Nuclear Physics. Part I: Introduction

In this lecture we introduce a few basic concepts about the nucleon and the nuclear force, such as its general properties, the existence of a separation of scales, the the derivation of nuclear forces and the strategies to deal with this problem, isospin symmetry and chiral symmetry.

Welcome to the lecture notes of the post-graduate course on nuclear physics (Spring 2019). Notice that the lecture notes do not necessarily cover all the topics covered in the actual lectures, while in a few cases they might contain things that were not actually covered in the lectures.

There is no unique textbook that we are following for this course. The contents are derived from a few textbooks, reviews and research papers in the field of nuclear physics. For the textbooks we can recommend

- *Nuclear Physics in a Nutshell* from C. Bertulani (for the general topics, but also for the nuclear models),
- *Advanced Quantum Mechanics* from J.J. Sakurai (for scattering theory),
- *Quarks and Leptons* from Halzen and Martin (for topics related to the standard model),
- *Lectures in Scattering Theory* from A.G. Sitenko (for scattering theory and the three body problem, wonderful classical russian soviet style book that does not shy away from complicated stuff, but hard to find however)

and maybe a few others. For recent reviews, the ones by R. Machleidt are particularly recommended, where one can easily find them in google or inspire-hep. R. Machleidt also has an excellent review in scholarpedia about nuclear forces. A few reseach papers will be referenced in these lectures notes, as you will see at the end of each handout.

For the evaluation of the course, which is always a hot topic: a few points will be given for assistance, a few for doing exercises and a few points will come from the exam. The exact details of how much points come from each of these three sources will be agreed upon in the classes, but exercises will be a big contribution. For the exercise part, I will grade you on the basis of a total of 10 exercises, or the equivalent: there are a few exercise that counts as double or triple, which is indicated by *two points* or *three points* written next to the exercise or at the end of the exercise. The exercises can be found directly within the lecture notes. They are also listed at the end of each of the lectures, but the list might not be complete. Besides, if you found mistakes in the equations or derivations written in the lecture notes and tell me, it will count as an exercise done (this is for equations and derivations, typos in the text do not count).

I. GENERAL CONSIDERATIONS

Nowadays we know that ordinary matter is made of atoms. In turn atoms are composed of a nucleus and a cloud of electrons around it, which bind owing to the electromagnetic interaction. The nucleus, which was discovered by Rutherford in 1911, is extremely small and contains most of the mass of the atom. We also know that nuclei are composed of neutrons and protons. They bind together due to the nuclear force to form the more or less 4000 nuclei that are experimentally known as of today. From deep inelastic scattering experiments we also know that neutrons and protons are not solid particles but are instead composed of three quarks, which are held together by the strong force. At distances below a small fraction of a fermi ($1\,\mathrm{fm} = 10^{-15}\,\mathrm{m}$) the strong force is completely analogous to the electromagnetic force, expect for the fact that instead of one type of electric charge there are three types of strong charge, which we call *colors*. However at distances of about $1\,\mathrm{fm}$ — the typical separation of neutrons and protons inside a nucleus — the strong force becomes terribly complicated and not mathematically solvable except by mammoth numerical simulations, which even nowadays can not be done gracefully. As a consequence the derivation of nuclear forces from first principles, i.e. from the strong force, is the most important open problem in nuclear physics.

Before starting, I can recommend a few references about the nuclear force which might be useful. An author that I particularly recommend is Machleidt, which has excellent reviews about the nuclear force: from the one boson exchange model [1], to the modern chiral approaches [2, 3], including a very recent historical perspective that is really nice to read [4]. Other review centered in the modern chiral approach is [5]. For the current approaches based on effective field theory, a very nice review is [6].

A. Why Nuclear Physics is Difficult : Separation of Scales

The character of nuclear physics depends on the *scales* involved in it. The term scale refers to the typical distance (or momentum) at which a physical process happens. To better understand the idea of a scale we will consider the case of atomic physics. The characteristic length scale of atomic physics, which describes the size of the atoms to give an example, can be determined from first principles. In particular from the Schrödinger equation and the Coulomb potential. The Schrödinger equation for an electron orbiting a nucleus reads

$$\left[-\frac{1}{2\mu}\nabla^2 - Z\,\frac{\alpha}{r}\right]\Psi(\vec{r}) = -B\,\Psi(\vec{r}) \tag{1}$$

where B is the binding energy, Z the number of protons in a nucleus and α the fine structure constant $1/\alpha \simeq 137$ and μ the reduced mass of the system

$$\frac{1}{\mu} = \frac{1}{m_e} + \frac{1}{M_A}\,, \tag{2}$$

where m_e and M_A are the electron and nucleus mass. Owing to $M_A \ll m_e$, we can see that the reduced mass basically coincides with the mass of the electron. Rearraging this equation we have

$$\left[-\nabla^2 - Z\,\frac{2}{a_B\,r}\right]\Psi(\vec{r}) = -\gamma^2\,\Psi(\vec{r}) \tag{3}$$

where $\gamma = \sqrt{2\mu B}$ is the wave number of the bound state and a_B a length scale that is given by

$$a_B = \frac{1}{\mu\,\alpha} \simeq 5.29\cdot 10^4\,\mathrm{fm}\,. \tag{4}$$

We see that the only dimensionful quantities that we have in the equation are a_B and γ, yet they play a different role: a_B is a given parameter of the theory, while γ is a prediction of the theory.

The length scale a_B, which is known as the Bohr radius, is really interesting because is the only dimensionful number that we can build from the Coulomb interaction and the electron mass. What are the implications of this? Easy, every physical quantity that we predict for the Schödinger equation can be written in units of a_B. Not only that, we expect these physical quantities to be *natural* in units of the Borh radius. What do we mean by natural? Let us consider the properties of the fundamental state of hydrogen-like atoms. Two examples are the binding momentum and the mean square radius, which we can write as

$$\gamma_A = c_A\,\frac{Z}{a_B}\,, \tag{5}$$

$$\sqrt{\langle r^2\rangle} = d_A\,\frac{a_B}{Z} \tag{6}$$

where c_A and d_A are numbers. According to the idea of naturalness, the numbers c_A and d_A should be of $\mathcal{O}(1)$, that is, most likely something like 1/3 or 3. On the contraty it is fairly unprobale that c_A and d_A are of the order of 1/300 or 300. These values are *unnatural* and if they happen they require a good explanation. Now let see how the hypothesis of naturalness stands against reality. If we solve the Schrödinger equation for a hydrogen-like atom we find the fundamental wave function

$$\Psi(\vec{r}) = \frac{1}{\sqrt{4\pi}}\frac{2}{a_B^{3/2}}\,e^{-r/a_B} \tag{7}$$

with binding momentum

$$\gamma_A = \frac{Z}{a_B} \qquad \text{that is,} \qquad c_A = 1\,, \tag{8}$$

and mean square radius

$$\sqrt{\langle r^2\rangle} = \sqrt{3}\,\frac{a_B}{Z} \qquad \text{that is,} \qquad d_A = \sqrt{3}\,. \tag{9}$$

In short, naturalness definitely works for atomic physics.

Now one might ask: but what happen with the other scales in the system? For instance: relativistic corrections? the finite size of the nucleus? The expectation is that if these effects have the length scale R_S, the relative size of the corrections coming from them should be

$$\mathcal{O}\left(\frac{R_S}{a_B}\right)\,. \tag{10}$$

If we take the example of relativistic corrections, they are evident if the momentum of the electron is of the order of its mass. As a consequence the length scale for this short-range effect is

$$R_S^{\rm rel} = \frac{1}{m_e} \simeq 3.86 \cdot 10^2 \,{\rm fm} \quad \text{that is,} \quad \left(\frac{R_S^{\rm rel}}{a_B}\right)^2 \simeq 0.5 \cdot 10^{-4} \,, \tag{11}$$

where we use $(R_S/a_B)^2$, instead of the expected (R_S/a_B), because relativistic corrections usually enter as a square (for instance in $\sqrt{p^2+m^2} = m + p^2/2m + 3p^4/8m + ...$). As a matter of fact this agrees well with the size of fine structure corrections in the hydrogen atom, which are of the order of 10^{-4}. In addition for the finite size of the proton we have

$$R_S^{\rm p} = \frac{1}{M_p} \simeq 0.2 \,{\rm fm} \quad \text{that is,} \quad \frac{R_S^{\rm p}}{a_B} \simeq 3.8 \cdot 10^{-6} \,. \tag{12}$$

The hyperfine splitting of the hydrogen atom is about 10^{-6} times the size of the ground state energy. It is due to the magnetic interaction of the proton and electron magnetic moments and as we see its relative size agrees with the really simple estimation we have done.

What we have learned with this example is the following

(i) physical systems have a characteristic scale a.

(ii) physical quantities are usually (but not always) natural in units of a.

(iii) there are corrections of relative size R_S/a, where R_S it the charecteristic scale of short-range physics.

For atomic physics naturalness works and in addition there is a very large and clear-cut separation of scales. This is actually one of the reasons why incredibly accurate calculations in atomic physics are easy and can be taught in undergraduate courses.

What about nuclear forces? Here things are a bit more complicated. Experimentally we know that the radius of the nucleus ranges from about 1 to 10 fm. This is in line with the first idea about the origin of nuclear forces. In 1938 Yukawa proposed that nuclear forces is mediated by the exchange of a meson. Indeed this is analogous to how the Coulomb force is generated by the exchange of a virtual photon, which generates the potential

$$V_C(r) = \frac{e^2}{4\pi r} = \frac{\alpha}{r} \,. \tag{13}$$

A photon is a massless vector boson, with quantum number $J^P = 1^-$ while the pion — the particle exchanged in nuclear forces — is a massive boson with $J^P = 0^-$. Originally Yukawa simply proposed a scalar meson, in which case if we calculate the potential that such a meson generates we find

$$V(r) = -g_H^2 \, \frac{e^{-mr}}{4\pi r} \,, \tag{14}$$

where g_H is a coupling constant and m is the mass of the meson. The exchanged meson is the pion which has a mass of 140 MeV. Unlike the Coulomb force that has an infinitely long range, the range of pion exchange is $1/m_\pi \simeq 1.4\,{\rm fm}$. This is the scale of nuclear forces that we were looking for, which is indeed of the order of the size of the nucleus.

