Exercises: part 1

This is the first exercise set for the nuclear physics course (Spring 2022), which is also the most difficult of the exercise sets (that is, do not panic if you find this particular set too difficult). To pass the course you are required to complete 10 points total, and the grade will depend on how well are these exercises done. A few comments are in order:

1. You can ask for hints on how to do exercises.
2. No copying (this is really important: you are already graduate students and are expected to do original research).

Each exercise has a different value in points: more difficult exercises give more points.

Regarding when to hand over the exercises: of course it must be before the school tell the teachers to upload the grades, but I do not know when that will exactly happen (but usually it is around the summer vacations). Besides, a few of the exercise sets are more difficult than others (for example, this set is probably the most complicated), which means that it is up to you to decide where to concentrate your efforts.

Exercises can be handed over in either Chinese or English (if in Chinese, use a really clear handwritting: the laoshi is only used to read printed characters). They can be a picture of your handwritten exercises, or they can be in pdf or in any other common format. You can send them via wechat or email (mpavon "at" buaa.edu.cn).

(1) As explained in the lectures, one of the reasons why we know that the nuclear force has a finite-range is saturation, i.e. the idea that the binding energy of a nucleus is proportional to the number of nucleons

$$B(\text{nucleus}) \propto A \tag{1}$$

with A the number of nucleons. Had the nuclear force be infinite-ranged, as the Coulomb force or gravitation, the binding energy would have been proportional to A^2 instead of A.

To understand this idea a bit more directly let's consider the gravitational binding energy of a spherical distribution of mass, such as a planet. This binding energy can be calculated as follows

$$V_G = -\frac{G}{2} \int d^3\vec{r}_1 d^3\vec{r}_2 \, \frac{\rho(\vec{r}_1)\,\rho(\vec{r}_2)}{|\vec{r}_1 - \vec{r}_2|} \tag{2}$$

with ρ the mass density of the object and G Newton's constant. For a spherical mass distribution of constant density ($\rho(\vec{r}) = \rho$), show that the gravitational binding energy is in fact proportional to the mass squared (M^2) of the object (**1 point**).

Now, if you want to go a step further, assume that the graviton has a finite mass ϵ, in which case Newton's law of gravitation changes to

$$V = -G\frac{m_1 m_2}{r}\, e^{-\epsilon r}\,. \tag{3}$$

This will naturally change the expression for the gravitational binding energy that we had written before. Compute the gravitational binding energy in this case and show that if the product of the graviton mass with the radius of the planet is large ($R\epsilon \gg 1$) then the gravitational binding energy will be proportional to the mass of the planet (M), instead of the square of the mass as in the previous case **(2 points)**.

(2) Explain how a nucleon-nucleon potential that is strongly repulsive at short distances while attractive at larger distances will lead to a constant nucleon density in the nucleus **(1 point)**.

(3) Show that for a potential of the type

$$V(r) = -g_H^2 \left[a\, \vec{\sigma}_1 \cdot \vec{\sigma}_2 \, W_C(r) \pm b\, S_{12}(\hat{r})\, W_T(r) \right],$$

for which $a, b > 0$, then the sign of the quadrupole moment is the same as the sign between the central and tensor piece, i.e.

$$Q = \pm |Q|\,.$$

For showing this, first take into account that $Q \neq 0$ requires that the spin of nucleons 1 and 2 must add up to 1. If we consider the total spin $\vec{S} = 2\,(\vec{\sigma}_1 + \vec{\sigma}_2)$, this corresponds to taking σ_1 paralell to σ_2. **(3 points)**

(4) Fine tuning and the deuteron: the deuteron is a bound state of a neutron and a proton, with a binding energy of $E_B = -2.2\,\mathrm{MeV}$. Consider the following idea: try to describe the deuteron with a square well with a range of 1 fm:

$$V(r) = V_0 \theta(r - a)\,, \tag{4}$$

with $a = 1\,\mathrm{fm}$. Reproducing the location of a bound state will give you a particular value of V_0 (the depth of the potential). Now consider that the binding energy E_B is the sum of two contributions:

$$E_B = \langle T \rangle + \langle V \rangle\,, \tag{5}$$

that is, the sum of the kinetic and potential energy. Find the ratio $R_{FT} = |E_B|/(|\langle T \rangle| + |\langle V \rangle|)$, which describes the level of fine tuning (**1 point**).

Now, consider a two-body system that might be similar to the deuteron: the $X(3872)$. The $X(3872)$ is a resonance which was discovered in 2003 and which is suspected to be a $D^0 \bar{D}^{0*}$ bound state with a binding energy of $E_B = -0.1\,\mathrm{MeV}$. Taking into account that the masses of the D^0 and $\bar{D}^{0*}$ mesons are 1865 and 2007 MeV, respectively, how does their degree of fine-tuning compare with that of the deuteron? (**1 point**).

