NUCLEAR PHYSICS 19

a) THE LIQUID DROP MODEL ✓

b) THE SHELL MODEL ✓

RECAP!

NUCLEAR MODELS:

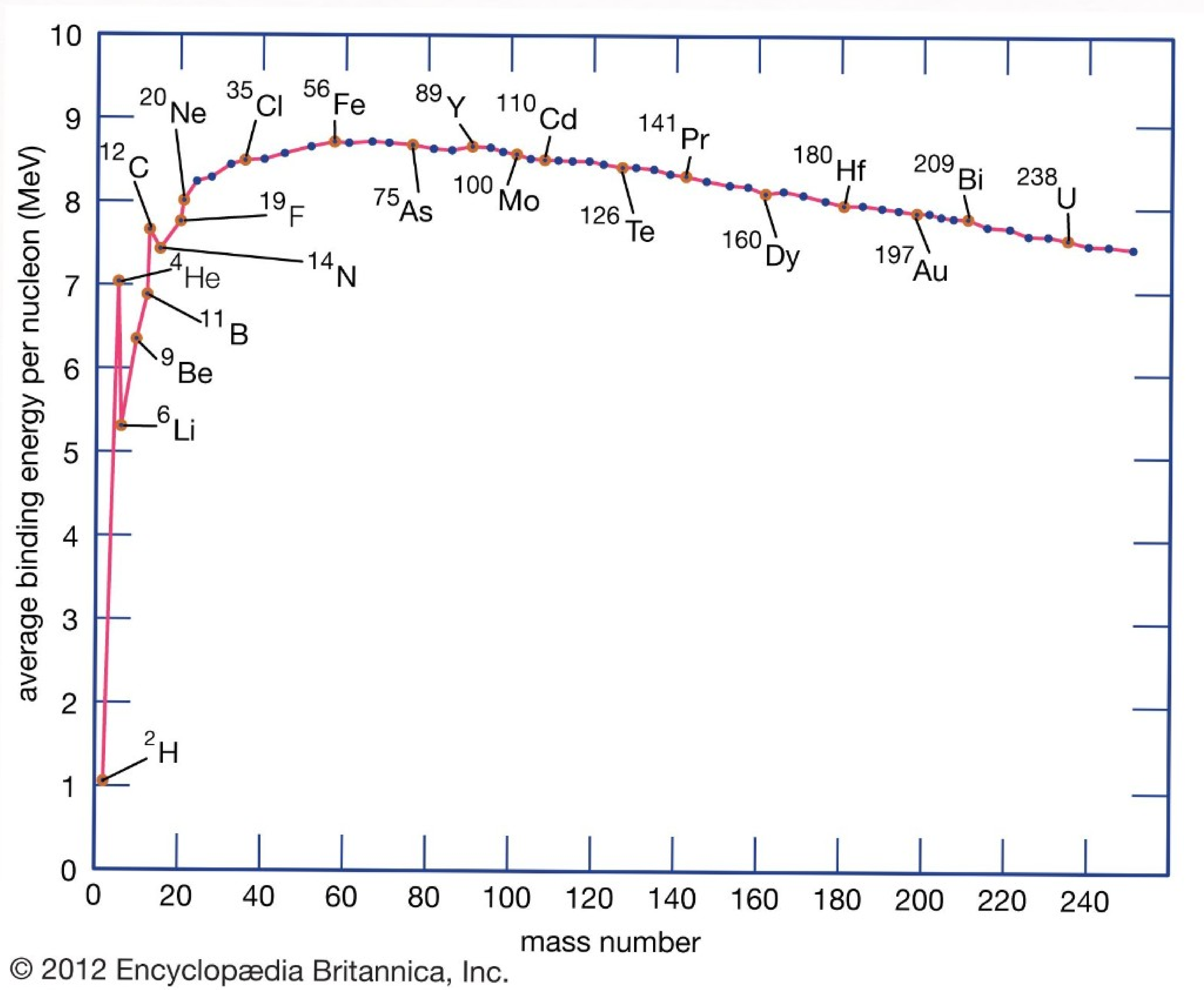
[Describe the qualitative & quantitative properties of nuclei, including:]

1) Nuclear binding $\Rightarrow$ $\underline{\underline{\frac{B}{A}}} \sim (8-9)$ MeV / nucleon
(saturation)
$\Downarrow$
Liquid drop model
2) Nuclear size
3) Angular momentum & parity
4) Electromagnetic momenta
5) Stability & Decay

LIQUID DROP MODEL (1)

=D It's main feature is that it explains binding (and saturation)

B/A



A

=D Why do we have saturation?

1) Finite-range of the nuclear force

2) Strong medium-range attraction

3) Stronger short-range repulsion

LIQUID DROP MODEL (2)

property of:

1) finite-range interaction
2) medium-range attraction
3) short-range repulsion

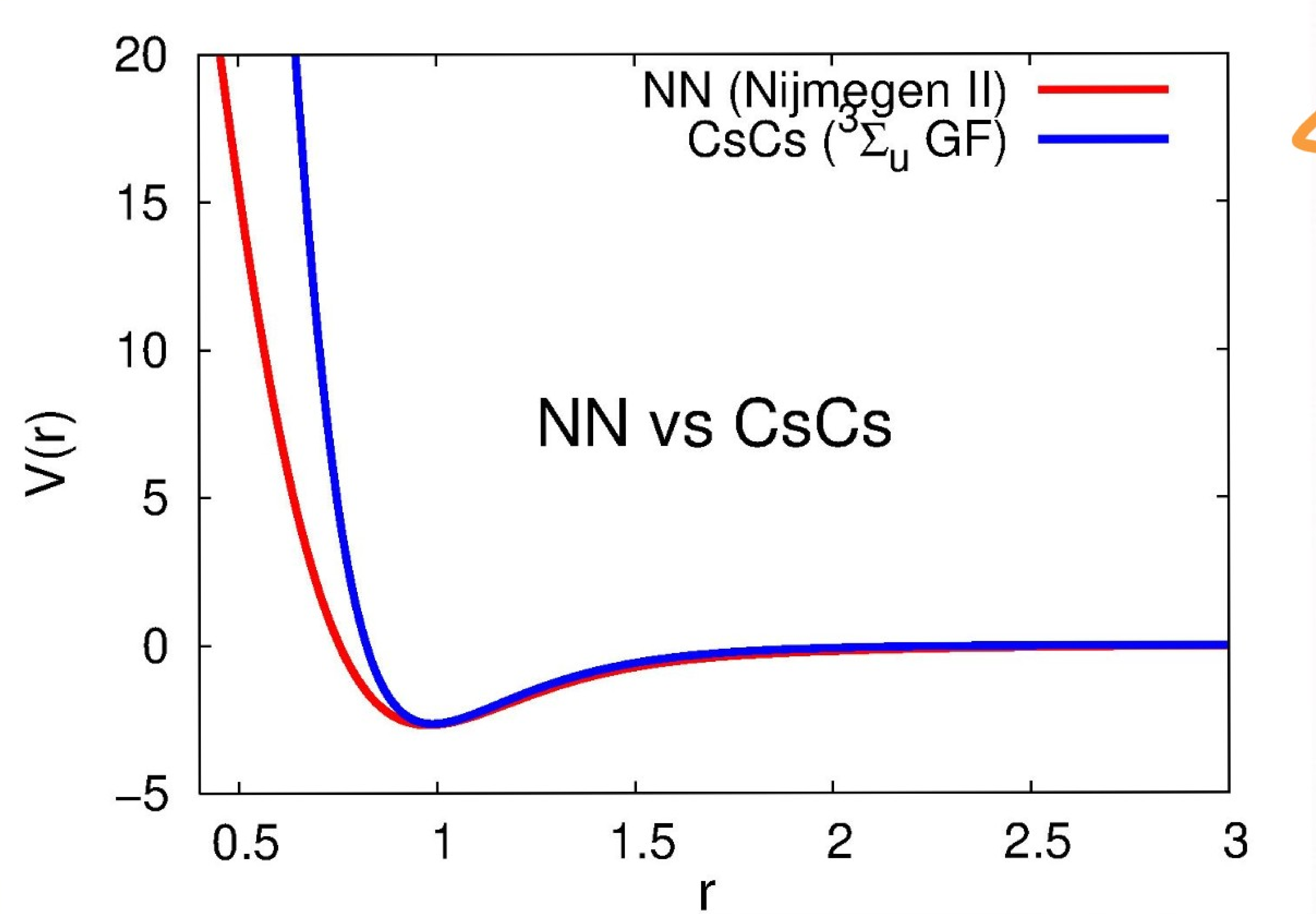
a) nuclear force
b) atomic forces (atom - atom interactions)

analogy

$\Rightarrow$ $\exists$ an analogy between a) & b),
and we can exploit it to create a nuclear model

LIQUID DROP MODEL ③

⇒ How good is the analogy between atomic & nuclear forces?



⇐ Example:

nucleon-nucleon interaction vs Cs-Cs interaction (arbitrary units)

⇒ pretty similar

LIQUID DROP MODEL (4)

⇒ How do we exploit this analogy?

$$\underbrace{\text{Group of atoms}}_{\substack{\text{they can form} \\ \text{a drop of a} \\ \text{liquid}}} \leftrightarrows \underbrace{\text{Group of nucleons}}_{\substack{\text{they will also form} \\ \text{a droplet of some} \\ \text{kind of nuclear liquid}}}$$

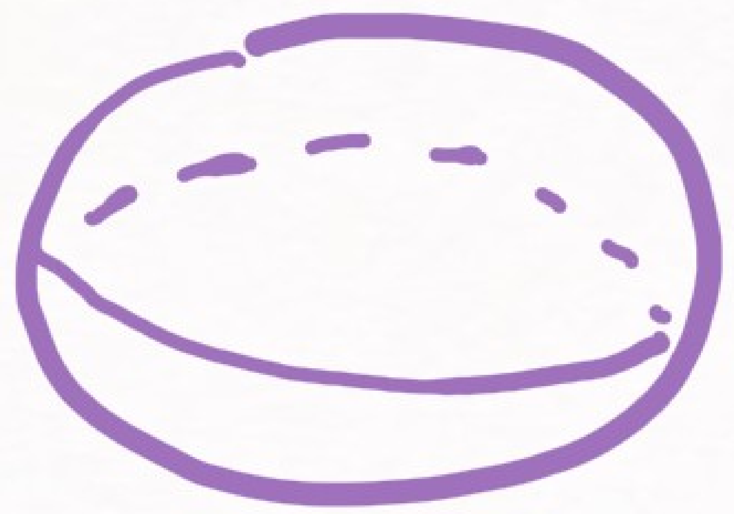
⇓

Derive properties of nuclei

LIQUID DROP MODEL (5)

⇒ We want to calculate $B(Z,A)$, how do we proceed?

1) Volume term: $$B(Z,A) = a_V A + \dots$$



$V \propto A$ ($A \rightarrow$ # of nucleons)

2) Surface term: $$B(Z,A) = a_V A - a_S A^{2/3} + \dots$$

$S \propto A^{2/3}$

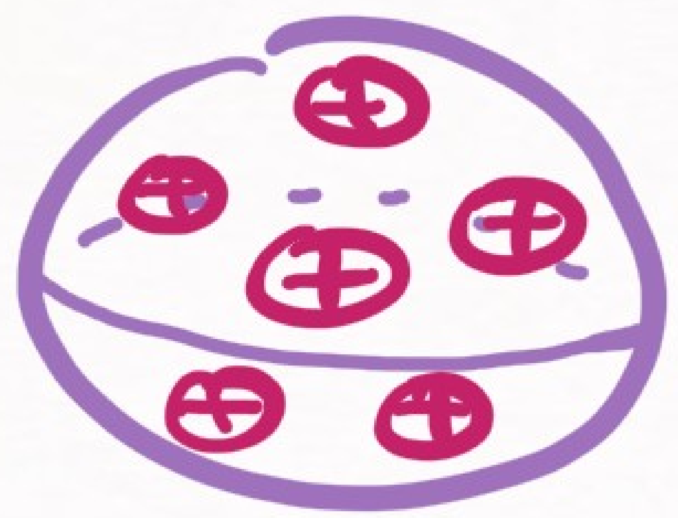
(akin to surface tension)

negative because liquids like to minimize the surface

LIQUID DROP MODEL ⑥

⟹ We want to calculate $B(Z,A)$, how do we proceed?

3) Coulomb term : $B(Z,A) = a_V A - a_S A^{2/3} - a_C \dfrac{Z(Z-1)}{A^{1/3}} + \cdots$



$q \propto Z$ ($Z \to$ # of protons)

1+2+3) → easy to understand

4+5) → they are going to be specific to nuclear physics

LIQUID DROP MODEL (7)

⇒ We want to calculate $B(Z,A)$, how do we proceed?

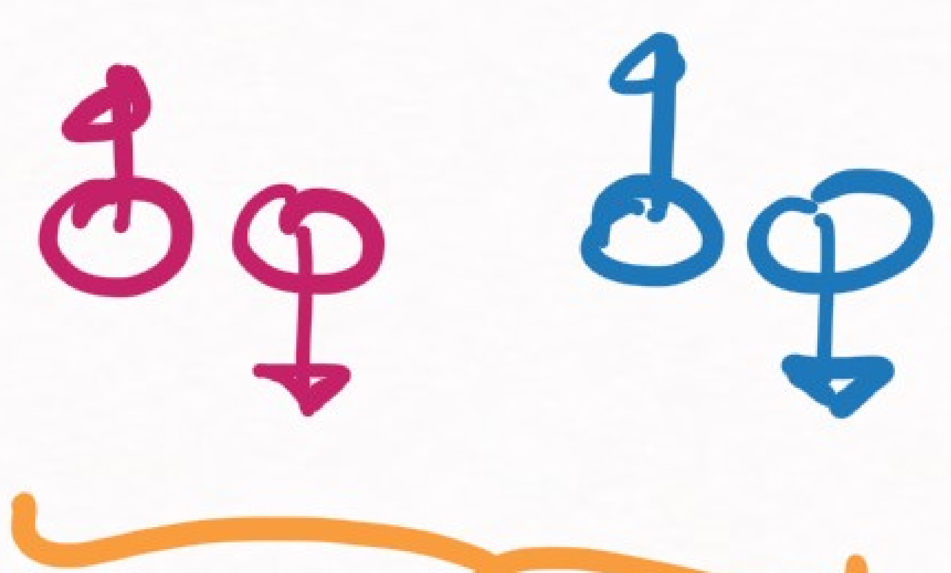
4) Asymmetry term: → nuclei like to have the same number of neutrons & protons

$$B(Z,A) = (\dots) - a_A \underbrace{\frac{(Z - A/2)^2}{A}} + \dots$$

⇒ min corresponds to $Z = N$

⇒ $(Z - A/2)^2 = 0$

5) Pairing term:



Remember these? (pairings)

$$B(Z,A) = (\dots) + a_P \underbrace{\frac{[(-1)^Z + (-1)^N]}{2A^{1/2}}}$$

min corresponds to Z, N even

LIQUID DROP MODEL ③

1) + 2) + 3) + 4) + 5) $\Rightarrow$ Liquid drop model

or

Semi-empirical mass formula

or

Bethe-Weizsäcker formula

Typical values:

