The Three Nucleon System

I. FADDEEV EQUATIONS

A. The Lippmann-Schwinger Equation and the Three Body System

We will consider a three-body system interacting via two-body forces

$$H = H_0 + V = \sum_{i=1}^{3} \frac{p_i^2}{2m_i} + V \,, \tag{1}$$

where the potential is

$$V = V_{12} + V_{23} + V_{31} \,. \tag{2}$$

We begin by considering the Lippmann-Schwinger equation for this system

$$T(Z) = V + V\, G_0(Z)\, T(Z) \,. \tag{3}$$

Alternatively we can consider the bound state equation

$$|\Psi_{3B}\rangle = G_0(Z)\, V\, |\Psi_{3B}\rangle \,. \tag{4}$$

With the potential above it is apparent that we can easily run into a very particular type of problem when solving the Lippmann-Schwinger equation. If we consider the matrix elements of the potential components V_{ij} we find the following

$$\langle \vec{p}_1{}'\, \vec{p}_2{}'\, \vec{p}_3{}' | V_{12} | \vec{p}_1\, \vec{p}_2\, \vec{p}_3 \rangle = (2\pi)^3\, \delta^{(3)}(\vec{P}_{12}{}' - \vec{P}_{12})\, \langle \vec{p}_{12}{}' | V_{12} | \vec{p}_{12} \rangle,\ (2\pi)^3\, \delta^{(3)}(\vec{p}_3{}' - \vec{p}_3) \,, \tag{5}$$

where $\vec{P}_{12}$ and $\vec{p}_{12}$ are the total and relative momentum of particles 1 and 2

$$\vec{P}_{12} = \vec{p}_1 + \vec{p}_2 \,, \tag{6}$$

$$\vec{p}_{12} = \frac{m_2\, \vec{p}_1 - m_1\, \vec{p}_2}{m_1 + m_2} \,, \tag{7}$$

plus similar expressions for V_{23} and V_{31}. The problem of the previous expressions is the existence of two deltas for the conservatation of the total momentum of particle pair ij and the expectator particle k, with $ij = 12$ and $k = 3$ in the example above. In the two-body system this is not a problem because we can globally the delta expressing the conservation of total momentum

$$\langle \vec{p}_1{}'\, \vec{p}_2{}' | T | \vec{p}_1\, \vec{p}_2 \rangle = (2\pi)^3\, \delta^{(3)}(\vec{P}_{12}{}' - \vec{P}_{12})\, \langle \vec{p}_{12}{}' | T | \vec{p}_{12} \rangle \,, \tag{8}$$

$$\langle \vec{p}_1{}'\, \vec{p}_2{}' | V | \vec{p}_1\, \vec{p}_2 \rangle = (2\pi)^3\, \delta^{(3)}(\vec{P}_{12}{}' - \vec{P}_{12})\, \langle \vec{p}_{12}{}' | V | \vec{p}_{12} \rangle \,, \tag{9}$$

$$\langle \vec{p}_1{}'\, \vec{p}_2{}' | G_0 | \vec{p}_1\, \vec{p}_2 \rangle = (2\pi)^3\, \delta^{(3)}(\vec{P}_{12}{}' - \vec{P}_{12})\, \frac{(2\pi)^3\, \delta^{(3)}(\vec{p}_{12}{}' - \vec{p}_{12})}{E - \frac{p_1^2}{2m_1} - \frac{p_2^2}{2m_2}} \,, \tag{10}$$

which means that at the end we are left with the equation

$$\langle \vec{p}\,' | T | \vec{p} \rangle = \langle \vec{p}\,' | V | \vec{p} \rangle + \int \frac{d^3\vec{q}}{(2\pi)^3}\, \langle \vec{p}\,' | V | \vec{p} \rangle\, \frac{1}{E_{cm} - \frac{q^2}{2\mu}}\, \langle \vec{p}\,' | T | \vec{p} \rangle \,, \tag{11}$$

where

$$E_{cm} = E - \frac{P^2}{2M} \quad , \quad M = m_1 + m_2 \quad \text{and} \quad \mu = \frac{m_1 m_2}{m_1 + m_2} \,. \tag{12}$$

But in the case of the three-body system this type of reduction is not possible. There is a global delta expressing the conservation of the total momentum

$$(2\pi)^3\, \delta^{(3)}(\vec{P}\,' - \vec{P}) \,, \tag{13}$$

with $\vec{P} = \vec{p}_1 + \vec{p}_2 + \vec{p}_3$. This delta is indeed easy to remove. But besides this, there will be terms including the conservation of momentum of particle 1, 2 and 3 that will make the resolution of the Lippmann-Schwinger equation quite troublesome. Actually these delta's expresses the conservation of when particle 1 scatters with cluster 23 (plus permutations), but yet the resulting equations are a mess not easy dealt with.

B. The Faddeev Equations

Here is where the idea of Faddeev enters. Faddeev proposed the following decomposition of the T-matrix

$$T(Z) = T^{(1)}(Z) + T^{(2)}(Z) + T^{(3)}(Z)\,, \tag{14}$$

where each of the $T^{(k)}$ follows the equation

$$T^{(k)}(Z) = V_{ij} + V_{ij}\,G_0(Z)\,T(Z)\,, \tag{15}$$

with $ijk = 123$ or the even permutations 231 and 312. These three equations still contain the problematic deltas, but we can get rid of them. For that we know consider the two-body T-matrices

$$T_{ij}(Z) = V_{ij} + V_{ij}\,G_0(Z)\,T_{ij}(Z)\,, \tag{16}$$

and the equivalent equation

$$(1 - V_{ij}\,G_0(Z))^{-1}\,T_{ij}(Z) = V_{ij}\,. \tag{17}$$

Now if we rearrange the equation for $T^{(k)}$ as follows

$$[1 - V_{ij}\,G_0(Z)]\;T^{(k)} = V_{ij} + V_{ij}\,G_0(Z)\,\left[T^{(i)}(Z) + T^{(j)}(Z)\right]\,, \tag{18}$$

and invert the $[1 - V_{ij}\,G_0(Z)]$ piece, we end up with *the Faddeev Equations*

$$T^{(k)}(Z) = T_{ij}(Z) + T_{ij}(Z)\,G_0(Z)\,\left[T^{(i)}(Z) + T^{(j)}(Z)\right]\,, \tag{19}$$

which can be solved. As a matter of fact there is still a delta for the conservation of momentum of particle k in T_{ij}, but not in the iterative term. This means that the equation above is actually solvable as an integral equation, just in the same way as the two-body Lippamann-Schwinger equation is.

