Exercises: part 2

This is the second exercise sheet for the nuclear physics course (Spring 2022). To pass the course you are required to complete 10 points total, and the grade will depend on how well are these exercises done. A few comments are in order:

1. You can ask for hints on how to do exercises.
2. No copying (this is really important: you are already graduate students and are expected to do original research).

Each exercise has a different value in points: more difficult exercises give more points.

Regarding when to hand over the exercises: of course it must be before the school tell the teachers to upload the grades, but I do not know when that will happen. I would recommend to try to have them ready one month after the classes finish. Besides, a few of the exercise sets are more difficult than others, which means that it is up to you to decide where to concentrate your efforts.

Exercises can be handed over in either Chinese or English (if in Chinese, use a really clear handwritting: the laoshi is only used to read printed characters). They can be a picture of your handwritten exercises, or they can be in pdf or in any other common format. You can send them via wechat or email (mpavon "at" buaa.edu.cn).

(1) Let's assume a two-body system in S-wave, which follows the reduced Schrödinger equation

$$-u_0''(r) + 2\mu V(r) u_0(r) = -\gamma^2 u_0(r) \, , \tag{1}$$

with $\gamma^2 = -2\mu E$ and $E < 0$ (i.e. we have a bound state). Assume a potential of the type

$$V(r) = 0 \quad \text{for} \quad r \neq 0 \, , \tag{2}$$

which generates a bound state (despite being a contact-range potential). Show that the wave function is

$$u_0(r) = A_S \, e^{-\gamma r} \, , \tag{3}$$

and compute the value of A_S. Show also that the mean square radius of such a bound state is $\langle r^2 \rangle = 1/(2\gamma^2)$. **(1 point)**

(2) Consider the l-wave reduced Schrödinger equation

$$-u_l''(r) + \left[2\mu V(r) + \frac{l(l+1)}{r^2} \right] u_l(r) = -\gamma^2 u_l(r) \, , \tag{4}$$

with $\gamma^2 = -2\mu E$, $E < 0$ and a contact-range potential

$$V(r) = 0 \quad \text{for} \quad r \neq 0 \, . \tag{5}$$

Find the complete form of the bound state solution for P- and D-waves (i.e. for $l = 1$ and $l = 2$)

$$u_1(r) \to A_P \, e^{-\gamma r} \left(1 + \dots \right) \, , \tag{6}$$

$$u_2(r) \to A_D \, e^{-\gamma r} \left(1 + \dots \right) \, , \tag{7}$$

that is, found the complete expression that goes inside the brackets. **(2 points)**

(3) Knowing that the $l = 0$, $k = 0$ solution of the wave function behaves asymptotically as

$$u_0(r) \to 1 - \frac{r}{a_0} \, , \tag{8}$$

with a_0 the scattering length, find the scattering length for the well-known square-well potential

$$2\mu \, V(r) = U(R) \theta(R - r) \, . \tag{9}$$

(1 point)

(4) Now, with the square-well potential, let us assume for simplicity that the scattering length is larger than the range of the square-well

$$\frac{\alpha_0}{R} \gg 1\,, \tag{10}$$

which means that we can do a few Taylor expansions that will simplify the results. Now imagine that you don't know the potencial $U(R)$ or R, but you know the value of the scattering length. For instance, in the two-nucleon system we have for the singlet channel ($S = 0$) that the scattering length is $\alpha_0 = -23.7\,\mathrm{fm}$ and for the triplet channel $\alpha_0 = 5.4\,\mathrm{fm}$. From this try to find $U = U(R; \alpha_0)$, i.e. U as a function of R and the scattering length. If you find this relation, you would have solved a renormalization group equation for the two-nucleon system. **(2 points)**

(5) Explain step-by-step the solutions of the reduced Schrödinger equation (s-wave) for the inverse-square potential

$$2\mu V(r) = \frac{g}{r^2}\,, \tag{11}$$

following the instructions in the main text. **(2 points)**

(6) Explain step-by-step the solutions of the reduced Schrödinger equation (s-wave) for the delta-shell potential

$$V(r; R_c) = \frac{C_0(R_c)}{4\pi R_c^2}\,\delta(r - R_c)\,, \tag{12}$$

following the instructions in the main text. **(2 points)**

(7) Solve the reduced Schrödinger equation (s-wave) for the delta-shell potential

$$V(r; R_c) = \frac{C_0(R_c)}{4\pi R_c^2}\,\delta(r - R_c)\,, \tag{13}$$

with the condition of fixing $C_0(R_c)$ as to reproduce the scattering length. i.e.