$a_V \simeq 16$ MeV

$a_S \simeq 18$ MeV

$a_C \simeq 0.7$ MeV

$a_A \sim 23$ MeV

$a_P \sim 11$ MeV

(but $\exists$ many fits)

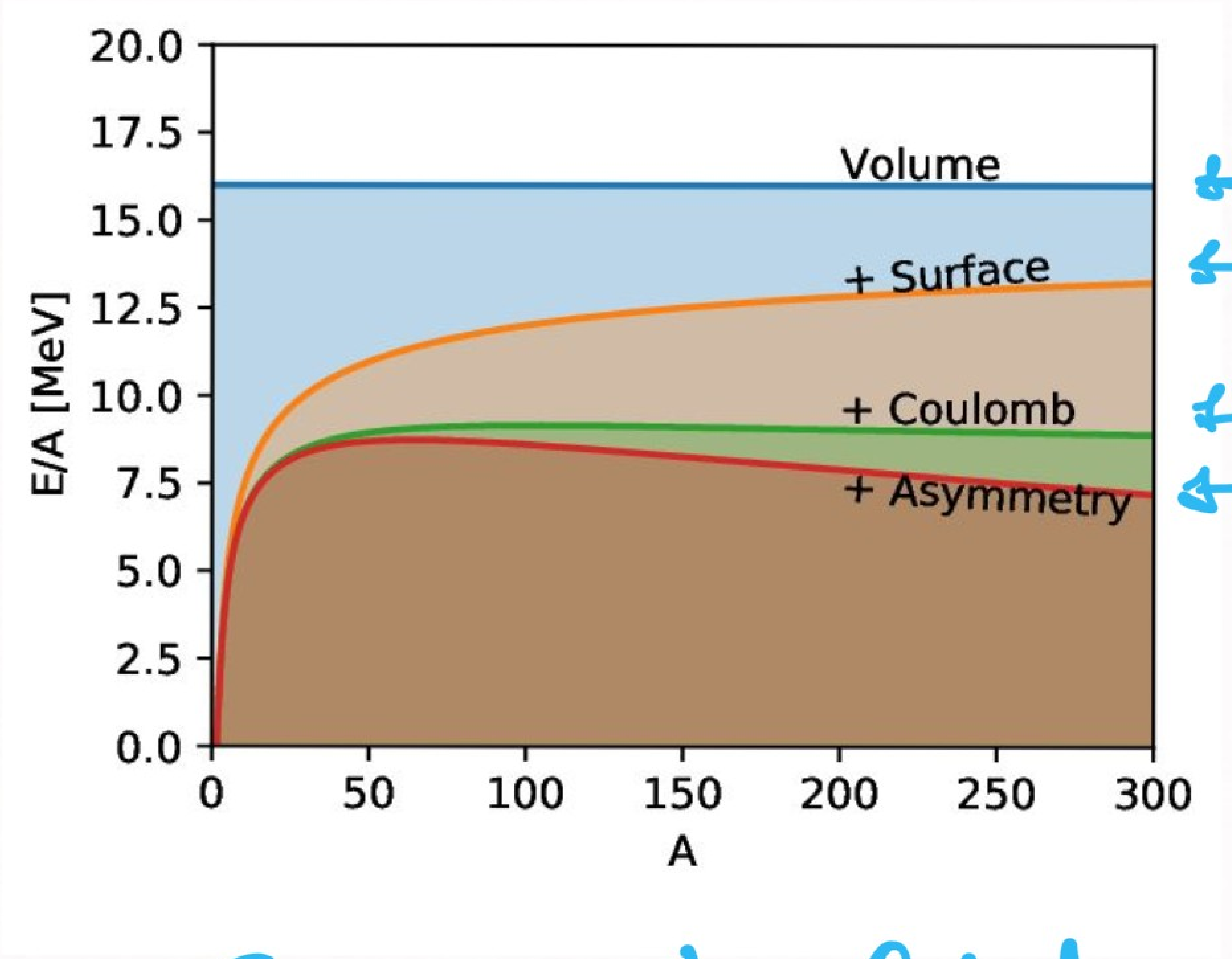
all of them are around those values

$$B(Z,A) = a_V A + a_S A^{2/3} + a_C \frac{Z(Z-1)}{A^{1/3}} - a_A \frac{(Z-N)^2}{A} + a_P \frac{[(-1)^Z + (-1)^N]}{2A^{1/2}}$$

LIQUID DROP MODEL ⑨

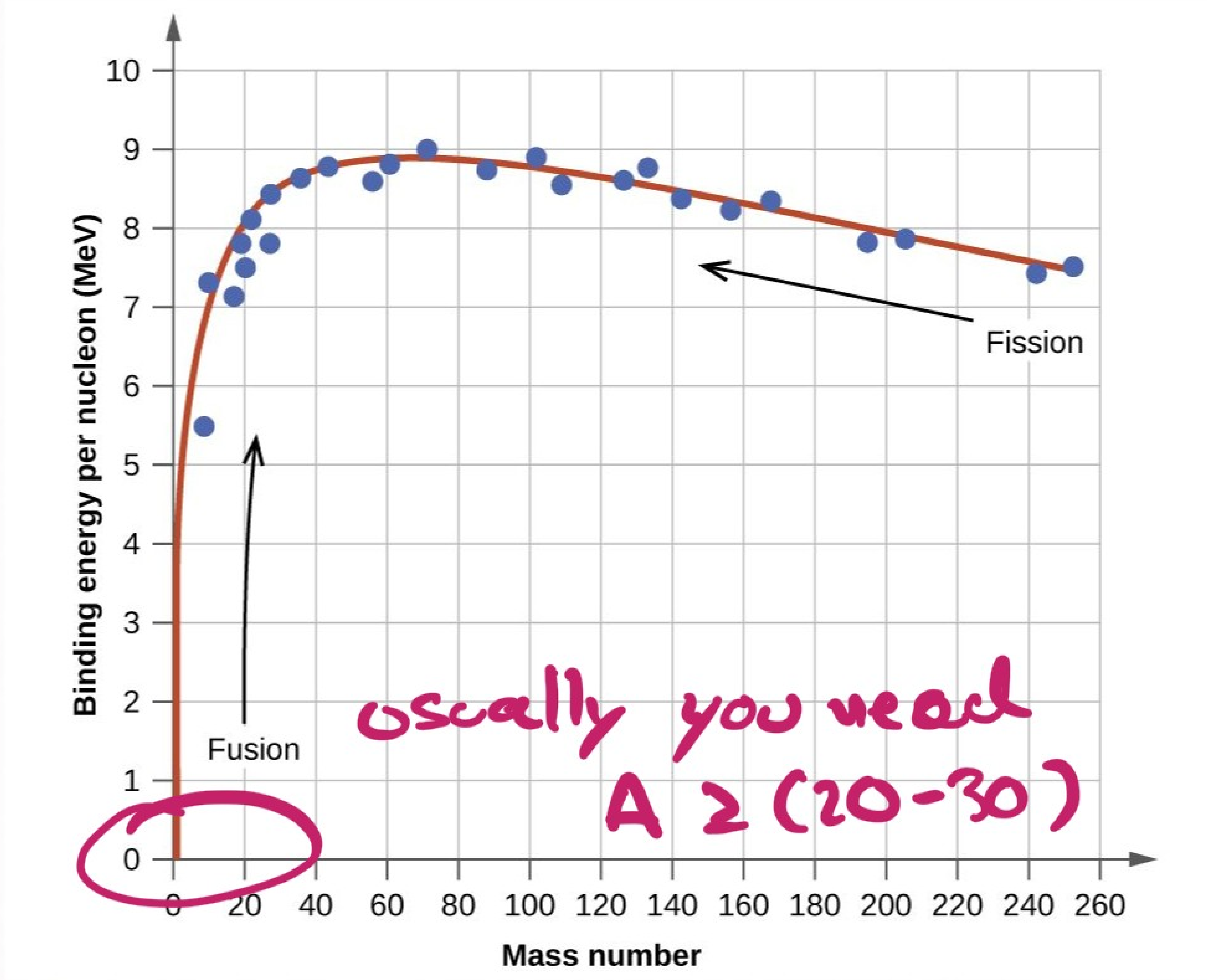
B(Z,A)/A right with an error of (0.5 - 1.0) MeV

⇒ A graphical representation:



(pairing not included in the figure)

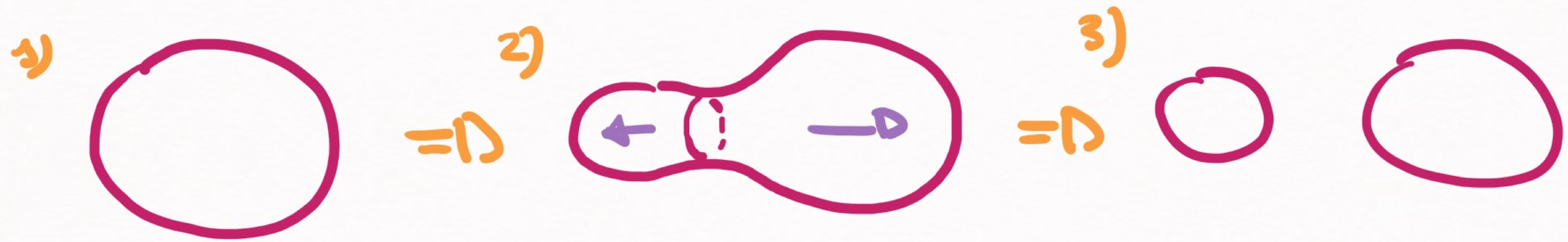
⇒ Also explains fission:



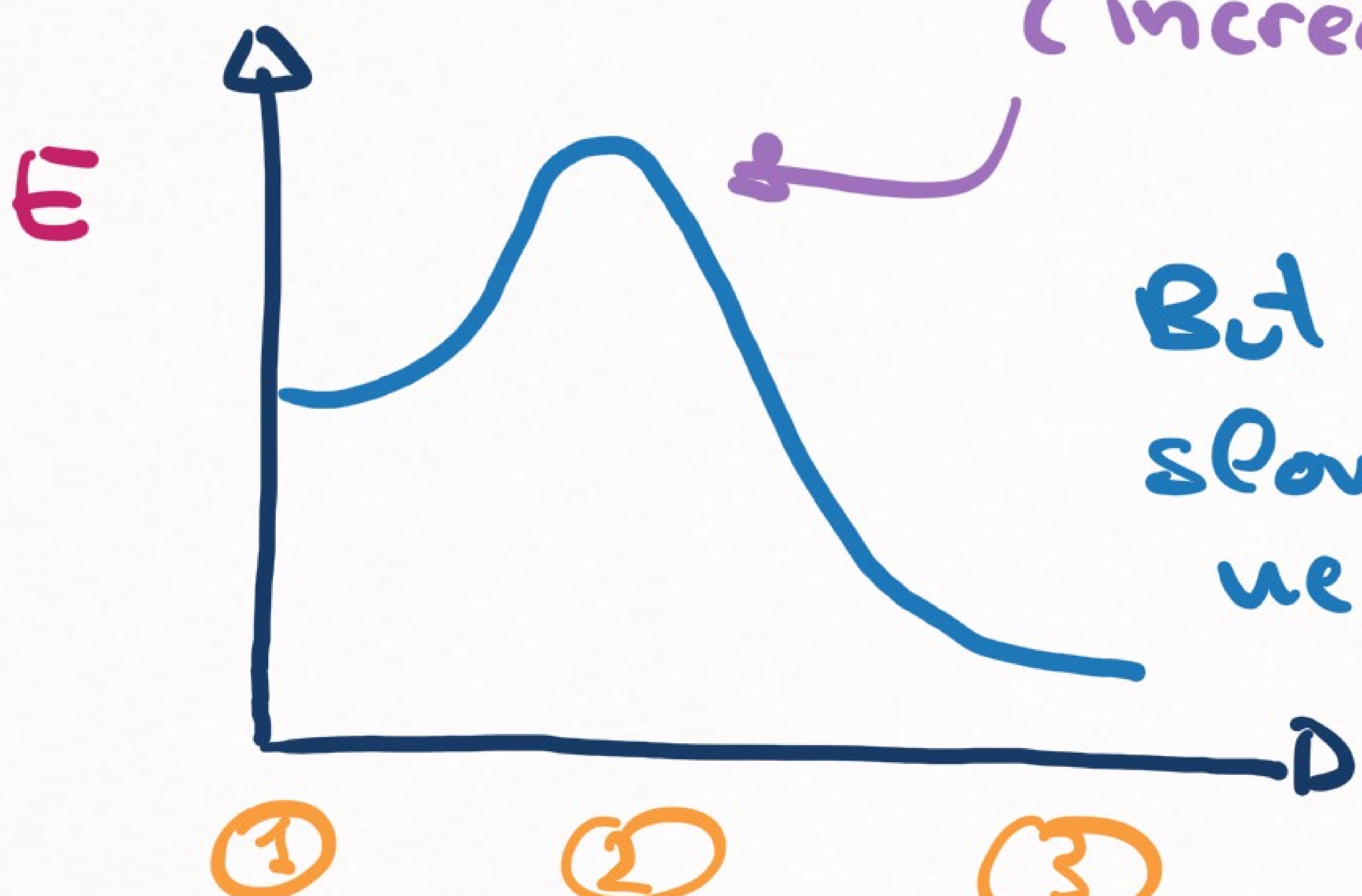
usually you need A ≳ (20-30)

LIQUID DROP MODEL (10)

⇒ Fission in the liquid drop model:



(increases surface)