C. Bound State Equations

We are mostly interested in bound states, which are the most simple thing to calculate with the Faddeev equations. For that we begin with the well-known relation

$$\lim_{Z\to E_{3B}} T(Z) \to G_0^{-1}(Z)\,\frac{|\Psi_{3B}\rangle\langle\Psi_{3B}|}{Z - E_{3B}}\,G_0^{-1}(Z)\,. \tag{20}$$

Now if we consider that

$$T(Z) = T^{(1)}(Z) + T^{(2)}(Z) + T^{(3)}(Z)\,, \tag{21}$$

a possible ansatz for the $T^{(k)}$ pole is the following

$$\lim_{Z\to E_{3B}} T^{(k)}(Z) \to G_0^{-1}(Z)\,\frac{|\psi^{(k)}\rangle\langle\Psi_{3B}|}{Z - E_{3B}}\,G_0^{-1}(Z)\,, \tag{22}$$

where we have

$$|\Psi_{3B}\rangle = |\psi^{(1)}\rangle + |\psi^{(2)}\rangle + |\psi^{(3)}\rangle\,, \tag{23}$$

which is called *the Faddeev decomposition* of the wave function. With this we end up with the equations

$$|\psi^{(1)}\rangle = G_0(Z)\,T_{23}(Z)\,\left[|\psi^{(2)}\rangle + |\psi^{(3)}\rangle\right]\,, \tag{24}$$

$$|\psi^{(2)}\rangle = G_0(Z)\,T_{31}(Z)\,\left[|\psi^{(3)}\rangle + |\psi^{(1)}\rangle\right]\,, \tag{25}$$

$$|\psi^{(3)}\rangle = G_0(Z)\,T_{12}(Z)\,\left[|\psi^{(1)}\rangle + |\psi^{(2)}\rangle\right]\,, \tag{26}$$

which are the Faddeev equations for the three-body bound state. These equations can also be written in matrix form as follows

$$\begin{pmatrix} |\psi^{(1)}\rangle \\ |\psi^{(2)}\rangle \\ |\psi^{(3)}\rangle \end{pmatrix} = G_0(Z)\,\begin{pmatrix} 0 & T_{23}(Z) & T_{23}(Z) \\ T_{31}(Z) & 0 & T_{31}(Z) \\ T_{12}(Z) & T_{12}(Z) & 0 \end{pmatrix}\begin{pmatrix} |\psi^{(1)}\rangle \\ |\psi^{(2)}\rangle \\ |\psi^{(3)}\rangle \end{pmatrix}\,. \tag{27}$$

D. Jacobi Coordinates

Now it would be really good to have a closed form of the previous equations that can be solved via discretization methods. For that the first step is to choose a suitable set of coordinates for the three body system. We would like to have in fact the three-body equaivalent of

$$\vec{P}_{12} = \vec{p}_1 + \vec{p}_2 \,, \tag{28}$$

$$\vec{p}_{12} = \frac{m_2\,\vec{p}_1 - m_1\,\vec{p}_2}{m_1 + m_2} \,, \tag{29}$$

which allows to write the free hamiltonian as

$$H_0 = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} \,, \tag{30}$$

$$= \frac{P_{12}^2}{2M_{12}} + \frac{p_{12}^2}{2\mu_{12}} \,, \tag{31}$$

with $M_{12} = m_1 + m_2$ and $\mu_{12} = m_1 m_2/(m_1 + m_2)$. The equivalent of this for the three body system are the Jacobi momenta, which read as follows

$$\vec{p}_1 = \frac{1}{M}\left\{(m_2 + m_3)\,\vec{k}_1 - m_1\,(\vec{k}_2 + \vec{k}_3)\right\} \tag{32}$$

$$\vec{k}_{23} = \frac{m_3\vec{k}_2 - m_2\vec{k}_3}{m_2 + m_3} \tag{33}$$

plus permutations, with $M = m_1 + m_2 + m_3$. The different permutations of the Jacobi momenta can be related with the following equations, which can actually be quite useful

$$\vec{p}_2 = -\frac{m_2}{m_2 + m_3}\,\vec{p}_1 + \vec{k}_{23} \,, \tag{34}$$

$$\vec{k}_{31} = -\frac{m_3 M}{(m_1 + m_3)(m_2 + m_3)}\,\vec{p}_1 - \frac{m_1}{m_1 + m_3}\,\vec{k}_{23} \tag{35}$$

$$\vec{p}_3 = -\frac{m_3}{m_2 + m_3}\,\vec{p}_1 - \vec{k}_{23} \,, \tag{36}$$

$$\vec{k}_{12} = \frac{m_2 M}{(m_1 + m_2)(m_2 + m_3)}\,\vec{p}_1 - \frac{m_1}{m_1 + m_2}\,\vec{k}_{23} \,, \tag{37}$$

plus permutations. The Jacobi momenta also have the following two cyclic properties

(i) $\vec{p}_1 + \vec{p}_2 + \vec{p}_3 = 0$,

(ii) $m_1(m_2 + m_3)\vec{k}_{23} + m_2(m_1 + m_3)\vec{k}_{31} + m_3(m_1 + m_2)\vec{k}_{12} = 0$.