There are however a few important difference with electrons in an atom. The most apparent one is the strength of the interaction:

$$\frac{e^2}{4\pi} \simeq \frac{1}{137} \quad \text{to be compared with} \quad \frac{g_H^2}{4\pi} \sim 15 \,. \tag{15}$$

That is, the strength of the one pion potential is much stronger than the electromagnetic interaction. The unwanted consequence of this insane strength will be that the naturalness hypothesis will not be always applicable to nuclear forces. The second difference lies in the separation of scales. The short-range length scale in nuclear physics is the size of the proton and the neutron, which is roughly $R_S \sim 0.5\,{\rm fm}$. As a consequence predictions on the basis of one pion exchange will be subjected to uncertainties of

$$\frac{R_S}{a_H} \sim \frac{1}{3} \,. \tag{16}$$

This is really big in comparison with electrons in atoms. These two problems, namely

(i) failure of naturalness and

(ii) poor separation of scales,

are the reasons why nuclear physics is complicated.

B. The Properties of the Nuclear Force

What do we know about nuclear forces? The following list comes to mind:

(i) The nuclear force is short-ranged: we know that because of a property that goes by the name of saturation. Saturation means that the binding energy per nucleon of a nucleus becomes roughly constant when the number of nucleons A increases. The binding energy for the deuteron is 2.2 MeV (1 MeV per nucleon), for the triton is 8.5 MeV (3 MeV per nucleon) and for the alpha particle is 28 MeV (7 MeV per nucleon). By then B/A basically reaches the saturation energy of 8 MeV per nucleon. If the range of the nuclear force was long, saturation will not be a necessary outcome. From the alpha particle Wigner deduced that the range of nuclear forces is about its size, namely about 1.7 fm give ot take.

(ii) The nuclear force is attractive at intermediate distances: we know this from the nucleon density of heavy nuclei, which is $0.17\,\mathrm{fm}^{-3}$. That gives an average separation of 1.8 fm. It is sensible to assume that this corresponds to a mimimum of the two-body force.

(iii) The nuclear force is repulsive at short distances: we know this partly from saturation, but more compellingly from the existence of a zero in the $^1S_0$ phase shift that can be nicely explained from a short-range repulsive core at 0.6 fm

(iv) The nuclear force does not distinguish neutrons and protons, a property usually referred as *charge independence*. That is why we usually talk about *nucleons* instead of neutrons and protons. We know this from the binding energy of nuclei with the same quantum numbers, same number of nucleons A but different number of protons. For example ^{3}H and ^{3}He, which have binding energies of 8.48 MeV and 7.72 MeV respectively (where the difference comes from the Coulomb repulsion between the two protons in ^{3}He).

(v) The nuclear force is not central, but has a bit more complicated structures such a spin-spin, tensor and spin-orbit forces. The tensor force is particularly important for explaining a few properties of the deuteron (its quadrupole and magnetic moments) and the spin-orbit force becomes important to explain shell structure in many nuclei.

C. The Origin of Nuclear Forces

Besides the properties of the nuclear force, the other important problem is the origin of the nuclear force. For that we begin with Yukawa's idea, which has been instrumental: the origin of the nucler force is the exchange of a meson, the pion. In particular Yukawa proposed the exchange of a scalar meson with quantum number $J^P = 0^+$. In the language of quantum field theory this requires a interaction of the type

$$\mathcal{L} = g_H\, \bar{\Psi}\phi\Psi\,, \tag{17}$$

where g_H is the coupling, Ψ is a Dirac field for the nucleon and ϕ is a Klein-Gordon field for the meson. This type of interaction yields a really simple and neat potential, the famous Yukawa potential

$$V_Y(r) = -g_H^2 \frac{e^{-mr}}{4\pi r}\,, \tag{18}$$

where m is the mass of the meson. This idea successfully explains that the nuclear forces are short-ranged.

The problem with this potential is that later it was discovered that the deuteron, the neutron-proton bound state, has an electric quadrupole moment. Let us remind that the charge, dipole moment, electric quadrupole moment and so on, simply refer to the way that different physical systems interact with an external electric field. If we have an external electric field Φ, the energy of some electrically charge object in that field will be

$$V = q\,\Phi + d_i\,\partial_i\Phi + \frac{1}{6}Q_{ij}\partial_{ij}\Phi + \dots \tag{19}$$

where the coefficients q, d_i, Q_{ij} are the charge, dipole and quadrupole moments respectively. If the object has a known charge distribution we can write these coefficients as

$$q = \int d^3\vec{r}\,\rho(\vec{r})\,, \tag{20}$$

$$d_i = \int d^3\vec{r}\,r_i\rho(\vec{r})\,, \tag{21}$$

$$Q_{ij} = \int d^3\vec{r}\,\left(3r_ir_j - r^2\delta_{ij}\right)\,\rho(\vec{r})\,, \tag{22}$$

where $\rho(\vec{r})$ is the charge distribution of the object we have put in the external electric field. Usually the quadrupole moment is referred with respect to a specific direction. For the deuteron we have that it is a spin 1 system, which means that is has a preferred direction: the direction where the spin is pointing to. If we call this direction z, then we can define the quadrupole moment of the deuteron relative to that direction

$$Q_d = \int d^3\vec{r}\,\left(3z^2 - r^2\right)\,\rho(\vec{r})\,, \tag{23}$$

where it is apparent that we have chosen $i = j = 3$. It happens experimentally that the quadrupole moment of the deuteron is positive. In particular we have $Q_d = 0.2859(3)\mathrm{fm}^2$. This also means that the deuteron is longer in the direction of its spin than in the other two directions: it is like a 油条 (youtiao), though the technical term would be "the deuteron is prolate". On the contrary a nucleus with a negative quadrupole moment is like a 包子 (baozi), the technical term being "oblate" in this case.

So why is this a problem for the hypothesis that the pion is a scalar? Well, a scalar meson generates a rotationally symmetric potential, which in turn generates a rotationally symmetric charge density, which in turns implies a quadrupole moment $Q_d = 0$. That means that the quantum number of the exchangeg meson have to be different than $J^P = 0^+$. Other possibility is that the pion is a vector meson with $J^P = 1^+$, i.e. some sort of heavy photon. In that case the lagrangian reads

$$\mathcal{L} = g_H\,\bar{\Psi}\gamma^\mu\,\phi_\mu\Psi + \frac{f_H}{4M_N}\bar{\Psi}\sigma_{\mu\nu}\Psi\left(\partial_\mu\phi_\nu - \partial_\nu\phi_\mu\right), \tag{24}$$

where now the interaction lagrangian is a bit more complicated because in principle it can contain two terms: an electric- and a magnetic-type interaction. The term with g_H is called electric because it yield a Coulomb-like force in the non-relativistic limit, while the term with f_H is called magnetic because it generates a potential that depend on the spin of the nucleons. In the expression above g_H and f_H are dimensionless coupling constants, γ_μ are the Dirac gamma matrices and $\sigma_{\mu\nu}$ are bilinears that can be constructed from the gamma matrices as

$$\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]\,. \tag{25}$$

Be it as it may, the important thing to notice is that the resulting potential is as follows

$$\begin{aligned} V(r) = &+ g_H^2\frac{e^{-mr}}{4\pi r} \\ &+ \left(\frac{g_H + f_H}{2M}\right)^2\left[\frac{2}{3}\,\vec{\sigma}_1\cdot\vec{\sigma}_2\,\frac{e^{-mr}}{4\pi r}\right. \\ &\left.-\frac{1}{3}\,\left(3\vec{\sigma}_1\cdot\hat{r}\,\vec{\sigma}_2\cdot\hat{r} - \vec{\sigma}_1\cdot\vec{\sigma}_2\right)\,\frac{e^{-mr}}{4\pi r}\,\left(1 + \frac{3}{mr} + \frac{3}{(mr)^2}\right)\right], \end{aligned} \tag{26}$$

which looks sort of complicated. We can make this expressions look a bit less menacing if we define

$$S_{12}(\hat{r}) = 3\vec{\sigma}_1\cdot\hat{r}\,\vec{\sigma}_2\cdot\hat{r} - \vec{\sigma}_1\cdot\vec{\sigma}_2\,, \tag{27}$$

$$W_C(r) = \frac{e^{-mr}}{4\pi r}\,, \tag{28}$$

$$W_T(r) = \frac{e^{-mr}}{4\pi r}\,\left(1 + \frac{3}{mr} + \frac{3}{(mr)^2}\right)\,, \tag{29}$$

from which we can rewrite

$$\begin{aligned} V(r, J^P = 1^+) = &+ g_H^2 W_C(r) \\ &+ \left(\frac{f_H}{2M}\right)^2\left[\frac{2}{3}\,\vec{\sigma}_1\cdot\vec{\sigma}_2\,W_C(r) - \frac{1}{3}\,S_{12}(\hat{r})\,W_T(r)\right]. \end{aligned} \tag{30}$$

We can see that the electric-type term is repulsive, which is to be expected as this is analogous to the exchange of a photon between identical charges. We also see that the magnetic-type term can be either attractive or repulsive depending on the alignment of the spins. The previous expression however is only correct for the two-neutron or two-proton potential: when there is a neutron and a proton, the pion can transform them into the other. The previous potential does not consider this possibility, yet the outcome is very simple: if we are considering the deuteron there should be a global minus factor in the potential. Actually the factor is a -3, i.e.

$$V_d(r) = -3\,V(r, J^p = 1^+)\,, \tag{31}$$

with the subscript d indicating we are talking about the deuteron case. We will learn that later when we study isospin. However what we are interested in is what type of quadrupole moment can be deduced from here. If happens that what we obtain is a negative quadrupole moment, but we are going to leave the proof as an **exercise**. This means that the possibility of a vector pion is not consistent with the experimental facts.

The **exercise** is as follows: show that for a potential of the type

$$V(r) = -g_H^2 \left[a\,\vec{\sigma}_1 \cdot \vec{\sigma}_2\, W_C(r) \pm b\, S_{12}(\hat{r})\, W_T(r) \right] . \tag{32}$$

for which $a > 0$, $b > 0$, then the sign of the quadrupole moment is the same as the sign between the central and tensor piece, i.e. $Q = \pm |Q|$. For that first take into account that $Q \neq 0$ requires that the spin of nucleons 1 and 2 must add up to $S = 1$. If we consider the total spin $\vec{S} = 2\,(\vec{\sigma}_1 + \vec{\sigma}_2)$, this corresponds to taking σ_1 paralell to σ_2.