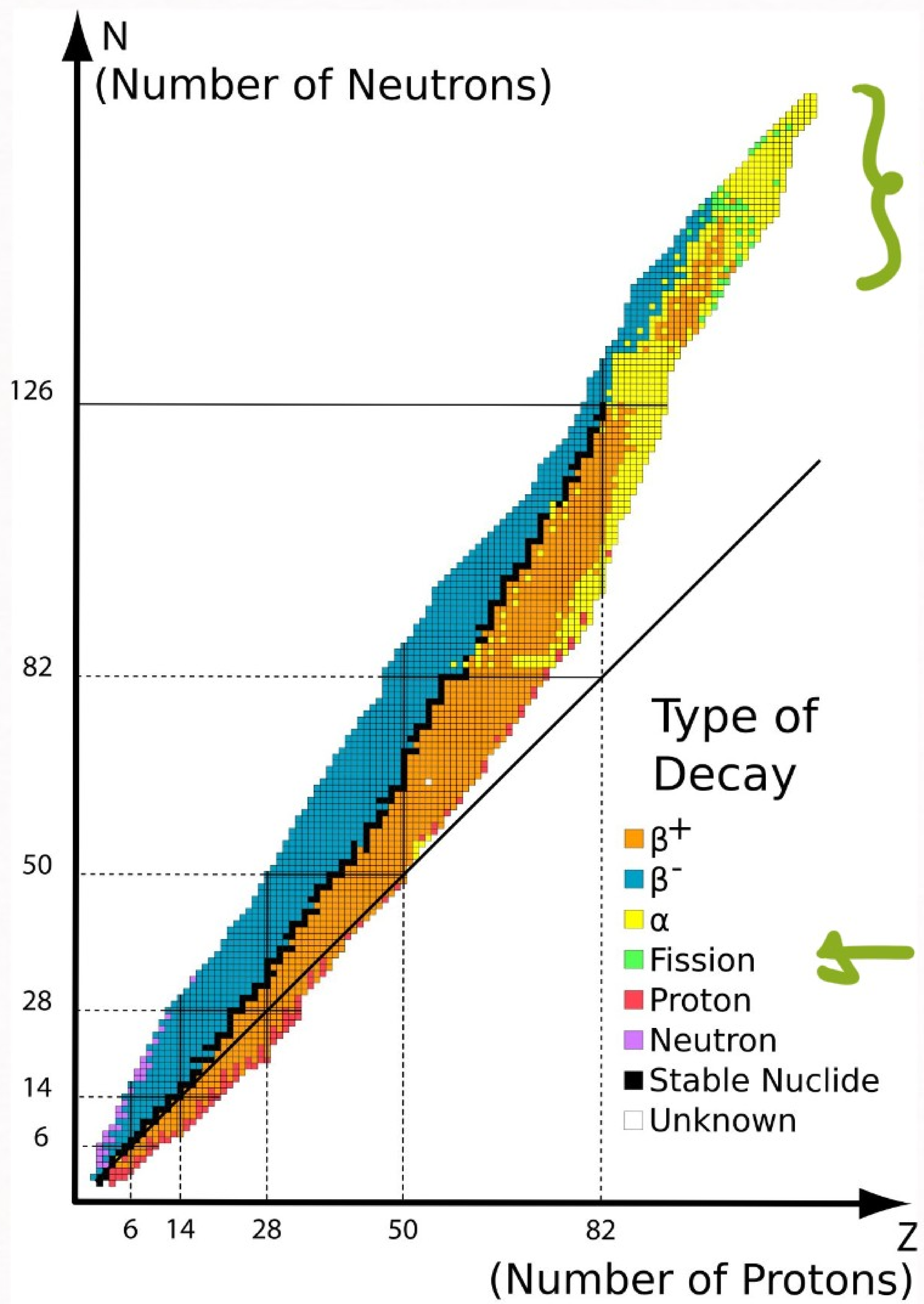


⇒ For very large A, this is energetically possible

But it will be slow, because we have to jump a potential barrier

It takes from decades to billions of years

LIQUID DROP MODEL (11)



here we have fission

$^{252}Cf \to 86y$

$^{235}U \to 3.5\cdot10^{17}y$

fission is not the main form of decay for ^{235}U:

^{235}U (α-decay): $7\cdot10^{8}y$

(fission-decay): $3.5\cdot10^{17}y$

SHELL MODEL ①

⇒ Relies on two assumptions:

1) nucleons are fermions (w.f. is antisymmetric)

2) nucleons generate some sort of a mean field

(e.g. gravitational field of the earth)

Let's explain the idea :

Microscopic description:
(takes into account ∀ degrees of freedom)

$$H = \sum_i \frac{p_i^2}{2m_i} + \sum_{i<j} V^{2B}(\vec{r}_i, \vec{r}_j) + \sum_{i<j<k} V^{3B}(\vec{r}_i, \vec{r}_j, \vec{r}_k) + \dots$$

$$H|\Psi_A\rangle = E_A|\Psi_A\rangle$$

SHELL MODEL ②

1) Microscopic description:

$$H = \sum_i \frac{p_i^2}{2m_i} + \sum_{i<j} V_{2B}(\vec{r}_i, \vec{r}_j) + \sum_{i<j<k} V_{3B}(\vec{r}_i, \vec{r}_j, \vec{r}_k) + \cdots$$

⇓

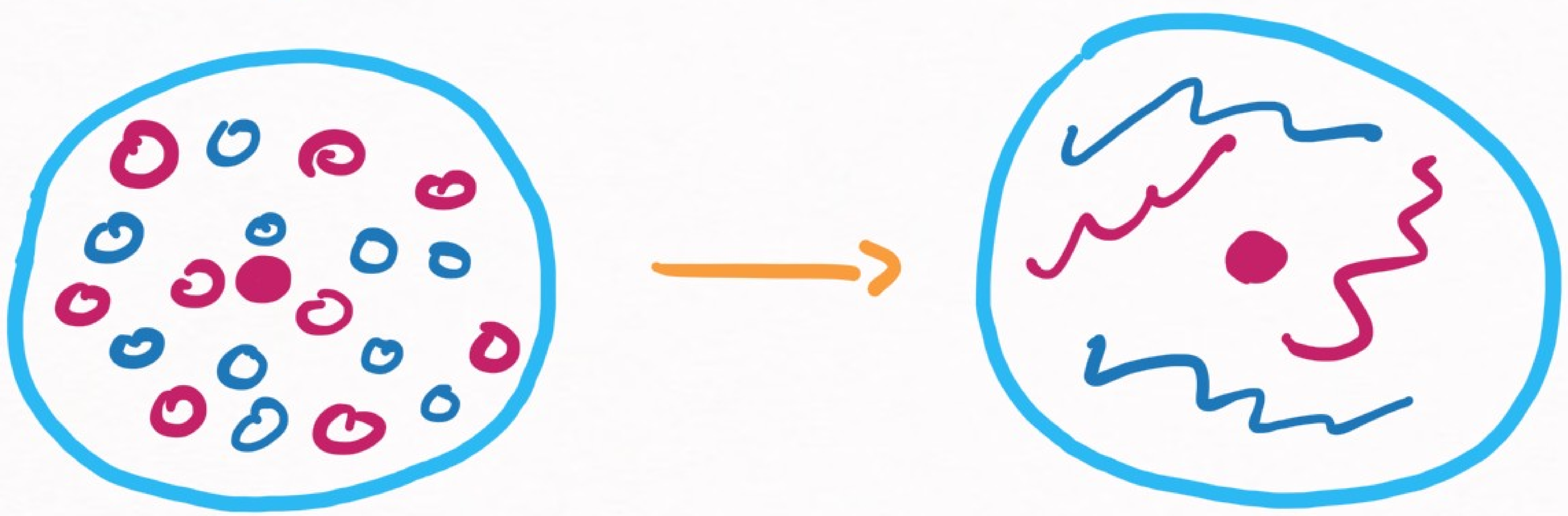
This problem is unmanageable computationally for large A

$$+ \quad \underbrace{H|\Psi_A\rangle = E_A|\Psi_A\rangle}_{\text{Exponentially difficult as } A \text{ increases}}$$

$\propto e^A$

⟶ Need simplification

2) Mean field:



Every nucleon will only feel the averaged pot of all other nucleons

SHELL MODEL ③

⇒ Basic idea: $\sum_{ij} V_{ij}^{2B} + \sum_{ijk} V_{ijk}^{3B} + \dots \longrightarrow \underbrace{\sum_i V_i^{MF}}_{}$

V_i^{MF}: mean-field

⇒ New Hamiltonian:

$$H = \sum_i \frac{p_i^2}{2m_i} + \sum_i V_i^{MF} + \Delta V \longrightarrow \Delta V = \underbrace{\sum_{ij} V_{ij}^{2B} + \sum_{ijk} V_{ijk}^{3B} + \dots - \sum_i V_i^{MF}}_{}$$

If V_i^{MF} is well-chosen,

ΔV will be a small perturbation

(we will ignore ΔV in a first approximation)

Residual interaction (we expect it to be small)

SHELL MODEL ④

$\Rightarrow$ Let's assume a good choice of the mean-field potential

$$H = \sum_i \left(\frac{p_i^2}{2m_i} + V_i^{MF} \right) + \underbrace{\cancel{\Delta V}}_{\text{this will be negligible}} \to 0$$

$$\Rightarrow \quad H \simeq \sum_i H_i \quad \Rightarrow \quad H_i \, \phi_i(r_i) = E_i \, \phi_i(r_i)$$

this hamiltonian factorizes

$$\Rightarrow \quad \Psi_A = \prod_{i=1}^{A} \phi_i(r_i)$$

this will be a solution

SHELL MODEL ⑤

$\Rightarrow$ However, nucleons are fermions:

$$\Psi_A = \prod_{i=1}^{A} \phi_i(r_i) \longrightarrow$$ this does not take into account that fermions are antisymmetric

$\Rightarrow$ we need an antisymmetric wave function:

$$\Psi_A = \prod_{i=1}^{A} \phi_i(r_i) \longrightarrow \Psi_A = \frac{1}{\sqrt{A!}} \sum_{\sigma} \prod_{i=1}^{A} (-1)^{[\sigma]} \phi_{\sigma(i)}(r_i)$$

$\sigma \to$ permutations

$[\sigma] \to$ sign of the permutation $\to$ decomposing a permutation in odd/even number of transpositions

$i \bullet \longleftrightarrow \bullet j$

$\boxed{\text{SHELL MODEL}}$ ⑥

$\Rightarrow$ Notice that $\Psi_A = \frac{1}{\sqrt{A!}} \sum_{\sigma} \prod_{i=1}^{A} (-1)^{[\sigma]} \phi_{\sigma(i)}(r_i)$

can also be written as:

Reminder:

$\det(A) = \sum_{\sigma} (-1)^{[\sigma]} a_{1\sigma(1)} \cdots a_{n\sigma(n)}$

$$\Psi_A = \frac{1}{\sqrt{A!}} \begin{vmatrix} \phi_1(r_1) & \phi_1(r_2) & \cdots & \phi_1(r_A) \\ \phi_2(r_1) & \phi_2(r_2) & \cdots & \phi_2(r_A) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_A(r_1) & \phi_A(r_2) & \cdots & \phi_A(r_A) \end{vmatrix}$$

[SLATER DETERMINANT]

$|\ldots| \rightarrow$ determinant

SHELL MODEL ②

⇒ Bottom-line: every nucleon is in a different state

→ Because nucleons are fermions

Example: $V^{MF}(r) = \frac{1}{2} m_N \omega^2 r^2$ (harmonic oscillator as our mean field)

↙

$H_i \phi_i = \epsilon_i \phi_i \qquad \epsilon_i = \epsilon(n_i, \ell_i)$

$\epsilon(n, \ell) = \omega\left(2n + \ell + \frac{3}{2}\right)$ $\begin{cases} n \to \text{energy level } (n = 0, 1, 2, \dots) \\ \ell \to \text{angular momentum } (\ell = 0, 1, 2, \dots) \end{cases}$

SHELL MODEL (8)

⇒ Example : $V_{MF}(r) = \frac{1}{2} m_N \omega^2 r^2$, $H_i \phi_i = \epsilon_i \phi_i$

$$\epsilon(n,\ell) = \omega(2n+\ell+3/2)$$

0) Ground state: $n=0$, $\ell=0$ ⟶ $\epsilon = \frac{3}{2}\omega$, $J^P = \frac{1}{2}^+$

↳ How many nucleons can I arrange in this energy level?

2 protons
2 nucleons

$\left.\begin{matrix} \\ \\ \end{matrix}\right\}$ $p\uparrow \; p\downarrow$
$n\uparrow \; n\downarrow$

$|n,\ell\rangle \, |\tfrac{1}{2} \pm \tfrac{1}{2}\rangle_S \, |\tfrac{1}{2} \pm \tfrac{1}{2}\rangle_I$

⇒ 4 nucleons in the ground state

SHELL MODEL (9)

1) Next energy level : $n = 0, \ell = 1 \longrightarrow \epsilon = \frac{5}{2}\omega$

Pauli exclusion principle $\longrightarrow$ $\underset{\text{spin}}{(2)} \times \underset{\text{orbital angular momentum}}{(2\ell+1)} = 6$ for $\ell = 1$

6 protons + 6 neutrons } 12 nucleons

SHELL MODEL (10)

2) Next energy level: $\left\{ \begin{array}{l} n=1,\ \ell=0 \\ n=0,\ \ell=2 \end{array} \right. \Rightarrow \underline{\underline{\epsilon = \frac{7}{2}\omega}}$

We can accommodate: $\underbrace{2}_{\ell=0} + \underbrace{10}_{\ell=2} = 12$ neutrons or protons

12 neutrons
+
12 protons

$\Rightarrow$ PATTERN: you just fill the energy shells (shell model ↗)

$\boxed{\text{SHELL MODEL}}$ 11

⇒ For the harmonic oscillator we have:

$$\epsilon = \frac{3}{2}\omega \longrightarrow 2p + 2n$$

$$\epsilon = \frac{5}{2}\omega \longrightarrow 6p + 6n$$

$$\epsilon = \frac{7}{2}\omega \longrightarrow 12p + 12n$$

⇒ This implies the existence of magic numbers:

$$\left.\begin{aligned} N &= 2, 8, 20, \dots \\ Z &= 2, 8, 20, \dots \end{aligned}\right\}$$

this is indeed a feature of <u>nuclei</u>

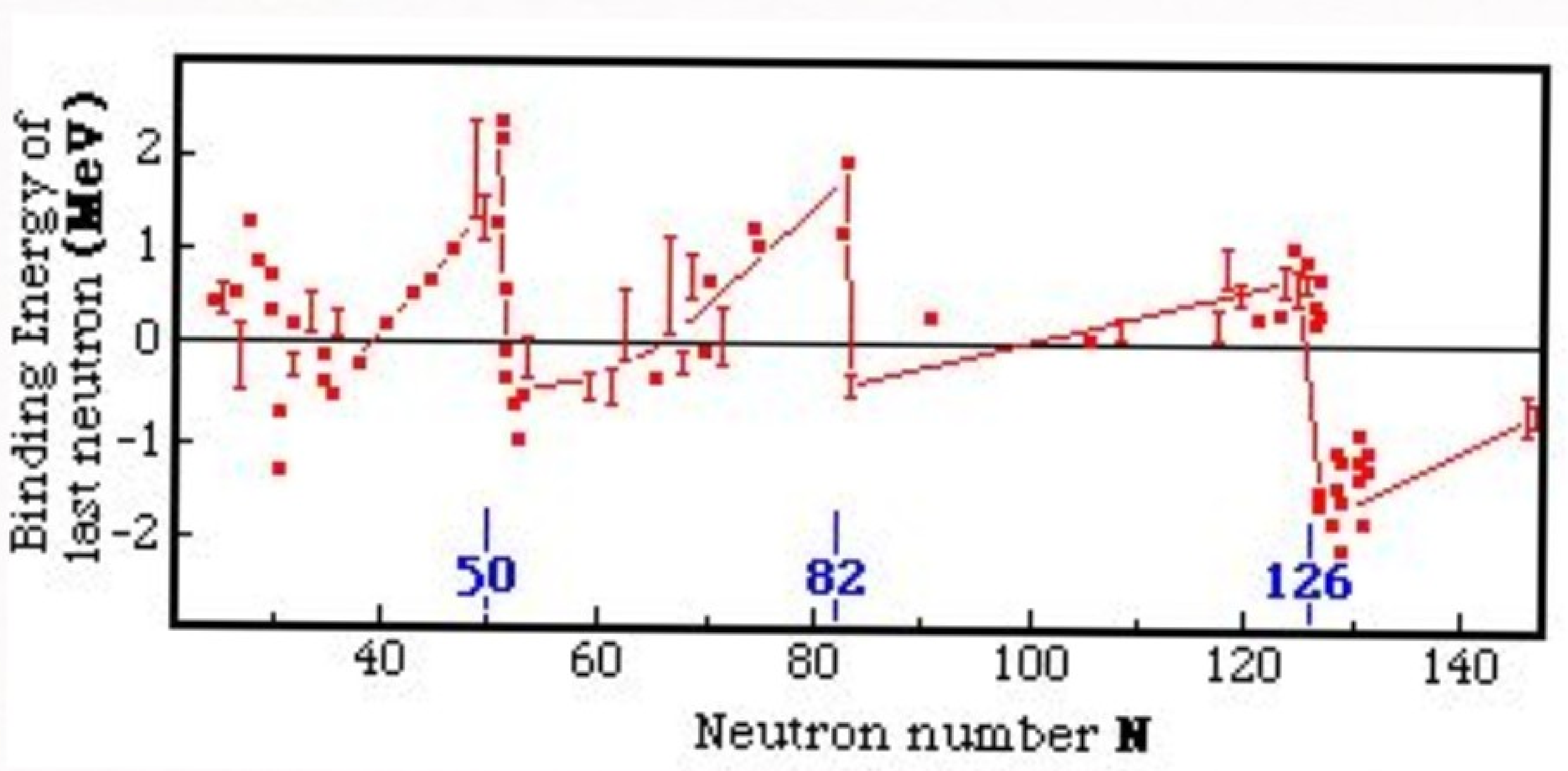
SHELL MODEL (12)