With the Jacobi momenta we can write the free hamiltonian as

$$H_0 = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + \frac{p_3^2}{2m_3}$$

$$= \frac{P^2}{2M} + \frac{p_1^2}{2\mu_1} + \frac{k_{23}^2}{2\mu_{23}} \,, \tag{38}$$

where the reduced masses are

$$\frac{1}{\mu_1} = \frac{1}{m_1} + \frac{1}{m_2 + m_3} \quad , \quad \frac{1}{\mu_{23}} = \frac{1}{m_2} + \frac{1}{m_3} \,. \tag{39}$$

Equivalently we can also consider the Jacobi coordinates

$$\rho_1 = r_1 - \frac{m_2 r_2 + m_3 r_3}{m_2 + m_3} \,, \tag{40}$$

$$r_{23} = r_2 - r_3 \,, \tag{41}$$

plus the relations

$$\rho_2 = -\frac{m_1}{m_1+m_3}\,\rho_1 + \frac{m_3\,M}{(m_1+m_3)(m_2+m_3)}\,r_{23}\,, \tag{42}$$

$$r_{31} = -\rho_1 - \frac{m_2}{m_2+m_3}\,r_{23}\,, \tag{43}$$

$$\rho_3 = -\frac{m_1}{m_1+m_2}\,\rho_1 - \frac{m_2\,M}{(m_1+m_2)(m_2+m_3)}\,r_{23}\,, \tag{44}$$

$$r_{12} = +\rho_1 - \frac{m_3}{m_2+m_3}\,r_{23}\,, \tag{45}$$

$$\tag{46}$$

plus permutations. The Jacobi coordinates have of course the property that the free hamiltonian can be written as

$$\begin{aligned} H_0 &= -\frac{\nabla_1^2}{2m_1} - \frac{\nabla_2^2}{2m_2} - \frac{\nabla_3^2}{2m_3} \\ &= -\frac{\nabla_R^2}{2M} - \frac{\nabla_{\rho_1}^2}{2\mu_1} - \frac{\nabla_{r_{23}}^2}{2\mu_{23}}\,. \end{aligned} \tag{47}$$

E. Integral Form of the Faddeev Equations

Now we can use the Jacobi coordinates to write the wave function as a sum of Faddeev components

$$\Psi = \psi^{(1)}(\vec{k}_{23},\vec{p}_1) + \psi^{(2)}(\vec{k}_{31},\vec{p}_2) + \psi^{(3)}(\vec{k}_{12},\vec{p}_3)\,, \tag{48}$$

from which we can write the Faddeev bound state equations as

$$\begin{aligned} \Psi^{(1)}(\vec{k}_{23},\vec{p}_1) = &\left(Z - \frac{k_{23}^2}{2\mu_{23}} - \frac{p_1^2}{2\mu_1}\right)^{-1} \int \frac{d^3\vec{k_{23}}'}{(2\pi)^3}\,\langle\vec{k}_{23}|t_{23}(Z - \frac{k_{23}^2}{2\mu_{23}})|\vec{k}_{23}{}'\rangle \\ &\times \left[\Psi^{(2)}(\vec{k}_{31}{}',\vec{p}_2{}') + \Psi^{(3)}(\vec{k}_{12}{}',\vec{p}_3{}')\right]\,, \end{aligned} \tag{49}$$

plus permutations. Notice that in this equation the meaning of $\vec{k}_{31}{}'$, $\vec{p}_2{}'$ and $\vec{k}_{12}{}'$, $\vec{p}_3{}'$ is

$$\vec{p}_2{}' = -\frac{m_2}{m_2+m_3}\,\vec{p}_1 + \vec{k}_{23}{}'\,, \tag{50}$$

$$\vec{k}_{31}{}' = -\frac{m_3 M}{(m_1+m_3)(m_2+m_3)}\,\vec{p}_1 - \frac{m_1}{m_1+m_3}\,\vec{k}_{23}{}' \tag{51}$$

$$\vec{p}_3{}' = -\frac{m_3}{m_2+m_3}\,\vec{p}_1 - \vec{k}_{23}{}'\,, \tag{52}$$

$$\vec{k}_{12}{}' = \frac{m_2 M}{(m_1+m_2)(m_2+m_3)}\,\vec{p}_1 - \frac{m_1}{m_1+m_2}\,\vec{k}_{23}{}'\,, \tag{53}$$

that is, the usual relation among Jacobi momenta but taking into account that we have to change k_{23} by $k_{23}{}'$.

II. SEPARABLE INTERACTIONS

The Faddeev equations are relatively easy to solve for separable interactions. If the two-body potentials are of the type

$$\langle\vec{k}\,'|V_{ij}|\vec{k}\rangle = \lambda_{ij}\,g_k(k)g_k(k')\,, \tag{54}$$

then the T-matrix is

$$\langle\vec{k}\,'|t_{ij}(Z)|\vec{k}\rangle = \tau_{ij}(Z)\,g_k(k)g_k(k')\,. \tag{55}$$

