Nuclear Physics. Part I: Introduction

In this lecture we introduce a few basic concepts about the nucleon and the nuclear force, such as its general properties, the existence of a separation of scales, the the derivation of nuclear forces and the strategies to deal with this problem, isospin symmetry and chiral symmetry.

Welcome to the lecture notes of the post-graduate course on nuclear physics (Spring 2022). Notice that the lecture notes do not necessarily include all the topics covered in the actual lectures, while in a few cases they might contain things that were not actually covered in the lectures.

There is no unique textbook that we are following for this course. The contents are derived from a few textbooks, reviews and research papers in the field of nuclear physics. For the textbooks we can recommend

- *Nuclear Physics in a Nutshell* from C. Bertulani (for the general topics, but also for the nuclear models),
- *Advanced Quantum Mechanics* from J.J. Sakurai (for scattering theory),
- *Quarks and Leptons* from Halzen and Martin (for topics related to the standard model),
- *Lectures in Scattering Theory* from A.G. Sitenko (for scattering theory and the three body problem, wonderful classical russian soviet style book that does not shy away from complicated stuff, but hard to find however)

and maybe a few others. For recent reviews, the ones by R. Machleidt are particularly recommended, where one can easily find them in google or inspire-hep. R. Machleidt also has an excellent review in scholarpedia about nuclear forces. A few reseach papers will be referenced in these lectures notes, as you will see at the end of each handout.

For the evaluation of the course, which is always a hot topic: a few points will be given for assistance, a few for doing exercises and a few points will come from the exam. The exact details of how much points come from each of these three sources will be agreed upon in the classes, but exercises will be a big contribution. For the exercise part, I will grade you on the basis of a total of 10-15 (excercise) points (the exact number depends on the year): each exercise is equivalent to a certain number of points, usually between one (for easy excersices) and three (for hard exercises). The exercises will be found in separate exercise sheets that I will distribute later, though the lecture notes will also contain a few exercises wherever it is pertinent.

I. GENERAL CONSIDERATIONS

Nowadays we know that ordinary matter is made of atoms. In turn atoms are composed of a nucleus and a cloud of electrons around it, which bind owing to the electromagnetic interaction. The nucleus, which was discovered by Rutherford in 1911, is extremely small and contains most of the mass of the atom. We also know that nuclei are composed of neutrons and protons. They bind together due to the nuclear force to form the more or less 4000 nuclei that are experimentally known as of today. From deep inelastic scattering experiments we also know that neutrons and protons are not solid particles but are instead composed of three quarks, which are held together by the strong force. At distances below a small fraction of a fermi ($1\,\text{fm} = 10^{-15}\,\text{m}$) the strong force is completely analogous to the electromagnetic force, expect for the fact that instead of one type of electric charge there are three types of strong charge, which we call *colors*. However at distances of about 1 fm — the typical separation of neutrons and protons inside a nucleus — the strong force becomes terribly complicated and not mathematically solvable except by mammoth numerical simulations, which even nowadays can not be done gracefully. As a consequence the derivation of nuclear forces from first principles, i.e. from the strong force, is the most important open problem in nuclear physics.

Before starting, I can recommend a few references about the nuclear force which might be useful. An author that I particularly recommend is Machleidt, which has excellent reviews about the nuclear force: from the one boson exchange model [1], to the modern chiral approaches [2, 3], including a very recent historical perspective that is really nice to read [4]. Other review centered in the modern chiral approach is [5]. For the current approaches based on effective field theory, a very nice review is [6].

A. Why Nuclear Physics is Difficult : Separation of Scales

The character of nuclear physics depends on the *scales* involved in it. The term scale refers to the typical distance (or momentum) at which a physical process happens.

1. The Hydrogen Atom

To better understand the idea of a scale we will consider the case of atomic physics. The characteristic length scale of atomic physics, which describes the size of the atoms to give an example, can be determined from first principles. In particular from the Schrödinger equation and the Coulomb potential. First, the Coulomb potential between an electron and a proton is given by

$$V(r) = -\frac{\alpha}{r}\,, \tag{1}$$

where $\alpha \simeq 1/137$ is the fine-structure constant and r the radius. From this potential, we can write the Schrödinger equation as

$$\left[-\frac{1}{2\mu}\nabla^2 - \frac{\alpha}{r}\right]\Psi(\vec{r}) = -B\,\Psi(\vec{r})\,, \tag{2}$$

where $\Psi(\vec{r})$ is the wave function, B is the binding energy and μ the reduced mass of the system

$$\frac{1}{\mu} = \frac{1}{m_e} + \frac{1}{m_p}\,, \tag{3}$$

where m_e and m_p are the electron and proton masses, respectively. Owing to $m_p \ll m_e$, we can see that the reduced mass basically coincides with the mass of the electron. Rearraging this equation we have

$$\left[-\nabla^2 - \frac{2}{a_B\,r}\right]\Psi(\vec{r}) = -\gamma^2\,\Psi(\vec{r})\,, \tag{4}$$

where $\gamma = \sqrt{2\mu B}$ is the wave number of the bound state and a_B a length scale that is given by

$$a_B = \frac{1}{\mu\,\alpha} \simeq 5.29 \cdot 10^4\,\text{fm}\,. \tag{5}$$

We see that the only dimensionful quantities that we have in the equation are a_B and γ, yet they play a different role: a_B is a given parameter of the theory, while γ is a prediction of the theory.

It is also worth noticing that we are using units of MeV and fm here, which are usual in nuclear physics (but not in atomic physics), and that we will convert energy and distance units by means of

$$\hbar c = 197.3\,\text{MeV fm}\,, \tag{6}$$

We have already used this conversion factor in the definition of the Bohr radius $a_B = 1/(\mu\,\alpha) \sim (1/m_e\,\alpha)$, where it is easy to see that $1/m_e = 1.96\,\text{MeV}^{-1} = 381.1\,\text{fm}$ (the electron mass is $m_e = 0.511\,\text{MeV}$).

2. Traditional approach to the hydrogen atom

Now we will review how the hydrogen atom is usually solved in atomic physics. It is apparent that the following wave function

$$\Psi(\vec{r}) = \frac{1}{\sqrt{4\pi}}\,\frac{2}{a_B^{3/2}}\,e^{-(r/a_B)}\,, \tag{7}$$

is indeed a solution to the Schrödinger equation for the hydrogen atom, i.e. Eq. (2). In fact, this is the ground state wave function of the hydrogen atom. Knowing this solution it is very easy to calculate a series of observables. The binding energy B in Eq. (2) will be given by

$$B = \frac{1}{2\mu}\,\frac{1}{a_B^2}\,, \tag{8}$$

while the wave number γ as define in Eq. (4) is even simpler

$$\gamma = \frac{1}{a_B}\,. \tag{9}$$

Of course, these are not the only observables we can calculate. For instance, we might be interested in the mean square radius:

$$\sqrt{\langle r^2\rangle} \quad \text{where} \quad \langle r^2\rangle = \langle\Psi|r^2|\Psi\rangle\,, \tag{10}$$

where a direct calculation with the ground state wave function yields

$$\sqrt{\langle r^2\rangle} = \sqrt{3}\,a_B\,. \tag{11}$$

Alternatively, we might also be interested in the mean square speed of the electron within the hydrogen atom

$$\sqrt{\langle v^2\rangle} = \frac{1}{m_e}\sqrt{\langle \hat{p}^2\rangle}\,, \tag{12}$$

where $\hat{p}^2$ is the square of the momentum operator, i.e. $\hat{p}^2 = -\nabla^2$ in r-space. In this case, a direct calculation shows that

$$\sqrt{\langle v^2\rangle} = \alpha \simeq \frac{1}{137}\,, \tag{13}$$

Again, here it is worth commenting that we are working in units for which $c = 1$ (c being the speed of light), which means that the previous speed actually means $c/137$. By changing the units of the speed of light to km/s, we will see that $\sqrt{\langle v^2\rangle} \simeq 2200\,\text{km/s}$.

Study recommendation: please check the solution to the ground state wave function of the hydrogen atom, as well as the calculation of the observables we have proposed here ($\langle r^2\rangle$ and $\langle p^2\rangle$).

3. Scales-based approach to the hydrogen atom

Actually, there is a shortcut to obtain the previous results. With this shortcut there will be no need to do the relatively lengthy calculations required to obtain the actual value of the observables we have considered (the wave number, mean square radius and mean square speed).

If one looks carefully at Eq. (4), it is interesting to notice that there is only one input parameter in the equation: a_B. Thus, every observable we compute must depend solely on a_B. Indeed, the length scale a_B, which is known as the Bohr radius, is really interesting because is the only dimensionful number that we can build from the Coulomb interaction and the electron mass. What are the implications of this? Easy, every physical quantity that we predict for the Schödinger equation can be written in units of a_B. Thus, we should have the following proportionality relations

$$\gamma \propto \frac{1}{a_B} \quad , \quad \sqrt{\langle r^2\rangle} \propto a_B \quad , \quad \sqrt{\langle p^2\rangle} \propto \frac{1}{a_B}\,. \tag{14}$$

where the power of a_B has been choosen according to the dimensions of the observable. That is, if we have an observable $\hat{\mathcal{O}}$ with dimensions of $[length]^n$

$$\left[\langle\Psi|\hat{\mathcal{O}}|\Psi\rangle\right] = [length]^n\,, \tag{15}$$

where in this equation the [...] brackets indicate that we are talking about dimensions, then the expected value of this observable should be proportional to a_B^n

$$\langle\Psi|\hat{\mathcal{O}}|\Psi\rangle \propto a_B^n\,. \tag{16}$$

Of course the previous is trival, it is just dimensional analysis. The problem is: *what is the proportionality constant?* Here we will include a new hypothesis: we expect physical observables to be *natural* in the characteristic scale of the system at hand. In the case of the hydrogen atom, this characteristic scale is the Borh radius.

But what do we mean by natural? Let us consider the properties of the ground state of hydrogen-like atoms, in particular the three observables we have previously seen

$$\gamma_A = c_A\,\frac{1}{a_B} \quad , \quad \sqrt{\langle r^2\rangle} = d_A\,a_B \quad , \quad \sqrt{\langle p^2\rangle} = e_A\,\frac{1}{a_B} \quad , \tag{17}$$

where c_A, d_A and e_A are pure numbers (i.e. they do not have dimensions). According to the idea of naturalness, these numbers should be of $\mathcal{O}(1)$

$$c_A\,,\,d_A\,,\,e_A = \mathcal{O}(1)\,, \tag{18}$$

that is, they are likely to have a value like 1/3, 1 or 3. On the contraty it is fairly improbale that c_A, d_A or e_A are of the order of 1/300 or 300: these values are *unnatural*. If they happen they will require a good explanation.

The take-to-home message is the following:

(i) Most physical systems have a characteristic scale.

(ii) Most observables can be expressed as powers of this characteristic scale

$$\langle\Psi|\hat{\mathcal{O}}|\Psi\rangle = c_{\mathcal{O}}\, a_B^n\,.$$

where the proportionality constant will be in general a number of order one, $c_{\mathcal{O}} = \mathcal{O}(1)$. These systems are said to be *natural*.

(iii) On the contrary, if $c_{\mathcal{O}} \neq \mathcal{O}(1)$, the system will be *unnatural* or *fine-tuned*.

As far as the physical system we are considering is natural, it will be relatively easy to deal with.

4. Scales and the boiling point of Cesium

To show the type of thought process of a scale-based approach to a physical system, we will consider the problem of calculating the boiling point of Cesium. This example is particularly interesting because in contrast to the hydrogen atom, here we do not know how to do a detailed calculation of the temperature at which Cesium boils. Instead, we will have to make approximations grounded on the different physical scales that are related to Cesium.

First, we will consider the properties of Cesium: for concreteness, we will consider Cs-133, one of its isotopes (the only stable one), with atomic number 133. Thus, the mass of a Cesium atom will be around $m_{\rm Cs} \simeq 130\,\text{GeV}$ give or take (i.e. a bit less than 1 GeV per nucleon within its nucleus). Second, the size of the Cesium atom is about $R_{\rm short} \simeq 7\,a_B$ (this is actually called the van der Waals radius, as there are several definitions of the size of an atom), with a_B the Bohr radius that we have already seen.

The next important thing to know is the potential between two Cesium atoms. This potential can be described as the sum of a short- and long-range part

$$V(r) = V_S(r) + V_L(r)\,, \tag{19}$$

where there is repulsion at short distances and attraction at long distances, i.e. $V_S > 0$ and $V_L < 0$. Indeed, one of the most standard parametrizations of the atom-atom potential is Lennard-Jones

$$V_{\rm LJ}(r) = \frac{C_{12}}{r^{12}} - \frac{C_6}{r^6}\,, \tag{20}$$

with $C_6, C_{12} > 0$. Yet, this is only one parametrization among many: actually the only point in common of most parametrizations is that the long-distance tail of the potentials is a van der Waals potential:

$$V_L(r) = V_{\rm vdW}(r) = -\frac{C_6}{r^6}\,. \tag{21}$$

This is also the part of the potential that is well-known, i.e. there are determinations of the C_6 coefficient. In particular, we will rewrite the van der Waals potential in a way that is scale-friendly

$$V_{\rm vdW}(r) = -\frac{C_6}{r^6} = -\frac{1}{2\mu}\,\frac{R_6^4}{r^6}\,, \tag{22}$$

where we have separated the 2μ reduced mass factor from the potential to better identify the scales. In this scale-friendly parametrization, we have that $R_6 = 202\,a_B$. Notice that this is exactly what we did with the Coulomb potential

$$V(r) = -\frac{\alpha}{r} = -\frac{1}{2\mu}\,\frac{2}{a_B r}\,, \tag{23}$$

though in this case we introduced a factor of 2 in the definition of the scale a_B for convenience (i.e. because we actually knew the solution and thus a_B happens to be a bit more convenient that $a_B/2$, which is what we would have obtained without the factor of 2).

Next, if we have a set of Cesium atoms the expectation is that they will occupy the minimum energy configuration. That is, in the absence of external factors perturbing the system (e.g. finite temperature), on average the distance between Cesium atoms will coincide with the distance at which the potential has a minimum. Thus we want to calculate $R_{\rm min}$ such that $V(R_{\rm min}) \leq V(r)$ for all r. The problem is that we do not know the potential at short distances, but we can make a quick and dirty estimation nonetheless. The size of the Cesium atoms is about $R_{\rm short} \simeq 7\,a_B$, so at $r \leq 7\,a_B$ the potential will be repulsive. However, at distances that are twice as large as the size of the Cesium atom

the potential will probably behave like an attractive van der Waals. That is, the distance at which the potential reaches a minimum will be somewhere in between $(7-14)\,a_B$.

A possible compromise is then to assume that the minimum of the potential can be estimated from the value of the van der Waals potential at 1.5 times the size of the Cesium atoms, which we will approximate by $R_{\rm min} = 10\,a_B$, i.e.

$$V_{\rm min} \simeq V_{\rm vdW}(R_{\rm min})\,. \tag{24}$$

This is obviously a *guesstimate*: we do not know very much of what happens at short distances with the Cesium atoms. However, this guesstimate allows us to define a characteristic scale for the Cesium atoms, for which we use the usual trick of factoring out the reduced mass

$$|V_{\rm min}| = \frac{1}{2\mu}\,\frac{1}{R_{\rm Cs}^2} = \frac{1}{2\mu}\,Q_{\rm Cs}^2\,, \tag{25}$$

where for convenience we will choose the momentum scale Q_{Cs} as the reference value (depending on the problem it might be more useful to use a length or a momentum/energy scale). This momentum scale is given by

$$Q_{\rm Cs} = \frac{R_6^2}{R_{\rm min}^3} \simeq \frac{41.2}{a_B} \quad \text{or} \quad 41.2\,Q_B\,. \tag{26}$$

With this, we can estimate the mean kinetic energy of a Cesium atom to be

$$E_{\rm kin} = \frac{Q_{\rm CS}^2}{2m_{\rm Cs}}\,, \tag{27}$$

which implicitly assumes that the atoms will tend to be in configurations with minimize the potential energy (a different argument not directly involving the scale $Q_{\rm Cs}$ could have been invoking the virial theorem). However, these configurations are only possible if there are no external factors that modify the movement of the atoms, like for instance a finite temperature. Indeed, if there is a finite temperature the atoms will have on average a thermal energy of

$$E_{\rm th} = k_B\,T\,, \tag{28}$$

with k_B the Boltzmann constant and T the temperature, where $k_B = 0.8617\cdot 10^{-10}\,\text{MeV/K}$, i.e. at approximately $11600\,K$ the average energy of a particle will be 1 eV, while at $11.6\cdot 10^9 K$ the average energy of a particle will reach 1 MeV. The existence of a thermal energy means that we can only expect the atoms to be in a non-gaseous form if

$$E_{\rm kin} > E_{\rm th}(T)\,, \tag{29}$$

because once the thermal energy is larger than this characteristic kinetic energy for a group of bound Cesium atoms, then these atoms cannot be held together by the potential and the atoms will transition to a gaseous state. The temperature at which we expect this to happen will be given by the condition $E_{\rm kin} = E_{\rm th}(T_c)$, or equivalently

$$T_c = \frac{1}{2}\,\frac{Q_{\rm Cs}^2}{m_{\rm Cs}\,k_B} \sim 1031\,K\,. \tag{30}$$

This is to be compared with the experimental boiling point of $944K$, i.e. we got it right within the 10% level. Admittedly, we were probably a bit lucky here: many of the approximations we made cancel with each other resulting in a temperature close to the experimental one. In most practical settings, it is enough to get things right within the order of magnitude level, so a result of $500\,K$ or $2000\,K$ would have also been pretty good.