$$C_0(R_c) \quad \text{such that} \quad k \cot\delta(k)\Big|_{k=0} = -\frac{1}{a_0}\,. \tag{14}$$

Show that the solution of $C_0(R_c)$ is

$$\frac{1}{C_0(R_c)} = \frac{\mu}{2\pi}\left(\frac{1}{a_0} - \frac{1}{R_c}\right)\,. \tag{15}$$

Then compute the effective range r_0 that we obtain with this potential (as fixed by the condition of reproducing the scattering length). Remberber that the effective range r_0 is the second term in the k^2 expansion of

$$k\,\cot\delta = -\frac{1}{a_0} + \frac{1}{2} r_0 k^2 + v_2 k^4 + \dots\,. \tag{16}$$

What happens to r_0 when we take $R_c \to 0$? **(2 points)**

(8) Consider a repulsive delta-shell of the type

$$2\mu V(r) = \lambda\delta(r - a)\,, \tag{17}$$

where $\lambda > 0$ and a is the range of the delta-shell. Show that when $\lambda \to \infty$ this potential, despite being repulsive, have bound states in the region $r < a$ where the bound state energy is

$$E_B = \frac{1}{2\mu}\left(n\frac{\pi}{a}\right)^2\,, \tag{18}$$

with $n = 1, 2, 3, \dots$. **(1 point)**

(9) A resonance is a quasi-bound state, i.e. a bound state at $E \geq 0$ that decays after some time. The way to check for the existence of a resonance is to look for solutions of the Schrödinger equation at complex energy

$$\left[-\frac{\nabla^2}{2\mu} + V(r)\right] \Psi(r) = E^* \Psi(r) \,, \tag{19}$$

where E^* is the energy of the resonance

$$E^* = E_R - i\frac{\Gamma_R}{2} \,, \tag{20}$$

where Γ_R is called the width of the resonance. The wave function of a resonant state behaves as

$$\Psi(r) \to \frac{e^{ik^* r}}{r} \, Y_{lm}(\hat{r}) \,, \tag{21}$$

with $k^* = \sqrt{2\mu E^*}$, which is the complex energy analog of the asymptotic behaviour of the bound state solution. Equivalently, if we consider the reduced wave function $u_l(r)$ instead of the full wave function, a resonance is a solution that behaves as

$$u_l(r) \to e^{ik^* r} \,, \tag{22}$$

with k^* a complex momentum.

To show how to calculate a resonant state solution, let us consider the repulsive delta-shell potential of the previous exercise

$$2\mu V(r) = \lambda \delta(r - a) \,, \tag{23}$$

where $\lambda > 0$ and a is the range of the delta-shell. For $\lambda \to \infty$ this potential has bound state solutions for $r < a$, where the binding energy is

$$E_B = \frac{1}{2\mu}\left(n\frac{\pi}{a}\right)^2 \,, \tag{24}$$

with n a non-zero positive integer. If λ is not infinite, it happens that these bound state solutions can *escape* the potential barrier provided by the repulsive delta-shell potential. Hence they adquire a width

$$E_B \to E^* = E_R - i\frac{\Gamma_R}{2} \,, \tag{25}$$

where the half-life of these bound states (or, more properly, resonances) is

$$\tau = \frac{1}{\Gamma_R} \,. \tag{26}$$

Show that for λ very big (but not infinite), we have that

$$E_R = E_B + \mathcal{O}(\frac{1}{\lambda^2}) \quad , \quad \Gamma_R = \mathcal{O}(\frac{1}{\lambda}) \,. \tag{27}$$

Compute the explicit expression of Γ_R at order $1/\lambda$. **(3 points)**

(10) The scattering wave function behaved asymptotically as

$$\Psi(\vec{r}) \to e^{i\vec{k}\cdot\vec{r}} + f(\Omega)\,\frac{e^{ikr}}{r} \,. \tag{28}$$

Alternatively, it can be written as

$$\Psi_k(\vec{r}) = \sum_l (2l+1) i^l \frac{u_l(r;k)}{r} P_l(\hat{k}\cdot\hat{r}) \,. \tag{29}$$