$\Rightarrow$ $N = 2, 8, 20, \ldots$
$Z = 2, 8, 20, \ldots$

there is an increase of ω in the energy required to remove the last nucleon

$\Downarrow$

This is related to the separation energies $S_p(N,Z)$ & $S_n(N,Z)$ which defined the magic numbers



$N, Z = \underbrace{2, 8, 20}, 28, 50, 82, 126$

we reproduce these

SHELL MODEL (13)

$\Rightarrow$ If you remember, we had the magic numbers

$$N, Z = 2, 8, 20, 28, 50, 82, 126, \dots \quad \text{(exp.)}$$

$\Rightarrow$ Our prediction with the shell model & a harmonic oscillator as a mean-field pot.:

$$N, Z = \underbrace{2, 8, 20}_{\text{OK}\checkmark}, \underbrace{40, 70, \dots}_{\text{NOT OK}\checkmark}$$

SHELL MODEL (14)

⇒ For the moment we have:

1) Mean-field potential
2) Fill the shells & get the magic numbers

⇒ With an harmonic oscillator

$$N, Z = 2, 8, 20, 40, 70, \ldots$$

we are close, but we are missing some ingredient of the mean-field potential

SHELL MODEL (15)

$\Rightarrow$ Choice of mean-field potential:

fundamental piece of the shell model

$V_F = \frac{1}{2} m_N \omega^2 r^2$ $\longrightarrow$ Really simple choice, it gets a few things right, but lacks something

$\downarrow$

Not good enough (spin-orbit force)

SHELL MODEL (16)

=▷ What we are missing is ... → Spin-orbit force

(Maria Goeppert Mayer)

&

with this you obtain the correct magic numbers & ordering of orbitals

=▷ Potential :

$$\left[\begin{array}{l} V_{MF} = \frac{1}{2} m_N \omega^2 r^2 - \xi \, \vec{\ell} \cdot \vec{s} \\ \quad \text{with } \xi > 0 \end{array} \right]$$

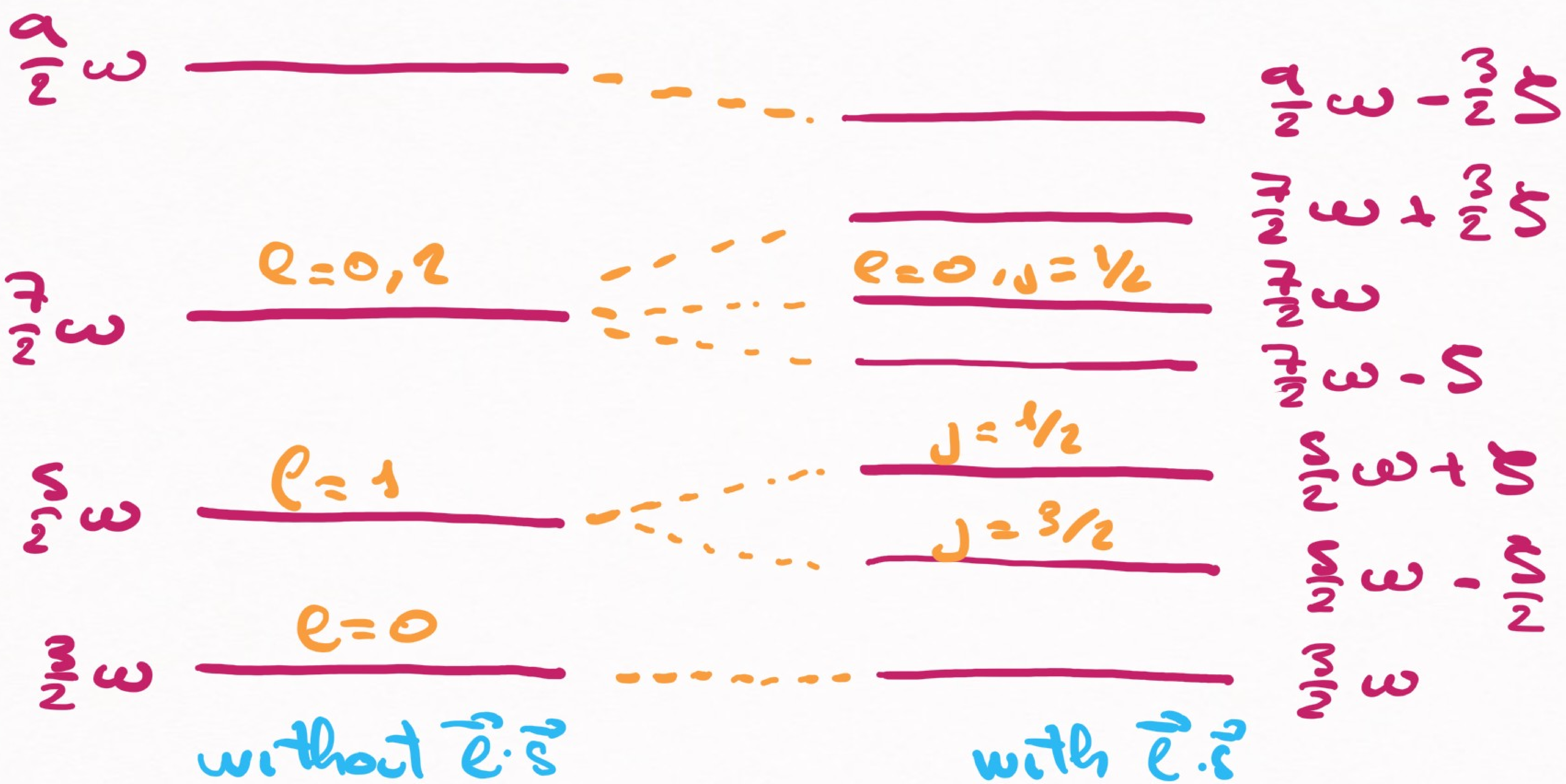
Let's see what happens...

SHELL MODEL (17)

$l, s \rightarrow$ orbital angular momentum, spin

$\Rightarrow$ Modifications : $\vec{l}\cdot\vec{s} = \frac{1}{2}[j(j+1) - l(l+1) - s(s+1)]$

$j \rightarrow$ total angular momentum



$\boxed{\text{SHELL MODEL}}$ 18

1) Shell-model without $\vec{\ell}\cdot\vec{s}$:

$$N, Z = 2, 8, 20, 40, 70, \dots$$

2) Shell-model with $\vec{\ell}\cdot\vec{s}$:

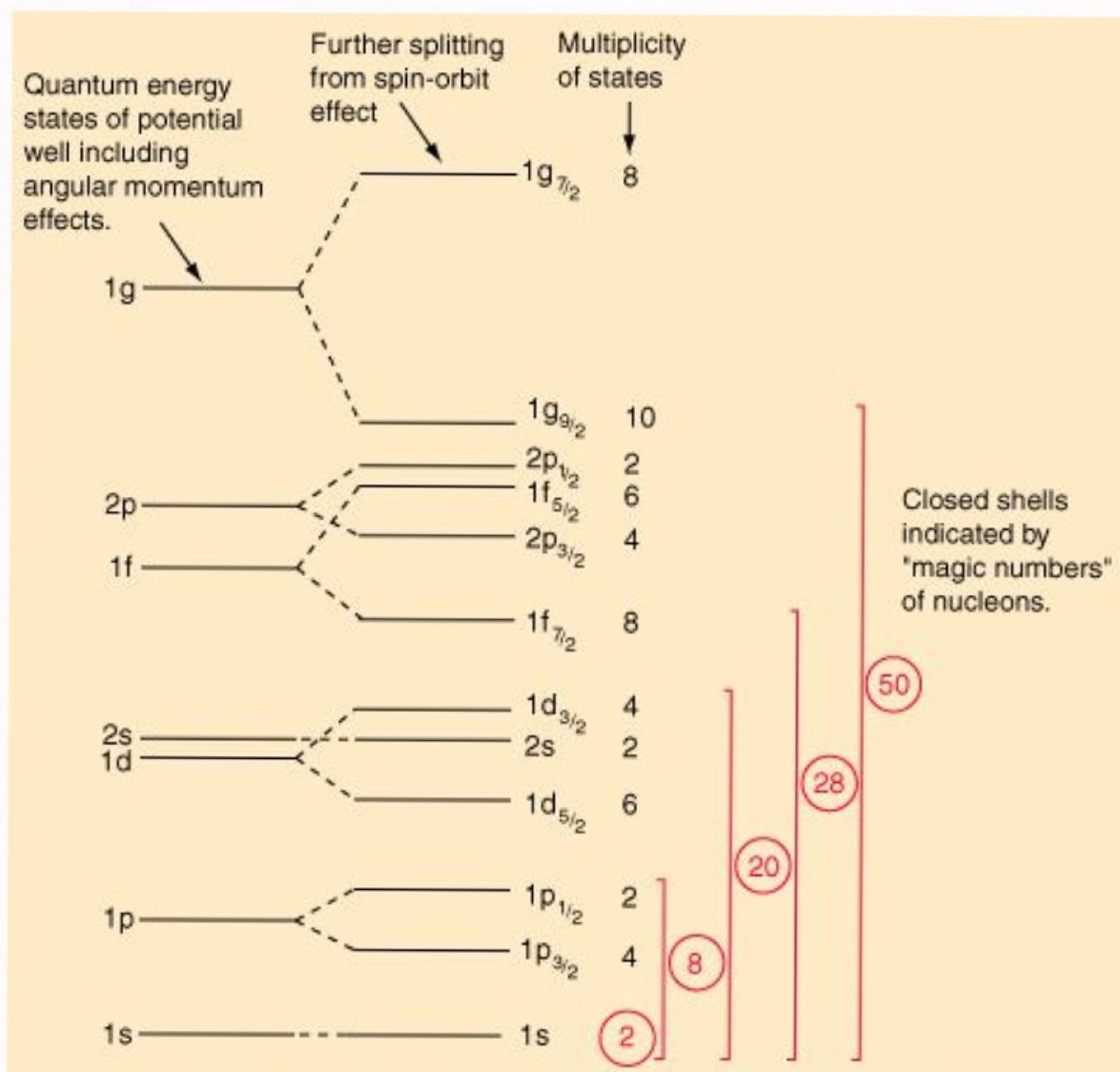
$$N, Z = 2, 8, 20, 28, 50, 82, \dots$$

$\Rightarrow$ We obtain the correct order

(spin-orbit is indeed the missing ingredient)

SHELL MODEL (19)

⇒ Summary of the orbitals:



Notation: $^{n}\ell_{J}$

n is the energy level

ℓ is the orbital angular momentum (s, p, d, f, ...)

J is total angular momentum

$N, Z = 2, 8, 20, 28, 50, 82, 126, \dots$

$\exists$ a really simple model:

$$U^{HF}(r) = \frac{1}{2} m_N \omega^2 r^2 - k\vec{\ell}^{\,2} - \zeta\, \vec{\ell}\cdot\vec{s}$$

improves even more

[SHELL MODEL: APPLICATIONS] ③

⇒ Filling the shells: ^4He : pretty simple (2 protons + 2 neutrons)

$$\underbrace{{}^{1}s_{1/2}\ \uparrow\downarrow}_{\text{protons}} \qquad \underbrace{\uparrow\downarrow\ {}^{1}s_{1/2}}_{\text{neutrons}}$$

⇒ Shell-model description of ^4He

⇓

Prediction for J^P:

pairings → $(\uparrow\downarrow)$ → $S=0$

$\left[J^P(^4He) = 0^+ \right]$ it is correct

$\ell = 0$ we call s

[SHELL MODEL: APPLICATIONS] 2

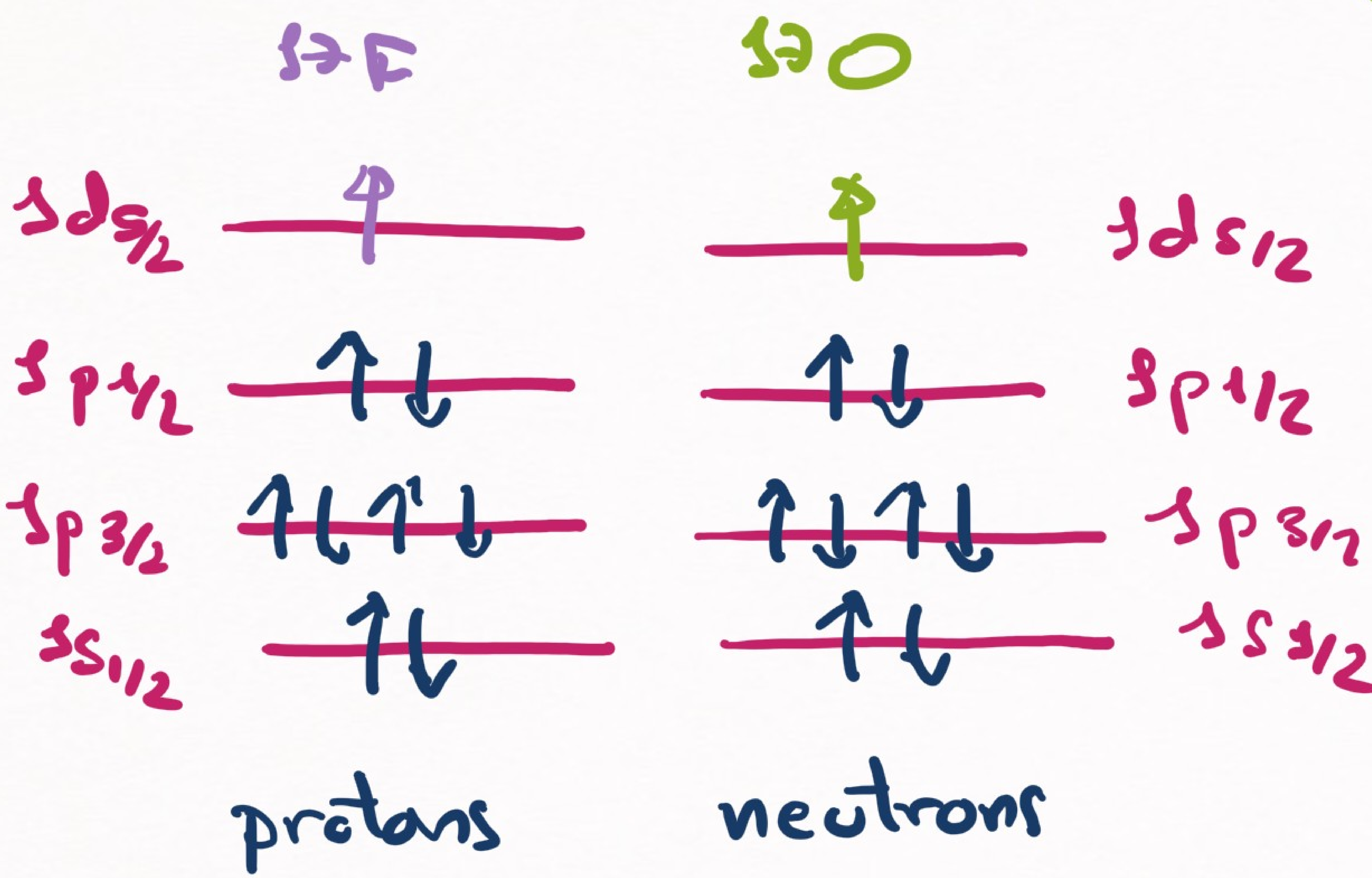
⇒ Filling the shells: ^{17}O / ^{17}F $\simeq$ (^{16}O core + n or p)

