

NUCLEAR PHYSICS ②6

- a) FORMAL SCATTERING THEORY (PART II)
⇒ Solutions for contact-range theories
- b) BOUND / VIRTUAL STATES & RESONANCES
- c) THE ONE BOSON EXCHANGE MODEL

RECAP

a) Formal scattering theory \Rightarrow Usual scattering theory, but we use the language of operators acting on a Hilbert space

$$\left. \begin{aligned} \psi(\vec{r}) \\ \left[-\frac{\nabla^2}{2\mu} + V(\vec{r}) \right] \psi(\vec{r}) \\ = E\psi(\vec{r}) \end{aligned} \right\} \Rightarrow \begin{aligned} | \psi \rangle \\ (H_0 + V) | \psi \rangle \\ = E | \psi \rangle \end{aligned}$$

b) This idea also applies to scattering:

$$\psi(\vec{r}) \rightarrow \underbrace{e^{i\vec{k}\cdot\vec{r}}}_{=} + P(\omega) \frac{e^{ikr}}{r} \quad \left. \right\} \Rightarrow | \psi_{\vec{k}} \rangle = | \vec{k} \rangle + G_0 V | \psi_{\vec{k}} \rangle$$

with $G_0(E) = \frac{1}{E - H_0}$

RECAP /

c) The scattering amplitude $f(\omega)$ is now the matrix element of an operator (T-matrix):

$$f(\omega) = -\frac{\mu}{2\pi} \langle \vec{k}' | T(E) | \vec{k} \rangle \text{ with } T(E) = V + V G_0(E) T(E)$$

(physical scattering: $E + i\epsilon$, $\epsilon \rightarrow 0^+$)

d) Bound states are poles in the T-matrix:

$$T(E) \xrightarrow{E \rightarrow E_B} \frac{V |\psi_B\rangle \langle \psi_B| V}{E - E_B}$$

$(E_B < 0)$

Bound state equation:

$$|\psi_B\rangle = \underline{G_0(E) V |\psi_B\rangle}$$

[T-MATRIX AND CONTACT-RANGE INTERACTIONS]

a) We begin w/ a reminder: separable interactions

$$\langle \vec{k}' | V | \vec{k} \rangle = g \underbrace{f(\vec{k}')} \underbrace{f(\vec{k})} \Rightarrow \langle \vec{k}' | T(E) | \vec{k} \rangle = T(E) f(\vec{k}') f(\vec{k})$$

$$T(E) = \frac{1}{\frac{1}{g} - \int \frac{d^3q}{(2\pi)^3} \frac{f^2(\vec{q})}{E - q^2/2\mu}}$$

b) And a second reminder: contact-range potentials

$$V_c(\vec{r}) = C_0 \delta^{(3)}(\vec{r} - \vec{r}') \longrightarrow V_c(\vec{q}) = C_0$$

Fourier transform

$$\langle \vec{k}' | V_c | \vec{k} \rangle = C_0$$

$$\vec{q} = \vec{k} - \vec{k}'$$

[T-MATRIX & CONTACTS]

c) Contacts have a really big problem: $\rho(\vec{k}) = 1$

$$\frac{1}{\tau(E)} = \frac{1}{\tau_0} - \underbrace{\int \frac{d^3\vec{q}}{(2\pi)^3} \frac{1}{E - \vec{q}^2/2\mu}}$$

$$\int \frac{d^3\vec{q}}{(2\pi)^3} \frac{1}{E - \vec{q}^2/2\mu} \sim -\frac{2\mu}{(2\pi)^3} \int \frac{d^3\vec{q}}{|\vec{q}^2|^2} \sim -\frac{\mu}{\pi^2} \int \frac{q^2 dq}{q^2} \sim -\frac{\mu}{\pi^2} \int dq$$

$$\int_0^\infty dq \rightarrow \infty = \text{D} \int^{\wedge} dq \sim \wedge$$

There is a linear divergence with this potential

[T-MATRIX AND CONTACTS]