The next theoretical possibility is that the pion is a pseudoscalar. The interaction lagrangian in this case is

$$\mathcal{L} = i g_H\, \bar{\Psi} \gamma^5 \phi \Psi \,, \tag{33}$$

and the potential one derives in this case is

$$V(r, J^P = 0^-) = \frac{g_H^2}{4 M_N^2}\, \left[\vec{\sigma}_1 \cdot \vec{\sigma}_2\, W_C(r) + S_{12}(\hat{r})\, W_T(r) \right] , \tag{34}$$

with M_N the nucleon mass. As in the previous case, we have to multiply by the mysterious factor of -3 to obtain the right potential for the neutron and proton inside the deuteron. A closer inspection on the line of the previous **exercise** shows that this potential generates a positive quadrupole moment. This implies that a pseudoscalar pion is compatible with the properties of the deuteron and we expect the pion to have $J^P = 0^-$.

The idea of Yukawa has also been extended in the past to explain the remaining properties of the nuclear force. For instance, the intermediate range attraction might be explain by the simultaneous exchange of two pions or by the exchange of a heavier meson. Originally in the 50's theoreticians tries the multi-pion theories, which however failed miserably. The reason for this failer is well-known today: chiral symmetry. This symmetry implies that the correct Lagrangian for the pion-nucleon interaction is not the one we wrote before, but one including derivative interactions:

$$\mathcal{L} = \frac{g_H}{m_H}\, \bar{\Psi} \gamma^5 \gamma^\mu \partial_\mu \phi \Psi \,, \tag{35}$$

which m_H a mass scale. Yet another reason why the multi-pion theories failed is the singular nature of the pion exchanges: as we can see from the formulas above, the one pion exchange (OPE) potential diverges as $1/r^3$ at distances $mr \leq 1$. This problem only gets worse once we consider two pion exchange (TPE). However in the 50's theoreticians considered that the correct path was the exchange of a scalar meson, the sigma, which could be regarded as a meson generated dynamically from the pion-pion interaction. The sigma meson indeed provides a respectable intermediate range attraction. For the short-range repulsion the standard explanation became the exchange of a vector meson, the omega meson, which interacted with the nucleon mostly by an electric type interaction. In addition there is another vector meson, the rho meson, which interacts mostly in a magnetic-like fashion. Yet the explanation of the short-range repulstion can also be more prosaic: if we consider that the nucleons are not point-like particles, but composed of quarks, which are fermions, then at distances smaller than the size of the nucleons these quarks will not like to be squeezed together owing to Fermi statistics. This also gives a strong short-range repulsion, but a type of repulsion that we can only understand once we know the nucleons are composite particles.

D. Putting a bit of order

A useful idea from Taketani, Nakamura and Sasaki (TNS) proposed to divide the nuclear force into three regions: long, medium and short-range (or classical, dynamical and phenomenological/core in their original words). For each of these regions we have

(i) The classical zone ($r \geq 2\,\mathrm{fm}$) is dominated by OPE.

(ii) The dynamical zone ($2\,\mathrm{fm} \geq r \geq 1\,\mathrm{fm}$) is dominated by TPE and other heavier mesons.

(iii) The phenomenological zone ($r \leq 1\,\mathrm{fm}$) where there are multi-pion exchanges, heavier mesons and all sort of other weird things we do not really understand.

This conception of the nuclear force is indeed very far sighted, because it advanced many of the concepts and ideas that we have today. For instance, it explains why it is okay to use form factors and other techniques to treat the short-range problems of the nuclear force. And it is also incredibly close to spirit to the effective field theory approaches that dominate the field nowadays and that we will explore later.

E. Nuclear Forces are Residual Forces

In the previous section we proposed an analogy between electrons in an atom and protons and neutrons in a nuclei to introduce ideas such as scales, naturalness and separation of scales. Yet electrons in an atom are not really that similar to protons and neutrons in a nuclei: electrons are bound to the nucleus owing to the electromagnetic force, a fundamental force of nature. However the strong force that is ultimately responsible for the binding of the nucleus does not mediate directly between protons and neutrons. Proton and neutrons are color neutral particles: they do not have color charge like the quarks composing them.

Actually the atomic physical system that is analogous to the proton and the neutron is the atom, which is electrically neutral. Yet neutral atoms can interact with other neutral atoms. A well-known atom-atom potential is the Lenard-Jones potential

$$V_{AA}(r) = -\frac{C_6}{r^6} + \frac{C_{12}}{r^{12}} \,. \tag{36}$$

Where does this potential comes from? The atom-atom potential is what we call a *residual force*: its origin is the electromagnetic interaction between the components of the atoms. Thus, though the atoms are neutral, there is still a force among. This force can in theory be computed from first principles if we know the internal structure of the atom in detail, though this calculation is certainly not trivial. It happens that the nuclear forces are a just like the Lenard-Jones potential: they are residual forces of the strong interaction of the quarks and gluons inside the neutrons and protons. In theory the nuclear force could also be derived from first principles, but unfortunately this type of derivation is incredibly more complex than the force between atoms for theoretical reasons we will explain in the next section.

II. THE DERIVATION OF NUCLEAR FORCES

All calculations in nuclear physics — properties of nuclei, excited states, nuclear reactions — ultimately depend on the nuclear force. This makes it very clear that the fundamental problem of nuclear physics is the derivation of nuclear forces from first principles. This problem is indeed as old as nuclear physics itself and theoreticians have been actively working in this for the last seven decades. The original idea of Yukawa was the starting point for this theoretical efforts. It was soon stablished that the meson responsible for nuclear forces is a pseudoscalar (this is necessary to obtain the correct quadrupole moment for the deuteron). Later in the 50s there were a few multipion theories, which failed however owing to the lack of a very important ingredient, while in the 60s and 70s the one boson exchange (OBE) model was develop. In the OBE model, besides the pion, the nuclear forces are also generated by the exchange of other mesons like the σ, the ρ and the ω to name the most important ones. However later, with deep inelastic scattering experiments, it was discover that the neutron and proton have an internal structure. They are composed of quarks and gluons, which interact by means of quantum chromodynamics (QCD), a quantum field theory that is extremely successful when it comes to explain the dynamics of these quarks and gluons (but which becomes terribly complicated when applies to neutrons, protons and pions). Owing to the existence of QCD, the fundamental theory of strong interactions, a serious derivation of nuclear forces should be grounded in this theory.

A. Asymptotic Freedom

Now we are going to explain why the derivation of nuclear forces from QCD looks like a hopeless task. The technical term for the reason of why this is so is *asymptotic freedom*. To explain it, we will go back again to quantum electrodynamics (QED) to give a more simple example.

QED is the quantum field theory of photons and electrons. The well-known QED lagrangian looks like this

$$\mathcal{L}_{\rm QED} = \bar{\Psi}(i\gamma^\mu D_\mu - m)\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \,, \tag{37}$$

where

$$D_\mu = \partial_\mu - ieA_\mu \,, \tag{38}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \,, \tag{39}$$

with A^μ the quantum field for the photon, Ψ the one for the electron, e the electron charge and m the electron mass. In principle we can use perturbation theory, i.e. Feynman diagrams, to compute quantities in QED. For instance, if we compute the potential between two electrons in the non-relativistic limit, we get the Coulomb potential

$$V_c(r) = \frac{\alpha}{r} \,. \tag{40}$$

The strength of all electron-photon interactions is set by the fine structure constant $\alpha = e^2/4\pi$. As already said here $\alpha \sim 1/137$, but this is only so at low energies. If the interactions between electrons and photons happen at high energies, the value of the fine structure constant will be subject to quantum corrections which will change its value. A graphical representation of this idea is that a high energy photon can fluctuate to an electron-prositron pair, which changes in fact the properties of the photon and the way photons couple to electrons. The running of the electromagnetic coupling can be compute in first order perturbation theory, leading to the well-known formula

$$\alpha(Q^2) = \frac{\alpha(\mu^2)}{1 - \frac{\alpha^2(\mu^2)}{3\pi}\,\log\frac{Q^2}{\mu^2}} \,, \tag{41}$$

where Q^2 and μ^2 are the energies of the photon emitted by the electron (if you want an interesting **exercise**, try to derive this formula; this will be three points as this is not exactly trivial). This implies that the strength of the electromagnetic couplings increases with the energy, as can be easily check from the formula. As a matter of fact the strength diverges at incredibly high energies, a phenomenon that is known as the Landau pole of QED, but the energies required for this are far above the Planck scale for which we do not think that QED will be valid. QED as any other physical theory is only applicable until the scale at which additional physics appear, its R_s. The point however is the following: for most practical energies α is pretty small and QED calculations can be done perturbatively.