$$J^P(^{17}O) = \left(\frac{5}{2}\right)^{+}$$

$+ \rightarrow$ d-wave

$\frac{5}{2} \rightarrow j = 5/2$

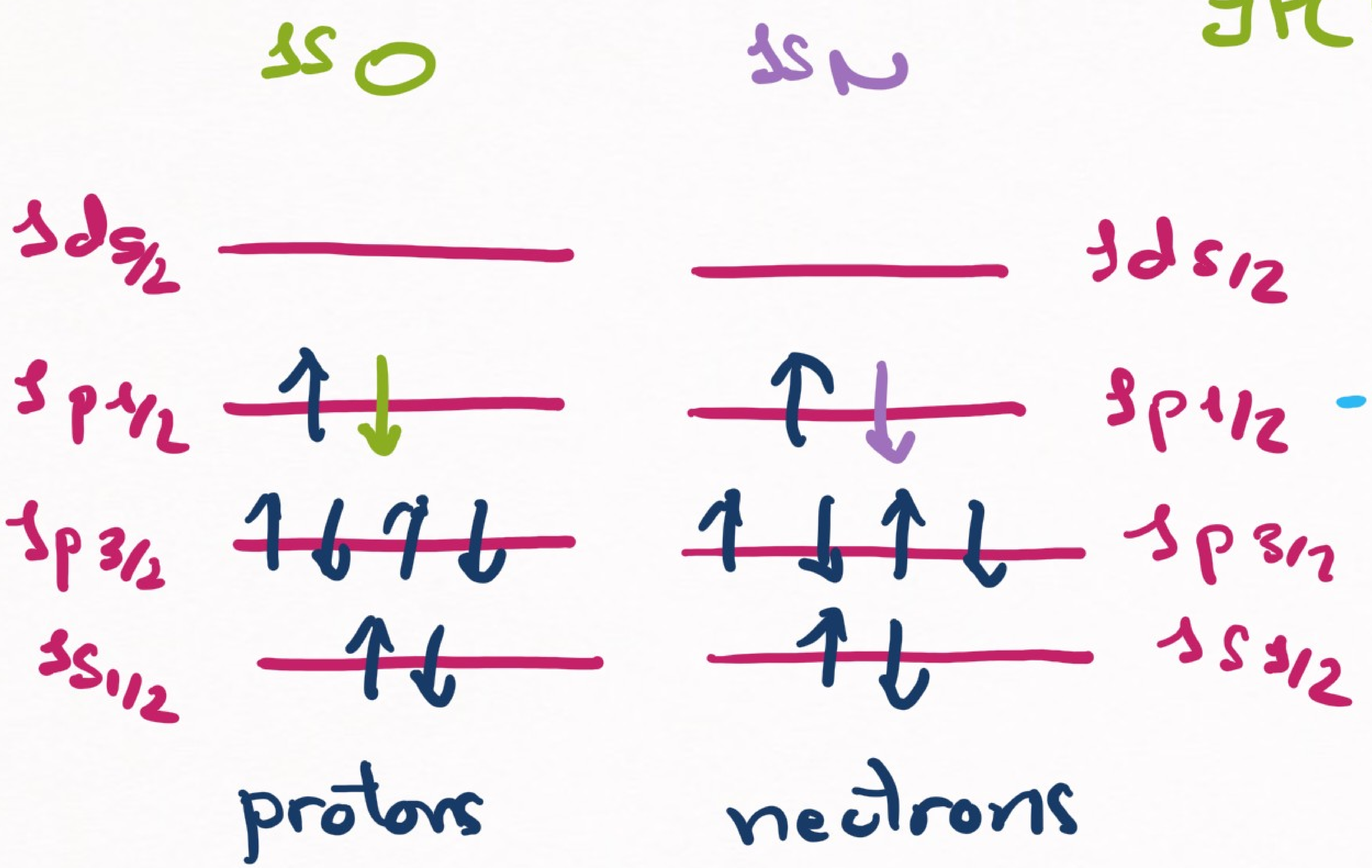
$$J^P(^{17}F) = \left(\frac{5}{2}\right)^{+}$$



^{16}O core

[SHELL MODEL: APPLICATIONS] 3

⇒ Leaving a shell not filled: ^{15}O / ^{15}N → (^{16}O core − (n or p))



$J^P(^{15}O) = J^P(\text{unpaired neutron})$

$= \left(\frac{1}{2}\right)^- \to$ p-wave

$\hookrightarrow j = 1/2$

$J^P(^{15}N) = \left(\frac{1}{2}\right)^-$

(unpaired proton)

[SHELL MODEL: APPLICATIONS] ④

⇒ The basic idea in these examples is:

1) ^{16}O → 8p, 8n (doubly magical) → just fill all shells

2) ^{17}O / ^{17}F → ^{16}O + neutron/proton → fill the next shell (which will be unpaired)

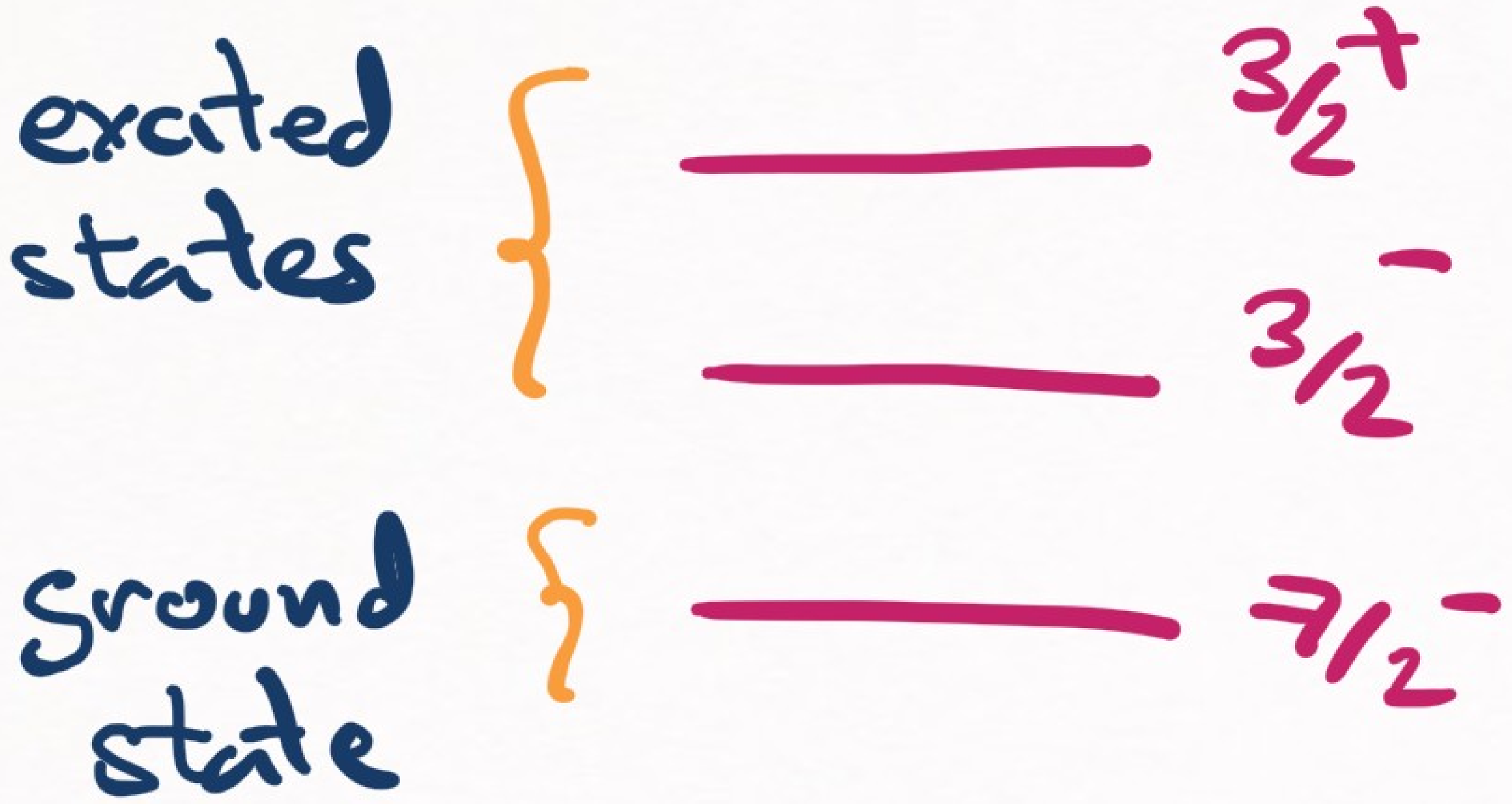
3) ^{15}O / ^{15}N → ^{16}O − neutron/proton
↓
hole in one of the last shells

it's always the unpaired nucleon that determines J^P

[SHELL MODEL: APPLICATIONS] 5

⇒ Next application: excited states

$^{41}Ca \rightarrow$ 20 protons, 21 neutrons

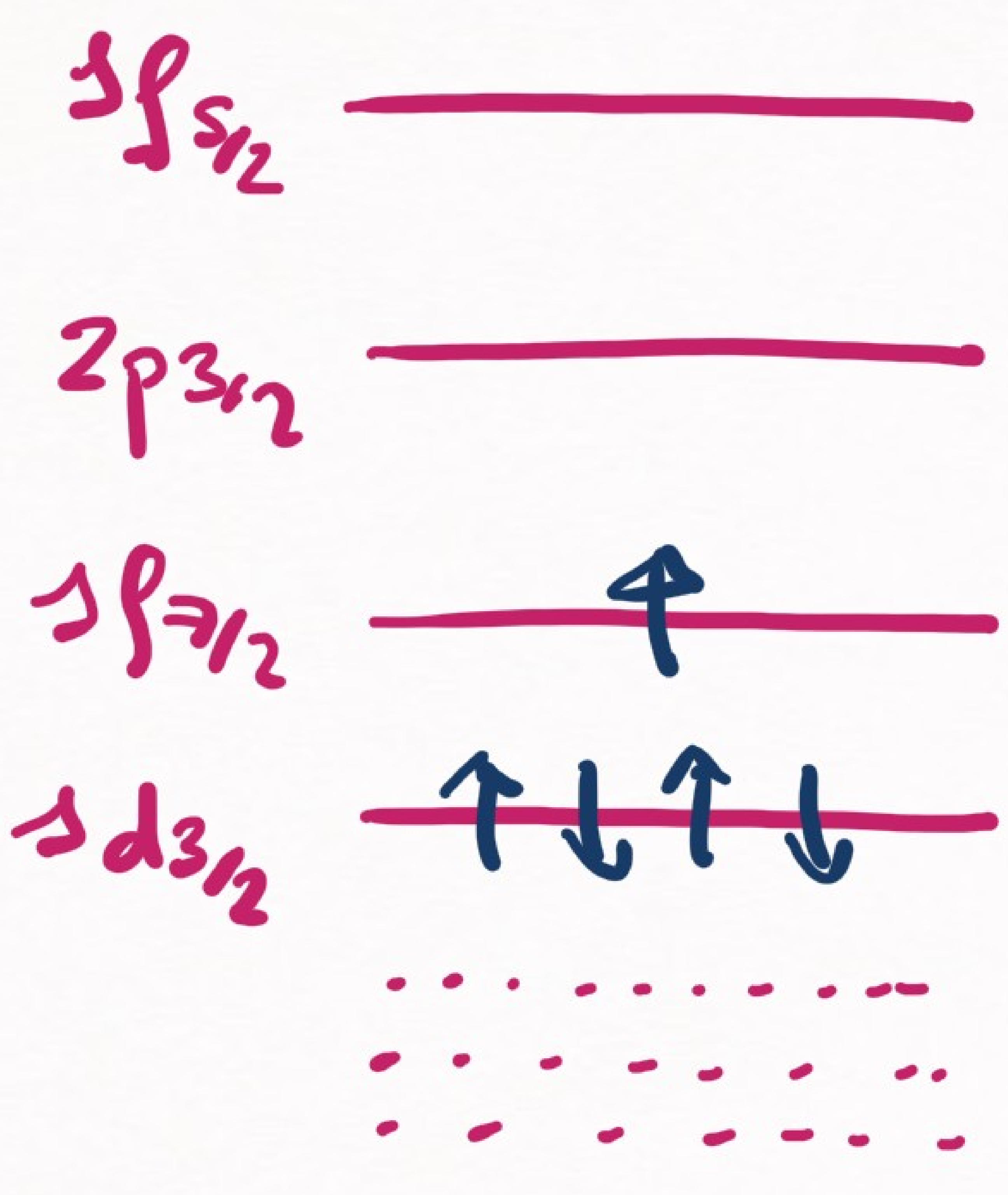


→ We want to reproduce this in the shell model

$$^{41}Ca \rightarrow (\underbrace{^{40}Ca \text{ core}}_{\substack{20+20 \\ J^P = 0^+}}) + (\underbrace{\text{a single neutron}}_{\text{this determines } J^P})$$

[SHELL MODEL: APPLICATIONS] 6

⇒ ^{41}Ca : ground state → $J^P = \frac{7}{2}^-$



The ground state should have:

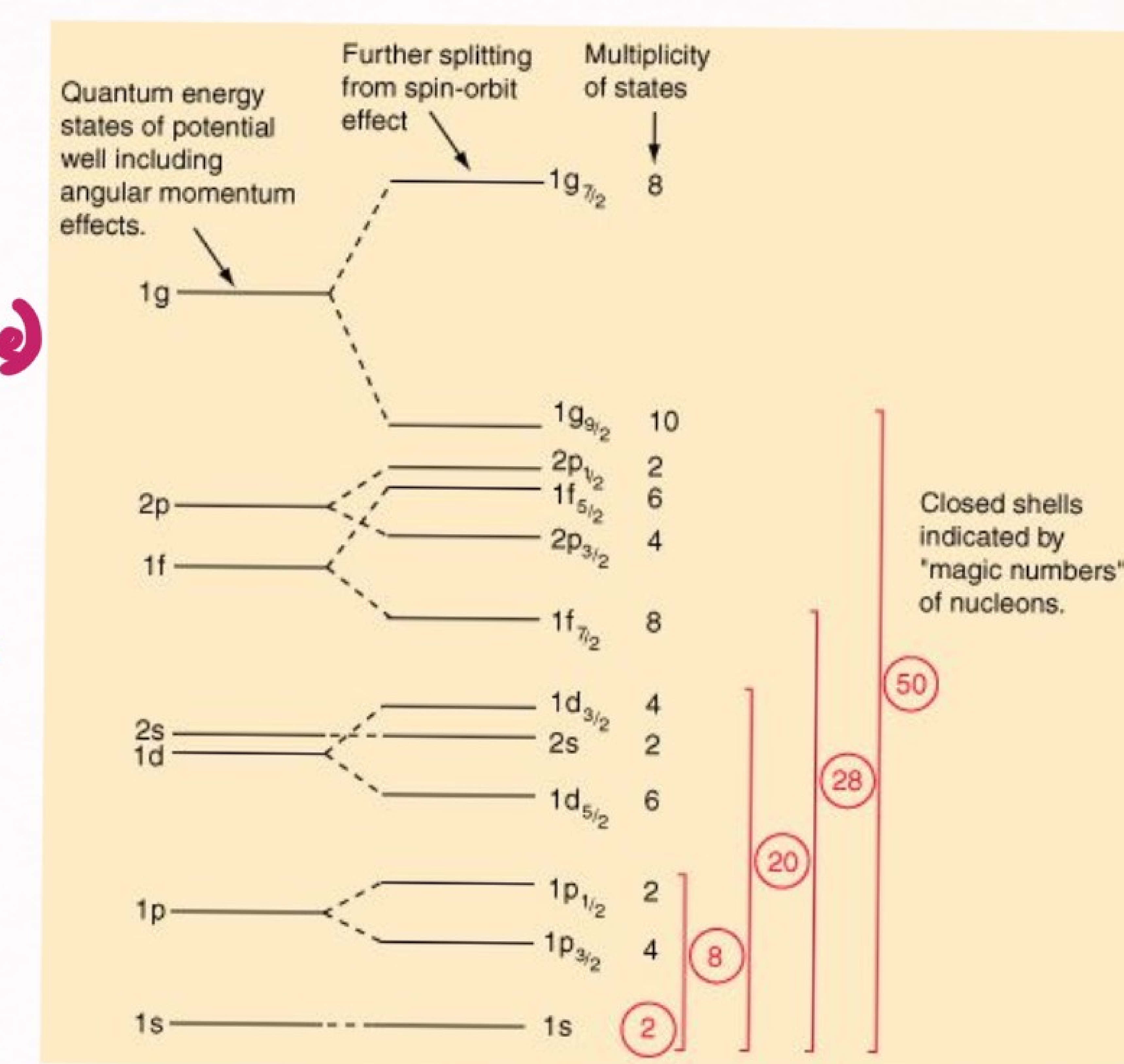
$J^P = \left(\frac{7}{2}\right)^-$ → $\ell = 3$ (f-wave)

↳ $j = 7/2$

filled shells } this makes 20

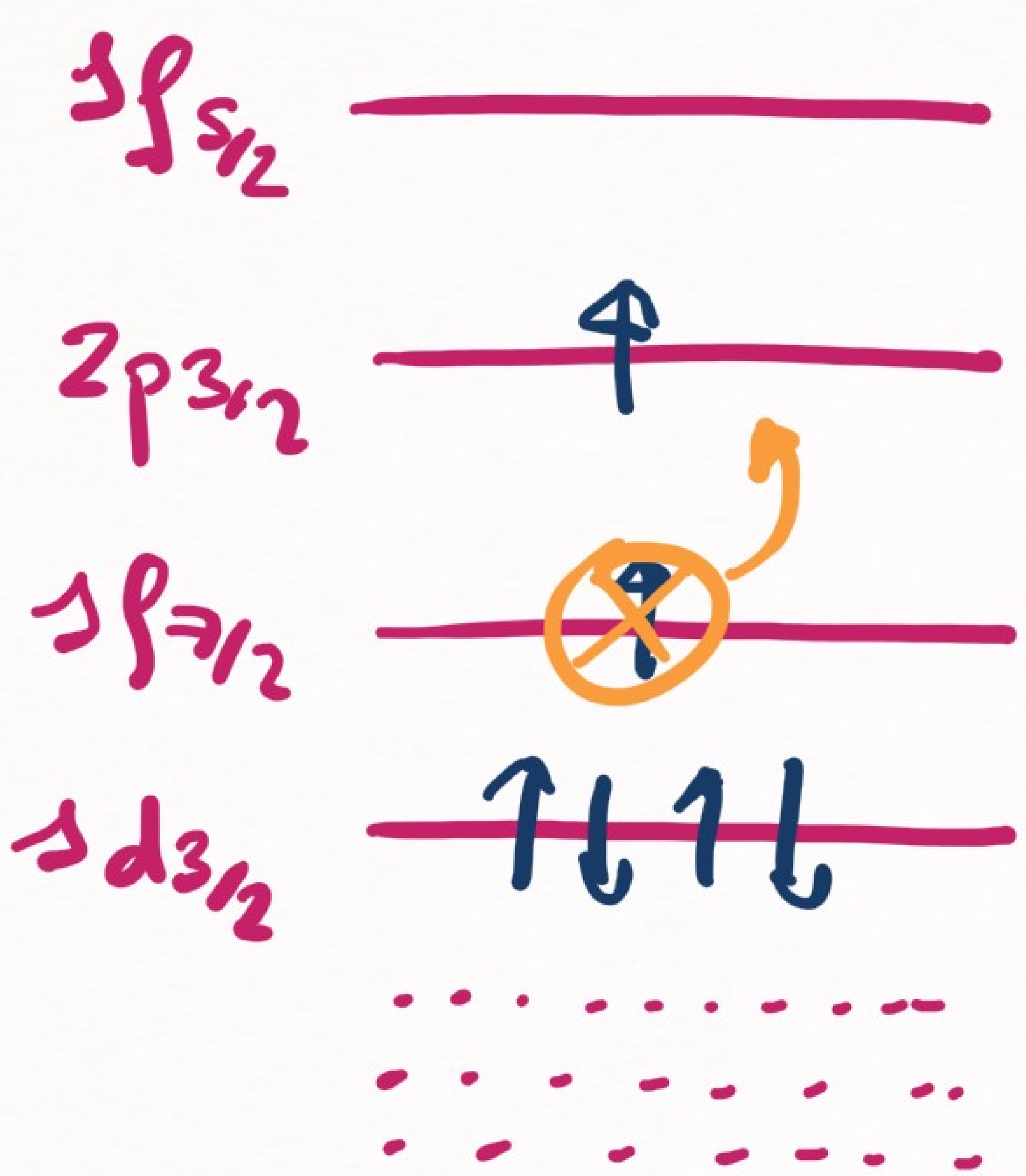
(neutrons)

Reminder



[SHELL MODEL: APPLICATIONS] 6

⇒ ^{41}Ca : first excited state $J^P = \frac{3}{2}^-$



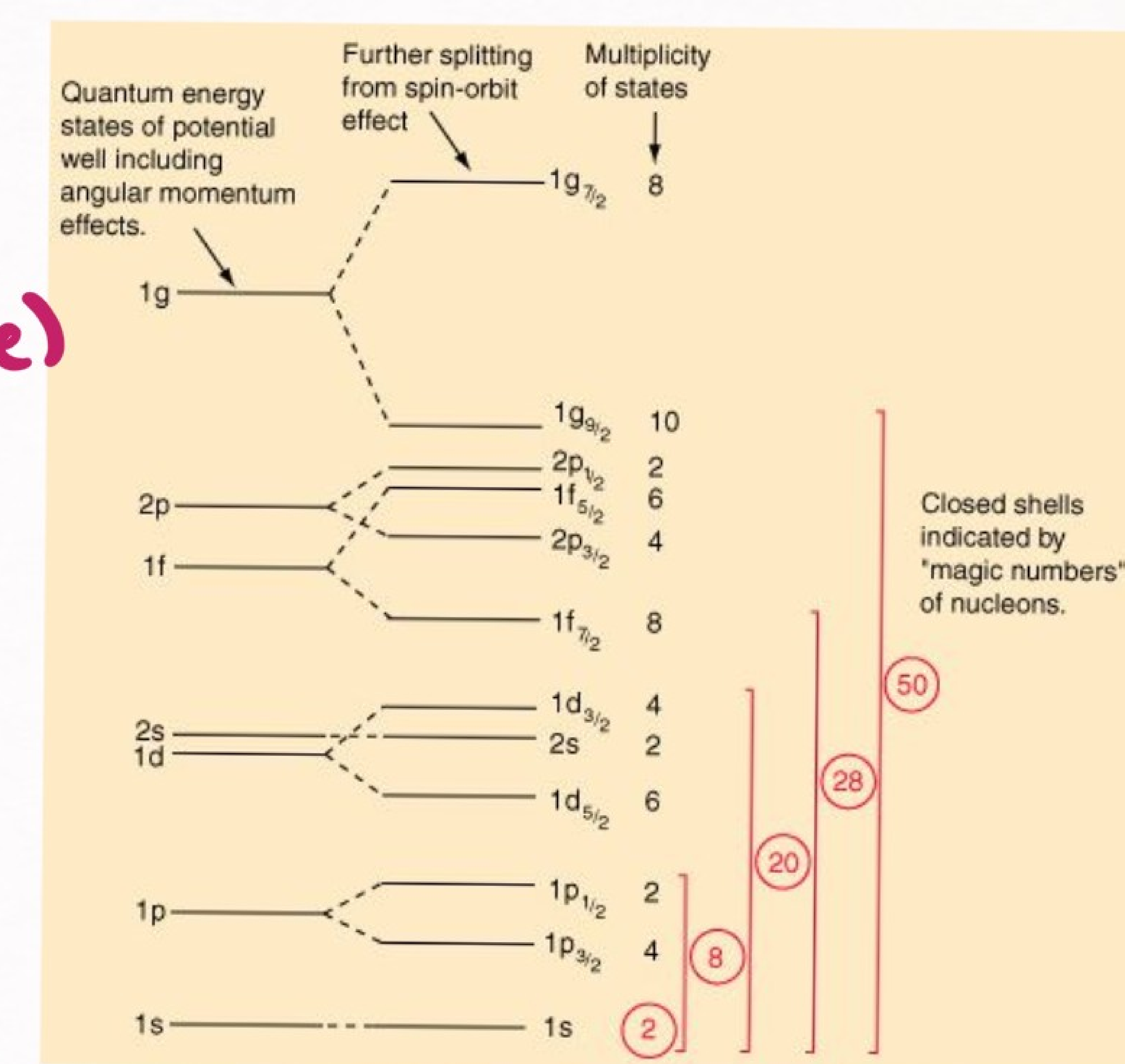
With this choice the prediction is:

$J^P = \left(\frac{3}{2}\right)^- \rightarrow \ell = 1$ (p-wave)

↳ $J = 3/2$

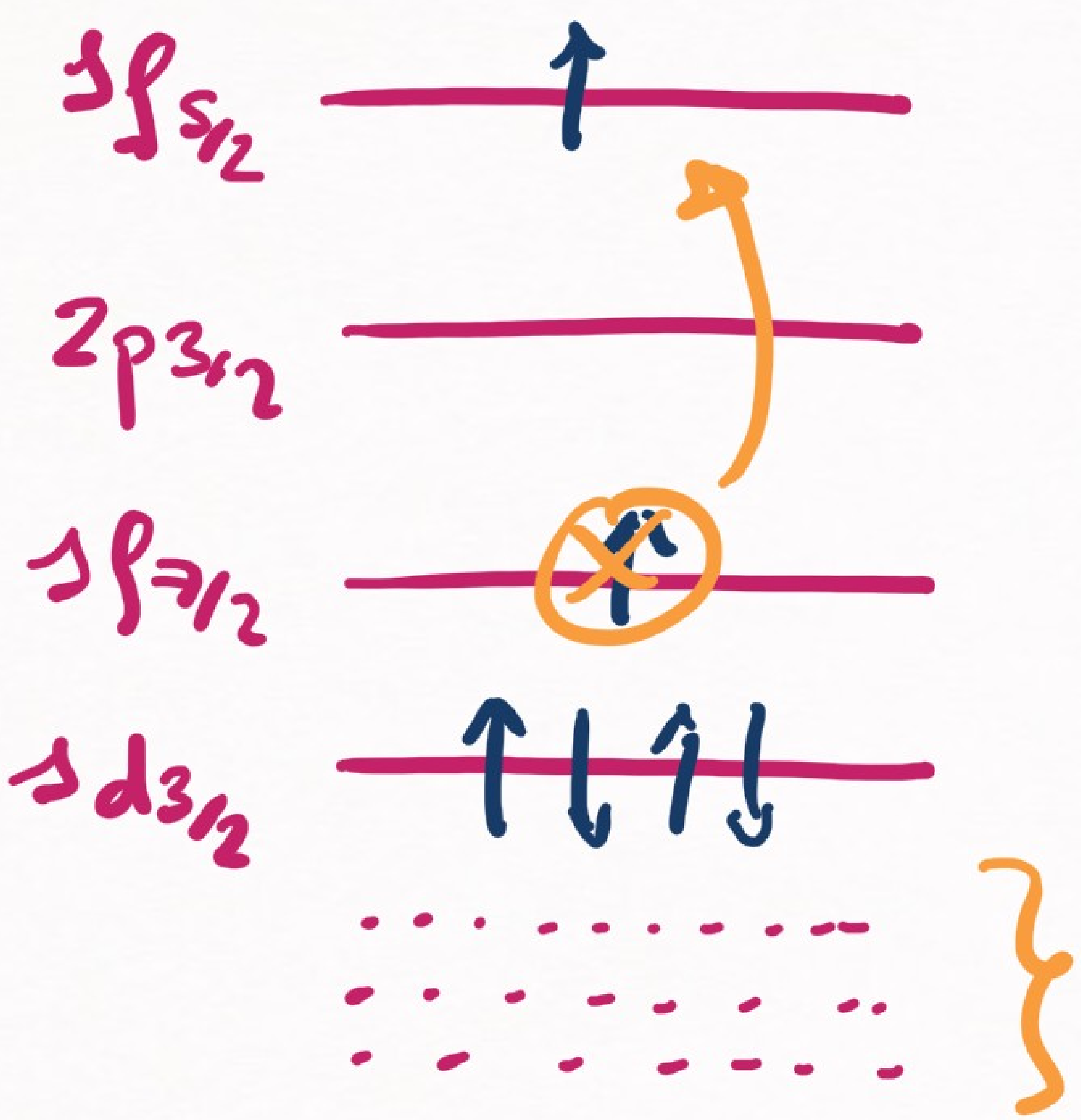
} all filled shells

Reminder



[SHELL MODEL: APPLICATIONS] (7)

⇒ ^{41}Ca : second excited state (a)