$f(x) \rightarrow$ a regulator function

d) But we can regularize the Dirac-delta:

$$\langle \vec{k}' | V_c | \vec{k} \rangle = C_0 \longrightarrow \langle \vec{k}' | V_c | \vec{k} \rangle = C_0 \underbrace{\rho\left(\frac{k'}{\Lambda}\right)}_{=} \underbrace{\rho\left(\frac{k}{\Lambda}\right)}_{=}$$

$f(x)$: d.1) $f(x) \xrightarrow{x \rightarrow 0} f$ (at low momenta: $\langle \vec{k}' | V_c | \vec{k} \rangle \rightarrow C_0$)

d.2) $f(x) \xrightarrow{x \rightarrow \infty} 0 \longrightarrow \int \frac{d^3 \vec{q}}{(2\pi)^3} \frac{1}{E - \vec{q}^2/2\mu} \rightarrow \int \frac{d^3 q}{(2\pi)^3} \frac{f^2(q)}{E - \vec{q}^2/2m}$

d.3) $\int dx f^2(x) < \infty \sim -\frac{\mu}{\pi^2} \int dq f^2(q) < \infty$

[T-MATRIX AND CONTACTS]

The two-body system collapses

e) For $\Lambda \rightarrow \infty$, we can see that this potential is singular

We consider $E = E_B$ (bound state) $E_B = -\frac{\gamma^2}{2\mu}$

$$x = g/\Lambda, \quad \tilde{\gamma} = \gamma/\Lambda$$

$$\int_0^\infty dx \rho^2(x) - \tilde{\gamma}^2 \int_0^\infty \frac{dx \rho^2(x)}{x^2 + \tilde{\gamma}^2} = -\frac{\pi^2 \Lambda}{\mu \Lambda C_0}$$

$$D \left[\frac{1}{\tau(E = E_B)} = 0 \right]$$

If $\Lambda \rightarrow \infty, \tilde{\gamma} \rightarrow \text{constant}$

$\rightarrow 0^+$
(for $C_0 < 0$)

$$\int \frac{d^3 \vec{q}}{(2\pi)^3} \frac{\rho^2(g/\Lambda)}{E_B - \frac{q^2}{2\mu}} = \frac{1}{C_0}$$

$\Rightarrow \gamma \propto \Lambda \rightarrow \infty$
 $E_B \rightarrow -\infty$ } COLLAPSE!

for C_0 attractive

[T-MATRIX AND CONTACTS]

8) How do we solve the collapse? $\Rightarrow C_0 = C_0(\Lambda)$

$$\frac{3}{C_0(\Lambda)} = -\frac{\mu}{\pi^2} \int_0^{\infty} \frac{q^2 dq}{\gamma^2 + q^2} \rho^2(q/\Lambda)$$

$$= -\frac{\mu}{\pi^2} \left[\Lambda \int_0^{\infty} \rho^2(x) dx - \gamma \int_0^{\infty} \frac{dx}{\gamma + x^2} \rho^2(x/\gamma) \right]$$

$$= \frac{\mu}{\pi^2} [\gamma F(\gamma/\Lambda) - \Lambda B_0]$$

$$= \frac{\mu}{2\pi} [\gamma - \Lambda G(\gamma/\Lambda)] \rightarrow$$

this reworking of the Eq. requires a bit of thinking, but it's really convenient

C_0 depends on Λ
 ρ is chosen to reproduce E_3/γ

$F(x), G(x)$
 are functions
 B_0 is a constant

[T-MATRIX AND CONTACTS] a sharp cutoff

\Rightarrow Exactly solvable example: $\rho(x) = \underline{\underline{\Theta(1-x)}}$

$$\frac{1}{C_0(\Lambda)} = -\frac{\mu}{\pi^2} \int_0^\Lambda \frac{q^2 dq}{q^2 + \gamma^2} = -\frac{\mu}{\pi^2} \left[\Lambda - \gamma \operatorname{atan} \frac{\Lambda}{\gamma} \right] \quad \begin{array}{l} \text{atan } x \\ = \frac{\pi}{2} - \operatorname{acot} x \end{array}$$

$$= \frac{\mu}{2\pi} \left[\gamma - \frac{2}{\pi} \Lambda \left(1 + \frac{\gamma}{\Lambda} \operatorname{acot} \frac{\Lambda}{\gamma} \right) \right]$$

