Exercises: part 3

This is the third exercise sheet for the nuclear physics course (Spring 2022). To pass the course you are required to complete 10 points total, and the grade will depend on how well are these exercises done. A few comments are in order:

1. You can ask for hints on how to do exercises.
2. No copying (this is really important: you are already graduate students and are expected to do original research).

Each exercise has a different value in points: more difficult exercises give more points.

Regarding when to hand over the exercises: of course it must be before the school tell the teachers to upload the grades, but I do not know when that will happen. Besides, a few of the exercise sets are more difficult than others, which means that it is up to you to decide where to concentrate your efforts.

Exercises can be handed over in either Chinese or English (if in Chinese, use a really clear handwritting: the laoshi is only used to read printed characters). They can be a picture of your handwritten exercises, or they can be in pdf or in any other common format. You can send them via wechat or email (mpavon "at" buaa.edu.cn).

(1) Assuming that the deuteron was a pure S-wave or a pure D-wave state — a $^3S_1$ or $^3D_1$ partial wave in the spectroscopic notation — show that the magnetic moment in units of the nuclear magneton should be

$$\mu(^3S_1) = \mu_p + \mu_n = 0.88 \tag{1}$$

$$\mu(^3D_1) = \frac{3}{4} - \frac{1}{2}\,(\mu_p + \mu_n) = 0.31\,. \tag{2}$$

(2 points)

(2) As we commented in the lectures, the magnetic moment of the deuteron also receives small contributions from two-body electromagnetic currents. For seeing how this works, it is useful to reconsider the calculation of the standard magnetic moment of the deuteron in a more detailed way. Indeed, the magnetic moment can be understood as the matrix element of a magnetic moment operator sandwiched between the deuteron wave functions

$$\mu_d = \langle \psi_d(11) | \hat{\mu} | \psi_d(11) \rangle\,, \tag{3}$$

where $(11) = (S m_S)$ refers to the spin state of the deuteron and the magnetic operator is given by

$$\hat{\mu} = \left[\mu_p \vec{\sigma}_p + \mu_n \vec{\sigma}_n\right]\, \delta^{(3)}(\vec{r_p}\,' - \vec{r_p})\, \delta^{(3)}(\vec{r_n}\,' - \vec{r_n})\,, \tag{4}$$

with $\vec{r}_{p(n)}$, $\vec{r}_{p(n)}\,'$ the positions of the proton (neutron) in the initial and final states, which do not change (hence the Dirac-delta's). Actually, these two Dirac-deltas can be rewritten in terms of the center-of-mass radius and the relative radius:

$$\delta^{(3)}(\vec{r_p}\,' - \vec{r_p})\, \delta^{(3)}(\vec{r_n}\,' - \vec{r_n}) = \delta^{(3)}(\vec{R}\,' - \vec{R})\, \delta^{(3)}(\vec{r}\,' - \vec{r})\,, \tag{5}$$

where $\vec{R} = (\vec{r}_p + \vec{r}_n)/2$ and $\vec{r} = \vec{r}_p - \vec{r}_n$ (we are assuming equal masses for the proton and neutron, i.e. $m_p = m_n$). Assuming a pure S-wave deuteron wave function for simplicity

$$\langle \vec{R}, \vec{r} | \Psi_d(S m_S) \rangle = e^{i\vec{K}\cdot\vec{R}}\, \frac{u(r)}{r} Y_{00}(\hat{r}) | S m_S \rangle\,, \tag{6}$$

where $\vec{K}$ is the total momentum of the system (which will disappear after taking the matrix elements). With this, reproduce the previous result for an S-wave deuteron:

$$\mu(^3S_1) = \mu_p + \mu_n\,. \tag{7}$$

(**1 point**)

Now, we can go further than this and include a two-body current for the magnetic operator

$$\hat{\mu} = \Big(\left[\mu_p \vec{\sigma}_p + \mu_n \vec{\sigma}_n\right]\, \delta^{(3)}(\vec{r}\,' - \vec{r}) + \mu_{2B} f(\vec{r}\,' - \vec{r}) \Big)\, \delta^{(3)}(\vec{R}\,' - \vec{R})\,. \tag{8}$$

Compute what will $\mu(^3S_1)$ look like once we include this new operator (**1 point**).