The lagrangian for the gluons and quarks is in fact extremely similar to the one of electrons and photons and reads:

$$\mathcal{L} = \sum_{i=1}^{n_F} \bar{q}_i\,(i\,\gamma^\mu D_\mu - m_i)q_i - \frac{1}{4}\,G^a_{\mu\nu}\,G^{a\mu\nu} \,, \tag{42}$$

with

$$D^a_\mu = \partial_\mu - ig\sum_a \frac{\lambda^a}{2}A^a_\mu \,, \tag{43}$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f_{abc} A^b_\mu A^c_\nu \,. \tag{44}$$

In the equation above q_i are the quark fields, i is a flavour index that runs from 1 to 6 to include the six types of quarks

$$q_i = \{u, d, s, c, b, t\} \,, \tag{45}$$

$a = 1, \ldots, 8$ is a color index, A^a_μ is the gluon field, λ^a are the SU(3) equivalent of the Pauli matrices (they are 3x3 matrices that go by the name of the Gell-Mann matrices) and f_{abc} are some indexes with the property $f_{abc} = -f_{acb}$. These are called structure constants and can also be obtained from

$$[\frac{\lambda^b}{2}, \frac{\lambda^c}{2}] = if^{abc}\frac{\lambda^a}{2} \,. \tag{46}$$

Actually this lagrangian can be considered as a direct extension of the electromagnetic one from one to three charges: that is why we say that electromagnetism is a $U(1)$ gauge theory and the strong force is an $SU(3)$ gauge theory. As with electromagnetism we can define a strong α_s, which also runs with energy. Howeve there is a fundamental difference with electromagnetism: while the electron does not carry electric charge, the gluons do indeed carry strong charge. As a consquence the gluons can interact with themselves directly. The outcome is that a calculation of the running of the strong coupling leads to this result

$$\alpha_s(Q^2) = \frac{\alpha_s(\mu^2)}{1 + \frac{\alpha_s(\mu^2)}{12\pi}\,(33 - 2n_f)\,\log\frac{Q^2}{\mu^2}} \,, \tag{47}$$

that is, the coupling diminishes for high energies (as in the previous case, this is an interesting **exercise**, three points). Alternatively we can define

$$\Lambda^2_{QCD} = \mu^2\,\exp\left[-\frac{12\pi}{(33 - 2n_f)\alpha_s(\mu^2)}\right] \,, \tag{48}$$

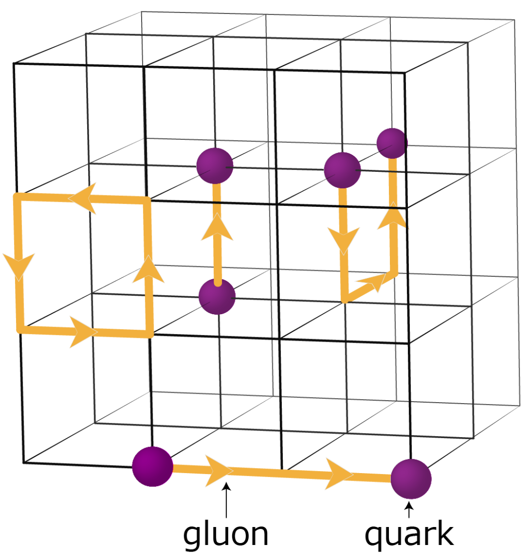


FIG. 1. In lattice QCD space-time is discretized in a lattice and QCD is solved in that lattice.

and write

$$\alpha_s(Q^2) = \frac{12\pi}{(33-2n_f)\log\frac{Q^2}{\Lambda^2_{QCD}}} \tag{49}$$

That is, QCD is only dependent on one dimensionful scale, which is Λ_{QCD}. It happens that $\Lambda_{QCD} \sim 200-300\,\mathrm{MeV}$.

Now the difference with QED should be apparent: as the enery Q^2 decreases, the coupling α_s increases. To make things worse, there is a particular Q^2 such that α_s diverges. This is $\Lambda_{\rm QCD}$, which is the natural energy scale of QCD. It happens that for energies larger than $\Lambda_{\rm QCD}$, α_s is small and QCD makes sense as a perturbative quantum field theory with can be solve with standard techniques such as Feynman diagrams. However for energies below Λ_{QCD} the coupling constant diverges and is apparent that perturbation theory cannot be used. What is the problem with this? Well, for starters in general most of our knowledge about solving QFT is perturbative. As a consequence we have no clear way of deriving the residual force among nucleons from QCD with analytical methods.

We are left with two strategies, one direct and one indirect, which are

(i) Lattice QCD and

(ii) Effective Field Theory.

The direct strategy is Lattice QCD and it amounts to use a computer to solve QCD numerically in the non-perturbative regime. As expected, this is anything but easy. The indirect strategy, effective field theory, amounts to solving QCD by not solving it at all. That is, to use a series of powerful techniques (renormalization, symmetries, power counting) to figure out how the most general low energy solution of QCD should look like. In the following lines we will explain a bit how they work.

B. Lattice QCD

Lattice QCD consist on making a calculation of a nuclear system on the basis of the quarks and gluons that form the nucleons. The reason why it has to be done in a computer is that these equations are simply not solvable analytically, owing to asymptotic freedom. However they could in principle be solved in a computer. This is done by discretizing space-time to form a lattice: we construct a cube with a side of few fm, put a grid inside with a grid size of a fraction of a fermi, then assume that there a few quarks inside and solve QCD on the lattice, see Fig. 1. The only problem is the technical difficulty. As a matter of fact the way to make the calculations easier is by changing the masses of the quarks: if the quarks are considerably lighter than $\Lambda_{\rm QCD}$ calculations become really difficult. Thus calculations are not always made for the physical world (where $m_u \sim 3\,\mathrm{MeV}$ and $m_d \sim 5\,\mathrm{MeV}$), but for a fictional world with different quark masses which leads to different pion and nucleon masses. Yet there are calculations at the physical pion mass for certain processes. The field is making quick progress.

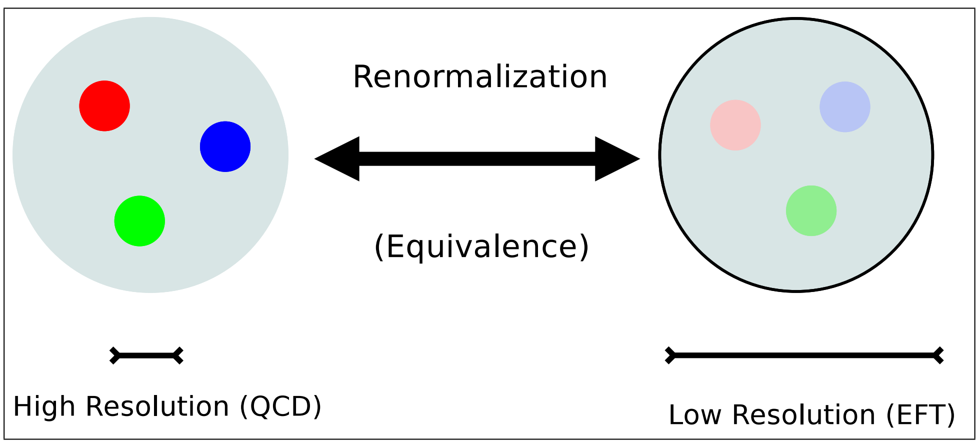
C. Effective Field Theory

The other way to solve QCD is paradoxically by not solving it. It happens that there is a very powerful physical idea called renormalization. The idea behind renormalization is the following one:

> Physics at long distances does not depend on the short distance details.

This is really important, it is the reason why physics itself is possible and we know it from countless example. We can describe chemistry, the chemical bond and the energy levels of the atom without known any details of the nucleus: the existence of the nucleus itself only manifest in the hyperfine structure and can be accounted for without knowing the details of what happens inside a nucleus. We can describe the solar system with great accuracy solely based on Newton's theory of gravity, the only exception being the precession of the perihelion of Mercury in which small corrections from general relativity enter. Yet the main correction can be parametrized with a L^2/r^3 type potential that is suppressed by a factor of c^2, where L is the angular momentum, r the radius and c the speed of light. There has been engineering and architecture before the discovery of quantum mechanics, despite the fact that the existence of solid matter can only be explained with quantum theory, in particular Fermi-Dirac statistics. In short, the list is endless and the existence of different branch of sciences boils down to the fact that we can analyze whatever happens at a particular distance scale without knowing what happens at shorter distance scales. Yet renormalization can go a bit farther than this and can be used to connect the different types of explanations that we have.

In the case of nuclear physics we know that at low energies we can describe the nucleons without knowning what happens to the quarks and gluons inside. At low energies (long distances) the world is composed of nucleons and we cannot see that they have any kind of structure. At high energies (short distances) the world is composed of quarks and gluons, which we can now see clearly, while nucleons disappear because they are just too big. In this context renormalization is the mathematical problem of uniting these two points of view:



1. *Renormalization*

Now we will explain renormalization in more detail, including how to formalize it. We begin by consider a physical system with a characteristic distance scale a. We know how to describe physics at distances of the order of a, but we know that this is not all there is to the physics of this system. At short distances R_S there might be a fundamental theory that we do not know yet, which might be able to give us a better description of this system. Or maybe we know this fundamental theory, but it is too complicated to solve and we want to obtain a more simple long distance description. Renormalization in general tells us that physics at the short distance R_S does not matter for the description of physic phenomena at distances a. Notice that here we have been talking about distance scales, but it is also possible to talk about momentum scales. In that case we will say that the fundamental theory happens at the momentum scale M ($\sim 1/R_S$) but we want to describe the world at the momentum scale Q ($\sim 1/a$).

Now the way to write this in a mathematical form is the following:

(i) Introduce a cut-off R_c, i.e. an extra scale that will mark the boundary between known and unknown physics

$$R_S < R_c < a \tag{50}$$

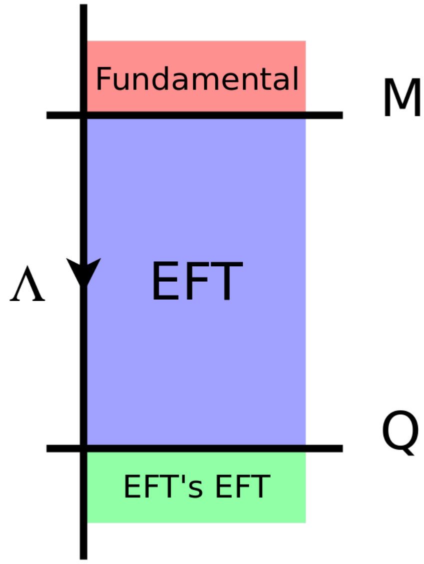


FIG. 2. The idea behind EFT is that it is possible to build a theory of low energy phenomena (at the energy scale Q, which is equivalent to the length scale $a \sim 1/Q$) without knowing the details of what happens at very high energy (at the energy scale M, with $M \ll Q$, where M is equivalent to the length scale $R_S \sim 1/M$). This is done by including a separation scale or cut-off and then requiring all observable quantities to be independent on the cut-off:

$$\frac{d}{d\Lambda} \langle \Psi | \mathcal{O} | \Psi \rangle = 0 \,,$$

where $|\Psi\rangle$ is the wave function and $\mathcal{O}$ is an operator representing an observable.

(ii) Impose that physical observables at long distances are independent of the cut-off:

$$\frac{d}{dR_c} \langle \Psi | \mathcal{O} | \Psi \rangle = 0 \,. \tag{51}$$

This condition is the basis of the renormalization group equations (RGEs), see Fig. 2 for a graphical representation.