Indeed, if we expand the previous formula in terms of the two scales involved (R_6 and $R_{\rm min}$), we find that

$$T_c = \frac{1}{2}\,\frac{R_6^4}{m_{\rm Cs}\,k_B\,R_{\rm min}^6}\,. \tag{31}$$

It is easy to see that this estimation is incredibly dependent on $R_{\rm min}$, which we have taken to be 1.55 times the ratio of what is called the van der Waals radius of an atom (about $6.5\,a_B$ for Cesium). Just a 10% variation in the choice of $R_{\rm min}$ will lead to up to a 90% difference in this quick-and-dirty estimation of the boiling point of Cesium. Indeed, if we repeated the calculation for other atoms by using the R_6 values listed for instance in Ref. [7], we will obtain the following boling points: $9000\,K$ for Lithium (exp. $1603\,K$), $2670\,K$ for sodium (exp. $1156\,K$), $2106\,K$ for potassium (exp. $1032\,K$), $3272\,K$ for calcium (exp. $1757\,K$) and $1421\,K$ for rubidium (exp. $916\,K$). In general they are of the right order of magnitude (expect for lithium, where the discrepancy can be solved by increasing the value of $R_{\rm min}$ by just a 35%), though not as good as in our example cesium.

5. Multiscale problems with a hierarchy: corrections

Now one might ask: but what happen with the other scales in the system? Most physical systems contain more than one scale and it would be interesting to know how to take them into account, or at least what type of deviation to expect from the existance of these scales.

To illustrate multiscale systems we will use again the hydrogen atom as a representative example. Indeed there are other scales besides the Bohr radius: the Bohr radius just happens to be the most easy to identify scale (and the longest range one too). However, on closer inspection of the problem we will quickly realize the existence of other scales. For instance: what about relativistic corrections? or the finite size of the nucleus?

Actually, the interesting thing about thinking in terms of scales is that we can estimate the size of these short range effects. The expectation is that if the characteristic length scale of these effects is of order R_S (where we assume that $R_S \ll a_B$), then the relative size of the corrections coming from them should be of the order of

$$\mathcal{O}\left(\frac{R_S}{a_B}\right). \tag{32}$$

If we take the example of relativistic corrections, they are evident if the momentum of the electron is of the order of its mass. As a consequence the length scale for this short-range effect is

$$R_S^{\rm rel} = \frac{1}{m_e} \simeq 3.86\cdot 10^2\,{\rm fm} \quad \text{that is,} \quad \left(\frac{R_S^{\rm rel}}{a_B}\right)^2 \simeq 5\cdot 10^{-5}\,, \tag{33}$$

where we use $(R_S/a_B)^2$, instead of the expected (R_S/a_B), because relativistic corrections usually enter as a square (for instance in $\sqrt{p^2+m^2} = m + p^2/2m + 3p^4/8m + ...$). Other possible interpretation of the previous scale is the distance at which we begin to notice that the electron has structure. By this we mean corrections coming from quantum field theory: at distances around $1/m_e$ it is possible to have electron-positron virtual pairs forming around the real electron. That is, at this distance we will begin to see what is usually called *vacuum polarization*. This effect does not exist in non-relativistic quantum mechanics and thus it can be interpreted as a relativistic correction, as we have done at first. Now, the main effect of the relativistic corrections is a spin-orbit term (i.e. a term of the form $\vec{l}\cdot\vec{s}$, where $\vec{l}$ and $\vec{s}$ are the orbital and intrinsic angular momentum of the electron) that gives rise to the fine structure corrections of the hydrogen atom spectrum. The spin-orbit term does not affect the ground state, which is an S-wave, but only those states with $l \geq 1$. For example in the first P-wave state, which is usually calle $2P$. The binding energy of the $2P$ state happens to be $1/4$ of that of the $1S$ (or ground) state

$$B(2P) \simeq 3.4\,{\rm eV}\,. \tag{34}$$

Meanwhile, the spin-orbit term implies that the energy of the $2P$ state will be different dependent on whether the total angular momentum j (where $\vec{j} = \vec{l} + \vec{s}$) of the electron is $j = \frac{1}{2}$ or $j = \frac{3}{2}$. We give these states the names $2P_{1/2}$ and $2P_{3/2}$ and their binding energy is slightly different

$$\Delta B(2P) = B(2P_{1/2}) - B(2P_{3/2}) \simeq 4.5\cdot 10^{-5}\,{\rm eV}\,. \tag{35}$$

From this we can see that the relative size of this relativistic correction is

$$\frac{\Delta B(2P)}{B(2P)}\Big|_{\rm fine} \simeq 1.3\cdot 10^{-5}\,, \tag{36}$$

which is to be compare with our estimation of $5\cdot 10^5$. That is, the a priori estimation based on the scale separation alone is only off by a factor of 4, which is pretty good if we consider that we have not done any calculation at all.

Another short-range effect is the finite size of the proton. In principle we could define this finite size in similar terms as with the electron, that is, we should expect to see some structure at distances about

$$R_S^{\rm p(point-like)} = \frac{1}{M_p} \simeq 0.2\,{\rm fm} \quad \text{, implying that:} \quad \frac{R_S^{\rm p(point-like)}}{a_B} \simeq 3.8\cdot 10^{-6}\,. \tag{37}$$

However the proton is not a fundamental particle and it happens that we will begin to see its structure much sooner than the previous estimation:

$$R_S^{\rm p} \simeq (0.5-1.0)\,{\rm fm} \quad \text{, from which we expect:} \quad \frac{R_S^{\rm p}}{a_B} \simeq (1-2)\cdot 10^{-5}\,. \tag{38}$$

Actually, the physical manifestation of the proton finite size is the hyperfine splitting of the hydrogen atom, which is a spin-spin effect and thus also happens in the S-wave ground state. This means that the energy of the ground state will be different depending on whether the spins of the proton and electron are parallel ($S = 1$) or not ($S = 0$). The size of this effect happens to be $\Delta B \simeq 5.9 \cdot 10^{-6}\, eV$, or in relative terms

$$\frac{\Delta B(1S)}{B(1S)}\Big|_{\text{hyperfine}} \simeq 4.3 \cdot 10^{-7}\,. \tag{39}$$

In this case our estimation did not work so well: the difference is a factor of $25 - 50$. Part of the reason is that the hyperfine splitting is a magnetic effect, the size of which is usually further suppressed with respect to electric effects (for the electron too, i.e. there is an extra suppression that we did not take into account). Yet, even a failure by a factor of $25 - 50$ is still acceptable when we consider the incredible difference in the relative sizes of these effects.

What we have learned with this example is the following

(i) physical systems have a characteristic scale R_{long}.

(ii) physical quantities are usually (but not always, as we will see) natural in units of R_{long}.

(iii) there are corrections of relative size $R_{\text{short}}/R_{\text{long}}$, where R_{short} it the charecteristic scale of short-range physics.

For atomic physics naturalness works and in addition there is a very large and clear-cut separation of scales. This is actually one of the reasons why incredibly accurate calculations in atomic physics are easy and can be taught in undergraduate courses. Next we will consider a toy system with two scales — the square well — and show how the interplay between these two scales can give rise to unnatural behavior (i.e. behavior that is clearly not $\mathcal{O}(1)$ in terms of the characteristic scale of the system).

6. Multiscale problems with fine tuning

Now we will consider a problem with two scales: the square well. The square well is defined by the potential:

$$V(\vec{r}) = V_0\, \theta(a - r)\,, \tag{40}$$

where V_0 is the strength of the well and a its range. As usual, if we factor out the reduced mass of the system we will get the length scale associated with the strength of the potential

$$V_0 = -\frac{1}{2\mu}\,\frac{1}{R_S^2}\,, \tag{41}$$

where we are assuming an attractive well and R_S is said strength scale.

This means that we have two scales — R_S and a — giving rise to three possible hierarchies:

(i) $R_S \ll a$: the strength of the square well is the most important factor in the description of the system. There will be bound states and their binding energies will be natural (i.e. similar to V_0 in size).

(ii) $R_S \sim a$: both the strength and range are balanced. In fact they can be balanced in such a way that there could be a zero-energy bound state (or a bound state with a binding energy considerably smaller than V_0). This balance has to be extremly precise, which is why it is often referred to as *fine tuning*.

(iii) $R_S \gg a$: the range of the square well is the most important factor. However in this case there are no bound states, so we will not consider this case.

First we will consider the natural case: $R_S \ll a$. This describes a well that is very deep. In particular, if we take the limit

$$\frac{R_S}{a} \to 0\,, \tag{42}$$

what we will have is that the wave function will be trapped within the confines of the $r < a$ region, from which it will be impossible to escape as there is an infinite potential wall surrounding that region. If we rewrite the binding energy B as

$$B = \frac{\gamma^2}{2\mu} = -V_0 - \frac{\kappa^2}{2\mu}\,, \tag{43}$$

where

$$\kappa^2 = \frac{1}{R_S^2} - \gamma^2 \,, \tag{44}$$

then we have that the wave function trapped within the region $r < a$ can be written as

$$\Psi(r) \propto \frac{\sin \kappa r}{r} \,, \tag{45}$$

where we are assuming an S-wave solution for simplicity. This wave function will be subjected to the boundary conditions:

$$\Psi(r=0) = 0 \quad \text{and} \quad \Psi(r=a) = 0 \,, \tag{46}$$

where the second condition indicates that we are taking the $R_S/a \to 0$ limit. If we relax this condition, the wave function will be able to escape a bit outside the $r < a$ region. But if we continue working in the $R_S/a \to 0$ limit we find that the boundary conditions imply the following:

$$\kappa\, a = n\, \pi \quad \text{with} \quad n = 1, 2, 3, \dots \,. \tag{47}$$

As a consequence the binding energy will be

$$B = -V_0 - \frac{1}{2\mu} \left(\frac{n\pi}{a}\right)^2 = -V_0 \, \left(1 - \frac{R_S^2}{a^2} \, (n\pi)^2\right) \,, \tag{48}$$

or, if we take into account that we are indeed working in the $R_S/a \to 0$:

$$B = -V_0 \,, \tag{49}$$

which is a surprissingly simple result.

Indeed, what is interesting about this example is that we have automatically calculated the first correction to the $R_S/a \to 0$ limit, i.e.

$$B = -V_0 \, \left(1 - \frac{R_S^2}{a^2} \, (n\pi)^2 + \mathcal{O}(\frac{R_S^3}{a^3})\right) \,. \tag{50}$$

For calculating further corrections we have to remember how we calculated the energy levels of a square well, which can be reduced to this eigenvalue equation

$$\kappa \, \cot(\kappa a) = -\gamma \quad \text{where} \quad \kappa = \sqrt{\frac{1}{R_S^2} - \gamma^2} \,. \tag{51}$$

Previously, what we have done by taking the $R_S/a \to 0$ limit is to solve this equation

$$\kappa \, \cot(\kappa a) \to \infty \quad \text{that is} \quad \kappa a = n\pi \,, \tag{52}$$

or, if we want to avoid infinities and have a well-defined expansion parameter

$$\frac{\tan \kappa a}{\kappa a} = -\frac{1}{\gamma a} = -\frac{R_S}{a} \, \frac{1}{1 - R_S^2 \kappa^2} = \frac{R_S}{a} \, \left(1 + \frac{1}{2} \, R_S^2 \kappa^2 + \mathcal{O}(R_S^4 \kappa^2)\right) \,. \tag{53}$$

From this, we can define the first correction to our calculation of κ as

$$\kappa a = n\pi + \frac{R_S}{a} \, \delta \,, \tag{54}$$

where δ gives this correction. Concrete calculations give $\delta = -n\pi$ and a binding energy of

$$B = -V_0 \, \left(1 - \frac{R_S^2}{a^2} \, (n\pi)^2 + 2 \, \frac{R_S^3}{a^3} \, (n\pi)^2 + \mathcal{O}(\frac{R_S^4}{a^4})\right) \,. \tag{55}$$

This results improves by one power of R_S/a the calculation we had in Eq. (50). The technical term by which we refer to this type of calculations in which we count powers of a ratio of scales such as R_S/a is *power counting*.

Now we will consider the second possible hierarchy of scales, $R_S \sim a$. In this case in principle it looks like we have to solve the full eigenvalue equation of the square well, i.e.

$$\kappa \, \cot \kappa a = -\gamma \,, \tag{56}$$

but actually, we can simplify the previous equation by proposing a new hierarchy of scales (or power counting) that includes γ

$$\gamma \ll \frac{1}{R_S}\,,\,\frac{1}{a}\,, \tag{57}$$

which means that we can solve the problem by expanding in powers of

$$\gamma R_S\,,\,\gamma a \ll 1\,. \tag{58}$$

If we take into account that $\kappa \sim 1/R_S$, the eigenvalue equation simplifies to

$$\begin{aligned} \kappa a\,\cot\kappa a &= -\gamma a \\ &= 0 + \mathcal{O}(\gamma a)\,, \end{aligned} \tag{59}$$

where in the second line we are taking into account the power counting we have just proposed. The solution of the previous equation is

$$\frac{a}{R_S} = (2n+1)\,\frac{\pi}{2} \quad \text{for} \quad n = 0, 1, 2, \ldots \tag{60}$$

or, if we concentrate in the first one of these solutions

$$\frac{a}{R_S} = \frac{\pi}{2}\,. \tag{61}$$

For this particular ratio of the scales a and R_S we will have that the wave number of the system is $\gamma = 0$ and its binding energy will also be zero.

This case is very interesting, because $\gamma = 0$ represents a bound state that it is incredibly large (the size of a bound state scales as $1/\gamma$, as we can check from the example of the hydrogen atom and its mean square radius). However, for having such a large bound state we really need to be close to the condition $a/R_S = \frac{\pi}{2}$. Indeed, if we allow for variations over this condition

$$\frac{a}{R_S} = \frac{\pi}{2}\,(1+x)\,, \tag{62}$$

where x is a small, positive number, we quickly realize the following

$$x = 0.0 \Rightarrow \gamma = 0\,, \tag{63}$$

$$x = 0.1 \Rightarrow \gamma = \frac{0.16}{R_S}\,, \tag{64}$$

$$x = 0.2 \Rightarrow \gamma = \frac{0.33}{R_S}\,, \tag{65}$$

$$x = 0.3 \Rightarrow \gamma = \frac{0.51}{R_S}\,, \tag{66}$$

$$x = 0.4 \Rightarrow \gamma = \frac{0.73}{R_S}\,. \tag{67}$$

That is, very quickly the wave number becomes natural. Thus the conditions for which we obtain a very large bound state require a very delicate balance between the two scales involved in this system, R_S and a. The moment their ratio deviates from $a/R_S = \pi/2$, the bound states begins to become natural. This is why a bound state of unnatural size is said to require fine tuning.

Study recommendation: at this point it would be an interesting exercise to calculate the explicit dependence of γ on the small parameter x that we have defined in Eq. (62). For this it is only necessary to keep $\mathcal{O}(x)$ terms, which will make the calculation analytic.

Here we can also consider a second point of view of how fine tuning works. The binding energy of a system is the sum of its kinetic and potential energies, i.e.

$$-B = \langle T \rangle + \langle V \rangle\,. \tag{68}$$

It so happens that if a bound state were to be at zero binding energy, i.e. $B = 0$, this will require a perfect cancellation between kinetic and potential energies. In this sense, fine-tuning refers to the problem of obtaining a small number as the difference of two big numbers: the most general outcome is that the difference of two random big numbers will be a big number. Only when the two big numbers are by chance very similar to each other we will obtain a small number.

