and so on. One of the predictions of this model is that $E(4^+)/E(2^+) = 3.3$, which happens to be fulfilled in a large number of heavy nuclei.

V. INTERACTING SHELL MODEL

Now we will briefly check how residual interactions are dealt with in the shell model. As already discussed, the basic idea of the shell model is that all the two-body, three-body, etc. interactions can be approximated by a mean

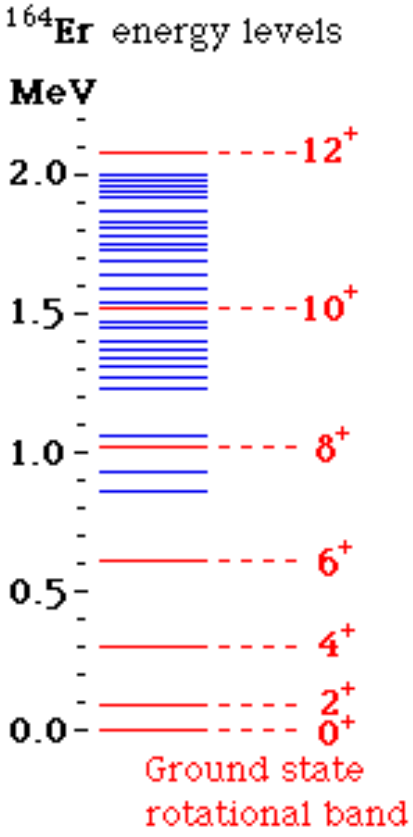


FIG. 5. Rotational bands of ^{164}Er, mixed with other non-rotational states. Yet the first few excited states are very clear examples of rotational bands.

field. That is, we begin with this Hamiltonian

$$H = \frac{1}{2M_N}\sum p_i^2 + \sum V_{2B}(r_i, r_j) + \sum V_{3B}(r_i, r_j, r_k) + \dots \tag{80}$$

and end up with the equivalent Hamiltonian

$$H = \frac{1}{2M_N}\sum p_i^2 + \sum V_{MF}(r_i) + \Delta V\,, \tag{81}$$

where the residual interaction ΔV is

$$\Delta V = \sum V_{2B}(r_i, r_j) + \sum V_{3B}(r_i, r_j, r_k) + \cdots - \sum V_{MF}(r_i)\,. \tag{82}$$

As far as the residual interaction is included, the two Hamiltonians will yield equivalent results. Of course the idea of the shell model is to simplify the original Hamiltonian by a good choice of the mean field that allows us to ignore the residual interaction.

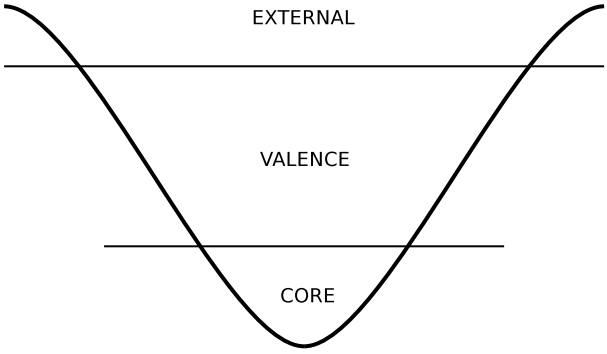
But this is merely a first approximation: if we want to improve the predictions of the shell model, we should include the residual interaction. This is usually called the interacting shell model. This is in general complicated, and the way to do it usually involves a series of simplifications. The most usual one is the following: if we consider a particular nucleus and want to include the residual interactions, we will divide the possible shells in which a nucleon can be located in three different types:

(a) Core: the core shells are always full.

(b) Valence: the valence shells can be occupied or empty.

(c) External: the external shells are always empty.

We can represent this idea graphically as



The way to do calculations in the interacting shell model is as follow. We begin with a mean field and the monoparticular wave functions generated by the mean field

$$h = \frac{1}{2M_N}p^2 + V_{MF}(r) \quad \text{with} \quad h\phi_i = \epsilon_i\phi_i \,. \tag{83}$$