In this case there is a simple ansatz for the Faddeev components

$$\psi^{(i)}(\vec{k},\vec{p}) = \mathcal{N}\,\frac{g_i(k)\,a_i(p)}{-Z + \frac{k^2}{2\mu_{ij}} + \frac{p^2}{2\mu_k}}\,, \tag{56}$$

with $\mathcal{N}$ a normalization constant and where the three $a_i(p)$ follow the equations

$$\begin{aligned} a_1(p) = -\tau_{23}(Z) \int \frac{d^3\vec{p}'}{(2\pi)^3} \Big[& \frac{g_1\left(\vec{p}' + \frac{m_2}{m_2+m_3}\vec{p}\right)\, g_2\left(-\vec{p} - \frac{m_1}{m_1+m_3}\vec{p}'\right)}{-Z + \frac{1}{2\mu_{31}}\left(\vec{p} + \frac{m_1}{m_1+m_3}\vec{p}'\right)^2 + \frac{p'^2}{2\mu_2}}\, a_2(p') \\ & + \frac{g_1\left(\vec{p}' - \frac{m_3}{m_2+m_3}\vec{p}\right)\, g_3\left(\vec{p} - \frac{m_1}{m_1+m_2}\vec{p}'\right)}{-Z + \frac{1}{2\mu_{12}}\left(\vec{p} - \frac{m_1}{m_1+m_2}\vec{p}'\right)^2 + \frac{p'^2}{2\mu_3}}\, a_3(p') \Big]\,, \end{aligned} \tag{57}$$

plus permutations. The previous equation can be further simplified by a convenient translation in the integral momentum $\vec{p}'$ plus a convenient renaming of the variables, leading to

$$\begin{aligned} a_1(p_1) &= \tau_{23}(Z) \int \frac{d^3\vec{p}_2}{(2\pi)^3}\, B_{12}(\vec{p}_1,\vec{p}_2)\, a_2(p_2) \\ &+ \tau_{23}(Z) \int \frac{d^3\vec{p}_3}{(2\pi)^3}\, B_{13}(\vec{p}_1,\vec{p}_3)\, a_3(p_3)\,, \end{aligned} \tag{58}$$

or in an even more general form

$$\begin{aligned} a_k(p_k) &= \tau_{ij}(Z_{ij}) \int \frac{d^3\vec{p}_i}{(2\pi)^3}\, B_{ki}(\vec{p}_k,\vec{p}_i)\, a_i(p_i) \\ &+ \tau_{ij}(Z_{ij}) \int \frac{d^3\vec{p}_3}{(2\pi)^3}\, B_{kj}(\vec{p}_k,\vec{p}_j)\, a_j(p_j)\,, \end{aligned} \tag{59}$$

with

$$Z_{ij} = Z - \frac{p_k^2}{2\mu_k}\,, \tag{60}$$

and where the driving terms B_{ij} are given by

$$B_{ij}(\vec{p}_i,\vec{p}_j) = \frac{g_i(\vec{q}_i) g_j(\vec{q}_j)}{Z - \frac{p_1^2}{2m_1} - \frac{p_2^2}{2m_2} - \frac{p_3^2}{2m_3}}\,, \tag{61}$$

with $\vec{p}_1 + \vec{p}_2 + \vec{p}_3 = 0$ and

$$\vec{q}_i = \frac{m_k\vec{q}_j - m_k\vec{q}_k}{m_j + m_k}\,, \tag{62}$$

where $ijk = 123,\ 231,\ 312$.

The point here is that the previous equations can be solved in exactly the same way as the Lippmann-Schwinger equation for bound states. That is, we first distinguish between the momentum and angular integral

$$\int \frac{d^3\vec{p}_k}{(2\pi)^3} = \int_0^\infty \frac{p_k^2 dp_k}{2\pi^2} \int \frac{d^2\hat{p}_k}{4\pi} \tag{63}$$

perform the angular integrals (s-wave in the example below)

$$b^0_{ik}(p_i,p_k) = \int \frac{d^2\hat{p}_k}{4\pi}\, B_{ik}(\vec{p}_i,\vec{p}_k)\,, \tag{64}$$

and discretize the momentum integral by the use of a suitable set of Gauss points

$$\int_0^\infty \frac{p_k^2 dp_k}{2\pi^2} \simeq \sum_{n=1}^{N} \frac{p_n^2}{2\pi^2}\, w_n\,. \tag{65}$$

After following these steps we will end up with a 3N x 3N eigenvalue equation of the type

$$a_{in} = F_{in,jm}(Z) a_{jm} \,, \tag{66}$$

where the bound state solutions are determined by the condition

$$\det \left(\delta_{in,jm} - F_{in,jm}(Z) \right) = 0 \,, \tag{67}$$

and the wave functions a_{in} are given by the eigenvector corresponding to the zero eigenvalue of the matrix from which we compute the determinant.

III. THE THREE BOSON SYSTEM AND THE EFIMOV EFFECT

Now I consider the case of three identical bosons, for which the wave function is symmetric under permutations of the particles. This in turn implies that

$$\psi^{(1)}(\vec{k},\vec{p}) = \psi^{(2)}(\vec{k},\vec{p}) = \psi^{(3)}(\vec{k},\vec{p}) = \psi(\vec{k},\vec{p}) \,, \tag{68}$$

plus $\psi(\vec{k},\vec{p}) = \psi(-\vec{k},\vec{p})$. The wave function can be written as

$$\Psi = \psi(\vec{k}_{23},\vec{p}_1) + \psi(\vec{k}_{31},\vec{p}_2) + \psi(\vec{k}_{12},\vec{p}_3) \,, \tag{69}$$

where the Faddeev equation for ψ is

$$\begin{aligned} \psi(\vec{k},\vec{p}) = \left(Z - \frac{3}{4}\frac{p^2}{m^2} - \frac{k^2}{m^2} \right)^{-1} \int \frac{d^3p'}{(2\pi)^3} \Big[\langle \vec{k} | t(Z - \frac{3}{4}\frac{p^2}{m^2}) | \frac{\vec{p}}{2} + \vec{p}\,' \rangle \\ + \langle \vec{k} | t(Z - \frac{3}{4}\frac{p^2}{m^2}) | -\frac{\vec{p}}{2} - \vec{p}\,' \rangle \Big] \, \psi(\vec{p} + \frac{\vec{p}\,'}{2}, \vec{p}\,') \,. \end{aligned} \tag{70}$$