7. Nuclear forces

What about nuclear forces? Here things are a bit more complicated. There will be two problems, namely

(i) failure of naturalness and

(ii) poor separation of scales,

which are the reason why nuclear physics will be much more complicated than atomic physics.

The first of this problem, failure of naturalness, can be understood from different perspectives. One is a square-well analogy, where we consider the necessary fine-tuning between the kinetic and potential energy for generating the deuteron, assuming the neutron-proton potential can be modeled by a square-well. The deuteron is a spin-1 neutron-proton bound state with a binding energy of about 2.2 MeV. Meanwhile, if we model the nuclear force as a square-well, by assuming a range of $a \sim (1-2)\,\text{fm}$ we will need a square well of a depth of about 50 MeV (but this depends on the specific value of a used). This requires a cancellation of the order of

$$-\underbrace{B}_{2\,\text{MeV}} = \underbrace{\langle T\rangle}_{+48\,\text{MeV}} + \underbrace{\langle V\rangle}_{-50\,\text{MeV}} , \tag{69}$$

that is, of about one part in 25 (from the ratio between the binding energy and the potential energy). That is, there is a considerable degree of fine-tuning for the deuteron to be so weakly bound (though care should be taken as there is no standard quantification for the amount of fine-tuning: depending on the author you will find different values).

The second problem, a poor separation of scales, can be illustrated by trying to understand the neutron-proton potential as well as the expected corrections coming from short-distance scales such as the size of the nucleons. Regarding the potential, in 1938 Yukawa proposed that nuclear forces is mediated by the exchange of a meson. Indeed this is analogous to how the Coulomb force is generated by the exchange of a virtual photon, which generates the potential

$$V_C(r) = -\frac{e^2}{4\pi r} = -\frac{\alpha}{r} = -\frac{1}{2\mu}\,\frac{2}{a_B r} . \tag{70}$$

A photon is a massless vector boson, with quantum number $J^P = 1^-$ while the pion — the particle exchanged in nuclear forces — is a massive boson with $J^P = 0^-$. Originally Yukawa simply proposed a scalar meson ($J^P = 0^+$), in which case if we calculate the potential that such a meson generates we will find

$$V(r) = -g_Y^2\,\frac{e^{-mr}}{4\pi r} , \tag{71}$$

where g_Y is a coupling constant and m is the mass of the meson. It happens that this potential contains two scales (range and strength, just like in the case of the square well), of which the range is given by $1/m$ and the strength by

$$V(r) = -\frac{1}{2\mu}\,\frac{2}{R_Y\, r}\,e^{-mr} , \tag{72}$$

where for defining the *Yukawa scale* R_Y we have simply followed the same criterion as with the Coulomb potential. It happens that the pion mass is about 140 MeV, giving a range of around 1.4 fm, and that $g_Y \sim (0.5-0.6)$, yielding $R_Y \sim (4-5)\,\text{fm}$ (which means that the exchange of a pion provides a somewhat weak force and that it is probably not the cause for the deuteron binding). From comparing these two scales it is clear that the important one is the range of the pion, $1/m_\pi$. However, once we compare this scale with the size of the nucleon, $(0.5-1.0)\,\text{fm}$, it is apparent that the separation of scales is far from ideal. If we consider $R_\pi = 1/m_\pi$ and compare it with the proton size

$$\frac{R_\pi}{R_p} \sim \frac{1}{3} - \frac{2}{3} , \tag{73}$$

we immediately see that the description of the deuteron is not going to be as easy as the description of the hydrogen atom. The only good point is that actual calculations show that the previous ratio of scales is probably closer to 1/3 (instead of the unmanageable 2/3). That is, it will be possible to make controlled expansions of the nuclear force that are similar in nature to the one that we have previously shown for the square well. These expansions are however much more technical and complicated, as they involved a series of calculations in quantum field theory followed by a process called renormalization, which removes the ambiguities generated by the fact that we do not have a good knowledge of the short-range physics.

B. The Properties of the Nuclear Force

What do we know about nuclear forces? Well, a lot of things actually: physicists have been studying the nuclear forces for more than half of a century. A possible list of the most notable features of the nuclear force is the following:

(i) The nuclear force is short-ranged:

This property means that the nuclear potential decreases dramatically as the distance increase. Unlike the Coulomb potential, which extends indefinitely (even though it also becomes weaker the longer the distance), the nuclear potential only acts at distances of the order of a few fm. In particular, the nuclear force diminishes exponentially with the distance (like the Yukawa potential of Eq. (71)).

How do we know this? Because of a property that goes by the name of saturation. Saturation means that the binding energy per nucleon of a nucleus becomes roughly constant when the number of nucleons A increases. The binding energy for the deuteron is 2.2 MeV (1 MeV per nucleon), for the triton is 8.5 MeV (3 MeV per nucleon) and for the alpha particle is 28 MeV (7 MeV per nucleon). By then the saturation energy per nucleon reaches

$$\frac{B}{A} \sim (8-9)\,\text{MeV}\,, \tag{74}$$

i.e. about 8 or 9 MeV per nucleon. If the range of the nuclear force was long, saturation will not be a necessary outcome: instead, the binding energy will increase quadratically with A (not linearly, which is what we find).

From the alpha particle, Wigner deduced that the range of nuclear forces is about its size, namely about 1.7 fm give ot take. Notice that this coincides with the previous estimation that we made bases on the range of pion exchange, $1/m_\pi \simeq 1.4\,\text{fm}$.

(ii) The nuclear force is attractive at intermediate distances:

We know this from the nucleon density of heavy nuclei, which is approximately constant and known to be about $0.17\,\text{fm}^{-3}$, i.e. 0.17 nucleons per fm^3. That gives an average separation of about 1.8 fm (from $1/(1.8)^3 \sim 0.17$). It is sensible to assume that this corresponds to a mimimum of the two-body force.

(iii) The nuclear force is repulsive at short distances:

We know this partly from saturation, from the constant density of heavy nuclei (which implies a minimum of the potential and hence that the nuclear force should be repulsive at distances below that minimum), but more compellingly from the existence of a zero in the $^1S_0$ phase shift that can be nicely explained from a short-range repulsive core at 0.6 fm

Notice that the size of this repulsive core is similar to the size of the nucleons: it can be partly explained by the fact that nucleons are composed of quarks and that quarks are fermions. As a consequence, when one tries to squeeze nucleons togethers, the quarks within them will try to avoid to occupy the same position/state (Pauli exclusion principle), generating a very strong repulsive force.

(iv) The nuclear force does not distinguish neutrons and protons, a property usually referred as *charge independence*. That is why we usually talk about *nucleons* instead of neutrons and protons.

We know this from the binding energy of nuclei with the same quantum numbers, same number of nucleons A but different number of protons. For example ^{3}H and ^{3}He, which have binding energies of 8.48 MeV and 7.72 MeV respectively (where the difference in the binding energies mostly comes from the Coulomb repulsion between the two protons in ^{3}He).

(v) The nuclear force is not central, but involves a more complicated operator structure.

That is, instead of simply having $V_{\text{NN}} = V_C(\vec{r})$, we will have a series of more complicated operators including spin-spin, tensor, or spin-orbit forces. The tensor force is particularly important for explaining a few properties of the deuteron (its quadrupole and magnetic moments) and the spin-orbit force becomes important to explain shell structure in many nuclei.

C. The Origin of Nuclear Forces

1. Yukawa's idea: the exchange of a meson

Once we know the general properties of the nuclear force, the most important and pressing problem is to understand its origin. For this, we will begin with Yukawa's idea, which has been instrumental in the development of nuclear physics: the origin of the nucler force is the exchange of a meson, the pion. In particular, Yukawa proposed the exchange of a scalar meson with quantum numbers $J^P = 0^+$ (i.e. a meson with spin $J = 0$ and positive parity). In the language of quantum field theory this requires a interaction of the type

$$\mathcal{L} = g_Y \, \bar{\Psi}_N \, \phi \, \Psi_N \,, \tag{75}$$

where g_Y is the coupling, Ψ_N is a Dirac field for the nucleon and ϕ is a Klein-Gordon field for the meson. This type of interaction yields a really simple and neat potential, the famous Yukawa potential

$$V_Y(r) = -g_Y^2 \frac{e^{-mr}}{4\pi r} \,, \tag{76}$$

where m is the mass of the meson. This idea successfully explains that the nuclear forces are short-ranged.

For the derivation of this potential, the most convenient method is to do it in the quantum field theory formalism (a recommended reading will be Chapter 4 of "An Introduction to Quantum Field Theory" by Peskin & Schröder). Yet, it is in principle possible to derive the Yukawa potential in quantum mechanics by means of adapting the second quantization formalism for the electromagnetic field to the scalar meson field and then calculating the potential in second order perturbation theory. This second method is detailed in Appendix A.

2. The quadrupole moment of the deuteron and the quantum numbers of the pion

The problem with the Yukawa potential is that later it was discovered that the deuteron — the neutron-proton bound state — has an electric quadrupole moment. Let us remind that the charge, dipole moment, electric quadrupole moment and so on, simply refer to the way that different physical systems interact with an external electric field. If we have an external scalar potential Φ (generating an electric field $\vec{E} = \vec{\nabla}\Phi$), the energy of some electrically charge object in that field will be

$$V = q\,\Phi + d_i\,\partial_i\Phi + \frac{1}{6}Q_{ij}\partial_{ij}\Phi + \frac{1}{6}\,\langle r^2\rangle\nabla^2\Phi + \dots \tag{77}$$

where the coefficients q, d_i, Q_{ij} are the charge, dipole and quadrupole moments, respectively, and $\langle r^2\rangle$ the electromagnetic radius (this last term can be ignored if the external charge distribution is zero, as this will imply $\nabla^2\Phi = 0$). If the object has a known charge distribution we can write these coefficients as

$$q = \int d^3\vec{r}\,\rho(\vec{r}) \,, \tag{78}$$

$$d_i = \int d^3\vec{r}\,r_i\rho(\vec{r}) \,, \tag{79}$$

$$Q_{ij} = \int d^3\vec{r}\,\left(3r_ir_j - r^2\delta_{ij}\right)\,\rho(\vec{r}) \,, \tag{80}$$

$$\langle r^2\rangle = \int d^3\vec{r}\,r^2\,\rho(\vec{r}) \,, \tag{81}$$

where $\rho(\vec{r})$ is the charge distribution of the object we have put in the external electric field. Usually the quadrupole moment is referred with respect to a specific direction. For the deuteron we have that it is a spin-1 system, which means that is has a preferred direction: the direction where the spin is pointing to. If we call this direction z, then we can define the quadrupole moment of the deuteron relative to that direction

$$Q_d = \int d^3\vec{r}\,\left(3z^2 - r^2\right)\,\rho(\vec{r}) \,, \tag{82}$$

where it is apparent that we have chosen $i = j = 3$. It happens experimentally that the quadrupole moment of the deuteron is positive. In particular we have $Q_d = 0.2859(3)\text{fm}^2$. This also means that the deuteron is longer in the direction of its spin than in the other two directions: it is like a 油条 (youtiao), though the technical term would be

"the deuteron is prolate". On the contrary a nucleus with a negative quadrupole moment is like a 包子 (baozi), the technical term being "oblate" in this case.

So why is this a problem for the hypothesis that the pion is a scalar? Well, a scalar meson generates a rotationally symmetric potential, which in turn generates a rotationally symmetric charge density, which in turns implies a quadrupole moment $Q_d = 0$. That means that the quantum number of the exchangeg meson have to be different than $J^P = 0^+$. Other possibility is that the pion is a vector meson with $J^P = 1^+$, i.e. some sort of heavy photon. In that case the lagrangian reads

$$\mathcal{L} = g_V\, \bar{\Psi}\gamma^\mu\, \phi_\mu \Psi + \frac{f_V}{4M_N} \bar{\Psi}\sigma_{\mu\nu}\Psi\, (\partial_\mu\phi_\nu - \partial_\nu\phi_\mu)\,, \tag{83}$$

where now the interaction lagrangian is a bit more complicated because in principle it can contain two terms: an electric- and a magnetic-type interaction. The term with g_V is called electric because it yield a Coulomb-like force in the non-relativistic limit, while the term with f_V is called magnetic because it generates a potential that depend on the spin of the nucleons. In the expression above g_V and f_V are dimensionless coupling constants, γ_μ are the Dirac gamma matrices and $\sigma_{\mu\nu}$ are bilinears that can be constructed from the gamma matrices as

$$\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]\,. \tag{84}$$

Be it as it may, the important thing to notice is that the resulting potential is as follows

$$\begin{aligned} V(r) = {} & g_V^2 \frac{e^{-mr}}{4\pi r} \\ & + \left(\frac{g_V + f_V}{2M}\right)^2 \left[\frac{2}{3}\,\vec{\sigma}_1\cdot\vec{\sigma}_2\,\frac{e^{-mr}}{4\pi r}\right. \\ & \left. -\frac{1}{3}\,(3\vec{\sigma}_1\cdot\hat{r}\,\vec{\sigma}_2\cdot\hat{r} - \vec{\sigma}_1\cdot\vec{\sigma}_2)\,\frac{e^{-mr}}{4\pi r}\,\left(1 + \frac{3}{mr} + \frac{3}{(mr)^2}\right)\right], \end{aligned} \tag{85}$$

which looks more complex than the potential for a scalar meson. We can make this expressions look a bit less menacing if we define

$$S_{12}(\hat{r}) = 3\vec{\sigma}_1\cdot\hat{r}\,\vec{\sigma}_2\cdot\hat{r} - \vec{\sigma}_1\cdot\vec{\sigma}_2\,, \tag{86}$$

$$W_C(r) = \frac{e^{-mr}}{4\pi r}\,, \tag{87}$$

$$W_T(r) = \frac{e^{-mr}}{4\pi r}\,\left(1 + \frac{3}{mr} + \frac{3}{(mr)^2}\right)\,, \tag{88}$$

from which we can rewrite

$$\begin{aligned} V(r, J^P = 1^+) = {} & g_V^2\, W_C(r) \\ & + \left(\frac{g_V + f_V}{2M}\right)^2 \left[\frac{2}{3}\,\vec{\sigma}_1\cdot\vec{\sigma}_2\, W_C(r) - \frac{1}{3}\, S_{12}(\hat{r})\, W_T(r)\right]. \end{aligned} \tag{89}$$

We can see that the electric-type term is repulsive, which is to be expected as this is analogous to the exchange of a photon between identical charges. We also see that the magnetic-type term can be either attractive or repulsive depending on the alignment of the spins.

However, the previous expression is only correct for the two-neutron or two-proton potential: when there is a neutron and a proton, the pion can transform them into the other. The previous potential does not consider this possibility, yet it is incredibly simple to take it into account by including isospin. Indeed, the general two-nucleon potential should contain a $\vec{\tau}_1\cdot\vec{\tau}_2$ factor, where $\vec{\tau}_1$ and $\vec{\tau}_2$ are the Pauli matrices as applied to the isospin wave functions of nucleons 1 and 2. This might sound a bit esoteric at first, but it is just the isospin equivalent of the well-known $\vec{\sigma}_1\cdot\vec{\sigma}_2$ spin-spin operator. That is, the actual potential generated from the exchange of an isospin-1 $J^P = 0^+$ meson will be

$$V(r, I = 1, J^P = 0^+) = -\,\vec{\tau}_1\cdot\vec{\tau}_2\, g_Y^2 \frac{e^{-mr}}{4\pi r}\,, \tag{90}$$

while for an isospin-1 and $J^P = 1^-$ meson it will be

$$\begin{aligned} V(r, I = 1, J^P = 1^+) = {} & \vec{\tau}_1\cdot\vec{\tau}_2\, g_V^2 W_C(r) \\ & + \vec{\tau}_1\cdot\vec{\tau}_2 \left(\frac{g_V + f_V}{2M}\right)^2 \left[\frac{2}{3}\,\vec{\sigma}_1\cdot\vec{\sigma}_2\, W_C(r) - \frac{1}{3}\, S_{12}(\hat{r})\, W_T(r)\right]. \end{aligned} \tag{91}$$

The pattern is clear though: if we exchange an isospin-1 meson there will be a multiplicative $\vec{\tau}_1 \cdot \vec{\tau}_2$ factor in front of the potential. We will review this point in more detail later when we study isospin.