If we expand the plane wave and the scattering amplitude in partial waves

$$e^{i\vec{k}\cdot\vec{r}} = \sum_l (2l+1) i^l j_l(kr) P_l(\hat{k}\cdot\hat{r}) \,, \tag{30}$$

$$f(\hat{k}\cdot\hat{r}) = \sum_l (2l+1) f_l(k) P_l(\hat{k}\cdot\hat{r}) \,, \tag{31}$$

show that by taking

$$\frac{u_l(k;r)}{r} \to e^{i\delta_l} \, \left[\cos\delta_l(k)\, j_l(kr) - \sin\delta_l(k) y_l(kr)\right] \,, \tag{32}$$

for the asymptotic behaviour of the partial waves in the scattering wave function, we can manipulate the expressions to obtain

$$f_l(k) = \frac{e^{i\delta_l}\sin\delta_l}{k} = \frac{1}{k\,\cot\delta_l(k) - ik} \,, \tag{33}$$

which is pretty simple. **(2 points)**

(11) When we derived the effective range expansion, we used the following trick: building Wronskian identities from two different Schrödinger equations. In this exercise we will calculate how the scattering length changes when the two-body potential changes, i.e.

$$V \to V + \Delta\, V \quad \Rightarrow \quad a_0 \to a_0 + \Delta\, a_0 \,, \tag{34}$$

where it can be shown that the change in the scattering length follows the equation

$$\frac{\Delta a_0}{a_0^2} = 2\mu \int_0^\infty \Delta V(r)\, u_0^2(r)\, dr + \mathcal{O}((\Delta V)^2) \,, \tag{35}$$

where $u_0(r)$ is the zero energy wave function in a normalization for which $u_0(r) \to (1 - r/a_0)$ for $r \to \infty$. For this we will follow a similar logic to the one behing the derivation of the effective range expansion, that is, we will consider two Schrödinger equations

$$-u_0''(r) + 2\mu\, V_0(r)\, u_0(r) = 0 \,, \tag{36}$$

$$-u_1''(r) + 2\mu\, V_1(r)\, u_1(r) = 0 \,, \tag{37}$$

where V_0 and V_1 are two potentials and u_0, u_1 are their respective zero-energy solutions normalized as

$$u_0(r) \to 1 - \frac{r}{a_0} \quad \text{and} \quad u_1(r) \to 1 - \frac{r}{a_1} \,, \tag{38}$$

for $r \to \infty$. Thus your aim will be to construct a suitable Wronskian identity between this two solutions, and then assuming that $V_1 = V_0 + \Delta V_0$, $a_1 = a_0 + \Delta a_0$, etc., arrive to the previous relation between the change in the potential and the change in the scattering length **(3 points)**.

(12) Consider a weakly bound system for which the wave number γ $(= \sqrt{2\mu B})$ is considerably smaller than the exponential decay in the potential, that is, $V(r) \propto f(r) e^{-mr}$ and $\gamma \ll m$ (equivalently, the size of the wave function is much larger than the range of the potential). In this situation the reduced wave function of the system can be approximated by

$$u_\gamma(r) \simeq A_S e^{-\gamma r} \,. \tag{39}$$

Actually, the same will be true for the positive energy (i.e. scattering state) reduced wave function, which can be approximated by

$$u_k(r) \simeq \cos(kr) + \cot\delta(k)\, \sin(kr) \,. \tag{40}$$

Using the orthogonality of the bound and scattering state wave functions, i.e.

$$\int_0^\infty u_\gamma(r) u_k(r) dr = 0 \,, \tag{41}$$

find the value of $\cot\delta(k)$ in this case. **(1 point)**

(13) The Lippmann-Schwinger equation

$$T(E) = V + VG_0T(E)\,, \tag{42}$$

can be easily solved for a potential of the type

$$\langle \vec{p}'|V|\vec{p}\rangle = C(\Lambda)\,\theta(\Lambda - |\vec{p}'|)\,\theta(\Lambda - |\vec{p}|)\,, \tag{43}$$

as explained in these lectures. We have also explain in the lectures that the T-matrix has poles that corresponds to bound states

$$\lim_{E\to E_B} T(E) \to \frac{\mathrm{Res}T}{E - E_B}\,, \tag{44}$$

where Res$T$ refers to the residue of the T-matrix at the pole. In turn this residue is related to the wave function in momentum space by the relation

$$\begin{aligned}\mathrm{Res}T &= V|\Psi_B\rangle\langle\Psi_B|V \\ &= G_0^{-1}(E = E_B)|\Psi_B\rangle\langle\Psi_B|G_0^{-1}(E = E_B)\,.\end{aligned} \tag{45}$$