For separable interactions, we can again write t and ψ as

$$\langle \vec{k}\,' | t(Z) | \vec{k} \rangle = \tau(Z)\, g(k')\, g(k) \,, \tag{71}$$

$$\psi(\vec{k},\vec{p}) = \frac{a(\vec{p}) g(k)}{k^2 + \frac{3}{4}p^2 + \gamma^2} \,, \tag{72}$$

from which it follows that the equation for $a(\vec{p})$ is

$$a(\vec{p}) = -2m\, \tau(Z - \frac{3}{4}\frac{p^2}{m}) \int \frac{d^3\vec{p}\,'}{(2\pi)^3} \frac{g\left(\left| \vec{p} + \frac{\vec{p}\,'}{2} \right| \right) \, g\left(\left| \frac{\vec{p}}{2} + \vec{p}\,' \right| \right)}{\gamma^2 + p^2 + {p'}^2 + \vec{p} \cdot \vec{p}\,'} \, a(\vec{p}\,') \,. \tag{73}$$

This can be further simplified for contact-range interactions. If we ignore the finite cut-off we have $g(k) = 1$. For $\tau(Z)$, we have

$$\tau(Z) = \frac{4\pi}{m} \frac{1}{\frac{1}{a_2} + i\sqrt{mZ}} \quad \text{for } Z > 0 \,, \tag{74}$$

$$\tau(Z) = \frac{4\pi}{m} \frac{1}{\frac{1}{a_2} - \sqrt{-mZ}} \quad \text{for } Z < 0 \,, \tag{75}$$

where a_2 is the two-body scattering length. Including this into the previous equation, it now further simplifies to

$$p\, a(p) = -\frac{2}{\pi} \frac{1}{\frac{1}{a_2} - \sqrt{\frac{3}{4}p^2 - mZ}} \int_0^\infty dp' \, \log\left(\frac{\gamma^2 + p^2 + p'^2 + pp'}{\gamma^2 + p^2 + p'^2 - pp'} \right) p' a(p') \,, \tag{76}$$

where we have assumed that $a(\vec{p}) = a(p)$ (s-wave). Now we can analyze the large p limit of this equation, i.e. $p \gg \frac{1}{a_2}$ and $p \gg \sqrt{-mZ}$, which is

$$p^2\, a(p) = \frac{4}{\pi\sqrt{3}} \int_0^\infty \frac{dp'}{p'} \, \log\left(\frac{\gamma^2 + p^2 + p'^2 + pp'}{\gamma^2 + p^2 + p'^2 - pp'} \right) {p'}^2 a(p') \,. \tag{77}$$

If we define $b(p) = p^2\, a(p)$, we have the following

$$b(p) = \frac{4}{\pi\,\sqrt{3}} \int_0^\infty dp'\, \frac{b(p')}{p'}\, \log\left(\frac{p^2 + p'^2 + pp'}{p^2 + p'^2 - pp'}\right) . \tag{78}$$

The equation for $b(p)$ is scale invariant and admits a solution of the type $b(p) = p^s$, which leads to

$$1 = \frac{4}{\pi\sqrt{3}} \int_0^\infty dx\, x^{s-1}\, \log\left(\frac{x^2 + x + 1}{x^2 - x + 1}\right) = \frac{4}{\pi\sqrt{3}}\, I_{\rm Efimov}(s) . \tag{79}$$

Taking into account that

$$I_{\rm Efimov}(s) = \frac{2\pi}{s}\, \frac{\sin{(\pi s/6)}}{\cos{(\pi s/2)}} , \tag{80}$$

we end up with

$$1 - \frac{8}{\sqrt{3}\, s}\, \frac{\sin\frac{\pi s}{6}}{\cos\frac{\pi s}{2}} = 0 , \tag{81}$$

for which there is the solution $s = \pm i s_0$ with $s_0 = 1.00624$. Putting all the pieces together, we have a wave function solution determined by the Faddeev component ψ:

$$\psi(k,p) = \mathcal{N}\, \frac{b(p)}{p^2}\, \frac{1}{\gamma_3^2 + k^2 + \frac{3}{4} p^2} , \tag{82}$$

with $\mathcal{N}$ a normalization constant and γ_3 the three-body bound state binding momentum. We have that $b(p)$ behaves as

$$b(p) \to \sin\left[s_0 \log\left(\frac{p}{\Lambda^*}\right)\right] , \tag{83}$$

for $p \to \infty$ (or more accurately, for $p\, a_2 \gg 1$). In the expression above Λ^* is a scale we have to introduce for fixing the solution. It is also easy to check that for $p\, a_2 \gg 1$ (or for $a_2 \to 0$) the wave function displays discrete scale invariance

$$\psi(\lambda_0 k, \lambda_0 p, \lambda_0 \gamma_3) = \frac{1}{\lambda_0^4}\, \psi(k, p, \gamma_3) , \tag{84}$$

with $\lambda_0 = e^{\pi/s_0} \simeq 22.69$. This property in turn implies the existence of an infinite number of bound states. Why? If we do the change

$$\gamma_3 \to \gamma_3' = \lambda_0\, \gamma_3 , \tag{85}$$

then we can construct a wave function that is a solution for γ_3'. This means, that if we have a bound state at E_0 then we have a second bound state at $\lambda_0^2 E_0$ and also at E_0/λ_0^2. As a consequence we have in principle an infinite number of bound states. This is called the Efimov effect.