If we are considering the particular case of the deuteron — a neutron-proton bound state with total isospin $I = 0$ — there should be a global minus factor in the potential. Actually the factor is a -3 (because $\vec{\tau}_1 \cdot \vec{\tau}_2 = -3$ for $I = 0$), i.e.

$$V_d(r) = -3\, V(r, J^p = 1^+)\,, \tag{92}$$

with the subscript d indicating we are talking about the deuteron case, and $V(r, J^p = 1^+)$ to the potential for a vector meson without isospin that we wrote in Eq. (89). However, what we are interested in is what type of quadrupole moment can be deduced from here. If happens that what we obtain is a negative quadrupole moment, but we are going to leave the proof as an **exercise**. This means that the possibility of a vector pion is not consistent with the experimental facts.

The **exercise** is as follows: show that for a potential of the type

$$V(r) = -g^2 \left[a\, \vec{\sigma}_1 \cdot \vec{\sigma}_2\, W_C(r) \pm b\, S_{12}(\hat{r})\, W_T(r) \right]. \tag{93}$$

for which $a > 0$, $b > 0$, then the sign of the quadrupole moment is the same as the sign between the central and tensor piece, i.e. $Q = \pm|Q|$. For that first take into account that $Q \neq 0$ requires that the spin of nucleons 1 and 2 must add up to $S = 1$. If we consider the total spin $\vec{S} = 2\,(\vec{\sigma}_1 + \vec{\sigma}_2)$, this corresponds to taking σ_1 paralell to σ_2.

The next theoretical possibility is that the pion is a pseudoscalar. The interaction lagrangian in this case is

$$\mathcal{L} = i g_{PS}\, \bar{\Psi} \gamma^5 \phi \Psi\,, \tag{94}$$

and the potential one derives in this case is

$$V(r, J^P = 0^-) = \frac{g_{PS}^2}{4M_N^2}\, \left[\vec{\sigma}_1 \cdot \vec{\sigma}_2\, W_C(r) + S_{12}(\hat{r})\, W_T(r) \right]\,, \tag{95}$$

with M_N the nucleon mass. As previously discussed, the pion is an isospin-1 meson and thus we still have to multiply by a factor of $\vec{\tau}_1 \cdot \vec{\tau}_2$ to obtain the right potential:

$$V(r, I = 1, J^P = 0^-) = V_\pi(r) = \vec{\tau}_1 \cdot \vec{\tau}_2\, \frac{g_{PS}^2}{4M_N^2}\, \left[\vec{\sigma}_1 \cdot \vec{\sigma}_2\, W_C(r) + S_{12}(\hat{r})\, W_T(r) \right]\,, \tag{96}$$

A closer inspection on the line of the previous **exercise** shows that this potential generates a positive quadrupole moment. This implies that a pseudoscalar pion is compatible with the properties of the deuteron and we expect the pion to have $J^P = 0^-$.

3. Beyond the pion

Yukawa's idea — the exchange of a pion — indeed explains a few important properties of the nuclear force: its finite range, the existence of attraction at certain distances and the sign of the quadrupole moment of the deuteron. Yet, by itself, the one pion exchange (OPE) potential thus generated is not enough for a complete description of the nuclear force. Two possible improvements over the OPE potential come to mind

(i) To include the exchange of more pions (two-pion exchange, three-pion exchange, etc.)

(ii) To include the exchange of other mesons.

Both of these ideas have a series of strong and weak points, which we will discuss now.

The most obvious extension to OPE is two-pion exchange (TPE) or even multi-pion exchanges. Indeed, this was originally discussed in the 50's by theoreticians, but it so happened that this early multi-pion theories failed miserably. The reason for this failure is well-known today: they were not considering the correct dynamics for the pion. This might seem odd at first sight: how can it be that the pion exchange that we have derived is based on wrong assumptions? Well, the problem here is that "pion dynamics" are not directly observable. Different types of pion dynamics will generate the same OPE potential. In terms of quantum field theory, the pion-nucleon Lagrangian we have considered is the following one:

$$\mathcal{L} = i g_{\pi NN}\, \bar{\Psi}_N \gamma^5 \tau_a \pi_a \Psi_N\,, \tag{97}$$

where π_a is the pion field and $a = 1, 2, 3$ a isospin index (as the pion is an isospin-1 particle), which ends up in the OPE potential

$$V_{\rm OPE}(\vec{q}) = -\frac{g_{\pi NN}^2}{4M_N^2}\,\vec{\tau}_1\cdot\vec{\tau}_2\,\frac{\vec{\sigma}_1\cdot\vec{q}\,\vec{\sigma}_2\cdot\vec{q}}{m_\pi^2+\vec{q}^{\,2}}\,. \tag{98}$$

But it happens that the previous is not the only Lagrangian from which one can obtain the OPE potential. The following Lagrangian will also work

$$\mathcal{L} = \frac{g_{\pi NN}}{2M_N}\,\bar{\Psi}_N\gamma^5\gamma^\mu\partial_\mu\pi_a\tau_a\Psi_N\,, \tag{99}$$

and will generate exactly the same potential as Eq. (97).

The only difference between these two Lagrangians is that one does not contain derivatives of the pion field (i.e. we have π_a in the Lagrangian), while the oder does (i.e. we have $\partial_\mu\,\pi_a$ in the Lagrangian). They generate the same OPE potential, but when they are used to calculate multi-pion exchange potentials they will generate different potentials. This in turn is reminiscent of a well-known problem in quantum mechanics: the "inverse scattering problem", which states that there are an infinite number of potentials that generate the same scattering amplitude.

Nowadays we know that the correct pion dynamics is derivative in nature. The reason is chiral symmetry breaking, a symmetry (appearing in quantum chromodynamics) of the light quarks composing the nucleons, which at the end implies a derivative interaction. As a consequence the correct pion-nucleon Lagrangian is the one given by Eq. (99), though the usual way to write it is in terms of the following constants

$$\mathcal{L} = \frac{g_A}{2f_\pi}\,\bar{\Psi}_N\gamma^5\gamma^\mu\partial_\mu\pi_a\tau_a\Psi_N\,, \tag{100}$$

where $g_A = 1.26$ is called the pion axial coupling and $f_\pi = 92.4\,\text{MeV}$ the pion weak decay constant (as it is indeed the constant that appears when calculating the $\pi^- \to \mu^-\bar{\nu}_\mu$ decay, while g_A appears for instance in the weak decay of the neutron $n \to pe^-\bar{\nu}_e$). The relation between the two sets of constants is

$$\frac{g_A}{f_\pi} = \frac{g_{\pi NN}}{M_N}\,, \tag{101}$$

which is called the Goldberger-Treiman relation.

However in the 50's theoreticians did not know about chiral symmetry or the derivative nature of pion interactions and reached the conclusion that multi-pion theories were a dead end. Instead, they considered that the correct path to advance their knowledge of nuclear forces was to include the exchange of other mesons, in particular the following three mesons: the sigma (σ), the rho (ρ) and the omega (ω). These resonances also appear in pion-pion scattering and could be interpreted as pion rescattering effects that might have not been properly included when considering multi-pion exchanges. For instance, we now know that the sigma can indeed be regarded as a meson that is generated dynamically from the pion-pion interaction. Thus, the exchange of mesons heavier than the pion provided a reason of why the multi-pion theories failed.

Within the meson-exchange theories, each of the mesons have a particular role:

(i) the sigma meson provides the strong intermediate range attraction that is necessary to explain saturation,

(ii) the rho meson (a vector meson) provides a tensor force of the opposite sign of that from the pion, check Eqs. (91) and (95), which helps to reproduce the exact value of the deuteron quadrupole moment and,

(iii) the omega meson provides the strong short-range repulsion that is necessary to further explain saturation and the $^1S_0$ phase shift.

Yet, nowadays we know that (i) and (ii) can be correctly explained in terms of the two-pion exchange potential (once we take into account that the pions require a derivative interaction), while (iii) is easily explained from the fact that nucleons are not point-like particles, but composed of quarks (which are fermions, then at distances smaller than the size of the nucleons these quarks will avoid being squeezed together owing to Fermi statistics). This also gives a strong short-range repulsion, but a type of repulsion that we can only understand once we know the nucleons are composite particles (and this was not known until the sixties).

D. Putting a bit of order

A useful idea from Taketani, Nakamura and Sasaki (TNS) proposed to divide the nuclear force into three regions: long, medium and short-range (or classical, dynamical and phenomenological/core in their original words). For each of these regions we have

(i) The classical zone ($r \geq 2\,\mathrm{fm}$) is dominated by OPE.

(ii) The dynamical zone ($2\,\mathrm{fm} \geq r \geq 1\,\mathrm{fm}$) is dominated by TPE and other heavier mesons.

(iii) The phenomenological zone ($r \leq 1\,\mathrm{fm}$) where there are multi-pion exchanges, heavier mesons and all sort of other weird things we do not really understand.

This conception of the nuclear force is indeed very far sighted, because it advanced many of the concepts and ideas that we have today. For instance, it explains why it is okay to use form factors and other techniques to treat the short-range problems of the nuclear force. And it is also incredibly close to spirit to the effective field theory approaches that dominate the field nowadays and that we will explore later.

E. Nuclear Forces are Residual Forces

In the previous section we proposed an analogy between electrons in an atom and protons and neutrons in a nuclei to introduce ideas such as scales, naturalness and separation of scales. Yet electrons in an atom are not really that similar to protons and neutrons in a nuclei: electrons are bound to the nucleus owing to the electromagnetic force, a fundamental force of nature. However the strong force that is ultimately responsible for the binding of the nucleus does not mediate directly between protons and neutrons, but between quarks and gluons, particles which have *color* (the strong charge). Proton and neutrons are color neutral particles: they do not have color charge like the quarks composing them.

Actually the atomic physical system that is analogous to the proton and the neutron is the atom, which is electrically neutral. Yet neutral atoms can interact with other neutral atoms. A well-known atom-atom potential is the Lenard-Jones potential

$$V_{AA}(r) = -\frac{C_6}{r^6} + \frac{C_{12}}{r^{12}}\,. \tag{102}$$

Where does this potential comes from? The atom-atom potential is what we call a *residual force*: its origin is the electromagnetic interaction between the components of the atoms. Thus, though the atoms are neutral, there is still a force among them. This force can in theory be computed from first principles if we know the internal structure of the atom in detail, though this calculation is certainly not trivial. It happens that the nuclear forces are a just like the Lenard-Jones potential: they are residual forces of the strong interaction of the quarks and gluons inside the neutrons and protons. In theory the nuclear force could also be derived from first principles, but unfortunately this type of derivation is incredibly more complex than the force between atoms for theoretical reasons we will explain in the next section.

II. THE DERIVATION OF NUCLEAR FORCES

All calculations in nuclear physics — properties of nuclei, excited states, nuclear reactions — ultimately depend on the nuclear force. This makes it very clear that the fundamental problem of nuclear physics is the derivation of nuclear forces from first principles. This problem is indeed as old as nuclear physics itself and theoreticians have been actively working in this for the last seven decades. The original idea of Yukawa was the starting point for these theoretical efforts. It was soon stablished that the meson responsible for nuclear forces is a pseudoscalar (this is necessary to obtain the correct quadrupole moment for the deuteron). Later in the 50s there were a few multipion theories, which failed however owing to the lack of a very important ingredient, chiral symmetry (which provides the correct pion-nucleon dynamics), while in the 60s and 70s the one boson exchange (OBE) model was developed. In the OBE model, besides the pion, the nuclear forces are also generated by the exchange of other mesons like the σ, the ρ and the ω to name the most important ones. However later, with deep inelastic scattering experiments, it was discovered that the neutron and proton have an internal structure. They are composed of quarks and gluons, which interact by means of quantum chromodynamics (QCD), a quantum field theory that is extremely successful when it comes to explain the dynamics of these quarks and gluons (but which becomes terribly complicated when applied to neutrons, protons and pions). Owing to the existence of QCD, the fundamental theory of strong interactions, a serious derivation of nuclear forces should be grounded in this theory.

A. Global and local symmetries and gauges theories

We will briefly explain quantum electrodymanics (QED) and QCD from the point of view of their symmetries. We will begin with QED, which despite its apparent dissimilarities with QCD happens to be extremely similar in structure to QCD.

First, we will consider a theory containing a free Dirac field

$$\mathcal{L}_{\text{Dirac}} = \bar{\Psi}(i\gamma_\mu \partial^\mu - m)\Psi \,, \tag{103}$$

where γ_μ are the Dirac matrices, Ψ the Dirac field and $\bar{\Psi} = \Psi^\dagger \gamma_0$. It happens that this theory contains a symmetry that leaves the Lagrangian unchanged:

$$\Psi(x) \to e^{i\phi}\Psi(x) \quad \Longrightarrow \quad \mathcal{L}_{\text{Dirac}} \to \mathcal{L}_{\text{Dirac}} \,, \tag{104}$$

that is, a symmetry that consists simply in a change of phase of the Dirac field. This symmetry is what we call a *global symmetry*, which means that it is the same regardless of the position of the field: the phase $e^{i\phi}$ is the same for every x.

The name of this global symmetry is $U(1)$, as it corresponds with the *unitary group* of dimension $N = 1$, where we remind that the unitary group is the set of $N \times N$ matrices such that when a matrix is multiplied by its conjugate transpose it gives the identity matrix:

$$U(N) = \left\{ A \in \text{GL}(N, \mathbb{C}) / AA^\dagger = 1 \right\} \,, \tag{105}$$

where $\text{GL}(N, \mathbb{C})$ is the *general linear group*, i.e. the group of all $N \times N$ complex matrices (group under the product of matrices) such that their determinant is different from zero. Here it is worth to also include a reminder of other groups that are important in physics, such as the special unitary group

$$SU(N) = \{ A \in U(N) / \det A = 1 \} \,, \tag{106}$$

and the orthogonal and special orthogonal unitary groups

$$O(N) = \left\{ A \in \text{GL}(N, \mathbb{R}) / AA^T = 1 \right\} \,, \tag{107}$$

$$SO(N) = \{ A \in O(N) / \det A = 1 \} \,, \tag{108}$$

which are among the most common groups appearing in physics, where $\text{GL}(N, \mathbb{R})$ refers to the general linear group of *real* matrices.

After reviewing the most common groups, we will continue now with the problem of what happens when the symmetry, instead of being global, becomes local

$$\Psi(x) \to e^{i\phi(x)}\Psi(x) \,, \tag{109}$$

in which case we quickly realize that this is no longer a symmetry of the Lagrangian

$$\mathcal{L}_{\text{Dirac}} \not\to \mathcal{L}_{\text{Dirac}} \,. \tag{110}$$

Indeed, if we consider the two terms (mass and kinetic) of the Lagrangian, we see that the one that fails is the kinetic one

$$\bar{\Psi}\Psi \to \bar{\Psi}\Psi \quad \text{but} \quad \bar{\Psi}\partial_\mu \Psi \to \bar{\Psi}\partial_\mu \Psi + i(\partial_\mu \phi)\, \bar{\Psi}\, \Psi \,, \tag{111}$$

where the kinetic term is generating a new term. So, can we fix this problem in some way? Actually, yes, by the inclusion of a new field A_μ with the transformation property

$$\{\Psi(x), A_\mu(x)\} \to \left\{ e^{i\phi(x)}\Psi(x), A_\mu(x) + \frac{1}{e}\,\partial_\mu \phi \right\} \,, \tag{112}$$

where e is a parameter we have introduced for defining how much A_μ changes when we do a local change of phase in $\Psi(x)$. In addition, we will change the derivative ∂_μ in the kinetic term by a new derivative D_μ

$$\bar{\Psi} D_\mu \, \Psi \quad \text{with} \quad D_\mu = \partial_\mu - i\, e\, A_\mu \,. \tag{113}$$

Now, if we consider the Lagrangian we obtain with the new derivative D_μ

$$\mathcal{L}'_{\text{Dirac}} = \bar{\Psi}(i\gamma_\mu D^\mu - m)\Psi \,, \tag{114}$$

it is easy to check that it is invariant under the local U(1) transformation

$$\mathcal{L}'_{\rm Dirac} \to \mathcal{L}'_{\rm Dirac}\,, \tag{115}$$

which means that we have worked out a modification of the original Lagrangian with the desired symmetric properties.