To understand this better, let us imagine a theory of two interacting particles. At high energies they interact by means of a Yukawa potential that comes from the exchange of a heavy meson with mass M

$$V_S(r) = -g^2 \frac{e^{-Mr}}{4\pi r} \,. \tag{52}$$

However at low energies / large distances it is not possible to distinguish the particle that generates this interaction. If we have a ruler in units of a and we try to determine what type of potential there is between particles 1 and 2 we will get to the conclusion that the potential is a *contact-type interaction*. This means a potential that

$$V(\vec{r}) = 0 \quad \text{if} \quad \vec{r} \neq 0 \,. \tag{53}$$

That is, we will get to the concusion that the particles have to touch each other in oder to interact. The only way to see the internal structure of the potential V_S is to use a ruler of size $R_S \sim 1/M$, otherwise we will be unable to see any detail at all.

The mathematical way to describe a potential that is zero if $r \neq 0$, but is not-zero if $r = 0$ is with a Dirac delta. In one dimension we can describe the Dirac delta as follows:

$$\begin{aligned} &\delta(x) = 0 \quad \text{if} \quad x \neq 0 \quad \text{and} \,, \\ &\delta(x) = \infty \quad \text{if} \quad x = 0 \,. \end{aligned} \tag{54}$$

Besides this $\delta(x)$ also fulfills the condition

$$\int_{-\infty}^{+\infty} f(x)\, \delta(x)\, dx = f(0) \,. \tag{55}$$

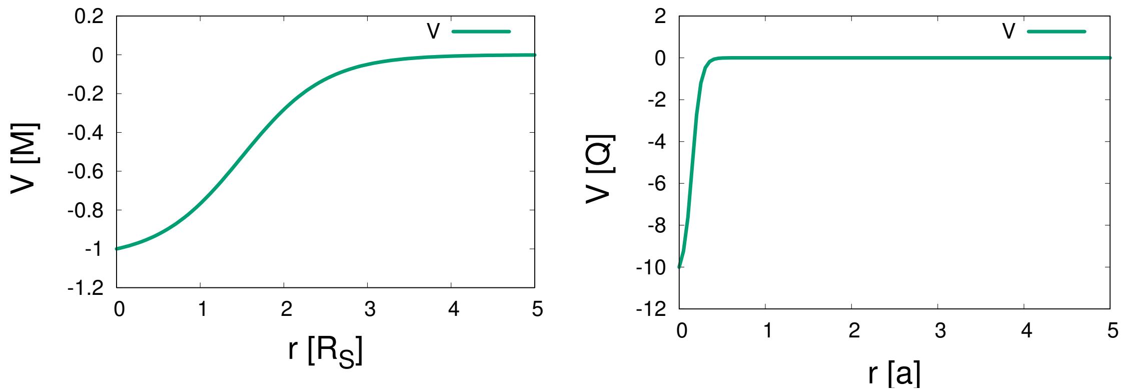


FIG. 3. The potential between two particles looked at different resolutions. In the left panel we see the potential when measured in units of M and then the distances are measure in units of R_S. In the right panel we see the potential in units of Q and where the distance is expressed in unit of a. We have taken $a = 10\, R_S$. In this second case we are not able to distinguish so much structure of the potential as in the first case.

This condition defines the $\delta(x)$ is a more concrete way. The delta can also be viewed as the limit of some different function when the range of that function goes to zero, for instance

$$\delta(x) = \lim_{R_c \to 0} \frac{\pi}{|R_c|} \frac{1}{x^2 + R_c^2} \,, \tag{56}$$

$$\delta(x) = \lim_{R_c \to 0} \frac{1}{\sqrt{\pi}|R_c|} e^{-(x/R_c)^2} \,, \tag{57}$$

to give just two examples (actually there are infinite ways to represent the delta). A quantum mechanical potential is defined in three dimensions, and hence we will need a three dimensional delta:

$$\delta^{(3)}(\vec{r}) = \delta(x)\, \delta(y)\, \delta(z) \,, \tag{58}$$

such that

$$\int d^3\vec{r} f(\vec{r}) \delta^{(3)}(\vec{r}) = f(0) \,. \tag{59}$$

Knowing this, for $a \gg R_S$ the potential that we are able to see is not the original potential, but an effective potential that looks like a δ

$$V(\vec{r}) \to V_{\rm eff}(\vec{r}) = C \delta^3(\vec{r}) \,. \tag{60}$$

However the EFT and renormalization ideas require the inclusion of a cut-off R_c. This means that we should not be using a δ, but rather some function for which the $R_c \to 0$ is a δ:

$$V_{\rm eff}(\vec{r}) \to V_{\rm eff}(\vec{r}, R_c) = C(R_c)\, \delta^3_{R_c}(\vec{r}) \quad \text{such that} \quad \lim_{R_c \to 0} \delta^3_{R_c}(\vec{r}) = \delta^3(\vec{r}) \,. \tag{61}$$

This is usually called a *smeared* delta function. Besides, notice that the strength of the effective potential is now a function of the cut-off:

$$C = C(R_c) \,. \tag{62}$$

For the three dimensional delta a few examples of a smeared delta are

$$\delta^{(3)}_{R_c}(\vec{r}) = \frac{\pi^2}{4 R_c^3} \frac{1}{(r^2 + R_c^2)^3} \,, \tag{63}$$

$$\delta^{(3)}_{R_c}(\vec{r}) = \frac{e^{-(r/R_C)^2}}{\pi^{3/2} R_C^3} \,. \tag{64}$$

$$\delta^{(3)}_{R_c}(\vec{r}) = \frac{\delta(r - R_C)}{4\pi R_C^2} \,. \tag{65}$$

where the third one is called a *delta-shell* and is particularly interesting because it simplifies calculations in an incredible way. Notice that the meaning of the $\delta(r - R_c)$ is straightforward:

$$\int_0^\infty dr f(r)\delta(r - R_c) = f(R_c)\,. \tag{66}$$

By the way, here you have an easy **exercise**: check the *normalization* of the previous smeared delta functions, i.e. the funny factors of π and powers of R_c that appear in the three expressions above.

Now we continue with the delta-shell regularization, for which the effective potential is now

$$V_{\rm eff}(r, R_C) = C(R_C)\frac{\delta(r - R_C)}{4\pi R_C^2}\,. \tag{67}$$

We can write a RGE for the coupling C in this equation as follows

$$\frac{d}{dR_C}\langle\Psi|V_{\rm eff}|\Psi\rangle = 0\,, \tag{68}$$

where Ψ is the wave function. Notice that for a wave function that is radially symmetric, i.e.

$$\Psi(\vec{r}) = \Psi(r)\,, \tag{69}$$

the evaluation of the previous matrix element is straighforward:

$$\langle\Psi|V_{\rm eff}|\Psi\rangle = C(R_c)\,|\Psi(R_c)|^2\,. \tag{70}$$

But the solution of the renormalization group equation (RGE) is not unique: it depends on the form of the wave function at long distances. There are actually two family of solutions

(i) the long range physics is perturbative or

(ii) the long range physics is non-perturbative.

In the first case the two-body system at large distances is basically a free (i.e. non-interacting) system and the two-body wave function is a free wave

$$\langle\vec{r}|\Psi\rangle = e^{i\vec{k}\cdot\vec{r}}\,, \tag{71}$$

for which we obtain

$$\langle\Psi|V_{\rm eff}|\Psi\rangle = C(R_C) + \mathcal{O}((kR_C)^2)\,. \tag{72}$$

As a consequence the RGE reads

$$\frac{d}{d\,R_C}\left[C(R_c)\right] = 0 + \mathcal{O}(k^2 R_C)\,, \tag{73}$$

with the solution

$$C(R_C) \propto 1\,. \tag{74}$$

In the second case the two-body wave function is of the type

$$\langle\vec{r}|\Psi\rangle = \frac{1}{\sqrt{4\pi}}\frac{e^{-\gamma r}}{r}\,, \tag{75}$$

which leads to the RGE

$$\frac{d}{d\,R_C}\left[\frac{C(R_c)}{R_c^2}\right] = 0 + \mathcal{O}(\gamma)\,, \tag{76}$$

and the solution

$$C(R_C) \propto R_C^2\,. \tag{77}$$

The first of these solutions is called the attractive fixed point of the renormalization group (RG), while the second is the repulsive fixed point. Alternatively, the first solution describes a natural system while the second solution describes an unnatural system.

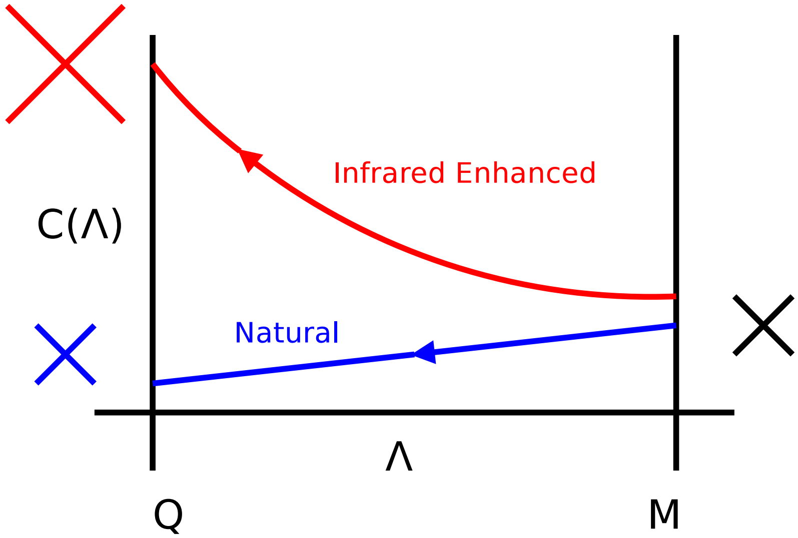


FIG. 4. Within the EFT description of low energy physics we can simplify the true potential of a system by a *smeared* delta, i.e. a contact-interaction with a range of the order of the cut-off R_c:

$$V_{\rm eff}(\vec{r}; R_c) = C(R_c)\,\delta^{(3)}_{R_c}(\vec{r})\,.$$

The strength of the effective potential is given by the coupling constant $C(R_c)$, but the RGE of this coupling with R_c is not unique. For two-body systems which at low energies behave as free particles plus corrections, the coupling evolves as a constant

$$C(R_c) \propto 1\,,$$

while for two-body systems which interact strongly at low energies the evolution is different

$$C(R_c) \propto R_c^2\,,$$

and the coupling becomes stronger and stronger as $R_c \to a$. The first case leads to a relatively simple EFT, while the second to a more complicated EFT. They are sometimes called “natural” and “unnatural” / “infrared enhanced” scaling.