With this information, determine the value of the coupling constant

$$C(\Lambda) \quad \text{such that } T \text{ has a pole at } E_B = -B. \tag{46}$$

that is, the value of $C(\Lambda)$ such that we have a bound state at $E_B = -B = -\frac{\gamma^2}{2\mu}$. Determine the wave function too, and show that in the $\Lambda \to \infty$ limit

$$\Psi_B(p) = \frac{\mathcal{N}}{p^2 + \gamma^2}\,, \tag{47}$$

with $\mathcal{N}$ a normalization constant. Which is the value of $\mathcal{N}$? Also, which is the Fourier-transform of the previous wave function into r-space? **(2 points)**

(14) The Yamaguchi potential is a separable potential proposed a long time ago for describing the nucleon-nucleon interaction. It takes the form

$$\langle p'|V_Y|p\rangle = \frac{\lambda_Y}{2\mu}\,g(p)g(p')\,, \tag{48}$$

with λ_Y a couping constant, μ the reduced mass of the system and the function g given by

$$g(x) = \frac{1}{\beta^2 + x^2}\,. \tag{49}$$

For $\lambda_Y < 0$ this potential will have a bound state with $B = -\frac{\gamma^2}{2\mu}$. Find the relation between λ_Y, β and γ for the Yamaguchi potential. **(2 points)**

(15) Find the scattering length for the Yamaguchi potential (shown in the previous exercise). **(1 point)**

(16) We have already studied that for a separable potential of the type

$$\langle p'|V|p\rangle = C_0\,g(p')g(p)\,, \tag{50}$$

the solution for the T-matrix takes the form

$$\langle p'|T(E)|p\rangle = \tau(E)\,g(p')g(p)\,, \tag{51}$$

with $\tau(E)$ given by

$$\tau(E) = \frac{1}{\frac{1}{C_0} - \int \frac{d^3q}{(2\pi)^3} \frac{g^2(q)}{E - \frac{q^2}{2\mu}}} \,. \tag{52}$$

Now consider the following potential

$$\langle p'|V|p\rangle = \left[C_0 + C_2(p^2 + p'^2)\right] \, g(p')g(p) \,, \tag{53}$$

which is actually a sum of two separable potentials. Find an anzatz for the T-matrix that solves the Lippmann-Schwinger equation for the previous potential and show the explicit solution in terms of C_0, C_2 and g. **(4 points)**

(17) Up till now we have only considered S-wave separable potentials. For this exercise we will consider the P-wave separable potential

$$\langle \vec{p}\,'|V|\vec{p}\rangle = C_1 \, \vec{p}\,' \cdot \vec{p} \, g(p')g(p) \,. \tag{54}$$

Find a suitable ansatz for the T-matrix and its solution. **(3 points)**

(18) Within formal scattering theory, we can characterize the Born approximation as

$$T^{\rm Born} = V \,. \tag{55}$$

Using the Born approximation, calculate the scattering length for the one pion exchange potential:

$$\begin{aligned} V_{\rm OPE}(\vec{q}) &= -\frac{g^2}{4f_\pi^2} \, \vec{\tau}_1 \cdot \vec{\tau}_2 \, \frac{\vec{\sigma}_1 \cdot \vec{q} \, \vec{\sigma}_2 \cdot \vec{q}}{\vec{q}^2 + m_\pi^2} \\ &= -\frac{g^2}{4f_\pi^2} \, \vec{\tau}_1 \cdot \vec{\tau}_2 \, \frac{\frac{1}{3} \, \vec{\sigma}_1 \cdot \vec{\sigma}_2 \, q^2}{\vec{q}^2 + m_\pi^2} + (\text{tensor piece}) \,, \end{aligned} \tag{56}$$

where in the second line we have explicitly separated the spin-spin piece (which contributes to the S-wave scattering length) from the tensor piece (which does not). Using that $g = 1.26$, $f_\pi = 92.4\,\text{MeV}$ and $m_\pi = 138\,\text{MeV}$, calculate the S-wave scattering length in the singlet and triplet channels from one pion exchange. **(1 point)**

(19) Besides the T-matrix formalism (as in the previous question), a different way to reproduce the Born approximation is the Wronskian trick we used to derive the effective range expansion. For that we consider two different wave functions u_0 and u_1, such that

$$-u_0''(r) = k^2 u_0(r) \,, \tag{57}$$

$$-u_1''(r) + 2\mu V(r) u_1(r) = k^2 u_1(r) \,. \tag{58}$$

By using suitable wave functions and their Wronskian identities, deduce the Born approximation for the S-wave phase shifts. **(2 points)**

(20) It is well known that for a potential that decays exponentially at long distances

$$V(r) = f(r) \, e^{-mr} \,, \tag{59}$$

the effective range expansion only converges for $k < (m/2)$. Find the reason why this is the case. **(5 points)**