However, we have had to introduce a new vector field A_μ for achieving local U(1) symmetry, which generates a new problem that we did not have before: how do we have to modify the Lagrangian to include the new field A_μ? This is indeed an interesting problem: the new field should have at least a kinetic term and maybe a mass term. But the mass term violates the symmetry of the A_μ field, as we have

$$A_\mu \to A_\mu + \frac{1}{e}\,\partial_\mu\phi \quad \Longrightarrow \quad m_A^2 A_\mu A^\mu \to m_A^2\left[A_\mu A^\mu + \frac{2}{e}A_\mu\partial^\mu\phi + \frac{1}{e^2}\partial_\mu\phi\,\partial^\mu\phi\right]\,, \tag{116}$$

which means that we are required to have $m_A = 0$, i.e. the new A_μ field has to be massless. Next, for the kinetic term, we first have to construct a term with derivatives of the A_μ field that is invariant under local U(1). The most obvious answer, $\partial_\nu A_\mu$, does not work

$$\partial_\nu A_\mu \to \partial_\nu A_\mu + \frac{1}{2}\partial_\nu\partial_\mu\phi\,. \tag{117}$$

However, the following choice works

$$\partial_\mu A_\nu - \partial_\nu A_\mu \to \partial_\mu A_\nu - \partial_\nu A_\mu\,, \tag{118}$$

which we will call $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. From this, the Lagrangian with one Dirac field that is symmetric under local U(1) has to take the form

$$\mathcal{L}_{U(1)-{\rm local}} = \bar\Psi(i\gamma_\mu D^\mu - m)\Psi + \lambda F_{\mu\nu}F^{\mu\nu}\,, \tag{119}$$

where the only thing that is left is to determine the correct proportionality constant λ for the kinetic term of A_μ. This proportionality constant can be done by a comparison with the real Klein-Gordon field, where its Lagrangian is

$$\mathcal{L}_{\rm KG} = \frac{1}{2}\partial^\mu\Phi\partial_\mu\Phi - \frac{1}{2}m^2\Phi^2\,. \tag{120}$$

It is worth noticing that the Lagrangian is not unique, what is unique is the action

$$S = \int d^4x\,\mathcal{L}\,, \tag{121}$$

and if we change the Lagrangian in such a way that the action does not change, it is totally okay. This observation allows us to rewrite the kinetic term in the KG equation as

$$\mathcal{L}_{\rm KG-kin} = \frac{1}{2}\partial^\mu\Phi\partial_\mu\Phi \quad \to \quad \mathcal{L}'_{\rm KG-kin} = -\frac{1}{2}\Phi\partial^\mu\partial_\mu\Phi\,, \tag{122}$$

because the difference when integrated to obtain the action is a global surface term that vanishes when the appropriate boundary conditions are imposed. So, the problem is basically to find the vector-field equivalent of the $\mathcal{L}'_{\rm KG-kin}$ kinetic term. This is easier if we first consider that $A_\mu = (A_0, \vec{A})$, where for the spatial component $\vec{A}$ we expect a kinetic term of the type

$$\mathcal{L}'_{\rm kin} = -\frac{1}{2}\vec{A}\cdot\left(\partial^\mu\partial_\mu\vec{A}\right) + (\text{the term with } A_0)\,. \tag{123}$$

The reason for this choice is that $\vec{A}$ is the non-relativistic limit for a vector field (at least for a massive vector field, which is not the case; yet, this is not important because the kinetic term will be the same for a massive or a massless field), from which is easier to understand that the non-relativistic equivalent of the scalar field square Φ^2 is the scalar product of the vector field $\vec{A}^{\,2}$. Finally, the inclusion of the A_0 piece depends on the definition of the metric. If we use $g_{\mu\nu} = {\rm diag}(1,-1,-1,-1)$, then we should have

$$\mathcal{L}'_{\rm kin} = \frac{1}{2}A_\nu\left(\partial^\mu\partial_\mu A^\nu\right)\,, \tag{124}$$

and from this and a proper comparison with our kinetic term $\lambda F_{\mu\nu}F^{\mu\nu}$, we quickly find that $\lambda = -1/4$.

Putting all the pieces together, we arrive at

$$\mathcal{L}_{U(1)-\text{local}} = \bar{\Psi}(i\gamma_\mu D^\mu - m)\Psi - \frac{1}{4}\, F_{\mu\nu}F^{\mu\nu}\,, \tag{125}$$

which describes the physics of a Dirac field interacting with a massless vector field in such a way that it preserves a type of local U(1) symmetry. Now we can connect the dots and notice that we already know a vector field in physics — the photon — which interacts with charged particles, among them the electron — a Dirac field — and all this means that the Lagrangian we have obtained describes the interaction of electron with photons. That is, we have found the Lagrangian for quantum electrodynamics (QED).

B. Asymptotic Freedom

Now we are going to explain why the derivation of nuclear forces from QCD looks like a hopeless task. The technical term for the reason of why this is so is *asymptotic freedom*. To explain it, we will go back again to quantum electrodynamics (QED) to give a more simple example.

QED is the quantum field theory of photons and electrons. The QED lagrangian is exactly the one we derived a few lines before (which we repeat here for clarity), that is

$$\mathcal{L}_{\text{QED}} = \bar{\Psi}(i\gamma^\mu D_\mu - m)\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\,, \tag{126}$$

where

$$D_\mu = \partial_\mu - ieA_\mu\,, \tag{127}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,, \tag{128}$$

with A^μ the quantum field for the photon, Ψ the one for the electron, e the electron charge and m the electron mass. In principle we can use perturbation theory, i.e. Feynman diagrams, to compute quantities in QED. For instance, if we compute the potential between two electrons in the non-relativistic limit, we get the Coulomb potential

$$V_c(r) = \frac{\alpha}{r}\,. \tag{129}$$

The strength of all electron-photon interactions is set by the fine structure constant $\alpha = e^2/4\pi$. As already said here $\alpha \sim 1/137$, but this is only so at low energies. If the interactions between electrons and photons happen at high energies, the value of the fine structure constant will be subject to quantum corrections which will change its value. A graphical representation of this idea is that a high energy photon can fluctuate to an electron-positron pair, which changes in fact the properties of the photon and the way photons couple to electrons. The running of the electromagnetic coupling can be compute in first order perturbation theory, leading to the well-known formula

$$\alpha(Q^2) = \frac{\alpha(\mu^2)}{1 - \frac{\alpha^2(\mu^2)}{3\pi}\,\log\frac{Q^2}{\mu^2}}\,, \tag{130}$$

where Q^2 and μ^2 are the energies of the photon emitted by the electron (if you want to increase your knowledge of physics, it will be very much recommended to learn how this formula is derived). This implies that the strength of the electromagnetic couplings increases with the energy, as can be easily checked from the formula. As a matter of fact the strength diverges at incredibly high energies, a phenomenon that is known as the Landau pole of QED, but the energies required for this are far above the Planck scale for which we do not think that QED will be valid. QED as any other physical theory is only applicable until the scale at which additional physics appear, its R_{short}. The point however is the following: for most practical energies α is pretty small and QED calculations can be done perturbatively.

The lagrangian for the gluons and quarks is in fact extremely similar to the one of electrons and photons and reads:

$$\mathcal{L} = \sum_{i=1}^{n_F} \bar{q}_i\,(i\,\gamma^\mu D_\mu - m_i)q_i - \frac{1}{4}\,G^a_{\mu\nu}\,G^{a\mu\nu}\,, \tag{131}$$

with

$$D^a_\mu = \partial_\mu - ig\sum_a \frac{\lambda^a}{2}A^a_\mu\,, \tag{132}$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f_{abc} A^b_\mu A^c_\nu\,. \tag{133}$$

In the equation above q_i are the quark fields, i is a flavour index that runs from 1 to 6 to include the six types of quarks

$$q_i = \{u, d, s, c, b, t\}\,, \tag{134}$$

$a = 1, \ldots, 8$ is a color index, A^a_μ is the gluon field, λ^a are the SU(3) equivalent of the Pauli matrices (they are 3x3 matrices that go by the name of the Gell-Mann matrices) and f_{abc} are some indexes with the property $f_{abc} = -f_{acb}$. These are called structure constants and can also be obtained from

$$[\frac{\lambda^b}{2}, \frac{\lambda^c}{2}] = if^{abc}\frac{\lambda^a}{2}\,. \tag{135}$$

Actually this lagrangian can be considered as a direct extension of the electromagnetic one from one to three charges: that is why we say that electromagnetism is a $U(1)$ gauge theory and the strong force is an $SU(3)$ gauge theory. As with electromagnetism we can define a strong α_s, which also runs with energy. Howeve there is a fundamental difference with electromagnetism: while the electron does not carry electric charge, the gluons do indeed carry strong charge. As a consquence the gluons can interact with themselves directly. The outcome is that a calculation of the running of the strong coupling leads to this result

$$\alpha_s(Q^2) = \frac{\alpha_s(\mu^2)}{1 + \frac{\alpha_s(\mu^2)}{12\pi}(33 - 2n_f)\,\log\frac{Q^2}{\mu^2}}\,, \tag{136}$$

that is, the coupling diminishes for high energies (as in the previous case, this is an important result and it is worth learning its derivation from your quantum field theory textbook). Alternatively we can define

$$\Lambda^2_{QCD} = \mu^2\,\exp\left[-\frac{12\pi}{(33 - 2n_f)\alpha_s(\mu^2)}\right], \tag{137}$$

and write

$$\alpha_s(Q^2) = \frac{12\pi}{(33 - 2n_f)\log\frac{Q^2}{\Lambda^2_{QCD}}} \tag{138}$$

That is, QCD is only dependent on one dimensionful scale, which is Λ_{QCD}. It happens that $\Lambda_{QCD} \sim 200 - 300\,\mathrm{MeV}$.

Now the difference with QED should be apparent: as the enery Q^2 decreases, the coupling α_s increases. To make things worse, there is a particular Q^2 such that α_s diverges. This is Λ_{QCD}, which is the natural energy scale of QCD. It happens that for energies larger than Λ_{QCD}, α_s is small and QCD makes sense as a perturbative quantum field theory that can be solved with standard techniques such as Feynman diagrams. However for energies below Λ_{QCD} the coupling constant diverges and is apparent that perturbation theory cannot be used. What is the problem with this? Well, for starters in general most of our knowledge about solving QFT is perturbative. As a consequence we have no clear way of deriving the residual force among nucleons from QCD with analytical methods.

We are left with two strategies, one direct and one indirect, which are

(i) Lattice QCD and

(ii) Effective Field Theory.

The direct strategy is Lattice QCD and it amounts to use a computer to solve QCD numerically in the non-perturbative regime. As expected, this is anything but easy. The indirect strategy, effective field theory, amounts to solving QCD by not solving it at all. That is, to use a series of powerful techniques (renormalization, symmetries, power counting) to figure out how the most general low energy solution of QCD should look like. In the following lines we will explain a bit how they work.

C. Lattice QCD

Lattice QCD consist on making a calculation of a nuclear system on the basis of the quarks and gluons that form the nucleons. The reason why it has to be done in a computer is that these equations are simply not solvable analytically, owing to asymptotic freedom. However they could in principle be solved in a computer. This is done by discretizing space-time to form a lattice: we construct a cube with a side of few fm, put a grid inside with a grid size of a fraction of a fermi, then assume that there a few quarks inside and solve QCD on the lattice, see Fig. 1. The only problem is the technical difficulty. As a matter of fact the way to make the calculations easier is by changing the masses of the quarks: if the quarks are considerably lighter than Λ_{QCD} calculations become really difficult. Thus calculations are not always made for the physical world (where $m_u \sim 3\,\mathrm{MeV}$ and $m_d \sim 5\,\mathrm{MeV}$), but for a fictional world with different quark masses which leads to different pion and nucleon masses. Yet there are calculations at the physical pion mass for certain processes. The field is making quick progress.

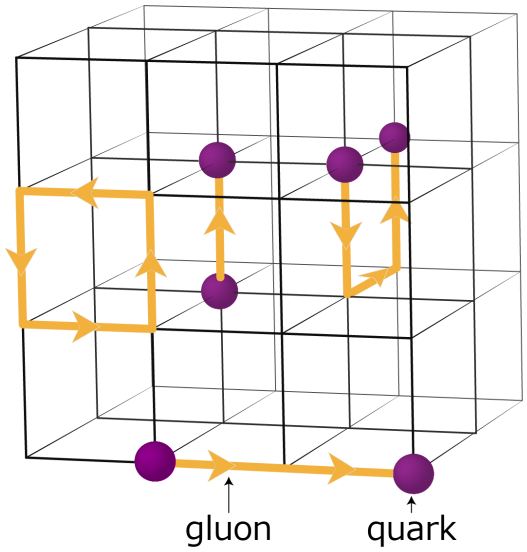


FIG. 1. In lattice QCD space-time is discretized in a lattice and QCD is solved in that lattice.

D. Effective Field Theory

The other way to solve QCD is paradoxically by not solving it. It happens that there is a very powerful physical idea called renormalization. The idea behind renormalization is the following one:

> Physics at long distances does not depend on the short distance details.

This is really important, it is the reason why physics itself is possible and we know it from countless example. We can describe chemistry, the chemical bond and the energy levels of the atom without knowing any details of the nucleus: the existence of the nucleus itself only manifest in the hyperfine structure and can be accounted for without knowing the details of what happens inside a nucleus. We can describe the solar system with great accuracy solely based on Newton's theory of gravity, the only exception being the precession of the perihelion of Mercury in which small corrections from general relativity enter. Yet the main correction can be parametrized with a L^2/r^3 type potential that is suppressed by a factor of c^2, where L is the angular momentum, r the radius and c the speed of light. There has been engineering and architecture before the discovery of quantum mechanics, despite the fact that the existence of solid matter can only be explained with quantum theory, in particular Fermi-Dirac statistics. In short, the list is endless and the existence of different branch of sciences boils down to the fact that we can analyze whatever happens at a particular distance scale without knowing what happens at shorter distance scales. Yet renormalization can go a bit farther than this and can be used to connect the different types of explanations that we have.

In the case of nuclear physics we know that at low energies we can describe the nucleons without knowning what happens to the quarks and gluons inside. At low energies (long distances) the world is composed of nucleons and we cannot see that they have any kind of structure. At high energies (short distances) the world is composed of quarks and gluons, which we can now see clearly, while nucleons disappear because they are just too big. In this context renormalization is the mathematical problem of uniting these two points of view:

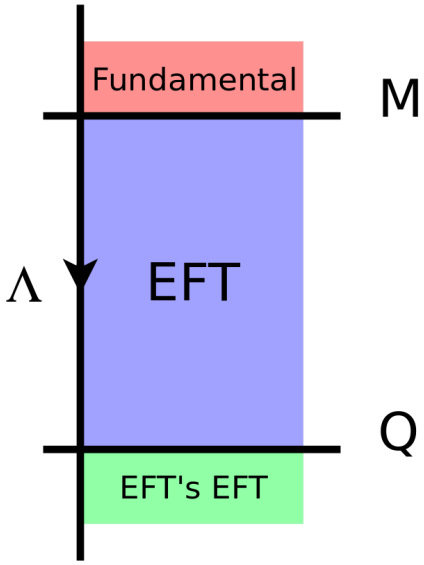
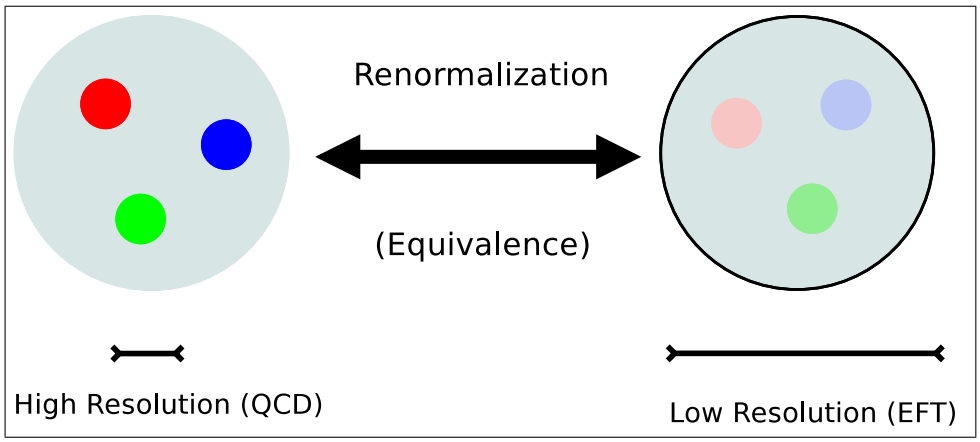


FIG. 2. The idea behind EFT is that it is possible to build a theory of low energy phenomena (at the energy scale Q, which is equivalent to the length scale $a \sim 1/Q$) without knowing the details of what happens at very high energy (at the energy scale M, with $M \ll Q$, where M is equivalent to the length scale $R_S \sim 1/M$). This is done by including a separation scale or cut-off and then requiring all observable quantities to be independent on the cut-off:

$$\frac{d}{d\Lambda} \langle \Psi | \mathcal{O} | \Psi \rangle = 0 \,,$$

where $|\Psi\rangle$ is the wave function and $\mathcal{O}$ is an operator representing an observable.