IV. THE THREE NUCLEON SYSTEM

Now we can write the Faddeev equations for the three nucleon system with separable interactions, e.g. contact interactions. The first step is to remember that a nucleon is characterized by its spin and isospin quantum numbers, i.e.

$$|N\rangle = |\frac{1}{2} m_S\rangle_S\, |\frac{1}{2} m_I\rangle_I , \tag{86}$$

where we have factored the nucleon into a spin and isospin wave function. When we have a two nucleon system in the S-wave, owing to the antisymmetry of the wave function, we can write

$$|NN(S)\rangle = |00\rangle_S\, |1 m_I\rangle_I , \tag{87}$$

$$|NN(T)\rangle = |1 m_S\rangle_S\, |00\rangle_I , \tag{88}$$

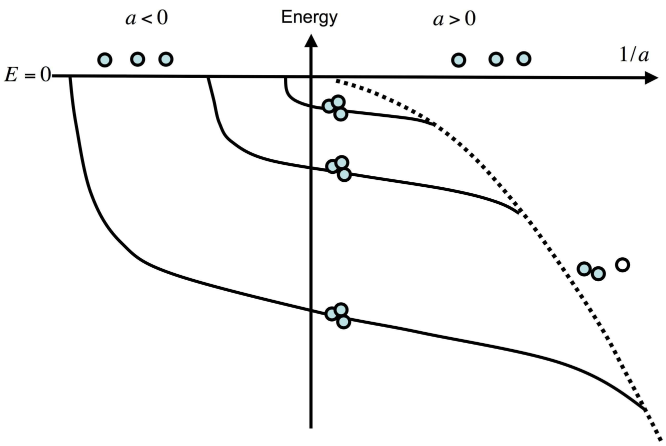

FIG. 1. A graphical representation of the Efimov effect: if the scattering length of a boson pair in a three boson system goes to infinity, $a \to \infty$, then a tower of three body bound state appears. The binding energy of the n-th and $(n+1)$-th excited state are related by $E_n \simeq 521 E_{n+1}$.

where the first line represents the singlet $^1S_0$ channel and the second the triplet $^3S_1$ channel. A separable potential in the singlet and triplet channels can be written as

$$V_S = \lambda_S\, g(k')g(k)\,, \tag{89}$$

$$V_T = \lambda_T\, g(k')g(k)\,, \tag{90}$$

where the triplet interaction must be tuned as to reproduce the deuteron bound state. For the three body case, it happens that the triton quantum numbers are $S = \frac{1}{2}$ and $I = \frac{1}{2}$. Besides, for the triton it is more probable to bind if the spatial wave function is symmetric, which implies that we have to construct a spin-isospin wave function that is antisymmetric. Yet this is not so trivial as it might appear at first look. Thus we will take a constructive approach and we will begin with a wave function that is symmetric with respect to particles 1 and 2. The first step is to construct the spin and isospin wave functions of the three nucleon system. For that we first couple spin of particles 1 and 2 to obtain S_{12}, which is then coupled to spin of particle 3 to form the total spin $S_T = \frac{1}{2}$:

$$|S_T = \frac{1}{2}\rangle = |S_{12} \otimes \frac{1}{2}\rangle\,. \tag{91}$$

We also do exactly the same with the isospin wave function

$$|I_T = \frac{1}{2}\rangle = |I_{12} \otimes \frac{1}{2}\rangle\,. \tag{92}$$

Now if we want to have a symmetric spatial wave function for particles 1 and 2 we are left with the two spin-isospin combinations that are antisymmetric:

$$|1_{12} \otimes \frac{1}{2}\rangle_S\, |0_{12} \otimes \frac{1}{2}\rangle_I \quad \text{and} \quad |0_{12} \otimes \frac{1}{2}\rangle_S\, |1_{12} \otimes \frac{1}{2}\rangle_I\,. \tag{93}$$

From this we can write the wave function as

$$\begin{aligned} \Psi_{3N} &= \left[\psi_S(k_{23},p_1) + \psi_S(k_{31},p_2) + \phi_S(k_{12},p_3)\right]\, |0_{12} \otimes \frac{1}{2}\rangle_S\, |1_{12} \otimes \frac{1}{2}\rangle_I \\ &+ \left[\psi_T(k_{23},p_1) + \psi_T(k_{31},p_2) + \phi_T(k_{12},p_3)\right]\, |1_{12} \otimes \frac{1}{2}\rangle_S\, |0_{12} \otimes \frac{1}{2}\rangle_I \end{aligned} \tag{94}$$

which is antisymmetric with respect to Π_{12}

$$\Pi_{12}|\Psi_{3N}\rangle = -|\Psi_{3N}\rangle\,. \tag{95}$$

Now to make the wave function fully antisymmetric, we can proceed in the following way: we make the wave function to be symmetric under the permutation Π_{123}

$$\Pi_{123}|\Psi_{3N}\rangle = |\Psi_{3N}\rangle \,. \tag{96}$$

Why? The permutation Π_{123} consist on making the changes $1 \to 2$, $2 \to 3$ and $3 \to 1$. This is in turn equivalent to do two permutations

$$\Pi_{123} = \Pi_{23}\Pi_{12} \,, \tag{97}$$

which can be checked to be the same. Now the thing is that with Π_{12} and Π_{123} we can construct any permutation of the system. This can be proven by just building $¶_{23}$ and $¶_{31}$

$$\Pi_{23} = \Pi_{12}\,\Pi_{123} \,, \tag{98}$$
$$\Pi_{31} = \Pi_{23}\,\Pi_{123} = \Pi_{12}^2\,\Pi_{123} \,. \tag{99}$$

Now we can do the permutation of the wave function, which is not so difficult

$$\begin{aligned}\Pi_{123}\,|\Psi_{3N}\rangle &= [\psi_S(k_{31},p_2) + \psi_S(k_{12},p_1) + \phi_S(k_{23},p_2)]\; |\frac{1}{2}\otimes 0_{23}\rangle_S\,|\frac{1}{2}\otimes 1_{23}\rangle_I \,, \\ &+ [\psi_T(k_{31},p_2) + \psi_T(k_{12},p_1) + \phi_T(k_{23},p_2)]\; |\frac{1}{2}\otimes 1_{23}\rangle_S\,|\frac{1}{2}\otimes 0_{23}\rangle_I ,\end{aligned} \tag{100}$$

which is to be compared with the original wave function. Except that there is a catch: the spin and isospin wave functions are not the original ones. The original spin (isospin) wave functions first coupled particles 12 and then proceed to couple particle 3. The new spin (isospin) wave function are constructed in a different order and hence they are different than the original ones. Rewritting them in terms of the original wave functions is the tricky part.