The discovery of the attractive and repulsive fixed points of the type we are discussing here for two body scattering can be consulted in the original publication by Birse, McGovern and Richardson [7]. The treatment is however more involved than here because the analysis is done is momentum space and contains much more information than the scaling of the coupling $C(R_c)$ with the cut-off. A more simple explanation can be found in a recent review by myself [8], which uses a standard quantum mechanical approach in coordinate space.

Now here we can propose an interesting **exercise** (three points). Actually, the RG equation for non-perturbative systems is not the one we have written above (or the one written in the publication by Birse, McGovern and Richardson [7]). The real RG equation is the following one:

$$\frac{d}{d\,R_C}\left[\frac{C(R_c)}{R_c}\right] = 0 + \mathcal{O}(\gamma)\,, \tag{78}$$

Find an explanation of why it is this way. As a historical comment: the two versions of the RG evolution of $C(R_c)$ for non-perturbative systems, $C(R_c) \propto R_c^2$ and $C(R_c) \propto R_c$ and their interpretation have cause real headaches within the nuclear EFT community. For this exercise you do not have to go as far as making an interpretation of them, it is just enough to find how one can obtain the equation above.

III. THE ONE NUCLEON SECTOR

A. Isospin Symmetry

The proton and the neutron are spin 1/2 fermions, which masses of

$$M_p = 938.272\,\text{MeV} \quad , \quad M_n = 939.565\,\text{MeV}\,. \tag{79}$$

Their masses are indeed almost identical. Not only that, from the point of view of the nuclear force they behave in almost the same way, a feature that we have already commented goes by the name of *charge invariance*. This has prompted theoreticians to view the neutron and the proton as two states of the same particle, the nucleon

$$N = \begin{pmatrix} p \\ n \end{pmatrix}\,, \tag{80}$$

where the quantum number that makes these two states different is the *isospin*. The isospin is actually a copy of spin. In the isospin formalism the nucleon is a isospin $I = 1/2$ particle, of which the $m_I = +1/2$ state is the proton and the $m_I = -1/2$ one is the neutron, that is

$$|I = \frac{1}{2}\,, m_I = +\frac{1}{2}\rangle = |p\rangle \quad \text{and} \quad |I = \frac{1}{2}\,, m_I = -\frac{1}{2}\rangle = |n\rangle \tag{81}$$

The thing is that isospin behaves in exactly the same way as spin. If we have a system of two nucleons we can group them in isospin 0 and 1 configurations

$$|NN(I = 0, M = 0)\rangle = \frac{1}{\sqrt{2}}[|pn\rangle - |np\rangle\,, \tag{82}$$

$$|NN(I = 1, M = +1)\rangle = |pp\rangle\,, \tag{83}$$

$$|NN(I = 1, M = 0)\rangle = \frac{1}{\sqrt{2}}[|pn\rangle + |np\rangle\,, \tag{84}$$

$$|NN(I = 1, M = -1)\rangle = |nn\rangle\,. \tag{85}$$

When we include isospin symmetry, the fermionic character of the neutron and proton is taken into account by formulating a sort of extended Fermi-Dirac statistics. That is, the wave function of A nucleons should be antisymmetric. For a system of two nucleons in S-wave, this means that there are only two possible spin-isospin configurations

(i) the singlet: $S = 0$, $I = 1$ and

(ii) the triplet: $S = 1$, $I = 0$.

The deuteron, the most simple nucleus, is in the triplet state. Yet isospin symmetry requires that the interaction in the three isospin states of the singlet should be approximately the same (in particular the nn and pp interactions). In fact this happens to be the case, but this idea can also be extended to heavier nuclei. For example:

(i) the binding energies of ^3H and ^3He ($I = 1/2$) (8.48 MeV and 7.72 MeV),

(ii) the binding energies of ^6He, ^6Li and ^6Be ($I = 1$) (29.27, 31.99 and 26.92 MeV)

In the second example however we refer to the excited state of ^6Li. The three nuclei ^6He, ^6Li and ^6Be can be visualized as a ^4He core ($I = 0$) that is surrounded by a two-nucleon pair. In the case of ^6Li the two nucleon pair can be either in a isoscalar ($I = 0$) or isovector ($I = 1$) configuration, of which only the isovector one can be related to ^6He and ^6Be by isospin symmetry.

That also means that the nucleon-nucleon interaction in S-wave can be written as a sum of spin and isospin operators

$$V_{NN}(r) = (V_C + \vec{\tau}_1 \cdot \vec{\tau}_2 W_c) + \vec{\sigma}_1 \cdot \vec{\sigma}_2\,(V_S + \vec{\tau}_1 \cdot \vec{\tau}_2 W_S) + S_{12}(\hat{r})\,(V_T + \vec{\tau}_1 \cdot \vec{\tau}_2 W_T)\,. \tag{86}$$

We can now revisit the one pion exchange potential in view of the isospin formalism. The idea of the isospin is connected with the fact that the nucleon is composed of u and d quarks and can be extended to any other particle containing these quarks, including the pion. In fact we know that there are three types of pions — π^+, π^0 and π^- — and that their masses are also pretty similar:

$$m_{\pi^\pm} = 139.570\,\text{MeV} \qquad m_{\pi^0} = 134.977\,\text{MeV}\,. \tag{87}$$

As a consequence we can group the three pions into an isospin 1 state

$$\pi = \begin{pmatrix} \pi^+ \\ \pi^0 \\ \pi^- \end{pmatrix} . \tag{88}$$

This also means that we can update the interaction lagrangian between the pion and the nucleon in a more correct way

$$\mathcal{L} = i g_H \bar{N} \gamma^5 \vec{\tau} \cdot \vec{\pi} N \,, \tag{89}$$

which leads to the following OPE potential

$$V_{\rm OPE}(r) = \frac{g_H^2}{4 M_N^2} \, \vec{\tau}_1 \cdot \vec{\tau}_2 \, \left[\vec{\sigma}_1 \cdot \vec{\sigma}_2 \, W_C(r) + S_{12}(\hat{r}) \, W_T(r) \right] . \tag{90}$$

Other isovector meson is the ρ meson, which can also be arranged as

$$\rho = \begin{pmatrix} \rho^+ \\ \rho^0 \\ \rho^- \end{pmatrix} . \tag{91}$$

This also means that the contributions of the ρ meson to the nuclear force will also display a $\vec{\tau}_1 \cdot \vec{\tau}_2$ factor. Last, the σ and ω mesons are isoscalars, i.e. they are like a $|00\rangle$ vector in isospin space:

$$\sigma = |00\rangle_I \quad , \quad \omega = |00\rangle_I \,. \tag{92}$$

They will give rise to forces that do not contain a $\vec{\tau}_1 \cdot \vec{\tau}_2$ factor.

Other example of the use of isospin are nuclear reactions. For instance, if we consider the probabilities of the reactions

$$P(pp \to d\pi^+) \quad \text{and} \quad P(np \to d\pi^0) \,, \tag{93}$$

then these probabilites can be related by means of Clebsch-Gordan coefficients corresponding to the coupling of different isospins (notice that the isospin of the deuteron is $I = 0$)

$$P(pp \to d\pi^+) \propto \left| \langle \frac{1}{2}\frac{1}{2}, \frac{1}{2}\frac{1}{2} | 11 \rangle \right|^2 = 1 \,, \tag{94}$$

$$P(np \to d\pi^0) \propto \left| \langle \frac{1}{2} - \frac{1}{2}, \frac{1}{2}\frac{1}{2} | 10 \rangle \right|^2 = \frac{1}{2} \,, \tag{95}$$

from which we derive

$$\frac{P(pp \to d\pi^+)}{P(np \to d\pi^0)} = 2 \,. \tag{96}$$

B. SU(3)-Flavour Symmetry

As we know there are six types of quarks — u, d, s, s, b and t — of which only the five first ones are important for hadron physics. The top quark decays two quickly into the bottom quark and as a consequence it does not have enough time to form hadrons. Isospin symmetry rest on the idea that the u and d quarks are very light and their mass difference are actually really small in comparison with the nucleon or the pion masses. The idea of isospin can be extended to other of the quark species, the s quark, in which case we end up with flavour symmetry. However this symmetry is much more approximate in nature than isospin. While isospin symmetry is conserved at the few percent level, the violations of flavour symmetry are usually of the order of twenty to thirty percent.

C. The Linear Sigma Model and the Goldstone Theorem

Let us consider the pion from a different point of view. For that we will consider this Lagrangian proposed by Gell-Mann and Levi in the 60's [9]:

$$\mathcal{L} = i\bar{N}\gamma^\mu\partial_\mu N + g\,\bar{N}(\phi^0 + i\gamma^5\vec{\tau}\cdot\vec{\phi})N + \frac{1}{2}\sum_i \partial_\mu\phi_i\partial^\mu\phi_i - V(\phi)\,, \tag{97}$$

$$V(\phi) = \frac{1}{2}\mu^2\,(\sum_i \phi_i^2) + \frac{\lambda}{4}\,(\sum_i \phi_i^2)^2\,. \tag{98}$$

This lagrangian contain a massless nucleon field and four bosonic fields ϕ_i, of which ϕ_0 is a scalar and ϕ_i with $i = 1, 2, 3$ pseudoscalars. The lagrangian has $O(4)$ symmetry in the fields ϕ_i with $i = 0, 1, 2, 3$, that is, the transformation

$$\phi_i \to R_{ij}\phi_j \tag{99}$$

where R_{ij} are 4x4 matrices such that $R^T R = 1$ leaves the lagrangian unchanged. As a reminder, the goup $O(n)$ is the group of nxn matrices such that

$$O(n) = \left\{R\ (\text{nxn})\text{ such that } R^T R = 1\right\}\,, \tag{100}$$

and it has $n(n-1)/2$ independent generators.