1. *Renormalization*

Now we will explain renormalization in more detail, including how to formalize it. We begin by considering a physical system with a characteristic distance scale a. We know how to describe physics at distances of the order of a, but we know that this is not all there is to the physics of this system. At short distances R_S there might be a fundamental theory that we do not know of yet, which might be able to give us a better description of this system. Or maybe we know this fundamental theory, but it is too complicated to solve and we want to obtain a more simple long distance description. Renormalization in general tells us that physics at the short distance R_S does not matter for the description of physic phenomena at distances a. Notice that here we have been talking about distance scales, but it is also possible to talk about momentum scales. In that case we will say that the fundamental theory happens at the momentum scale M ($\sim 1/R_S$) but we want to describe the world at the momentum scale Q ($\sim 1/a$).

Now the way to write this in a mathematical form is the following:

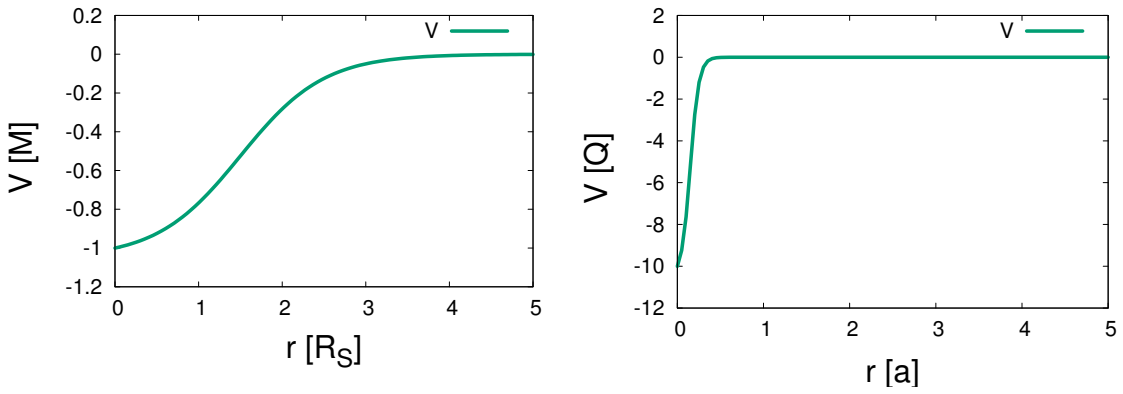


FIG. 3. The potential between two particles looked at different resolutions. In the left panel we see the potential when measured in units of M and then the distances are measure in units of R_S. In the right panel we see the potential in units of Q and where the distance is expressed in unit of a. We have taken $a = 10\,R_S$. In this second case we are not able to distinguish so much structure of the potential as in the first case.

(i) Introduce a cut-off R_c, i.e. an extra scale that will mark the boundary between known and unknown physics

$$R_S < R_c < a \tag{139}$$

(ii) Impose that physical observables at long distances are independent of the cut-off:

$$\frac{d}{dR_c}\,\langle\Psi|\mathcal{O}|\Psi\rangle = 0\,. \tag{140}$$

This condition is the basis of the renormalization group equations (RGEs), see Fig. II D 1 for a graphical representation.

To understand this better, let us imagine a theory of two interacting particles. At high energies they interact by means of a Yukawa potential that comes from the exchange of a heavy meson with mass M

$$V_S(r) = -g^2\frac{e^{-Mr}}{4\pi r}\,. \tag{141}$$

However at low energies / large distances it is not possible to distinguish the particle that generates this interaction. If we have a ruler in units of a (such that $Ma \gg 1$) and we try to determine what type of potential there is between particles 1 and 2 we will get to the conclusion that the potential is a *contact-type interaction*. This means a potential that

$$V(\vec{r}) = 0 \quad \text{if} \quad \vec{r} \neq 0\,. \tag{142}$$

That is, we will get to the concusion that the particles have to touch each other in oder to interact. The only way to see the internal structure of the potential V_S is to use a ruler of size $R_S \sim 1/M$, otherwise we will be unable to see any detail at all.

The mathematical way to describe a potential that is zero if $r \neq 0$, but is not-zero if $r = 0$ is with a Dirac delta. In one dimension we can describe the Dirac delta as follows:

$$\begin{aligned}&\delta(x) = 0 \quad \text{if} \quad x \neq 0 \quad \text{and}\,,\\ &\delta(x) = \infty \quad \text{if} \quad x = 0\,.\end{aligned} \tag{143}$$

Besides this $\delta(x)$ also fulfills the condition

$$\int_{-\infty}^{+\infty} f(x)\,\delta(x)\,dx = f(0)\,. \tag{144}$$

This condition defines the $\delta(x)$ is a more concrete way. The delta can also be viewed as the limit of some different function when the range of that function goes to zero, for instance

$$\delta(x) = \lim_{R_c\to 0} \frac{\pi}{|R_c|}\frac{1}{x^2+R_c^2}\,, \tag{145}$$

$$\delta(x) = \lim_{R_c\to 0} \frac{1}{\sqrt{\pi}|R_c|}e^{-(x/R_c)^2}\,, \tag{146}$$

to give just two examples (actually there are infinite ways to represent the delta). A quantum mechanical potential is defined in three dimensions, and hence we will need a three dimensional delta:

$$\delta^{(3)}(\vec{r}) = \delta(x)\,\delta(y)\,\delta(z)\,, \tag{147}$$

such that

$$\int d^3\vec{r} f(\vec{r})\delta^{(3)}(\vec{r}) = f(0)\,. \tag{148}$$

Knowing this, for $a \gg R_S$ the potential that we are able to see is not the original potential, but an effective potential that looks like a δ

$$V(\vec{r}) \to V_{\rm eff}(\vec{r}) = C\delta^3(\vec{r})\,. \tag{149}$$

However the EFT and renormalization ideas require the inclusion of a cut-off R_c. This means that we should not be using a δ, but rather some function for which the $R_c \to 0$ is a δ:

$$V_{\rm eff}(\vec{r}) \to V_{\rm eff}(\vec{r},R_c) = C(R_c)\,\delta^3_{R_c}(\vec{r}) \quad \text{such that} \quad \lim_{R_c\to 0}\delta^3_{R_c}(\vec{r}) = \delta^3(\vec{r})\,. \tag{150}$$

This is usually called a *smeared* delta function. Besides, notice that the strength of the effective potential is now a function of the cut-off:

$$C = C(R_c)\,. \tag{151}$$

For the three dimensional delta a few examples of a smeared delta are

$$\delta^{(3)}_{R_c}(\vec{r}) = \frac{\pi^2}{4R_c^3}\frac{1}{(r^2+R_c^2)^3}\,, \tag{152}$$

$$\delta^{(3)}_{R_c}(\vec{r}) = \frac{e^{-(r/R_C)^2}}{\pi^{3/2}R_C^3}\,. \tag{153}$$

$$\delta^{(3)}_{R_c}(\vec{r}) = \frac{\delta(r-R_C)}{4\pi R_C^2}\,. \tag{154}$$

where the third one is called a *delta-shell* and is particularly interesting because it simplifies calculations in an incredible way. Notice that the meaning of the $\delta(r-R_c)$ is straightforward:

$$\int_0^\infty dr f(r)\delta(r-R_c) = f(R_c)\,. \tag{155}$$

By the way, here you have an easy **exercise**: check the *normalization* of the previous smeared delta functions, i.e. the funny factors of π and powers of R_c that appear in the three expressions above.

Now we continue with the delta-shell regularization, for which the effective potential is now

$$V_{\rm eff}(r,R_C) = C(R_C)\frac{\delta(r-R_C)}{4\pi R_C^2}\,. \tag{156}$$

We can write a RGE for the coupling C in this equation as follows

$$\frac{d}{dR_C}\langle\Psi|V_{\rm eff}|\Psi\rangle = 0\,, \tag{157}$$

where Ψ is the wave function. Notice that for a wave function that is radially symmetric, i.e.

$$\Psi(\vec{r}) = \Psi(r)\,, \tag{158}$$

the evaluation of the previous matrix element is straighforward:

$$\langle\Psi|V_{\rm eff}|\Psi\rangle = C(R_c)\,|\Psi(R_c)|^2\,. \tag{159}$$

But the solution of the renormalization group equation (RGE) is not unique: it depends on the form of the wave function at long distances. There are actually two family of solutions

(i) the long range physics is perturbative or

(ii) the long range physics is non-perturbative.

In the first case the two-body system at large distances is basically a free (i.e. non-interacting) system and the two-body wave function is a free wave

$$\langle \vec{r}|\Psi\rangle = e^{i\vec{k}\cdot\vec{r}}\,, \tag{160}$$

for which we obtain

$$\langle \Psi|V_{\rm eff}|\Psi\rangle = C(R_C) + \mathcal{O}((kR_C)^2)\,. \tag{161}$$

As a consequence the RGE reads

$$\frac{d}{d\,R_C}\left[C(R_c)\right] = 0 + \mathcal{O}(k^2 R_C)\,, \tag{162}$$

with the solution

$$C(R_C) \propto 1\,. \tag{163}$$

In the second case the two-body wave function is of the type

$$\langle \vec{r}|\Psi\rangle = \frac{1}{\sqrt{4\pi}}\,\frac{e^{-\gamma r}}{r}\,, \tag{164}$$

which leads to the RGE

$$\frac{d}{d\,R_C}\left[\frac{C(R_c)}{R_c^2}\right] = 0 + \mathcal{O}(\gamma)\,, \tag{165}$$

and the solution

$$C(R_C) \propto R_C^2\,. \tag{166}$$

The first of these solutions is called the attractive fixed point of the renormalization group (RG), while the second is the repulsive fixed point. Alternatively, the first solution describes a natural system while the second solution describes an unnatural system.

The discovery of the attractive and repulsive fixed points of the type we are discussing here for two body scattering can be consulted in the original publication by Birse, McGovern and Richardson [8]. The treatment is however more involved than here because the analysis is done is momentum space and contains much more information than the scaling of the coupling $C(R_c)$ with the cut-off. A more simple explanation can be found in a recent review by myself [9], which uses a standard quantum mechanical approach in coordinate space.

Now here we can propose an interesting **exercise** (three points). Actually, the RG equation for non-perturbative systems is not the one we have written above (or the one written in the publication by Birse, McGovern and Richardson [8]). The real RG equation is the following one:

$$\frac{d}{d\,R_C}\left[\frac{C(R_c)}{R_c}\right] = 0 + \mathcal{O}(\gamma)\,, \tag{167}$$

Find an explanation of why it is this way. As a historical comment: the two versions of the RG evolution of $C(R_c)$ for non-perturbative systems, $C(R_c) \propto R_c^2$ and $C(R_c) \propto R_c$ and their interpretation have cause real headaches within the nuclear EFT community. For this exercise you do not have to go as far as making an interpretation of them, it is just enough to find how one can obtain the equation above.

III. THE ONE NUCLEON SECTOR

A. Isospin Symmetry

The proton and the neutron are spin 1/2 fermions, which masses of

$$M_p = 938.272\,{\rm MeV} \quad , \quad M_n = 939.565\,{\rm MeV}\,. \tag{168}$$

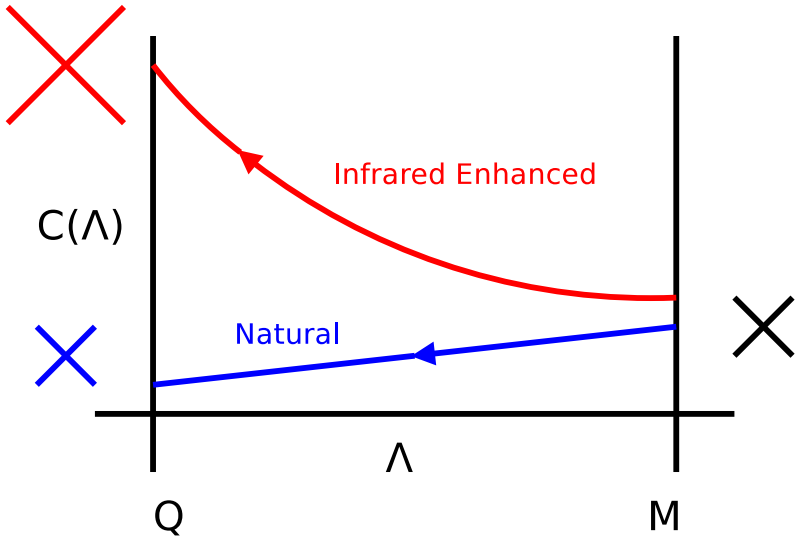


FIG. 4. Within the EFT description of low energy physics we can simplify the true potential of a system by a *smeared* delta, i.e. a contact-interaction with a range of the order of the cut-off R_c:

$$V_{\rm eff}(\vec{r};R_c) = C(R_c)\,\delta^{(3)}_{R_c}(\vec{r})\,.$$

The strength of the effective potential is given by the coupling constant $C(R_c)$, but the RGE of this coupling with R_c is not unique. For two-body systems which at low energies behave as free particles plus corrections, the coupling evolves as a constant

$$C(R_c) \propto 1\,,$$

while for two-body systems which interact strongly at low energies the evolution is different

$$C(R_c) \propto R_c^2\,,$$

and the coupling becomes stronger and stronger as $R_c \to a$. The first case leads to a relatively simple EFT, while the second to a more complicated EFT. They are sometimes called "natural" and "unnatural" / "infrared enhanced" scaling.

Their masses are indeed almost identical. Not only that, from the point of view of the nuclear force they behave in almost the same way, a feature that we have already commented goes by the name of *charge invariance*. This has prompted theoreticians to view the neutron and the proton as two states of the same particle, the nucleon

$$N = \begin{pmatrix} p \\ n \end{pmatrix}\,, \tag{169}$$

where the quantum number that makes these two states different is the *isospin*. The isospin is actually a copy of spin. In the isospin formalism the nucleon is a isospin $I = 1/2$ particle, of which the $m_I = +1/2$ state is the proton and the $m_I = -1/2$ one is the neutron, that is

$$|I = \frac{1}{2}\,,\, m_I = +\frac{1}{2}\rangle = |p\rangle \quad \text{and} \quad |I = \frac{1}{2}\,,\, m_I = -\frac{1}{2}\rangle = |n\rangle\,. \tag{170}$$

The thing is that isospin behaves in exactly the same way as spin. If we have a system of two nucleons we can group them in isospin 0 and 1 configurations

$$|NN(I=0, M=\ \ 0)\rangle = \frac{1}{\sqrt{2}}\left[|pn\rangle - |np\rangle\right]\,, \tag{171}$$

$$|NN(I=1, M=+1)\rangle = |pp\rangle\,, \tag{172}$$

$$|NN(I=1, M=\ \ 0)\rangle = \frac{1}{\sqrt{2}}\left[|pn\rangle + |np\rangle\right]\,, \tag{173}$$

$$|NN(I=1, M=-1)\rangle = |nn\rangle\,. \tag{174}$$

When we include isospin symmetry, the fermionic character of the neutron and proton is taken into account by formulating a sort of extended Fermi-Dirac statistics. That is, the wave function of A nucleons should be antisymmetric. For a system of two nucleons in S-wave, this means that there are only two possible spin-isospin configurations

(i) the singlet: $S = 0$, $I = 1$ and

(ii) the triplet: $S = 1$, $I = 0$.

The deuteron, the most simple nucleus, is in the triplet state. Yet isospin symmetry requires that the interaction in the three isospin states of the singlet should be approximately the same (in particular the nn and pp interactions). In fact this happens to be the case, but this idea can also be extended to heavier nuclei. For example:

(i) the binding energies of ${}^3H$ and ${}^3He$ ($I = 1/2$) (8.48 MeV and 7.72 MeV),

(ii) the binding energies of ${}^6He$, ${}^6Li$ and ${}^6Be$ ($I = 1$) (29.27, 31.99 and 26.92 MeV)

In the second example however we refer to the excited state of ${}^6Li$. The three nuclei ${}^6He$, ${}^6Li$ and ${}^6Be$ can be visualized as a ${}^4He$ core ($I = 0$) that is surrounded by a two-nucleon pair. In the case of ${}^6Li$ the two nucleon pair can be either in a isoscalar ($I = 0$) or isovector ($I = 1$) configuration, of which only the isovector one can be related to ${}^6He$ and ${}^6Be$ by isospin symmetry.