Now the thing is to figure out how it works this thing of exchanging the spin and isospin couplings. We can begin with a very simple example: three spin $\frac{1}{2}$ particles coupled to $S = \frac{3}{2}$. In this case, if we couple first particle 12 and then add particle 3 the outcome is pretty simple

$$|1_{12}\otimes\frac{1}{2}\,(S=\frac{3}{2})\rangle \,, \tag{101}$$

where we are explicitly indicating that the total spin is $\frac{3}{2}$. In principle if we permute this state, we have that

$$\Pi_{123}|1_{12}\otimes\frac{1}{2}\,(S=\frac{3}{2})\rangle = |\frac{1}{2}\otimes 1_{23}\otimes\,(S=\frac{3}{2})\rangle \,. \tag{102}$$

that is, we change the order of how we couple the states. But if we go back to the original $S = \frac{3}{2}$ state and consider the case for which with $M_S = \frac{3}{2}$ we will see that

$$|1_{12}\otimes\frac{1}{2}\,(S=\frac{3}{2},M_s=\frac{3}{2})\rangle = |+++\rangle \,, \tag{103}$$

where $|+\rangle = |\frac{1}{2}\frac{1}{2}\rangle$ is the spin "up" state. From this we see that this state is completely symmetric. In fact we can check that the symmetry of the $S = \frac{3}{2}$ state is independent of M_S. This means that when we apply the permutation Π_{123} we will get the following

$$\Pi_{123}|1_{12}\otimes\frac{1}{2}\,(S=\frac{3}{2})\rangle = |1_{12}\otimes\frac{1}{2}\,(S=\frac{3}{2})\rangle = |\frac{1}{2}\otimes 1_{23}\otimes\,(S=\frac{3}{2})\rangle \,, \tag{104}$$

and we are left with the conclusion that

$$|1_{12}\otimes\frac{1}{2}\,(S=\frac{3}{2})\rangle = |\frac{1}{2}\otimes 1_{23}\otimes\,(S=\frac{3}{2})\rangle \,. \tag{105}$$

The cases with spin $\frac{1}{2}$ are more complicated because they are not eigenvalues of the permutation operator and hence a permutation will in general mix them

$$\Pi_{123}|0_{12}\otimes\frac{1}{2}\rangle_{\frac{1}{2}} = |\frac{1}{2}\otimes 0_{12}\rangle_{\frac{1}{2}} = a_0|0_{12}\otimes\frac{1}{2}\rangle_{\frac{1}{2}} + a_1|1_{12}\otimes\frac{1}{2}\rangle_{\frac{1}{2}} \,, \tag{106}$$

plus a similar equation for the $|1_{12}\otimes\frac{1}{2}\rangle$ case. Concrete calculations show that

$$\Pi_{123}\begin{pmatrix}|0_{12}\otimes\frac{1}{2}\rangle_{\frac{1}{2}} \\ |1_{12}\otimes\frac{1}{2}\rangle_{\frac{1}{2}}\end{pmatrix} = \begin{pmatrix}\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2}\end{pmatrix}\begin{pmatrix}|\frac{1}{2}\otimes 0_{23}\rangle_{\frac{1}{2}} \\ |\frac{1}{2}\otimes 1_{12}\rangle_{\frac{1}{2}}\end{pmatrix} \tag{107}$$

which you can try as an **exercise**. From now on we will be back to our usual notation, in which if we are coupling to total spin (isospin) $S = \frac{1}{2}$ ($I = \frac{1}{2}$) we will not indicate it. That is

$$|1_{12} \otimes \frac{1}{2}\rangle \equiv |1_{12} \otimes \frac{1}{2}\rangle_{\frac{1}{2}} \quad \text{or} \quad |1_{12} \otimes \frac{1}{2}(S = \frac{1}{2})\rangle \tag{108}$$

Now if we permute the combinations that appear in the wave function

$$\Pi_{123}\,|0_{12} \otimes \frac{1}{2}\rangle_S\,|1_{12} \otimes \frac{1}{2}\rangle_I = +\frac{1}{4}\,|\frac{1}{2} \otimes 0_{23}\rangle_S\,|\frac{1}{2} \otimes 1_{23}\rangle_I - \frac{3}{4}\,|\frac{1}{2} \otimes 1_{23}\rangle_S\,|\frac{1}{2} \otimes 0_{23}\rangle_I + \dots\,, \tag{109}$$

$$\Pi_{123}\,|1_{12} \otimes \frac{1}{2}\rangle_S\,|0_{12} \otimes \frac{1}{2}\rangle_I = -\frac{3}{4}\,|\frac{1}{2} \otimes 0_{23}\rangle_S\,|\frac{1}{2} \otimes 1_{23}\rangle_I + \frac{1}{4}\,|\frac{1}{2} \otimes 1_{23}\rangle_S\,|\frac{1}{2} \otimes 0_{23}\rangle_I + \dots\,, \tag{110}$$