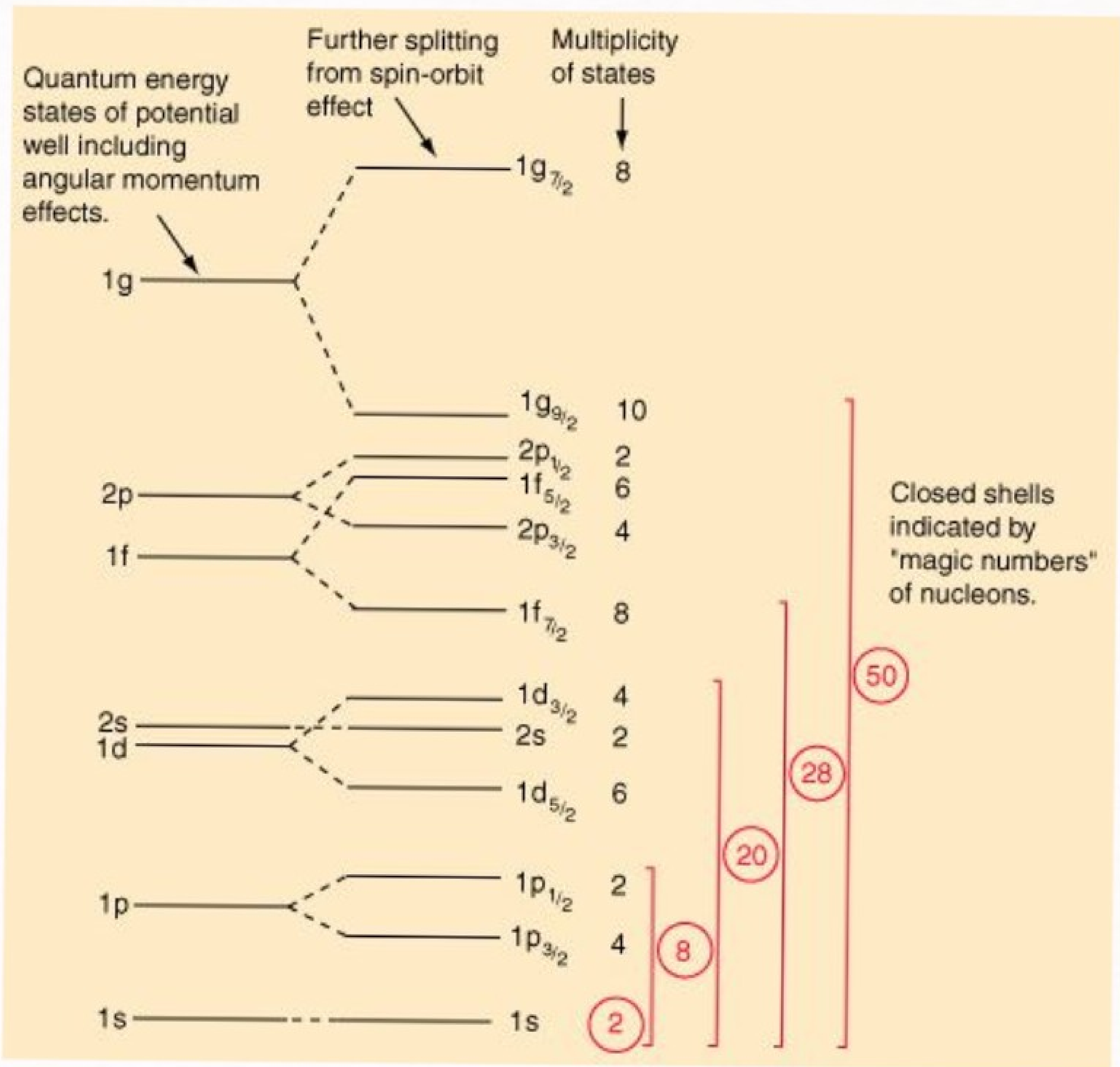
This choice predicts:

$$J^P = \left(\frac{5}{2}\right)^-$$

But $J^P_{exp} = \left(\frac{3}{2}\right)^+$

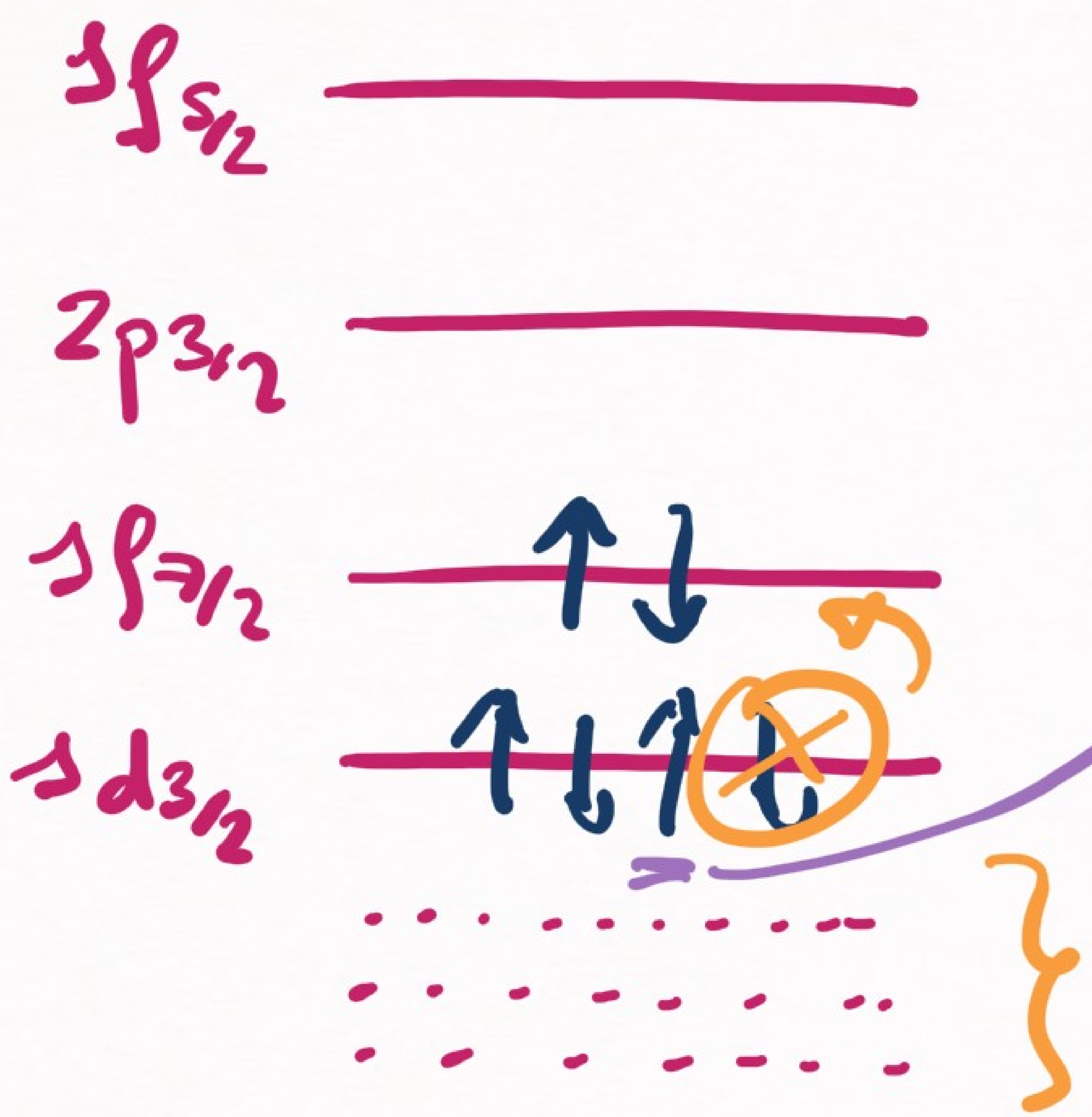
(something is not right w/ this explanation)

Reminder



[SHELL MODEL: APPLICATIONS] (7)

⇒ ^{41}Ca : second excited state (b) $J^P = \left(\frac{3}{2}\right)^+$



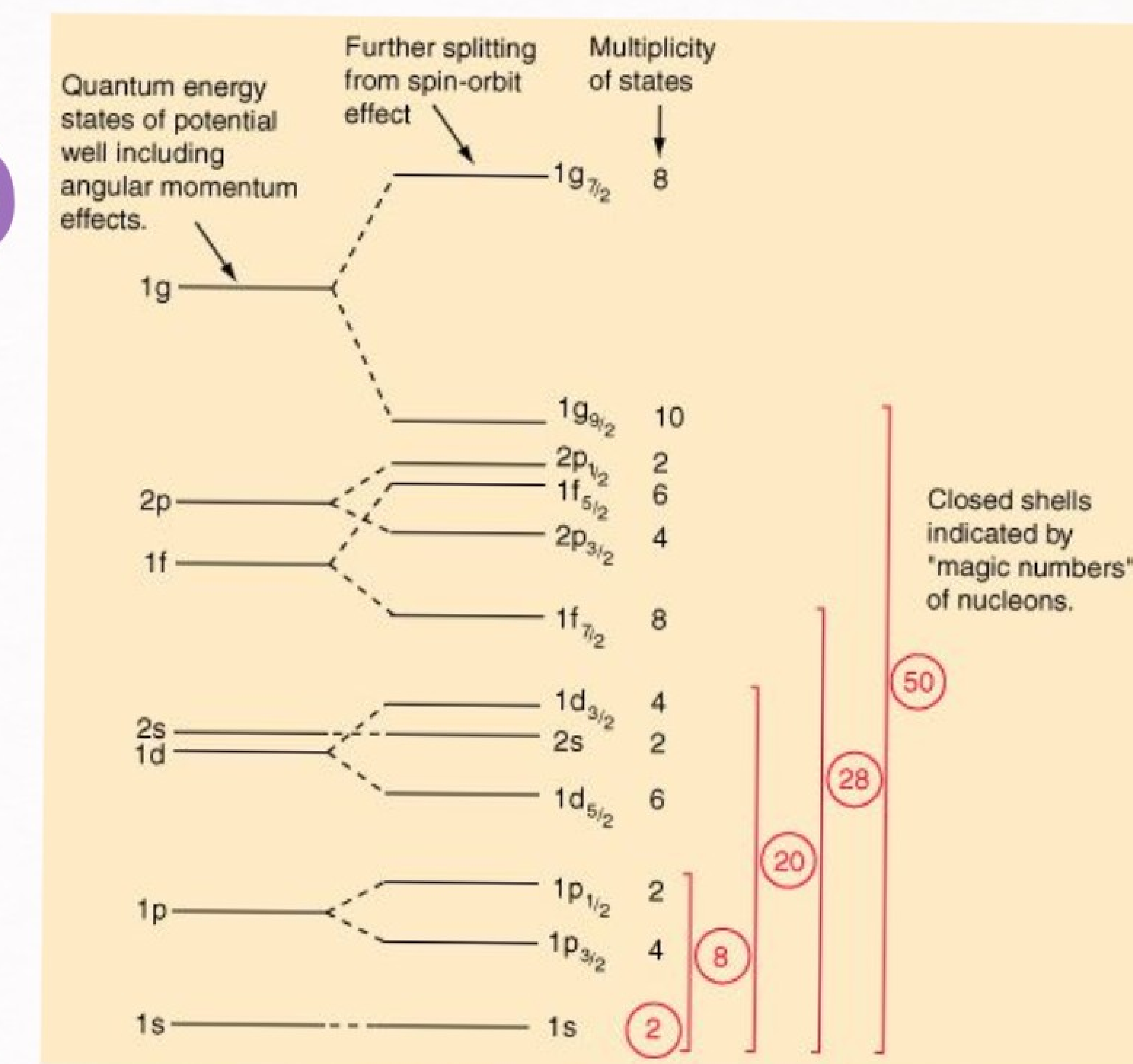
The prediction will be:

$J^P = \left(\frac{3}{2}\right)^+ \rightarrow \ell = 2$ (d-wave)