(5) As already explained, the standard Dirac-delta in three dimensions:

$$\delta^{(3)}(\vec{r})\,,$$

is rather inconvenient to use. For this reason we usually *smear* the delta, i.e. we make it a bit broader by including a cutoff

$$\delta^{(3)}(\vec{r}) \to \delta^{(3)}(\vec{r}; R_c)\,.$$

One example is

$$\delta^{(3)}(\vec{r}; R_c) = \frac{e^{-(r/R_c)^2}}{\pi^{3/2} R_c^3}\,. \tag{6}$$

Show that the normalization of this smeared delta is indeed the correct one, that is,

$$\int d^3\vec{r}\, \delta^{(3)}(\vec{r}; R_c) = 1\,. \tag{7}$$

(1 point)

(6) Running of the coupling constant: consider a delta-potential of the type

$$V_C(\vec{r}) = C\, \delta^{(3)}(\vec{r})\,,$$

which we regularize as

$$V_C(\vec{r}; R_c) = \frac{C(R_c)}{\frac{4}{3}\pi R_c^3} \theta(R_c - r)\,, \tag{8}$$

that is, we regularize it as a square well. We want this potential to reproduce a bound state with binding energy

$$E_B = -B = -\frac{\gamma^2}{2\mu}\,, \tag{9}$$

with μ the reduced mass of the two-body system.. Show the explicit running of $C(R_c)$ with respect to R_c and γ, that is, how $C(R_c)$ depends on R_c and γ. Show in particular that for $\gamma R_c \ll 1$:

$$C(R_c) \propto R_c\,. \tag{10}$$

(3 points)

(7) Running of the coupling constant II: now we will derive the RGE from the condition

$$\frac{d}{dR_c} \langle \Psi | V(R_c) | \Psi \rangle = 0 \,, \tag{11}$$

where V is the (effective) potential and Ψ the wave function, as we saw in the lectures. For the two-body wave function we can simply consider the asymptotic form of a standard bound state wave function

$$\Psi(r) = \frac{A_S}{\sqrt{4\pi}} \frac{e^{-\gamma r}}{r} \,. \tag{12}$$

For V_c we use a contact-range potential, i.e. a potential that is a regularized delta

$$V(\vec{r}; R_c) = C(R_c) \, \delta^{(3)}(\vec{r}; R_c) \,. \tag{13}$$

We could use for instance a regularized square-well, as in the previous exercise:

$$V_C(\vec{r}; R_c) = \frac{C(R_c)}{\frac{4}{3}\pi R_c^3} \, \theta(R_c - r) \,, \tag{14}$$

or any other regulator of your choice. **(1 point)** Show that for the previous regulator we find the following RG equation for $C(R_c)$

$$\frac{d}{dR_c} \left[\frac{C(R_c)}{R_c^2} \right] \simeq 0 \,. \tag{15}$$

Now, if you want to get an additional **1 point**, show that you can get the same RGE with a different regulator (you can use a Gaussian regulator, a delta-shell, or whatever you like). Finally, for an additional **1 point**, try to find a simple argument of why you will always find the same RGE: a good clue is to find an argument that does not rely on the explicit evaluation of the matrix element that one gets for each regulator.

(8) We have derived the RGE from the condition

$$\frac{d}{dR_c} \langle \Psi | V(R_c) | \Psi \rangle = 0 \,, \tag{16}$$

where V is the (effective) potential and Ψ the wave function. We have derived this equation in two ways: one is as in exercise (6), where we arrive at

$$\frac{d}{dR_c} \left[\frac{C(R_c)}{R_c} \right] \simeq 0 \,, \tag{17}$$

and the other is as in exercise (7), where we arrive at

$$\frac{d}{dR_c} \left[\frac{C(R_c)}{R_c^2} \right] \simeq 0 \,. \tag{18}$$

These two RGE will lead us to two different results

$$C(R_c) \propto R_c \quad \text{(exercise 6)} \,, \tag{19}$$

$$C(R_c) \propto R_c^2 \quad \text{(exercise 7)} \,. \tag{20}$$

Why are these two results different? What is the "mistake" that has been done when obtaining $C(R_c) \propto 1/R_c^2$? **(3 points)**

(9) Isospin and the one pion exchange potential: as seen in the lectures, if there were no isospin the one pion exchange potential would take the form

$$V_0(\vec{q}) = -\frac{g_A^2}{4f_\pi^2}\frac{\vec{\sigma}_1\cdot\vec{q}\,\vec{\sigma}_2\cdot\vec{q}}{\vec{q}^{\,2}+m_\pi^2}\,, \tag{21}$$

with $g_A = 1.26$ the axial coupling of the nucleon, $f_\pi = 92.4\,\text{MeV}$ the pion weak decay constant, $\vec{\sigma}_{1(2)}$ the Pauli spin operators as applied to nucleon 1(2) and m_π the pion mass. But as a matter of fact these details are irrelevant for this exercise. As we learned in the lecture notes, the addition of the isospin quantum numbers can be taken into account by

$$V(\vec{q}) = \vec{\tau}_1\cdot\vec{\tau}_2\,V_0(\vec{q})\,, \tag{22}$$

where the justification we gave on the lectures' slides was that (i) the nucleon is an isospinor in isospin space

$$N = \begin{pmatrix} p \\ n \end{pmatrix} \quad \text{or more explicitly} \quad |p\rangle = |\frac{1}{2}+\frac{1}{2}\rangle_I\,, |n\rangle = |\frac{1}{2}-\frac{1}{2}\rangle_I \tag{23}$$

while (ii) the pion could be considered a matrix in this isopinor space, where the matrix representing the pion in the Cartesian basis would correspond to the Pauli matrices.