where the dots indicate contributions that will be symmetric under the permutation of particles 23. These contributions are there, but will contribute to components of the wave function in which a two-body subsystem is in P-wave and the third body is also in relative P-wave with the two-body subsystem. These components are expected to be less important than the pure S-wave components that we are considering for the moment, yet they should be considered in a Faddeev calculation with long range forces. The previous permutation can also be written more compactly in matrix form

$$\Pi_{123} \begin{pmatrix} |0_{12} \otimes \frac{1}{2}\rangle_S\,|1_{12} \otimes \frac{1}{2}\rangle_I \\ |1_{12} \otimes \frac{1}{2}\rangle_S\,|0_{12} \otimes \frac{1}{2}\rangle_I \end{pmatrix} = \begin{pmatrix} +\frac{1}{4} & -\frac{3}{4} \\ -\frac{3}{4} & +\frac{1}{4} \end{pmatrix} \begin{pmatrix} |\frac{1}{2} \otimes 0_{23}\rangle_S\,|\frac{1}{2} \otimes 1_{23}\rangle_I \\ |\frac{1}{2} \otimes 1_{23}\rangle_S\,|\frac{1}{2} \otimes 0_{23}\rangle_I \end{pmatrix} + \dots\,, \tag{111}$$

plus the other combinations that require a P-wave spatial wave function. From this we obtain

$$\begin{aligned} &\Pi_{123}\,|\Psi_{3N}\rangle = \\ &\quad \left\{ +\frac{1}{4}\,[\psi_S(k_{31},p_2) + \psi_S(k_{12},p_1) + \phi_S(k_{23},p_2)] - \frac{3}{4}\,[\psi_T(k_{31},p_2) + \psi_T(k_{12},p_1) + \phi_T(k_{23},p_2)] \right\} |0 \otimes \frac{1}{2}\rangle_S\,|1 \otimes \frac{1}{2}\rangle_I\,, \\ &+ \left\{ -\frac{3}{4}\,[\psi_S(k_{31},p_2) + \psi_S(k_{12},p_1) + \phi_S(k_{23},p_2)] + \frac{1}{4}\,[\psi_T(k_{31},p_2) + \psi_T(k_{12},p_1) + \phi_T(k_{23},p_2)] \right\} |1 \otimes \frac{1}{2}\rangle_S\,|0 \otimes \frac{1}{2}\rangle_I\,. \end{aligned} \tag{112}$$

We remind that the symmetry/antisymmetry of the wave function requires

$$\Pi_{123}\,|\Psi_{3N}\rangle = |\Psi_{3N}\rangle\,, \tag{113}$$

where notice that for this type of permutation it does not matter whether we have bosons or fermions. From the antisymmetry condition we thus obtain

$$\psi_S(k,p) = +\frac{1}{4}\,\phi_S(k,p) - \frac{3}{4}\,\phi_T(k,p)\,, \tag{114}$$

$$\psi_T(k,p) = -\frac{3}{4}\,\phi_S(k,p) + \frac{1}{4}\,\phi_T(k,p)\,. \tag{115}$$

If we now propose the usual ansatz

$$\phi_S(k,p) = \frac{g(k)}{\gamma^2 + k^2 + \frac{3}{4}\,p^2}\,a_S(p)\,, \tag{116}$$

$$\phi_T(k,p) = \frac{g(k)}{\gamma^2 + k^2 + \frac{3}{4}\,p^2}\,a_T(p)\,, \tag{117}$$

we end up with the equations

$$a_S(p_3) = 2\,\tau_S(Z_{12}) \int \frac{d^3p_1}{(2\pi)^3}\,B_{31}(p_3,p_1)\,\left[+\frac{1}{4}\,a_S(p_1) - \frac{3}{4}\,a_T(p_1) \right]\,, \tag{118}$$

$$a_T(p_3) = 2\,\tau_T(Z_{12}) \int \frac{d^3p_1}{(2\pi)^3}\,B_{31}(p_3,p_1)\,\left[-\frac{3}{4}\,a_S(p_1) + \frac{1}{4}\,a_T(p_1) \right]\,, \tag{119}$$

which are indeed a little set of really cute equations.

A. Relation with the Efimov Effect

If we take the limit in which the singlet and triplet scattering lengths go to infinity

$$a_S \to \infty \quad \text{and} \quad a_T \to \infty \,, \tag{120}$$

then the bound state equations for the triton reduce to the ones for the three boson system in the unitary limit. In this previous limit

$$\tau_S(Z_{12}), \tau_T(Z_{12}) \to -\frac{4\pi}{m_N} \frac{1}{\sqrt{-m_N Z_{12}}} \tag{121}$$

which also means that $\tau_S = \tau_T$. In addition if we take $a_S(p) = -a_T(p) = a(p)$, then we trivially end up with the equation

$$p^2 \, a(p) = \frac{4}{\pi \sqrt{3}} \int_0^\infty \frac{dp'}{p'} \, \log\left(\frac{\gamma^2 + p^2 + p'^2 + pp'}{\gamma^2 + p^2 + p'^2 - pp'} \right) {p'}^2 a(p') \,, \tag{122}$$

that is, the equation we already obtained for the three boson system. This happens to be relevant to nuclear physics. As we have explained previously in this course, it happens that the two-nucleon scattering lengths are relatively big in comparison with the pion mass

$$m_\pi a_S \simeq -16.6 \quad \text{and} \quad m_\pi a_T \simeq 3.8 \,. \tag{123}$$

Though the triplet scattering length is not that big, actually we can try to take the limit $m_\pi a_T \to \infty$ and expect to be able to describe nuclear physics with a relative error of $1/(m_\pi a_T) \sim 30\,\%$, which is not bad at all considering the type of simple model we are using.