If we take $\mu^2 > 0$ (as usual) and $\lambda > 0$ (so we avoid vacuum decay), we have that the four bosonic fields are massive and their mass is μ. The reason is, of course, the mass term

$$\mathcal{L}_{\text{mass}} = -\frac{1}{2}\mu^2\,(\sum_i \phi_i^2)\,. \tag{101}$$

In addition they have a four boson interaction term. However if $\mu^2 < 0$ something interesting happens. The potential $V(\phi)$ now has a minimum at

$$\phi_0 = v = \sqrt{-\frac{\mu^2}{\lambda}}\,. \tag{102}$$

We can also make the following change of variables

$$\sigma = \phi_0 - v\,, \tag{103}$$

$$\pi_i = \phi_i\,, \tag{104}$$

from which we obtain (do it as **exercise**)

$$\mathcal{L} = \bar{N}(i\gamma^\mu\partial_\mu - gv)N + g\,\bar{N}(\sigma + i\gamma^5\vec{\tau}\cdot\vec{\pi})N + \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma + \frac{1}{2}\partial_\mu\vec{\pi}\cdot\partial^\mu\vec{\pi} - V(\sigma,\pi)\,, \tag{105}$$

$$V(\sigma,\pi) = \lambda v^2\sigma^2 + v\lambda\sigma^3 + \frac{\lambda}{4}(\sigma^4 - v^4) + \frac{\lambda}{4}\vec{\pi}^4 + v\lambda\vec{\pi}^2\sigma + \frac{\lambda}{2}\vec{\pi}^2\sigma^2\,. \tag{106}$$

This lagrangian is interesting for a series of reasons:

(a) First, we can see that the original $O(4)$ symmetry is spontaneously broken to $O(3)$, which only applies to the $\vec{\pi}$ field.

(b) Second, originally all the fields have equal mass $\mu^2 > 0$. But if we have $\mu^2 < 0$, the σ field is massive and has a mass of $\lambda v^2 = -\mu^2$ while the $\vec{\pi}$ fields are massless.

(c) Third, the N field now has a mass of $M_N = g\,v$

(d) Fourth, for the sake of clarity let's call v by its more common name, f_π: $v = f_\pi$, with $f_\pi = 92.4\,\text{MeV}$.

(e) Fifth, this model implies $g_\sigma = g = M_N/v = 10.2$ and $g_{\pi NN} = g_\sigma \sim 10.2$, which (although we have not explained it) happens to be really accurate. The actual $g_{\pi NN} = 13.4$ is a $25-30\%$ larger than the prediction done here as a consequence that the nucleon couples to pseudoscalars in a slightly different way than to scalar. In particular by making the correction

$$\bar{N}(\phi_0 + i\gamma_5\vec{\tau}\cdot\vec{\phi})N \to \bar{N}(\phi_0 + ig_A\gamma_5\vec{\tau}\cdot\vec{\phi})N\,, \tag{107}$$

with $g_A \simeq 1.26$ we can obtain the correct coupling to the pion (g_A is called the axial coupling).

Other interesting variation of this lagrangian is to include a small term in the original $O(4)$ symmetric lagrangian that breaks $O(4)$ symmetry, like for example

$$\Delta V(\phi) = -\frac{\epsilon}{v}\phi_0\,. \tag{108}$$

This term is interesting because it will slightly shift the mass of the σ meson and give a tiny mass to the pions

$$m_\sigma^2 = \lambda v^2 + \frac{\epsilon}{v}\,, \tag{109}$$

$$m_\pi^2 = \frac{\epsilon}{v}\,. \tag{110}$$

Yet the final $O(3)$ symmetry remains in the final lagrangian. As we will see, once we explain chiral symmetry, this is indeed very similar to what happens in the real world.

1. *A Brief Overview of the Goldstone Theorem*

The mechanism by which this lagrangian generates a series of massless bosons after a symmetry is spontaneously broken is the Goldstone theorem. A more abstract version than the example we have use is the following, which is taken from some lectures notes by Leutwyler [10]. Assume we have a hamiltonian H that is invariant under a Lie group G. If we denote the generators of the group G as Q_i, with $i = 1, \ldots, n_G$, with n_G the dimension of the Lie group, what we have is the following

$$[Q_i, H] = 0\,. \tag{111}$$

This symmetry is spontaneously broken if the ground state of H – the vacuum – is not invariant under G. That this, there are a few generators for which

$$Q_i|0\rangle \neq 0\,. \tag{112}$$

This implies that the vacuum is not unique: $[Q_i, H] = 0$ and therefore $Q_i|0\rangle$ describes a state with the same energy as the vacuum. Yet there will be a subset of Q_i that leave the vacuum invariant

$$J_i|0\rangle = 0\,. \tag{113}$$

These generators obey $[J_i, J_k] = 0$ and span a subgroup H of the original Lie group G. If the dimension of H is n_H, then there are still $n_G - n_H$ generators in the quotient group G/H that do not annihilate the vacuum. If we give the name K_i to the generators of G/H, then we have

$$K_i|0\rangle\,, \tag{114}$$

linearly independent states with the same energy as the fundamental states. These $n_G - n_H$ states are the Goldstone bosons.

Notice that in the example of the linear sigma model, $G = O(4)$ and $H = O(3)$. The dimension of $O(n)$ is $n(n-1)/2$, with means that $n_G = 6$ and $n_H = 3$. As a consequence we end up with $n_G - n_H = 3$ Goldstone bosons.

D. Chiral Symmetry

1. *The Massless Fermion*

Let us consider a Lagrangian containing a fermion field

$$\mathcal{L} = \bar{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi\,. \tag{115}$$

This lagrangian has a global $U(1)$ symmetry, which means that it is invariant under a change of phase of the field Ψ

$$\Psi(x) \rightarrow e^{i\alpha}\Psi(x)\,. \tag{116}$$

But when the fermion is massless, the lagrangian has an additional symmetry that goes by the name of chiral symmetry. This has to do with what are the right and left hand components of the field Ψ

$$\Psi_R = \frac{1}{2}\left(1 + \gamma_5\right)\Psi \tag{117}$$

$$\Psi_L = \frac{1}{2}\left(1 - \gamma_5\right)\Psi \tag{118}$$

In terms of the L and R fields, it happens that the derivative term in the lagrangian does not mix them together

$$\bar{\Psi}\, i\,\gamma^\mu \partial_\mu\, \Psi = \bar{\Psi}_L\, i\,\gamma^\mu \partial_\mu\, \Psi_L + \bar{\Psi}_R\, i\,\gamma^\mu \partial_\mu\, \Psi_R\,. \tag{119}$$

Meanwhile the mass term mixes terms of opposite chirality

$$\bar{\Psi}\, m\, \Psi = \bar{\Psi}_L\, m\, \Psi_R + \bar{\Psi}_R\, m\, \Psi_L\,, \tag{120}$$

which is why chiral symmetry is only a symmetry is the fermion is massless. In such a case there is a new global $U(1)$ chiral symmetry

$$\Psi(x) \to e^{i\alpha_5 \gamma_5}\Psi(x)\,. \tag{121}$$

Alternatively we can say that the fields Ψ_R and Ψ_L are each one invariant under a global $U(1)$ symmetry

$$\Psi_R \to e^{i\alpha_R}\Psi_R \quad , \quad \Psi_L \to e^{i\alpha_R}\Psi_L\,, \tag{122}$$

which can also be illuminating, as we will see.

2. The Massless Quarks

This idea apply after certain modifications to QCD. As already seen the QCD Lagrangian reads

$$\mathcal{L} = -\frac{1}{4}\, G^a_{\mu\nu}\, G^{a\mu\nu} + \bar{q}\, i\,\gamma^\mu D_\mu\, q - \bar{q}\mathcal{M}q \tag{123}$$

where we have written q as a vector

$$q = \begin{pmatrix} u \\ d \\ s \\ c \\ b \\ t \end{pmatrix}\,, \tag{124}$$

with $\mathcal{M}$ the quark mass matrix

$$\mathcal{M} = \begin{pmatrix} m_u & & & & & \\ & m_d & & & & \\ & & m_s & & & \\ & & & m_c & & \\ & & & & m_b & \\ & & & & & m_t \end{pmatrix}\,. \tag{125}$$

It happens that three of the quarks masses — m_u, m_d and m_s — are smaller than Λ_{QCD}, which is the natural scale of QCD. In particular two of the masses, m_u and m_d, are really small in comparison with Λ_{QCD}. As a consequence we expect that the approximation that the masses of these quarks are zero will be pretty good.

We can also define the L and R components of the quark fields as usual

$$q_R = \frac{1}{2}\,(1+\gamma_5)\,q\,, \tag{126}$$

$$q_L = \frac{1}{2}\,(1-\gamma_5)\,q\,, \tag{127}$$

where the derivative terms respect this decomposition

$$\bar{q}\, i\,\gamma^\mu D_\mu\, q = \bar{q}_L\, i\,\gamma^\mu D_\mu\, q_L + \bar{q}_R\, i\,\gamma^\mu D_\mu\, q_R \tag{128}$$

but the mass term does not

$$\bar{q}\, m_q\, q = \bar{q}_L\, m_q\, q_R + \bar{q}_R\, m_q\, q_L\,. \tag{129}$$

In the limit when N of the quark masses are zero, the lagrangian becomes invariant under the following transformations

$$q_R \to V_R q_R \quad \text{and} \quad q_L \to V_L q_L \tag{130}$$

where V_R and V_L belong to the special unitary group $U(N)$. That is, the Lagrangian is invariant under the group

$$G = U(N)_L \times U(N)_R \,. \tag{131}$$

It turns out however that in QCD there is something called the axial anomaly, which implies that the axial current $J^5_\mu = \bar{q}\gamma_\mu\gamma_5 q$ fails to be conserved

$$\partial^\mu J^5_\mu \neq 0 \,. \tag{132}$$

As a consequence the Lagrangian is only invariant under the group

$$G = SU(N)_L \times SU(N)_R \times U(1)_{L+R} \,. \tag{133}$$

It happens however that this symmetry is spontaneously broken: the vacuum is only invariant under the vector transformations of this group, i.e. by the subgroup

$$H = SU(N)_{L+R} \times U(1)_{L+R} \,. \tag{134}$$

This can be seen from the fact that positive and negative parity hadrons have different masses: for example, the nucleon $J^P = \frac{1}{2}^+$ has a mass of 940 MeV, while the lightest $J^P = \frac{1}{2}^-$ nucleon-like baryon happens at 1535 MeV. The ρ and ω mesons, with $J^{PC} = 1^{--}$ and masses of 770 MeV and 780 MeV respectively, have axial partners ($J^{PC} = 1^{+-}$) at 1230 and 1170 MeV respectively, the $b_1(1235)$ and $h_1(1170)$. In general the negative parity states are heavier, with the exception of the pion as a consequence of the chiral symmetry breaking we are explaining here. The fact that the vacuum is independent under the subgroup H instead of G, implies that the spectrum must contain $N^2 - 1$ Goldstone bosons which are related to the quotient group

$$\frac{G}{H} \simeq SU(N)_{L-R} \,, \tag{135}$$

These Goldstone bosons are related to axial currents and as a consquence they happen to have non-natural parity [1]: they end up being pseudoscalar mesons with $J^P = 0^-$.