$J = 3/2$

all filled shell

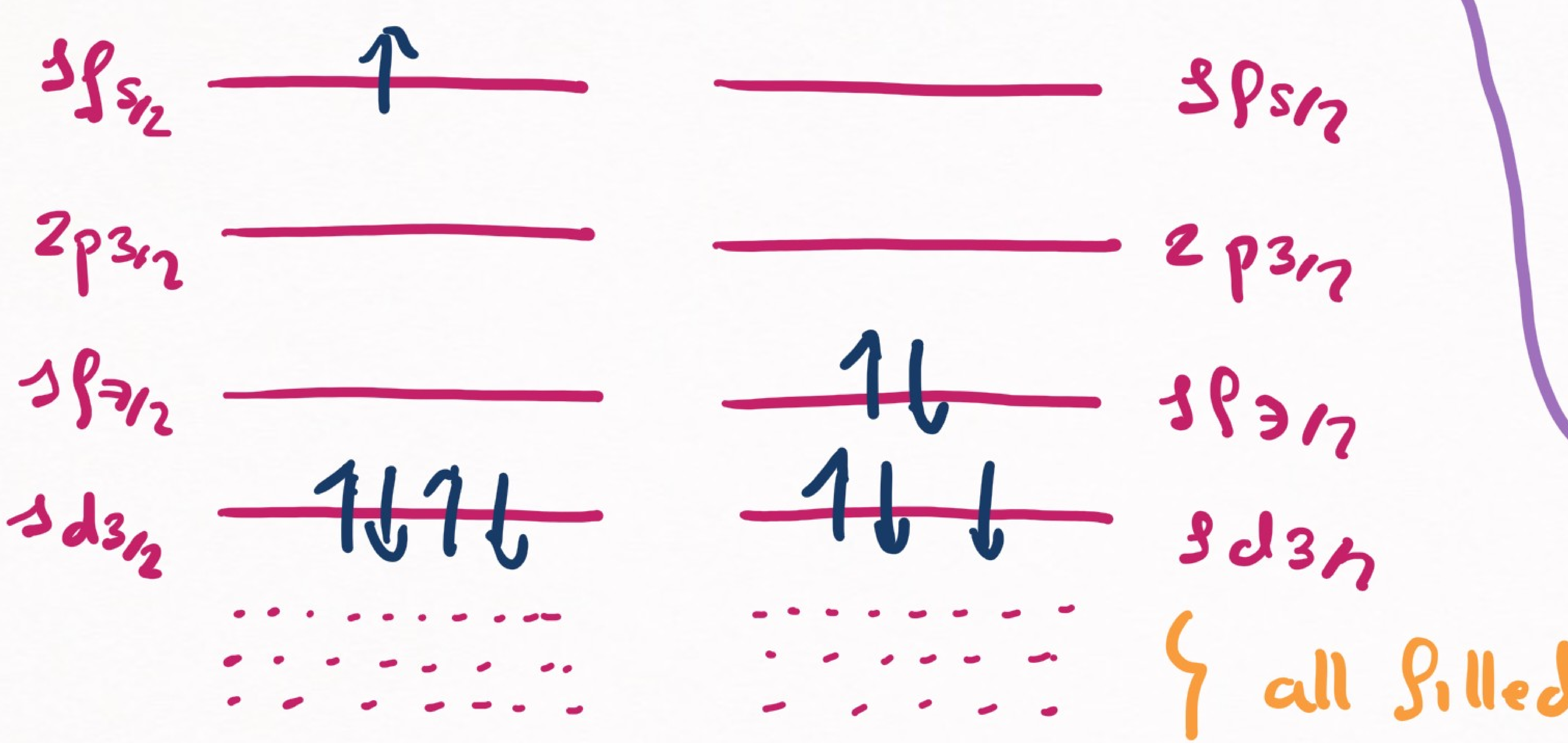
Reminder

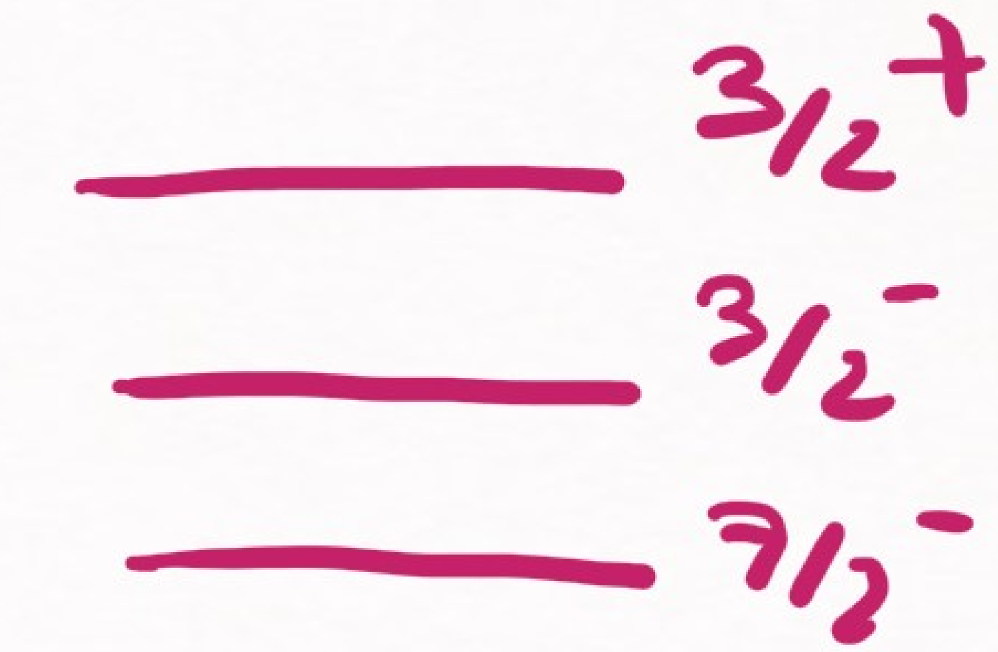


[SHELL MODEL: APPLICATIONS] (7)

⇒ ^{41}Ca : second excited state (c) → ∃ more than one way to explain a state

Explanation (a) Explanation (b)





⇒ this is the choice that nature takes

[SHELL MODEL: APPLICATIONS] 8

⇒ A more complicated example: $^{38}Ar \to$ 18 protons
20 neutrons

excited ——— 2^+
ground ——— 0^+
} this pattern is almost universal for even-even nuclei $(0^+, 2^+)$

⇒ Ground state:

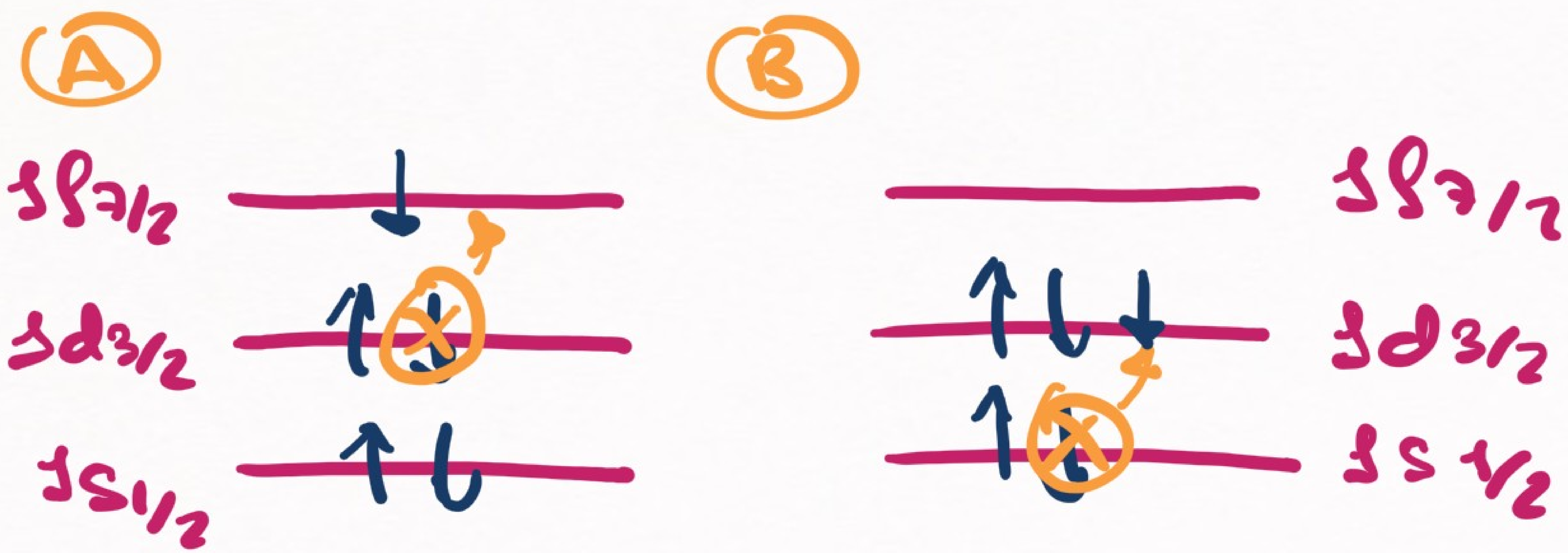
$1f_{7/2}$ ———
$1d_{3/2}$ —↑↓—
$1s_{1/2}$ —↑↓—
. . . . { full shells
(protons)

{ all protons are paired ⇒ $\boxed{J^P = 0^+}$

[SHELL MODEL: APPLICATIONS] 9

⇒ ^{38}Ar : excited states

[$J^P = 2^+$]



(protons)

A $J^P = \left(\frac{3}{2}^+ \otimes \frac{7}{2}^-\right)$

$= 2^- \oplus 3^- \oplus 4^- \oplus 5^-$

(one of these)

B $J^P = \left(\frac{1}{2}^+ \otimes \frac{3}{2}^+\right)$

$= 1^+ \oplus 2^+$

(one of these)

the choice that nature makes

[SHELL MODEL: PAIRING INTERACTION] ③

⇒ Reminder: $H = \sum_i \frac{p_i^2}{2m_i} + \sum_{ij} V_{ij}^{2B} + \sum_{ijk} V_{ijk}^{3B} + \dots$

[(try to guess a good V_i^{MF})] ⇒ $H = \sum_i \left(\frac{p_i^2}{2m_i} + V_i^{MF} \right) + \underline{\underline{\Delta V}}$

↳ we will always have a bit of residual $\underline{\underline{\text{interaction}}}$

⇒ Easiest example: pairing interaction

$$\langle \underbrace{jj}_{\text{two protons/neutrons in an orbital with } j}(JM) | V_{\text{pairing}} | jj(JM) \rangle = -\frac{1}{2} g (j+1) \delta_{J0} \delta_{M0}$$

two protons/neutrons in an orbital with j } → they couple to JM

[SHELL MODEL: PAIRING INTERACTION] (2)

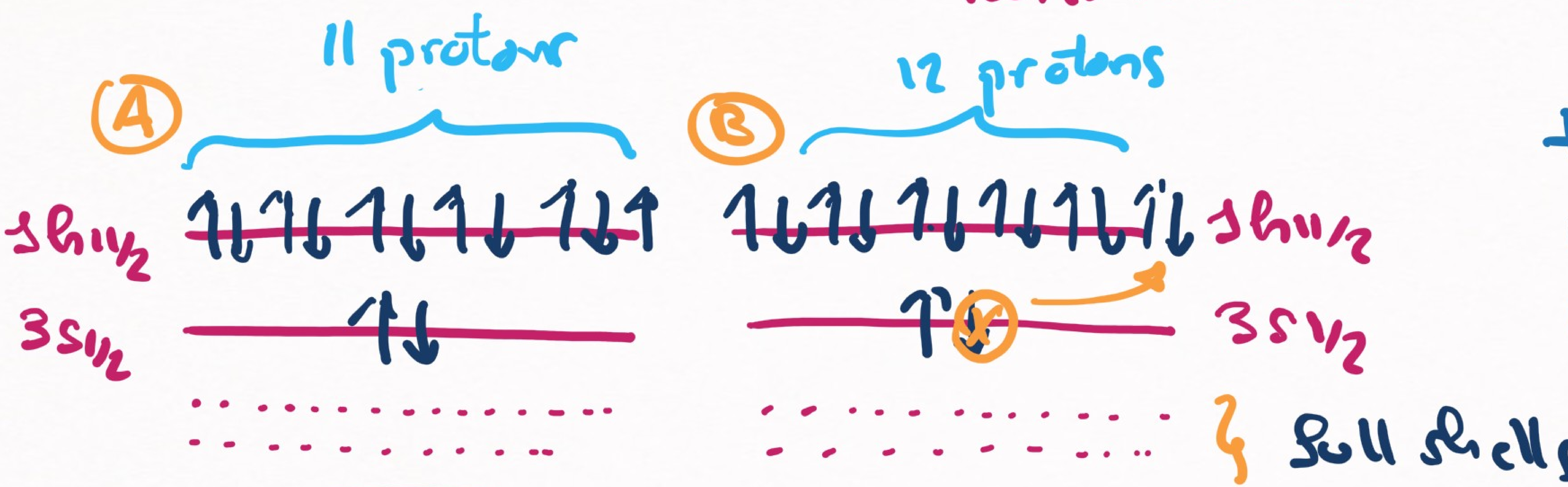
⇒ The pairing interaction is important
for shells with high "J"

Example: ^{203}Tl / ^{205}Tl 81 protons
122/124 neutrons

$\langle V_{pairing} \rangle \propto J$ $(j = \frac{11}{2})$

↓

this will determine min energy configuration if J is high



(A) → $J^P = (\frac{11}{2})^-$

(B) → $J^P = (\frac{1}{2})^+$

$E_B < E_A$

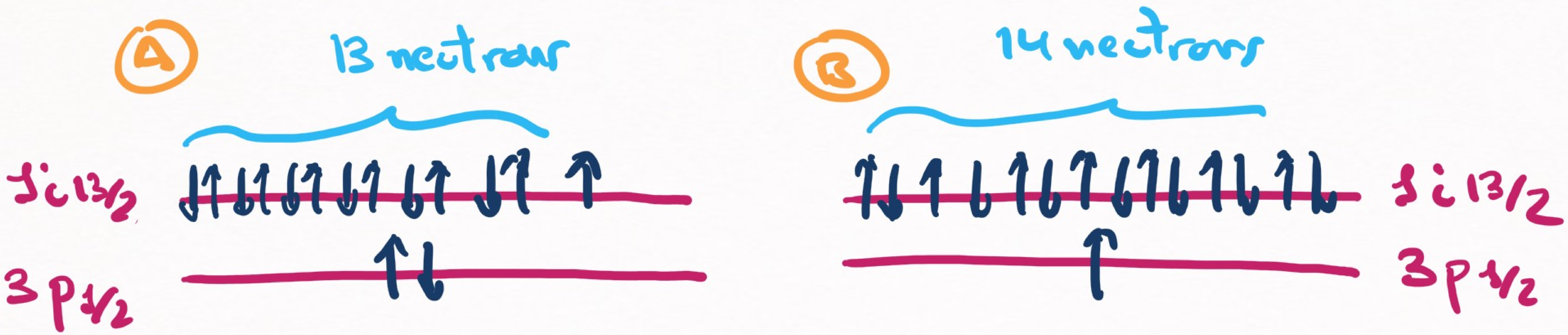
pairing

⇒ $J^P(^{203/205}Tl) = (\frac{1}{2})^+$

[SHELL MODEL: PAIRING INTERACTION] 3

Example 2: ^{207}Pb → 82 protons, 125 neutrons

we look at the neutrons

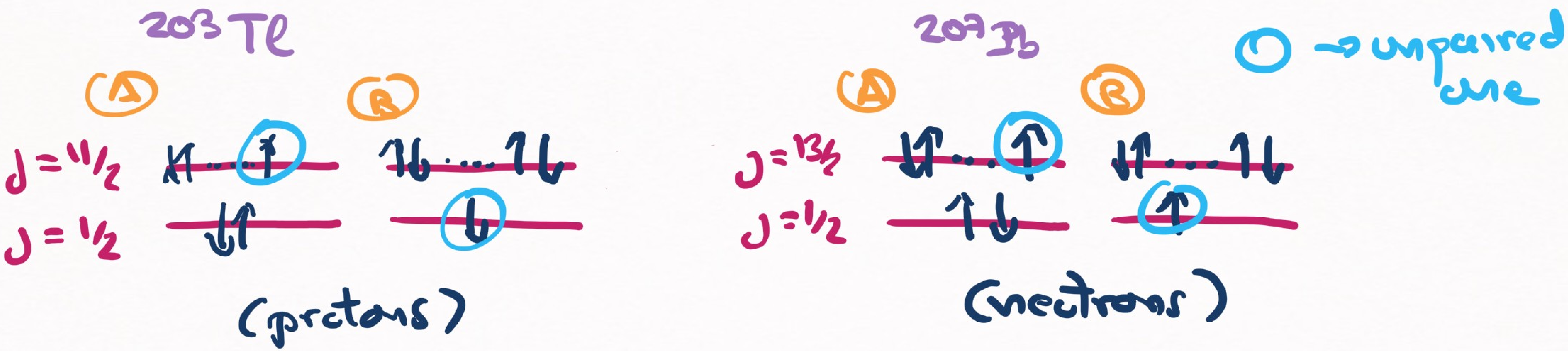


$$\boxed{E_B < E_A} \longrightarrow J^P(^{207}Pb) = \left(\frac{1}{2}\right)^+$$

$J_p = \left(\frac{1}{2}\right)^-$ $J_p = \left(\frac{13}{2}\right)^-$

[SHELL MODEL: PAIRING INTERACTION] (4)

$\Rightarrow$ In both cases the principle is the same



Pairing in the high j shell implies that $E_B < E_A$

$\Rightarrow$ Nature chooses option B

[SHELL MODEL: OUTLOOK]

⇒ These have been the basic features of the shell model

1) How to define V^{MF}?

2) How to deal with ΔV?

⇒ We will check this in a future lesson

RECAP !

1) Shell model: idea of a mean-field + fermi statistics

$$H = \sum_i H_i + \text{corrections} \Rightarrow H_i \Phi_i = \epsilon_i \Phi_i$$

$\Rightarrow$ We fill all the possible levels of ϵ_i

2) Spin-orbit interaction is required for magic numbers

3) Really simple mean field: $V_{MF} = \frac{1}{2} m_N \omega^2 r^2 - \xi \vec{\ell} \cdot \vec{s}$ $(\xi > 0)$

4) Pairing interaction: simplest example of residual interaction

See you on Tuesday

15:50