(3) The quadrupole moment of the deuteron can be calculated by using the well-known formula

$$Q_d = \frac{1}{20} \int_0^\infty dr\, r^2 w(r) \, (2\sqrt{2} u(r) - w(r)) \,, \tag{9}$$

with u, w the S- and D-wave components of the wave function. However, we have not studied the derivation of this formula, which actually makes for a good exercise. For understanding where this formula comes from, we begin with the definition of the quadrupole moment of the deuteron

$$Q_d = \langle \Psi_d(11) | \hat{Q}_{33} | \Psi_d(11) \rangle \,, \tag{10}$$

where $\Psi_d(S m_S)$ is the wave function for a deuteron state with spin S and third component of the spin m_S (for which we take $S = m_S = 1$). In turn the quadrupole operator is simply

$$\begin{aligned} \hat{Q}_{33} &= e_p(3z_p^2 - r_p^2) + e_n(3z_n^2 - r_p^2) \\ &= \frac{e_p}{4}(3z^2 - r^2) = \frac{1}{4}(3z^2 - r^2) \,, \end{aligned} \tag{11}$$

where in the first line we have written it in terms of the proton and neutron coordinates, while in the second we have use the relative coordinate $\vec{r} = \vec{r}_p - \vec{r}_n$ (with z the third component of $\vec{r}$); e_p and e_n refer to the proton and neutron electric charge, which we take to be $e_p = 1$ and $e_n = 0$. Notice that for simplicity we have omited the Dirac-delta factors and thus the matrix elements of $\hat{Q}_{33}$ are simply given by

$$\langle \Psi_d | \hat{Q}_{33} | \Psi_d \rangle = \int d^3 \vec{r}\, \Psi_d^\dagger(\vec{r}) \frac{1}{4}(3z^2 - r^2) \Psi_d(\vec{r}) \,. \tag{12}$$

The full deuteron wave function (including the D-wave) is

$$\Psi_d(\vec{r}) = \frac{u(r)}{r} \, \mathcal{Y}^{01}_{1 m_d}(\hat{r}) + \frac{w(r)}{r} \, \mathcal{Y}^{21}_{1 m_d}(\hat{r}) \,, \tag{13}$$

where $\mathcal{Y}^{ls}_{jm}(\hat{r})$ are generalized spherical harmonics that combine the spin and angular momentum wave functions of the deuteron, which are define as

$$\mathcal{Y}^{ls}_{jm}(\hat{r}) = \sum_{m_l m_s} Y_{l m_l}(\hat{r}) | s m_s \rangle \, \langle l m_l 1 m_s | j m \rangle \,, \tag{14}$$

with $Y_{l m_l}$ the standard spherical harmonics, $|s m_s\rangle$ the spin wave function and $\langle l m_l 1 m_s | j m \rangle$ a Clebsch-Gordan coefficient. Putting all the previous pieces together, deduce the formula for the deuteron quadrupole form factor that we wrote at the beginning of this exercise (**2 points**).

(4) Even-odd and odd-even nuclei can be understood as an even-even core with spin-parity $J^P = 0^+$ plus an impaired nucleon which generates the spin-parity and the magnetic moment of the nucleus. In particular for the magnetic moment we can write:

$$\mu(A) = \mu_{\rm core} + \mu_N = \mu_N \,, \tag{15}$$

because the core is 0^+ and its magnetic moment is zero. The unpaired nucleon has intrinsic spin $S = \frac{1}{2}$ and orbital angular momentum L, which can coupled to total angular momentum $J = L \pm \frac{1}{2}$. Show that the magnetic moment for the $J = L \pm \frac{1}{2}$ configurations is

$$\mu_N(J = L + \frac{1}{2}) = g_L (J - \frac{1}{2}) + \frac{1}{2} g_S \,, \tag{16}$$

$$\mu_N(J = L - \frac{1}{2}) = g_L \frac{J(J + \frac{3}{2})}{J+1} - \frac{J}{2(J+1)} g_S \,, \tag{17}$$

which are called the Schmidt values **(2 points)**.

(5) In the vibrational model, why if we have two quadrupolar phonons ($J_1, J_2 = 2$) their angular momentum does not couple to $J = 1$ and 3? **(2 points)** For an extra point, show explicitly why three quadrupolar phonons only couple to $J = 0, 2, 3, 4, 6$ but not $J = 1, 5$.

(6) In the Hartree-Fock method, the mean field potential is defined as

$$U(\vec{r}, \vec{r}') = \delta^3(\vec{r} - \vec{r}') \sum_{j=1}^{A} \int d^3\vec{r}' \phi_j^*(\vec{r}') V_{2B}(\vec{r}, \vec{r}') \phi_j(\vec{r}') - \sum_{j=1}^{A} V_{2B}(\vec{r}, \vec{r}') \phi_j^*(\vec{r}) \phi_j(\vec{r}') . \quad (18)$$

Show that for a pure contact-range potential

$$V_{2B}(\vec{r}, \vec{r}') = C \, \delta(\vec{r} - \vec{r}') , \quad (19)$$

the mean field potential vanishes (i.e. $U(\vec{r}, \vec{r}') = 0$). Why does this happens? How does the Skyrme force avoid this problem? **(2 points)**