That also means that the nucleon-nucleon interaction in S-wave can be written as a sum of spin and isospin operators

$$V_{NN}(r) = (V_C + \vec{\tau}_1 \cdot \vec{\tau}_2 W_c) + \vec{\sigma}_1 \cdot \vec{\sigma}_2 \, (V_S + \vec{\tau}_1 \cdot \vec{\tau}_2 W_S) + S_{12}(\hat{r}) \, (V_T + \vec{\tau}_1 \cdot \vec{\tau}_2 W_T) \, . \tag{175}$$

We can now revisit the one pion exchange potential in view of the isospin formalism. The idea of the isospin is connected with the fact that the nucleon is composed of u and d quarks and can be extended to any other particle containing these quarks, including the pion. In fact we know that there are three types of pions — π^+, π^0 and π^- — and that their masses are also pretty similar:

$$m_{\pi^\pm} = 139.570 \, \text{MeV} \qquad m_{\pi^0} = 134.977 \, \text{MeV} \, . \tag{176}$$

As a consequence we can group the three pions into an isospin 1 state

$$\pi = \begin{pmatrix} \pi^+ \\ \pi^0 \\ \pi^- \end{pmatrix} \, . \tag{177}$$

This also means that we can update the interaction lagrangian between the pion and the nucleon in a more correct way

$$\mathcal{L} = i g_H \bar{N} \gamma^5 \vec{\tau} \cdot \vec{\pi} N \, , \tag{178}$$

which leads to the following OPE potential

$$V_{\text{OPE}}(r) = \frac{g_H^2}{4 M_N^2} \, \vec{\tau}_1 \cdot \vec{\tau}_2 \, \left[\vec{\sigma}_1 \cdot \vec{\sigma}_2 \, W_C(r) + S_{12}(\hat{r}) \, W_T(r) \right] \, . \tag{179}$$

Other isovector meson is the ρ meson, which can also be arranged as

$$\rho = \begin{pmatrix} \rho^+ \\ \rho^0 \\ \rho^- \end{pmatrix} \, . \tag{180}$$

This also means that the contributions of the ρ meson to the nuclear force will also display a $\vec{\tau}_1 \cdot \vec{\tau}_2$ factor. Last, the σ and ω mesons are isoscalars, i.e. they are like a $|00\rangle$ vector in isospin space:

$$\sigma = |00\rangle_I \quad , \quad \omega = |00\rangle_I \, . \tag{181}$$

They will give rise to forces that do not contain a $\vec{\tau}_1 \cdot \vec{\tau}_2$ factor.

Other example of the use of isospin are nuclear reactions. For instance, if we consider the probabilities of the reactions

$$P(pp \to d\pi^+) \quad \text{and} \quad P(np \to d\pi^0) \, , \tag{182}$$

then these probabilites can be related by means of Clebsch-Gordan coefficients corresponding to the coupling of different isospins (notice that the isospin of the deuteron is $I = 0$)

$$P(pp \to d\pi^+) \propto \left| \langle \frac{1}{2}\frac{1}{2}, \frac{1}{2}\frac{1}{2} | 11 \rangle \right|^2 = 1 \,, \tag{183}$$

$$P(np \to d\pi^0) \propto \left| \langle \frac{1}{2} - \frac{1}{2}, \frac{1}{2}\frac{1}{2} | 10 \rangle \right|^2 = \frac{1}{2} \,, \tag{184}$$

from which we derive

$$\frac{P(pp \to d\pi^+)}{P(np \to d\pi^0)} = 2 \,. \tag{185}$$

B. SU(3)-Flavour Symmetry

As we know there are six types of quarks — u, d, s, s, b and t — of which only the five first ones are important for hadron physics. The top quark decays two quickly into the bottom quark and as a consequence it does not have enough time to form hadrons. Isospin symmetry rest on the idea that the u and d quarks are very light and their mass difference are actually really small in comparison with the nucleon or the pion masses. The idea of isospin can be extended to other of the quark species, the s quark, in which case we end up with flavour symmetry. However this symmetry is much more approximate in nature than isospin. While isospin symmetry is conserved at the few percent level, the violations of flavour symmetry are usually of the order of twenty to thirty percent.

C. The Linear Sigma Model and the Goldstone Theorem

Let us consider the pion from a different point of view. For that we will consider this Lagrangian proposed by Gell-Mann and Levi in the 60's [10]:

$$\mathcal{L} = i\bar{N}\gamma^\mu \partial_\mu N + g\,\bar{N}(\phi^0 + i\gamma^5 \vec{\tau}\cdot\vec{\phi})N + \frac{1}{2}\sum_i \partial_\mu \phi_i \partial^\mu \phi_i - V(\phi) \,, \tag{186}$$

$$V(\phi) = \frac{1}{2}\,\mu^2 \,(\sum_i \phi_i^2) + \frac{\lambda}{4}\,(\sum_i \phi_i^2)^2 \,. \tag{187}$$

This lagrangian contain a massless nucleon field and four bosonic fields ϕ_i, of which ϕ_0 is a scalar and ϕ_i with $i = 1, 2, 3$ pseudoscalars. The lagrangian has $O(4)$ symmetry in the fields ϕ_i with $i = 0, 1, 2, 3$, that is, the transformation

$$\phi_i \to R_{ij}\phi_j \tag{188}$$

where R_{ij} are 4x4 matrices such that $R^T R = 1$ leaves the lagrangian unchanged. As a reminder, the goup $O(n)$ is the group of nxn matrices such that

$$O(n) = \left\{ R \text{ (nxn) such that } R^T R = 1 \right\} \,, \tag{189}$$

and it has $n(n-1)/2$ independent generators.

If we take $\mu^2 > 0$ (as usual) and $\lambda > 0$ (so we avoid vacuum decay), we have that the four bosonic fields are massive and their mass is μ. The reason is, of course, the mass term

$$\mathcal{L}_{\text{mass}} = -\frac{1}{2}\,\mu^2 \,(\sum_i \phi_i^2) \,. \tag{190}$$

In addition they have a four boson interaction term. However if $\mu^2 < 0$ something interesting happens. The potential $V(\phi)$ now has a minimum at

$$\phi_0 = v = \sqrt{-\frac{\mu^2}{\lambda}} \,. \tag{191}$$

We can also make the following change of variables

$$\sigma = \phi_0 - v \,, \tag{192}$$

$$\pi_i = \phi_i \,, \tag{193}$$

from which we obtain (do it as **exercise**)

$$\mathcal{L} = \bar{N}(i\gamma^\mu\partial_\mu - gv)N + g\,\bar{N}(\sigma + i\gamma^5\vec{\tau}\cdot\vec{\pi})N + \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma + \frac{1}{2}\partial_\mu\vec{\pi}\cdot\partial^\mu\vec{\pi} - V(\sigma,\pi)\,, \tag{194}$$

$$V(\sigma,\pi) = \lambda v^2\sigma^2 + v\lambda\sigma^3 + \frac{\lambda}{4}(\sigma^4 - v^4) + \frac{\lambda}{4}\vec{\pi}^4 + v\lambda\vec{\pi}^2\sigma + \frac{\lambda}{2}\vec{\pi}^2\sigma^2\,. \tag{195}$$

This lagrangian is interesting for a series of reasons:

(a) First, we can see that the original $O(4)$ symmetry is spontaneously broken to $O(3)$, which only applies to the $\vec{\pi}$ field.

(b) Second, originally all the fields have equal mass $\mu^2 > 0$. But if we have $\mu^2 < 0$, the σ field is massive and has a mass of $\lambda v^2 = -\mu^2$ while the $\vec{\pi}$ fields are massless.

(c) Third, the N field now has a mass of $M_N = g\,v$

(d) Fourth, for the sake of clarity let's call v by its more common name, f_π: $v = f_\pi$, with $f_\pi = 92.4\,\mathrm{MeV}$.

(e) Fifth, this model implies $g_\sigma = g = M_N/v = 10.2$ and $g_{\pi NN} = g_\sigma \sim 10.2$, which (although we have not explained it) happens to be really accurate. The actual $g_{\pi NN} = 13.4$ is a $25-30\%$ larger than the prediction done here as a consequence that the nucleon couples to pseudoscalars in a slightly different way than to scalar. In particular by making the correction

$$\bar{N}(\phi_0 + i\gamma_5\vec{\tau}\cdot\vec{\phi})N \to \bar{N}(\phi_0 + ig_A\gamma_5\vec{\tau}\cdot\vec{\phi})N\,, \tag{196}$$

with $g_A \simeq 1.26$ we can obtain the correct coupling to the pion (g_A is called the axial coupling).

Other interesting variation of this lagrangian is to include a small term in the original $O(4)$ symmetric lagrangian that breaks $O(4)$ symmetry, like for example

$$\Delta V(\phi) = -\epsilon\, v^3\,\phi_0\,. \tag{197}$$

This term is interesting because it will slightly shift the mass of the σ meson and give a tiny mass to the pions

$$m_\sigma^2 = \lambda v^2 + a_0\,\epsilon\,, \tag{198}$$

$$m_\pi^2 = b_0\epsilon\,, \tag{199}$$

with a_0 and b_0 proportionality constants that you can try to calculate as an exercise. Yet the final $O(3)$ symmetry remains in the final lagrangian. As we will see, once we explain chiral symmetry, this is indeed very similar to what happens in the real world.

1. A Brief Overview of the Goldstone Theorem

The mechanism by which this lagrangian generates a series of massless bosons after a symmetry is spontaneously broken is the Goldstone theorem. A more abstract version than the example we have use is the following, which is taken from some lectures notes by Leutwyler [11]. Assume we have a hamiltonian H that is invariant under a Lie group G. If we denote the generators of the group G as Q_i, with $i = 1,\ldots,n_G$, with n_G the dimension of the Lie group, what we have is the following

$$[Q_i, H] = 0\,. \tag{200}$$

This symmetry is spontaneously broken if the ground state of H – the vacuum – is not invariant under G. That this, there are a few generators for which

$$Q_i|0\rangle \neq 0\,. \tag{201}$$

This implies that the vacuum is not unique: $[Q_i, H] = 0$ and therefore $Q_i|0\rangle$ describes a state with the same energy as the vacuum. Yet there will be a subset of Q_i that leave the vacuum invariant

$$J_i|0\rangle = 0\,. \tag{202}$$

These generators obey $[J_i, J_k] = 0$ and span a subgroup H of the original Lie group G. If the dimension of H is n_H, then there are still $n_G - n_H$ generators in the quotient group G/H that do not annihilate the vacuum. If we give the name K_i to the generators of G/H, then we have

$$K_i|0\rangle\,, \tag{203}$$

linearly independent states with the same energy as the fundamental states. These $n_G - n_H$ states are the Goldstone bosons.

Notice that in the example of the linear sigma model, $G = O(4)$ and $H = O(3)$. The dimension of $O(n)$ is $n(n-1)/2$, with means that $n_G = 6$ and $n_H = 3$. As a consequence we end up with $n_G - n_H = 3$ Goldstone bosons.

D. Chiral Symmetry

1. The Massless Fermion

Let us consider a Lagrangian containing a fermion field

$$\mathcal{L} = \bar{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi\,. \tag{204}$$

This lagrangian has a global $U(1)$ symmetry, which means that it is invariant under a change of phase of the field Ψ

$$\Psi(x) \to e^{i\alpha}\Psi(x)\,. \tag{205}$$

But when the fermion is massless, the lagrangian has an additional symmetry that goes by the name of chiral symmetry. This has to do with what are the right and left hand components of the field Ψ

$$\Psi_R = \frac{1}{2}\left(1 + \gamma_5\right)\Psi \tag{206}$$

$$\Psi_L = \frac{1}{2}\left(1 - \gamma_5\right)\Psi \tag{207}$$

In terms of the L and R fields, it happens that the derivative term in the lagrangian does not mix them together

$$\bar{\Psi}\, i\,\gamma^\mu\partial_\mu\,\Psi = \bar{\Psi}_L\, i\,\gamma^\mu\partial_\mu\,\Psi_L + \bar{\Psi}_R\, i\,\gamma^\mu\partial_\mu\,\Psi_R\,. \tag{208}$$

Meanwhile the mass term mixes terms of opposite chirality

$$\bar{\Psi}\, m\,\Psi = \bar{\Psi}_L\, m\,\Psi_R + \bar{\Psi}_R\, m\,\Psi_L\,, \tag{209}$$

which is why chiral symmetry is only a symmetry is the fermion is massless. In such a case there is a new global $U(1)$ chiral symmetry

$$\Psi(x) \to e^{i\alpha_5\gamma_5}\Psi(x)\,. \tag{210}$$

Alternatively we can say that the fields Ψ_R and Ψ_L are each one invariant under a global $U(1)$ symmetry

$$\Psi_R \to e^{i\alpha_R}\Psi_R \quad , \quad \Psi_L \to e^{i\alpha_R}\Psi_L\,, \tag{211}$$

which can also be illuminating, as we will see.

2. The Massless Quarks

This idea apply after certain modifications to QCD. As already seen the QCD Lagrangian reads

$$\mathcal{L} = -\frac{1}{4}\,G^a_{\mu\nu}\,G^{a\mu\nu} + \bar{q}\,i\,\gamma^\mu D_\mu\,q - \bar{q}\mathcal{M}q \tag{212}$$

where we have written q as a vector

$$q = \begin{pmatrix} u \\ d \\ s \\ c \\ b \\ t \end{pmatrix}\,, \tag{213}$$

with $\mathcal{M}$ the quark mass matrix

$$\mathcal{M} = \begin{pmatrix} m_u & & & & & & \\ & m_d & & & & & \\ & & m_s & & & & \\ & & & m_c & & & \\ & & & & m_b & & \\ & & & & & m_t & \end{pmatrix} . \tag{214}$$

It happens that three of the quarks masses — m_u, m_d and m_s — are smaller than $\Lambda_{\rm QCD}$, which is the natural scale of QCD. In particular two of the masses, m_u and m_d, are really small in comparison with $\Lambda_{\rm QCD}$. As a consequence we expect that the approximation that the masses of these quarks are zero will be pretty good.

We can also define the L and R components of the quark fields as usual

$$q_R = \frac{1}{2}\,(1+\gamma_5)\,q\,, \tag{215}$$

$$q_L = \frac{1}{2}\,(1-\gamma_5)\,q\,, \tag{216}$$

where the derivative terms respect this decomposition

$$\bar{q}\,i\,\gamma^\mu D_\mu\,q = \bar{q}_L\,i\,\gamma^\mu D_\mu\,q_L + \bar{q}_R\,i\,\gamma^\mu D_\mu\,q_R \tag{217}$$

but the mass term does not

$$\bar{q}\,m_q\,q = \bar{q}_L\,m_q\,q_R + \bar{q}_R\,m_q\,q_L\,. \tag{218}$$

In the limit when N of the quark masses are zero, the lagrangian becomes invariant under the following transformations

$$q_R \to V_R q_R \quad \text{and} \quad q_L \to V_L q_L \tag{219}$$

where V_R and V_L belong to the special unitary group $U(N)$. That is, the Lagrangian is invariant under the group

$$G = U(N)_L \times U(N)_R\,. \tag{220}$$

It turns out however that in QCD there is something called the axial anomaly, which implies that the axial current $J^5_\mu = \bar{q}\gamma_\mu\gamma_5 q$ fails to be conserved

$$\partial^\mu J^5_\mu \neq 0\,. \tag{221}$$

As a consequence the Lagrangian is only invariant under the group

$$G = SU(N)_L \times SU(N)_R \times U(1)_{L+R}\,. \tag{222}$$

It happens however that this symmetry is spontaneously broken: the vacuum is only invariant under the vector transformations of this group, i.e. by the subgroup

$$H = SU(N)_{L+R} \times U(1)_{L+R}\,. \tag{223}$$

This can be seen from the fact that positive and negative parity hadrons have different masses: for example, the nucleon $J^P = \frac{1}{2}^+$ has a mass of 940 MeV, while the lightest $J^P = \frac{1}{2}^-$ nucleon-like baryon happens at 1535 MeV. The ρ and ω mesons, with $J^{PC} = 1^{--}$ and masses of 770 MeV and 780 MeV respectively, have axial partners ($J^{PC} = 1^{++}$) at 1230 and 1280 MeV respectively, the $a_1(1260)$ and $f_1(1285)$. In general the negative parity states are heavier, with the exception of the pion as a consequence of the chiral symmetry breaking we are explaining here. The fact that the vacuum is independent under the subgroup H instead of G, implies that the spectrum must contain N^2-1 Goldstone bosons which are related to the quotient group

$$\frac{G}{H} \simeq SU(N)_{L-R}\,, \tag{224}$$

These Goldstone bosons are related to axial currents and as a consquence they happen to have non-natural parity [1]: they end up being pseudoscalar mesons with $J^P = 0^-$.