With this we can create the usual antisymmetric wave functions

$$\Psi_\alpha = \mathcal{A}[\Pi_i\phi_i] \,, \tag{84}$$

where $\mathcal{A}$ is the operator that antisymmetrizes the product of wave functions and the subscript α represent a particular combination of the monoparticular wave functions. In this α all the levels of the core are occuppied and all the levels of the external space are empty, while for the valence shells a few levels will be occuppied and a few empty. Thus it is possible to have several α's leading to the same spin and parities, with the full wave function a linear combination of them

$$\Psi = \sum_{\alpha=1}^{N_V} c_\alpha \Psi_\alpha \,, \tag{85}$$

where the dimension of the linear space spawned by the Ψ_α's is finite (and we call it N_V). Now for taking into account the residual interaction we simply diagonalize the full Hamiltonian in this basis:

$$H_{\alpha\beta} = \langle\Psi_\alpha|H|\Psi_\beta\rangle \longrightarrow (H_{\alpha\beta})\Psi = E\Psi \tag{86}$$

where $(H_{\alpha\beta})$ is actually a $N_V \times N_V$ matrix and Ψ a vector. That is, with the idea of a valence space we have reduced an infinite-dimensional A-body problem into a linear algebra problem. Of course this is an approximation: unless we remove the core and the external space, in which case we end up again with an infinite-dimensional problem, what we are doing is an approximation. Actually the way in which the N_V dimension of this valence space is calculated is already a quite involved procedure and usually we end up with N_V's of the order of a million independent states or more. Notice also that recently it is more usual to see calculations in the *no-core shell model*, in which the core is eliminated.

VI. DERIVING THE MEAN FIELD: THE HARTREE-FOCK METHOD

A different problem which is related with the shell-model is how we derive the mean field potential. When explaining the shell-model we simply took a phenomenological approach, in which we assume a simple interaction that was able to correctly reproduce the known shell ordering and magic numbers. But actually there are more sophisticated methods to derive the mean field potential.

A. An iterative method for the mean field

A very simple method is to derive the mean field iteratively. The way to do it is to begin with some mean-field V_0

$$H = \frac{1}{2M_N}\sum p_i^2 + \sum V_0(r_i) + \Delta V \,, \tag{87}$$

with the residual interaction ΔV being

$$\Delta V = \sum V_{2B}(r_i, r_j) - \sum V_0(r_i)\,. \tag{88}$$

where we have ignore three-body forces and higher. With this mean field V_0, we can construct the typical wave functions

$$\Psi = \mathcal{A}[\Pi_i \phi_i]\,, \tag{89}$$

which can then be used to calculate the *residual mean field*, that is, the residual potential that particle i will feel from the other particles (Hartree term)

$$\Delta V_0(r_i) = \sum_{j\neq i} \int d^3\vec{r}_j |\phi_j(r_j)|^2 V_{2B}(r_i, r_j)\,. \tag{90}$$

With this we can define a new mean field V_1 such that

$$V_1(r_i) = V_0(r_i) + \Delta V_0(r_i)\,, \tag{91}$$

and repeat the process again till we reach convergence.

B. The Hartree-Fock method

Hartree-Fock is a more complex version of the same idea. We begin with some mean field potential $V_{\rm MF}$ and then construct the wave function

$$\Psi = \mathcal{A}[\Pi_i \phi_i]\,. \tag{92}$$

With this wave function we go back to the original Hamiltonian H (which for simplicity we will only consider to have two-body forces) and we minimize the energy by means of the variational principle

$$\delta\left[\langle\Psi|H|\Psi\rangle - \sum_i \langle\Psi|h_i|\Psi\rangle\right] = 0\,, \tag{93}$$

where the second term is the evaluation of the monoparticular Hamiltonians ($h_i = T_i + V_{{\rm MF},i}$). That is, what we are asking with the previous equation is for the minimum value of the difference between the energy as calculated with the full Hamiltonian and the energy as calculated with a mean field. We notice that the evualuation of the Hamiltonian operator between the previous wave function yields

$$\begin{aligned}\langle\Psi|H|\Psi\rangle = &\sum_i \langle\phi_i|T_i|\phi_i\rangle \\ &+ \frac{1}{2}\sum_{ij}\langle\phi_i\phi_j|V|\phi_i\phi_j\rangle \\ &- \frac{1}{2}\sum_{ij}\langle\phi_i\phi_j|V|\phi_j\phi_i\rangle\,,\end{aligned} \tag{94}$$

where the last term has a minus sign because we are dealing with Fermions. With this, if we apply the variational principle we find the condition

$$\begin{aligned}&T_i|\phi_i\rangle + \sum_j \int d^3\vec{r}'\,\phi_j^*(\vec{r}')\,V_{2B}(\vec{r},\vec{r}')\,\phi_j(\vec{r}')\,\phi_i(\vec{r}) \\ &- \sum_j \int d^3\vec{r}'\,\phi_j^*(\vec{r}')\,V_{2B}(\vec{r},\vec{r}')\,\phi_i(\vec{r}')\,\phi_j(\vec{r}) = \epsilon_i\phi_i(\vec{r})\,.\end{aligned} \tag{95}$$