However this is just one possible way to introduce isospin. For this exercise we will consider a different point of view, in which the potential once we consider isospin can be written as

$$\langle\frac{1}{2}m'_{I1}\frac{1}{2}m'_{I2}|V(\vec{q})|\frac{1}{2}m_{I1}\frac{1}{2}m_{I2}\rangle = V_0(\vec{q})\sum_{m=-1}^{+1}\langle\frac{1}{2}m'_{I1}|1m\frac{1}{2}m_{I1}\rangle\,\langle\frac{1}{2}m'_{I2}1m|\frac{1}{2}m_{I2}\rangle\,, \tag{24}$$

where the sum is over the third component of the isospin wave function of the pion $|\pi\rangle = |1m\rangle$. From this try to obtain the matrix for the neutron-proton potential (a 2x2 matrix) and from this rework the isospin factor $\vec{\tau}_1\cdot\vec{\tau}_2$ (**1 point**).

Now, what will happen if instead of nucleons we would have Δ's? For your information, the Δ is a version of the nucleon with more mass, more spin ($S=\frac{3}{2}$) and more isospin ($I=\frac{3}{2}$). Thus there are four types of Δ with isospin wave functions: $|\Delta^{++}\rangle = |\frac{3}{2}+\frac{3}{2}\rangle$, $|\Delta^{+}\rangle = |\frac{3}{2}+\frac{1}{2}\rangle$, $|\Delta^{0}\rangle = |\frac{3}{2}-\frac{1}{2}\rangle$, $|\Delta^{-}\rangle = |\frac{3}{2}-\frac{3}{2}\rangle$. Obtain what will be the isospin factor of a $\Delta\Delta$ potential in term of the isospin-3/2 matrices. (**2 points**).

(10) In the linear σ model, the potential in the original Lagrangian is

$$V(\phi) = \frac{1}{2}\mu^2\sum_i\phi_i^2 + \frac{\lambda}{4}\left(\sum_i\phi_i^2\right)^2\,. \tag{25}$$

Show that after the change of variables

$$\sigma = \phi_0 - v \quad \text{and} \quad \vec{\pi} = \vec{\phi}\,, \tag{26}$$

with $v=\sqrt{-\mu^2/\lambda}$ and after rearranging the mass term of the σ, we end up with the potential

$$V(\sigma,\vec{\pi}) = \ldots \tag{27}$$

(please write down the calculations and the potential you obtain) **(1 point)**

(11) In the linear σ model, we can add a small term in the potential that breaks the original $O(4)$ symmetry

$$\Delta V(\phi) = -\epsilon\,v^3\,\phi_0\,, \tag{28}$$

with ϵ a small parameter. Show that after the change of variables to the σ and $\vec{\pi}$ fields, the mass of the σ changes slightly, while the pion acquires a finite mass

$$m_\sigma^2 = \lambda v^2 + a_0\,\epsilon \quad \text{and} \quad m_\pi^2 = b_0\,\epsilon\,, \tag{29}$$

and calculate the coefficients a_0 and b_0. **(2 points)**

(12) There is a second version of the σ model that is called the non-linear σ model. What is effectively done is this model is to take $\mu^2 \to -\infty$ but letting $v = \sqrt{-\mu^2/\lambda}$ fixed. This is equivalent to the condition

$$\phi_0^2 + \vec{\phi}^2 = v^2 \,. \tag{30}$$

After making the identification $\vec{\phi} = \pi$, derive the interaction lagrangian between (i) the nucleon and the pions and (ii) the pions alone, and check the differences with respect to the standar linear σ model. **(4 points)**

(13) A comparison between the electromagnetic, the nuclear and the strong force: when we consider a bound state in quantum mechanics, for instance the hydrogen atom, we see that the total mass of the bound state is a bit smaller than their components. For example, in the hydrogen atom we will have

$$m(H) = (m_e + m_p) - B_2^H < (m_e + m_p) \,, \tag{31}$$

where B_2 is the binding energy. The same happens with the deuteron

$$m(d) = (m_p + m_n) - B_2^d < (m_n + m_p) \,, \tag{32}$$

and to all other nuclei

$$m(A, Z) = (Z m_p + (A - Z) m_n) - B < (Z m_p + (A - Z) m_n) \,. \tag{33}$$

However, when we consider hadrons it happens exactly the contrary as here. For the pion we have

$$m_\pi > (m_u + m_d) \,, \tag{34}$$

where $m_\pi \simeq 140\,\mathrm{MeV}$ and $(m_u + m_d) \sim 8\,\mathrm{MeV}$, while for the proton we have instead

$$m_p > (2m_u + m_d) \,, \tag{35}$$

with $m_p \simeq 940\,\mathrm{MeV}$ and $(2m_u + m_d) \sim 11\,\mathrm{MeV}$. Explain why this is happening (**2 points**).