---

[1] Natural parity refers to the series 0^+, 1^-, 2^+, 3^- and so on.

In the real world the quark masses are

$$m_u = 2.3 \pm 0.7\,\text{MeV}\,, \tag{225}$$
$$m_d = 4.8 \pm 0.7\,\text{MeV}\,, \tag{226}$$
$$m_s = 95 \pm 5\,\text{MeV}\,, \tag{227}$$
$$m_c = 1.28 \pm 0.03\,\text{GeV}\,, \tag{228}$$
$$m_b = 4.18 \pm 0.04\,\text{GeV}\,. \tag{229}$$

What does it set a quark mass as light or heavy? The comparison with $\Lambda_{\text{QCD}} \sim 200 - 300\,\text{MeV}$, which is the natural scale of QCD. We have basically two options: to make the approximation of $m_u = m_d = 0$ or to include also the strange quark $m_u = m_d = m_s = 0$. The first option is $SU(2)$ chiral symmetry and the second $SU(3)$ chiral symmetry.

Now, what does this implies? The first thing is that in the limit of $m_q = 0$, the $SU(2) \times SU(2)$ symmetry is spontaneously broken. Therefore by the Golstone theorem there should be $N^2 - 1 = 3$ massless pseudo scalar bosons corresponding to this breaking of the symmetry. It happens the pions (π^+, π^0 and π^-) with a mass of about 140 MeV and 135 MeV for the charged and neutral cases respectively, are much lighter than any other hadron. Owing to the finite m_u and m_d masses these bosons adquire a finite mass. Indeed there is a formula for their mass, which is called the Gell-Mann, Oakes, Renner relation

$$m_\pi^2 = 2\,B\,(m_u + m_d) + \mathcal{O}(m_q^2)\,, \tag{230}$$

where B is given by

$$B = \left| \frac{\langle 0|u\bar{u}|0\rangle}{f_\pi^2} \right|, \tag{231}$$

with $\langle 0|u\bar{u}|0\rangle \sim -(250\,\text{MeV})^3$ a quantity called the quark condensate and $f_\pi = 92.4\,\text{MeV}$ the pion decay constant.

Now if we extend the approximation $m_q = 0$ to the strange quark we arrive at $SU(3)$ chiral symmetry. Next to the pions, we have another five pseudoscalars that are particularly light: the kaons, the antikaons and the η with a mass of about 495 MeV for the kaons and 545 MeV for the η. Together with the pions these are the lightest hadrons and in total they sum eight pseudoscalar with fits neatly with the idea that they are Goldstone bosons, as $N^2 - 1 = 8$ for $N = 3$.

The other very important aspect of chiral symmetry is that the pion-nucleon interaction has to be derivative. We previously arrived to the conclusion that the pion is a $J^P = 0^-$ pseudoscalar boson, that interacts with the nucleon by means of the lagrangian

$$\mathcal{L} = ig\bar{\Psi}_N \gamma^5 \vec{\tau} \cdot \vec{\pi} \Psi_N\,. \tag{232}$$

The Goldstone-boson nature of the pions actually implies that the correct interaction needs to include a derivative (a ∂^μ) of the pion field. The correct Lagrangian is

$$\mathcal{L} = \frac{g_A}{2f_\pi}\,\bar{\Psi}_N \gamma^5 \gamma_\mu \vec{\tau} \cdot \partial^\mu \vec{\pi} \Psi_N\,. \tag{233}$$

This little correction, though it leads to exactly the same type of OPE potential, is actually incredibly important when calculating the two pion exchange contributions to the nuclear forces. With the original pre-chiral lagrangian, the two pion exchange potential does not really work. But once we include chiral symmetry, the resulting two pion exchange potential indeed works and is able to describe the intermediate range region of the nuclear force.

Appendix A: Derivation of the Yukawa potential

At this point a problem that a few of you might be thinking about is the derivation of the Yukawa potential: it is probable that not all of you have studied quantum field theory. Yet, it is possible to derive it in quantum mechanics from what you already know about second order perturbation theory and second quantization (in particular the quantization of the electromagnetic field, which is usually studied in the context of explaining the decays of the excited states of the hydrogen atom). First, we consider a system of two nucleons and we want to calculate their energy shift when they can emit and absorb pions (or, more technically, scalar mesons). In the absence of scalar mesons, there will be no shift in energy. But if scalar mesons can couple to nucleons, then if they exists as intermediate states in the two-nucleon system they will generate an energy shift that can be calculated in second-order perturbation theory

$$E_{\text{NN}}^{(2)} = \sum_{NN\phi} \frac{\langle NN|H_{\text{pert}}|NN\phi\rangle\,\langle NN\phi|H_{\text{pert}}|NN\rangle}{E_{\text{NN}}^{(0)} - E_{\text{NN}\phi}^{(0)}}\,, \tag{A1}$$

where we sum over all possible intermediate states containing a scalar meson ϕ. In the previous formula $E^{(0)}_{\rm NN}$ is the energy of the two-nucleon system when there are no pions, which can be approximated by

$$E^{(0)}_{\rm NN} = 2m_N\,, \tag{A2}$$

i.e. twice the mass of the nucleon, where we have ignored corrections coming from the momentum of the nucleon as they will be small in comparison to the nucleon mass. Meanwhile $E^{(0)}_{{\rm NN}\phi}$ is the energy of a state containing two nucleons and a scalar meson ϕ, which will be

$$E^{(0)}_{{\rm NN}\phi} = 2m_N + \omega \quad \text{, where} \quad \omega = \sqrt{m^2+\vec{q}^2}\,. \tag{A3}$$

Here we notice that the scalar meson is much lighter than the nucleon, and thus we take into account its momentum and assume that this momentum can be so large as to make the meson relativistic. It is important to notice that the shift in energy $E^{(2)}_{\rm NN}$ can be interpreted as the nucleon-nucleon potential: if we assume heavy and static (i.e. they are not moving) nucleons separated by a distance r, then the potential is indeed the shift in their energies with respect to the situation where there is no potential (or when the distance between the nucleons is infinite). That is, we have that

$$V_{\rm NN}(r) = E^{(2)}_{\rm NN}\,. \tag{A4}$$

Next, we combine the previous observations to obtain

$$V_{\rm NN}(r) = -\sum_{NN\phi(\vec{q})} \frac{|\langle NN|H_{\rm pert}|NN\phi\rangle|^2}{\omega(\vec{q})}\,, \tag{A5}$$

which already give us a very important result: the exchange of a scalar meson will generate an attractive potential, as we can deduce from the minus sign. This in turns come from the fact that the intermediate states have a higher energy level than the states we are perturbing (i.e. the two-nucleon states), resulting in $E^{(0)}_{\rm NN} - E^{(0)}_{{\rm NN}\phi} < 0$ and hence an attractive potential.

The next step is evaluating the matrix elements of the perturbation $H_{\rm int}$, i.e. the interaction term between a nucleon and a scalar meson. For this we will work in analogy to the second quantization formalism that is usually develop to explain the decays of the excited states of the hydrogen atom. Indeed, in this formalism (which we will not review here, but only take its main results) the emission of a photon with momentum $\vec{q}$ and polarization λ was given by

$$\langle 1\gamma(\vec{q},\lambda)|\vec{A}(\vec{r})|0\gamma\rangle = \frac{1}{\sqrt{2\omega}}\, e^{-i\vec{q}\cdot\vec{r}}\, \vec{\epsilon}^{\,*}_{\lambda}(\vec{q})\,, \tag{A6}$$

where $\vec{A}(\vec{r})$ is the electromagnetic vector potential, which in the second quantization formalism can create or annihilate photons, and $\vec{\epsilon}_\lambda(\vec{q})$ is the polarization vector of a photon with polarization $\lambda = \pm 1$ (where its angular momentum has been quantized in the direction $\vec{q}$). Notice that in the previous expression we have use a different normalization that the one commonly found in textbooks about the second quantization: the reason for this is that here we are defining the electron charge as $\alpha = e^2/(4\pi)$, while in most textbooks explaining second quantization in quantum mechanics the convention is $\alpha = e^2$. This difference will give a factor of $\sqrt{4\pi}$.

For the case of a scalar meson, the quantization of its field can be done in exactly the same way. Indeed, knowing how it works for photons it is easy to guess that the only difference if we have scalar mesons is that there should be no polarization vector:

$$\langle 1\phi(\vec{q})|\phi(\vec{r})|0\phi\rangle = \frac{1}{\sqrt{2\omega}}\, e^{-i\vec{q}\cdot\vec{r}}\,, \tag{A7}$$

where $\vec{q}$ is the meson momentum and $\vec{r}$ the point where the meson is created or annihilated. From there, we can deduce the interaction Hamiltonian for the scalar meson from its interaction Lagrangian

$$-\mathcal{H}_{\rm int} = \mathcal{L}_{\rm int} = g_Y\, \bar{\Psi}_N \phi \Psi_N\,, \tag{A8}$$

and its non-relativistic limit will be

$$\langle N\phi(\vec{q})|H_{\rm int}|N\rangle = i\, g_Y\, \frac{1}{\sqrt{2\omega}}\, e^{-i\vec{q}\cdot\vec{r}}\,. \tag{A9}$$

Finally, if there were to be more than one nucleon in the system, it will be important to distingish to/from which nucleon is the meson emitted/absorbed. For instance, in the two-nucleon system we will write

$$\langle NN\phi_i(\vec{q})|H_{\rm int}|NN\rangle = i\, g_Y\, \frac{1}{\sqrt{2\omega}}\, e^{-i\vec{q}\cdot\vec{r}_i}\,. \tag{A10}$$

where $\vec{r}_i$ now has a subindex to indicate whether the meson comes from nucleon $i = 1$ or 2 (as there are two nucleons). However this distinction is only important if we are using quantum-mechanics style second order perturbation theory (in quantum field theory the formalism automatically takes care of these details).

To obtain the potential the only thing left is to put the pieces together

$$\begin{aligned} V(\vec{r}_1 - \vec{r}_2) &= -\sum_{ij=12,21} \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{\langle NN|H_{\rm pert}|NN\phi(\vec{r}_i)\rangle \, \langle NN\phi(\vec{r}_j)|H_{\rm pert}|NN\rangle}{\omega(\vec{q})} \\ &= -\sum_{ij=12,21} \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{e^{i\vec{q}\cdot(\vec{r}_i - \vec{r}_j)}}{2\omega^2(\vec{q})} \,, \end{aligned} \tag{A11}$$

where a few comments are in order. First, the sum of intermediate states has been rewritten as:

$$\sum_{NN\phi} = \sum_{ij=12,21} \int \frac{d^3\vec{p}}{(2\pi)^3} \,, \tag{A12}$$

where we have two sums: the first one refers to the nucleon from which the scalar meson is emited and then absorbed (it can be 12 or 21 and each of these possibilities should be counted independently), while the second one refers to the integral over all the possible momenta of the meson. The first sum is trivial, as it effectively becomes a factor of 2 (both $ij = 12$ or 21 give the same result), leaving us with

$$V(\vec{r}_1 - \vec{r}_2) = -g_Y^2 \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{e^{i\vec{q}\cdot(\vec{r}_1 - \vec{r}_2)}}{\omega^2(\vec{q})} = -g_Y^2 \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{e^{i\vec{q}\cdot(\vec{r}_1 - \vec{r}_2)}}{m^2 + \vec{q}^2} \,, \tag{A13}$$

which after integration of the momentum integral (which is basically a Fourier transform) leads to

$$V(\vec{r}) = -\frac{g_Y^2}{4\pi} \frac{e^{-mr}}{r} \,. \tag{A14}$$

This procedure can indeed be modified to calculate the potentials generated by the exchange of other particles, though there will be complications from the additional structures involved (isospin and spin operators, polarization of the exchanged meson if it is a vector meson, etc.). As a consequence, the potential that we obtain at the end is not necessarily always attractive, as suggested by the fact that they are derived from second order perturbation theory (reminder: second order perturbation theory always gives a negative energy contribution for the ground state).

The easiest example would be provided by the addition of isospin to the scalar meson we where considering, in which case:

$$\langle N\phi_a(\vec{r})|H_{\rm int}|N\rangle = i\, g_Y \frac{1}{\sqrt{2\omega}} \tau_a \, e^{-i\vec{q}\cdot\vec{r}} \,. \tag{A15}$$

where the meson ϕ now has an isospin index $a = 1, 2, 3$, which manifest as the Pauli matrix τ_a which is to be sanwhiched between the initial and final isospin wave function of the nucleon. As a consequence the potential ends up being

$$V(\vec{r}) = -\vec{\tau}_1 \cdot \vec{\tau}_2 \, g_Y^2 \frac{e^{-mr}}{4\pi r} \,, \tag{A16}$$

where there is now a $\vec{\tau}_1 \cdot \vec{\tau}_2$ factor that can change the sign of the potential. The addition of isospin is in general trivial, as it will always entail the inclusion of a $\vec{\tau}_1 \cdot \vec{\tau}_2$ multiplicative factor in the potential.

A more complex example will be to consider a pseudoscalar meson instead of a scalar one. In terms of the quantum field theory language this would mean to consider the interaction Lagrangian

$$-\mathcal{H}_{\rm int} = \mathcal{L}_{\rm int} = g_Y \, \bar{\Psi}_N i\gamma_5 \phi \Psi_N \,, \tag{A17}$$

where there is now a γ_5 factor, which usually signals a change of parity with respect to standard expectations. The non-relativistic limit of this interaction is

$$\langle N\phi(\vec{q})|H_{\rm int}|N\rangle = i\, g_Y \frac{1}{\sqrt{2\omega}} \frac{\vec{\sigma}\cdot\vec{\nabla}}{2M_N} \, e^{-i\vec{q}\cdot\vec{r}} \,, \tag{A18}$$

which now involves the spin of the nucleon. It is interesting to notice that the previous non-relativistic matrix element can also be derived in terms of quantum mechanics alone: if we begin by considering a scalar meson ($J^P = 0^+$), for which we have

$$\langle N\phi(\vec{q})|H_{\rm int}|N\rangle = i\, g_Y \frac{1}{\sqrt{2\omega}} \, [1] \, e^{-i\vec{q}\cdot\vec{r}} \,, \tag{A19}$$

the most direct way to modify this matrix element to suit a pseudoscalar meson ($J^P = 0^-$) is to change the quantity in brackets — the [1], which is a scalar with positive parity — into something with is also a scalar but has negative parity. For this, the most direct modification would be

$$[1] \to \left[\frac{1}{M}\,\vec{\sigma}\cdot\vec{\nabla}\right], \tag{A20}$$

with M a mass scale and where the rationale is straightforward: $\vec{\nabla}$ is a vector ($J^P = 1^-$) because it changes sign when we do the $\vec{r} \to -\vec{r}$ parity tranformation, $\vec{\sigma}$ is angular momentum and thus a pseudovector ($J^P = 1^+$) because it does not change sign when we do $\vec{r} \to -\vec{r}$ (remember that for orbital angular momentum $\vec{L} = \vec{r}\times\vec{p} = (-\vec{r})\times(-\vec{p})$ and as a consequence the same applies to intrinsic angular momentum) and, finally, the scalar product of two vectors is a scalar (so the scalar product of a vector and a pseudovector is a pseudoscalar). Independently of the derivation for the interaction of a pseudoscalar meson with nucleons, the end result is the following:

$$V(\vec{r}_1 - \vec{r}_2) = -\frac{g_Y^2}{4M_N^2}\int \frac{d^3\vec{p}}{(2\pi)^3}\,(\vec{\sigma}_1\cdot\vec{q})\,(\vec{\sigma}_2\cdot\vec{q})\,\frac{e^{i\vec{q}\cdot(\vec{r}_1-\vec{r}_2)}}{m^2+\vec{q}^2}\,, \tag{A21}$$

where, once we calculate the Fourier-transform, we indeed recover the one pion exchange potential.

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