In the real world the quark masses are

$$m_u = 2.3 \pm 0.7\,\text{MeV}\,, \tag{136}$$
$$m_d = 4.8 \pm 0.7\,\text{MeV}\,, \tag{137}$$
$$m_s = 95 \pm 5\,\text{MeV}\,, \tag{138}$$
$$m_c = 1.28 \pm 0.03\,\text{GeV}\,, \tag{139}$$
$$m_b = 4.18 \pm 0.04\,\text{GeV}\,. \tag{140}$$

What does it set a quark mass as light or heavy? The comparison with $\Lambda_{\text{QCD}} \sim 200 - 300\,\text{MeV}$, which is the natural scale of QCD. We have basically two options: to make the approximation of $m_u = m_d = 0$ or to include also the strange quark $m_u = m_d = m_s = 0$. The first option is $SU(2)$ chiral symmetry and the second $SU(3)$ chiral symmetry.

Now, what does this implies? The first thing is that in the limit of $m_q = 0$, the $SU(2) \times SU(2)$ symmetry is spontaneously broken. Therefore by the Golstone theorem there should be $N^2 - 1 = 3$ massless pseudo scalar bosons corresponding to this breaking of the symmetry. It happens the pions (π^+, π^0 and π^-) with a mass of about 140 MeV and 135 MeV for the charged and neutral cases respectively, are much lighter than any other hadron. Owing to the finite m_u and m_d masses these bosons adquire a finite mass. Indeed there is a formula for their mass, which is called the Gell-Mann, Oakes, Renner relation

$$m_\pi^2 = 2\,B\,(m_u + m_d) + \mathcal{O}(m_q^2)\,, \tag{141}$$

where B is given by

$$B = \left| \frac{\langle 0|u\bar{u}|0\rangle}{f_\pi^2} \right|, \tag{142}$$

with $\langle 0|u\bar{u}|0\rangle \sim -(250\,\text{MeV})^3$ a quantity called the quark condensate and $f_\pi = 92.4\,\text{MeV}$ the pion decay constant.

Now if we extend the approximation $m_q = 0$ to the strange quark we arrive at $SU(3)$ chiral symmetry. Next to the pions, we have another five pseudoscalars that are particularly light: the kaons, the antikaons and the η with a mass

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[1] Natural parity refers to the series 0^+, 1^-, 2^+, 3^- and so on.

of about 495 MeV for the kaons and 545 MeV for the η. Together with the pions these are the lightest hadrons and in total they sum eight pseudoscalar with fits neatly with the idea that they are Goldstone bosons, as $N^2 - 1 = 8$ for $N = 3$.

The other very important aspect of chiral symmetry is that the pion-nucleon interaction has to be derivative. We previously arrived to the conclusion that the pion is a $J^P = 0^-$ pseudoscalar boson, that interacts with the nucleon by means of the lagrangian

$$\mathcal{L} = ig\bar{\Psi}_N \gamma^5 \vec{\tau} \cdot \vec{\pi} \Psi_N \,. \tag{143}$$

The Goldstone-boson nature of the pions actually implies that the correct interaction needs to include a derivative (a ∂^μ) of the pion field. The correct Lagrangian is

$$\mathcal{L} = \frac{g_A}{2f_\pi} \, \bar{\Psi}_N \gamma^5 \gamma_\mu \vec{\tau} \cdot \partial^\mu \vec{\pi} \Psi_N \,. \tag{144}$$

This little correction, though it leads to exactly the same type of OPE potential, is actually incredibly important when calculating the two pion exchange contributions to the nuclear forces. With the original pre-chiral lagrangian, the two pion exchange potential does not really work. But once we include chiral symmetry, the resulting two pion exchange potential indeed works and is able to describe the intermediate range region of the nuclear force.

E. Exercises

(1) Show that for a potential of the type

$$V(r) = -g_H^2 \left[a \, \vec{\sigma}_1 \cdot \vec{\sigma}_2 \, W_C(r) \pm b \, S_{12}(\hat{r}) \, W_T(r) \right] .$$

for which $a, b > 0$, then the sign of the quadrupole moment is the same as the sign between the central and tensor piece, i.e.

$$Q = \pm |Q| \,.$$

For showing this, first take into account that $Q \neq 0$ requires that the spin of nucleons 1 and 2 must add up to 1. If we consider the total spin $\vec{S} = 2\,(\vec{\sigma}_1 + \vec{\sigma}_2)$, this corresponds to taking σ_1 paralell to σ_2. **(two points)**

(2) As already explained, the standard Dirac-delta in three dimensions:

$$\delta^{(3)}(\vec{r}) \,,$$

is rather inconvenient to use. For this reason we usually *smear* the delta, i.e. we make it a bit broader by including a cutoff

$$\delta^{(3)}(\vec{r}) \to \delta^{(3)}(\vec{r}; R_c) \,.$$

One example is

$$\delta^{(3)}(\vec{r}; R_c) = \frac{e^{-(r/R_c)^2}}{\pi^{3/2} R_c^3} \,. \tag{145}$$

Show that the normalization of this smeared delta is indeed the correct one, that is,

$$\int d^3\vec{r} \, \delta^{(3)}(\vec{r}; R_c) = 1 \,. \tag{146}$$

(one point)

(3) Running of the coupling constant: consider a delta-potential of the type

$$V_C(\vec{r}) = C\,\delta^{(3)}(\vec{r})\,,$$

which we regularize as

$$V_C(\vec{r};R_c) = \frac{C(R_c)}{\frac{4}{3}\pi R_c^3}\theta(R_c - r)\,, \tag{147}$$

that is, we regularize it as a square well. We want this potential to reproduce a bound state with binding energy

$$E_B = -B = -\frac{\gamma^2}{2\mu}\,, \tag{148}$$

with μ the reduced mass of the two-body system.. Show the explicit running of $C(R_c)$ with respect to R_c and γ, that is, how $C(R_c)$ depends on R_c and γ. Show in particular that for $\gamma R_c \ll 1$:

$$C(R_c) \propto 1/R_c\,. \tag{149}$$

(three points)

(4) We have derived the RGE from the condition

$$\frac{d}{dR_c}\langle\Psi|V(R_c)\Psi\rangle = 0; \tag{150}$$

where V is the (effective) potential and Ψ the wave function. For two-body systems with a bound state at low energies, the wave function can be written as

$$\Psi(r) = \frac{A_S}{\sqrt{4\pi}}\frac{e^{-\gamma r}}{r}\,, \tag{151}$$

which for a contact-range potential, i.e. a potential that is a regularized delta

$$V(\vec{r};R_c) = C(R_c)\,\delta^{(3)}(\vec{r};R_c)\,, \tag{152}$$

leads to the following equation for $C(R_c)$

$$\frac{d}{dR_c}\left[R_c^2 C(R_c)\right] \simeq 0\,, \tag{153}$$

from which we arrive at

$$C(R_c) \propto \frac{1}{R_c^2}\,. \tag{154}$$

However, if you have done the previous exercise, we obtained that for a system with a low energy bound state

$$C(R_c) \propto \frac{1}{R_c}\,. \tag{155}$$

Why are these two results different? What is the "mistake" that has been done when obtaining $C(R_c) \propto 1/R_c^2$? **(three points)**

(5) In the linear σ model, the potential in the original Lagrangian is

$$V(\phi) = \frac{1}{2}\mu^2 \sum_i \phi_i^2 + \frac{\lambda}{4}\left(\sum_i \phi_i^2\right)^2 . \tag{156}$$

Show that after the change of variables

$$\sigma = \phi_0 - v \quad \text{and} \quad \vec{\pi} = \vec{\phi} , \tag{157}$$

with $v = \sqrt{-\mu^2/\lambda}$ and after rearranging the mass term of the σ, we end up with the potential

$$V(\sigma, \vec{\pi}) = ... \tag{158}$$

(one point)

(6) In the linear σ model, we can add a small term in the potential that breaks the original $O(4)$ symmetry

$$\Delta V(\phi) = -\frac{\epsilon}{v}\,\phi_0 , \tag{159}$$

with ϵ a small parameter. Show that after the change of variables to the σ and $\vec{\pi}$ fields, the mass of the σ changes slightly, while the pion acquires a finite mass

$$m_\sigma^2 = \lambda v^2 + \frac{\epsilon}{v} \quad \text{and} \quad m_\pi^2 = \frac{\epsilon}{v} . \tag{160}$$

(two points)

(7) There is a second version of the σ model that is called the non-linear σ model. What is effectively done is this model is to take $\mu^2 \to -\infty$ but letting $v = \sqrt{-\mu^2/\lambda}$ fixed. This is equivalent to the condition

$$\phi_0^2 + \vec{\phi}^2 = v^2 . \tag{161}$$

After making the identification $\vec{\phi} = \pi$, derive the interaction lagrangian between (i) the nucleon and the pions and (ii) the pions alone, and check the differences with respect to the standar linear σ model. **(three points)**

(8) The running of the electromagnetic coupling is given by

$$\alpha(Q^2) = \frac{\alpha(\mu^2)}{1 - \frac{\alpha^2(\mu^2)}{3\pi}\,\log\frac{Q^2}{\mu^2}} , \tag{162}$$

where Q^2 and μ^2 are the energies of the photon emitted by the electron. If you have studied quantum field theory, derive the previous result. **(three points)**

(9) Derive the running of the strong coupling

$$\alpha_s(Q^2) = \frac{\alpha_s(\mu^2)}{1 + \frac{\alpha_s(\mu^2)}{12\pi}\,(33 - 2n_f)\,\log\frac{Q^2}{\mu^2}} , \tag{163}$$

where n_F is the number of flavours (the number of different quark fields). **(three points)**

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