This can be rewritten into a new equation for the monoparticular wave functions with a new, non-local mean field potential

$$T_i|\phi_i\rangle + \int d^3\vec{r}' U(\vec{r},\vec{r}')\phi_i(\vec{r}') = \epsilon_i\phi_i(\vec{r})\,, \tag{96}$$

where the mean field potential is defined as

$$U(\vec{r},\vec{r}') = \delta^3(\vec{r}-\vec{r}') \sum_{j=1}^{A} \int d^3\vec{r}'\, \phi_j^*(\vec{r}')\, V_{2B}(\vec{r},\vec{r}')\, \phi_j(\vec{r}') - \sum_{j=1}^{A} V_{2B}(\vec{r},\vec{r}')\, \phi_j^*(\vec{r})\, \phi_j(\vec{r}')\,. \tag{97}$$

The first term of this non-local potential is called the Hartree term and the second term is the Fock term. In general the Hartree-Fock equations will lead to non-local mean-field potentials, with an exception: if the two-body interaction we were beginning with is a contact-range interaction, the Fock term will be local. The reason is that for a contact-range potential we have

$$V_{2B}(\vec{r},\vec{r}') = C\, \delta^{(3)}(\vec{r}-\vec{r}')\,, \tag{98}$$

the Fock term reads

$$-\sum_{j=1}^{A} V_{2B}(\vec{r},\vec{r}')\, \phi_j^*(\vec{r})\, \phi_j(\vec{r}') = -C\, \delta^{(3)}(\vec{r}-\vec{r}') \sum_{j=1}^{A} |\phi_j(\vec{r})|^2\,, \tag{99}$$

which is local. Notice however that the contact-range potential that we have written leads to $U(\vec{r},\vec{r}')=0$. Why does this happen is left as an **exercise** (2 points).

C. The Skyrme interaction

As mention above, the Hartree-Fock mean field becomes local if the two-body interaction from which we begin with is a contact-range force. Skyrme proposed a particularly simple form for a two-body potential that happens to generate a relatively good mean field potential. The Skyrme two-body force takes the following form:

$$\begin{aligned} v(1,2) = {} & t_0\,(1+x_0P^\sigma)\,\delta^{(3)}(\vec{r}_1-\vec{r}_2) \\ & + \frac{1}{2}t_1\,\left[\delta^{(3)}(\vec{r}_1-\vec{r}_2)\,(-\nabla^2) + (-\nabla^2)\,\delta^{(3)}(\vec{r}_1-\vec{r}_2)\right] \\ & + t_2\,\vec{\nabla}\,\delta^{(3)}(\vec{r}_1-\vec{r}_2)\,\vec{\nabla} \\ & + iW_0\,(\vec{\sigma}_1+\vec{\sigma}_2)\cdot\left[\vec{\nabla}\times\left(\delta^{(3)}(\vec{r}_1-\vec{r}_2)\,\vec{\nabla}\right)\right]\,, \end{aligned} \tag{100}$$

where t_0, x_0, t_1, t_2 and W_0 are parameters and $P^\sigma = \frac{1}{2}(1+\vec{\sigma}_1\cdot\vec{\sigma}_2)$ (notice that we have spin dependence). This type of force is very similar to the contact-range interactions that are use in effective field theory, altough the motivation behind the Skyrme force is phenomenological. Actually there is also a three-body Skyrme force (because normally the Hartree-Fock method will be used with three-body forces), which reads

$$v(1,2,3) = t_3\,\delta^{(3)}(\vec{r}_1-\vec{r}_2)\,\delta^{(3)}(\vec{r}_2-\vec{r}_3)\,. \tag{101}$$

There are several Skyrme interactions, depending on the parameters we use. For instance, the Skyrme III interaction is defined by the following choices

$$\begin{aligned} t_0 = -1128.75\,\mathrm{MeV\,fm}^3 \quad , \quad t_2 = -95\,\mathrm{MeV\,fm}^5 \quad , \quad W_0 = +120\,\mathrm{MeV\,fm}^5\,, \\ t_1 = +395\,\mathrm{MeV\,fm}^5 \quad , \quad t_3 = +1.4\cdot 10^4\,\mathrm{MeV\,fm}^6 \quad , \quad x_0 = +0.45\,, \end{aligned}$$

but there are more than two hundred parametrizations of the Skyrme force. Besides Skyrme, other very popular two-body potential for Hartree-Fock is the Gogny force, which is basically like the Skyrme force plus a few terms with a finite range.

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