\Rightarrow For $\Lambda \rightarrow \infty$,
it simplifies to $\frac{1}{C_0(\Lambda)} \xrightarrow{\Lambda \rightarrow \infty} \frac{\mu}{2\pi} \left(\gamma - \frac{2}{\pi} \Lambda \right)$

[T-MATRIX AND CONTACTS] \Rightarrow SCATTERING STATES

\Rightarrow Now we solve $f(x) = \Theta(1-x)$ for $E > 0$:
 singular point

$$\int \frac{d^3 \vec{q}}{(2\pi)^3} \frac{f^2(q/\Lambda)}{E + i\epsilon - q^2/2\mu} = \frac{\mu}{\pi^2} \int_0^\Lambda \frac{q^2 dq}{k^2 + i\epsilon - q^2} \rightarrow \left[\frac{1}{k^2 - q^2} \text{ for } q=k \right]$$

\Rightarrow Trick: the principal value of an integral \leftarrow

$$\mathcal{P} \int_{-\infty}^{+\infty} f(x) dx = \lim_{\epsilon \rightarrow 0^+} \left[\int_{-\infty}^{-\epsilon} + \int_{-\epsilon}^{+\epsilon} + \int_{+\epsilon}^{+\infty} f(x) dx \right]$$

$$f(x) = \frac{c_0}{x} \Rightarrow \mathcal{P} \int_{-\infty}^{+\infty} f(x) dx = 0$$

\Rightarrow Exactly symmetric limit

[T-MATRIX AND CONTACTS] \Rightarrow SCATTERING STATES

\Rightarrow Principal value : $\mathcal{P} \int_{-\infty}^{\infty} \frac{dx}{x} = 0$ (example)

a) Useful formula: $\frac{1}{x \pm i\epsilon} = \mathcal{P} \left(\frac{1}{x} \right) \mp i\pi \delta(x)$

b) If we apply it to our integral:

$$\frac{\mu}{\pi^2} \int_0^{\Lambda} \frac{q^2 dq}{k^2 + i\epsilon - q^2} = \frac{\mu}{\pi^2} \left[\mp i\frac{\pi}{2} k - \Lambda + \frac{k}{2} \log \left| \frac{\Lambda - k}{\Lambda + k} \right| \right]$$

please check, using a)
+ b)

$$k = \sqrt{2\mu E}$$

[T-MATRIX AND CONTACTS] \Rightarrow SCATTERING STATES

\Rightarrow Now we can calculate $\tau(E)$:

$$\tau(E) = \frac{1}{\frac{1}{\omega(\lambda)} - \frac{\mu}{\pi} \int_0^{\infty} \frac{g^2 dg}{k^2 4i\epsilon - g^2}} = \frac{1}{\frac{1}{\omega(\lambda)} + \frac{\mu}{2\pi} \left[ik + \frac{2}{\pi} \Lambda - \frac{k}{\pi} \log \left| \frac{\Lambda - k}{\Lambda + k} \right| \right]}$$

\downarrow
 $E + i\epsilon$

$$f(\omega) = -\frac{\mu}{2\pi} \langle \vec{k}' | T(E + i\epsilon) | \vec{k} \rangle$$

Δ
 \downarrow matches the expressions

$$\hookrightarrow \tau(E) = -\frac{2\pi}{\mu} \frac{1}{k \omega(\lambda) - ik} = -\frac{2\pi}{\mu} \frac{1}{-\frac{2\pi}{\mu} \frac{1}{\omega(\lambda)} - \frac{2}{\pi} \Lambda + \dots - ik}$$

[T-MATRIX AND CONTACTS] \Rightarrow SCATTERING STATES

\Rightarrow Matching with $f_0(k)$ we arrive at: $k \cot \delta = -\frac{1}{a_0} + \dots$

$$k \cot \delta = -\frac{2\pi}{\mu C_0(\Lambda)} - \frac{2}{\pi} \Lambda + \dots \quad \Rightarrow \quad \frac{1}{C_0(\Lambda)} = \frac{\mu}{2\pi} \left(\frac{1}{a_0} - \frac{2}{\pi} \Lambda \right)$$

\Rightarrow For $\Lambda \rightarrow \infty$, we obtain:

Really neat equation

$$\langle \vec{k}' | T(E+i\epsilon) | \vec{k} \rangle \rightarrow \frac{2\pi}{\mu} \frac{1}{\frac{1}{a_0} + ik}$$

try to check this result

[T-MATRIX AND CONTACTS] \Rightarrow FROM SCATTERING TO BOUND STATES

a) For the $\Lambda \rightarrow \infty$ limit, we have:

$$\left[\langle \vec{k}' | T(E+i\epsilon) | \vec{k} \rangle = \frac{2\pi}{k} \frac{1}{\frac{1}{a} + ik}, \quad k = \sqrt{2\mu E} \right]$$

b) But, where are the bound states here? \rightarrow Poles

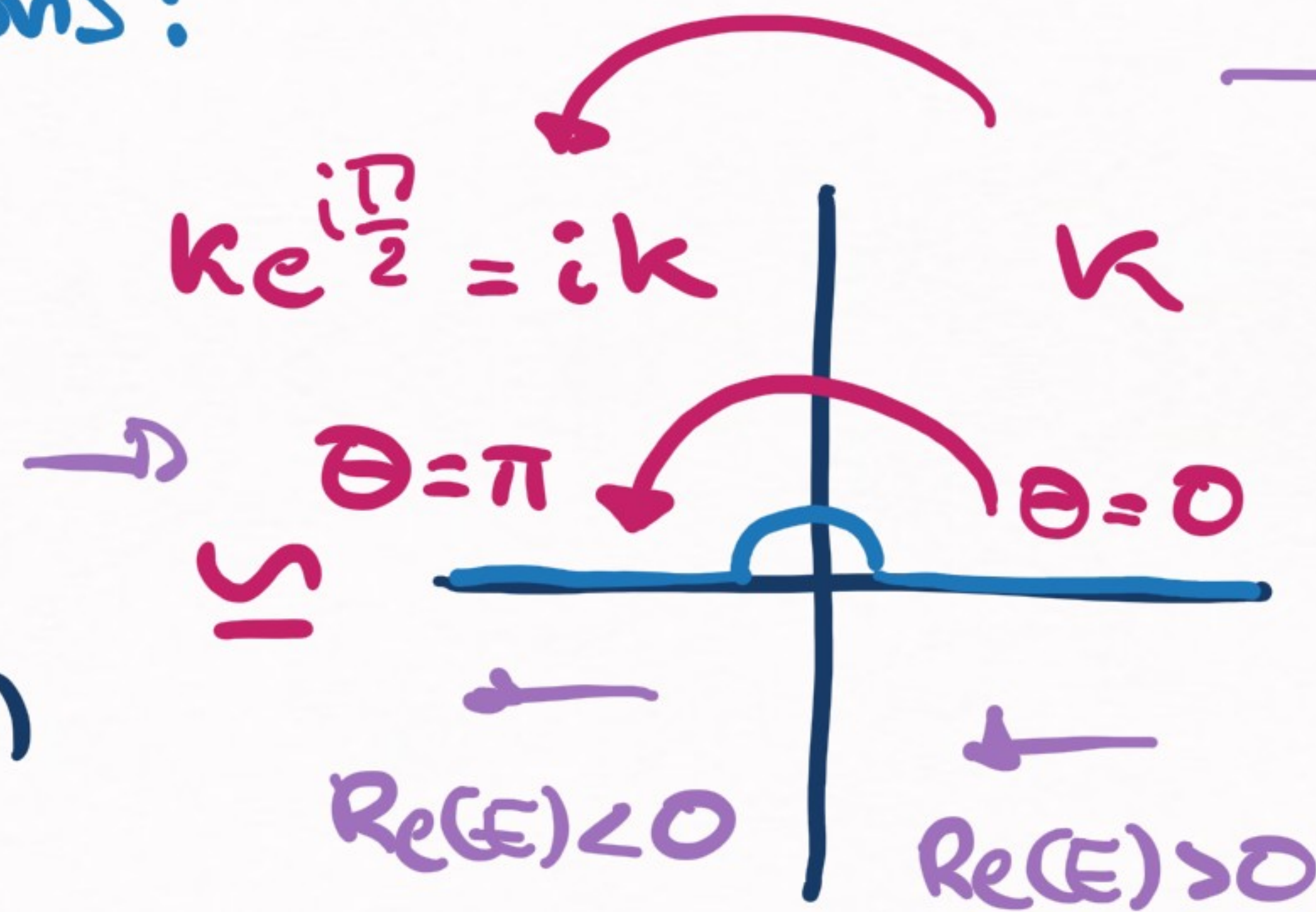
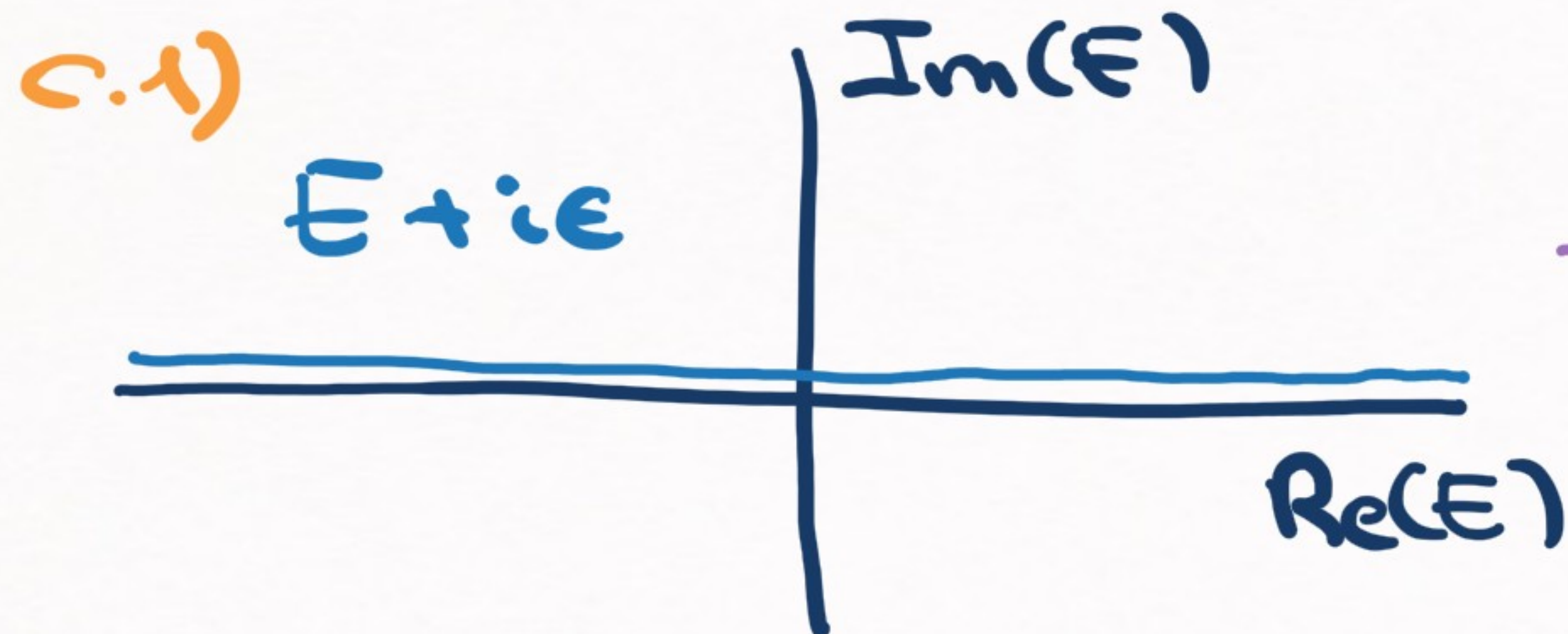
$$E_B = -B \quad (E_B < 0 \Rightarrow B > 0) \quad \Rightarrow \quad k = \sqrt{2\mu E} = \sqrt{-2\mu B}$$

If $B = \frac{\gamma^2}{2\mu} \Rightarrow \boxed{k = \pm i\gamma}$ \leftarrow Could be problematic

How do we choose the correct root?

[T-MATRIX AND CONTACTS] \Rightarrow FROM SCATTERING TO BOUND STATES

c) The $E \pm i\epsilon$ prescriptions:



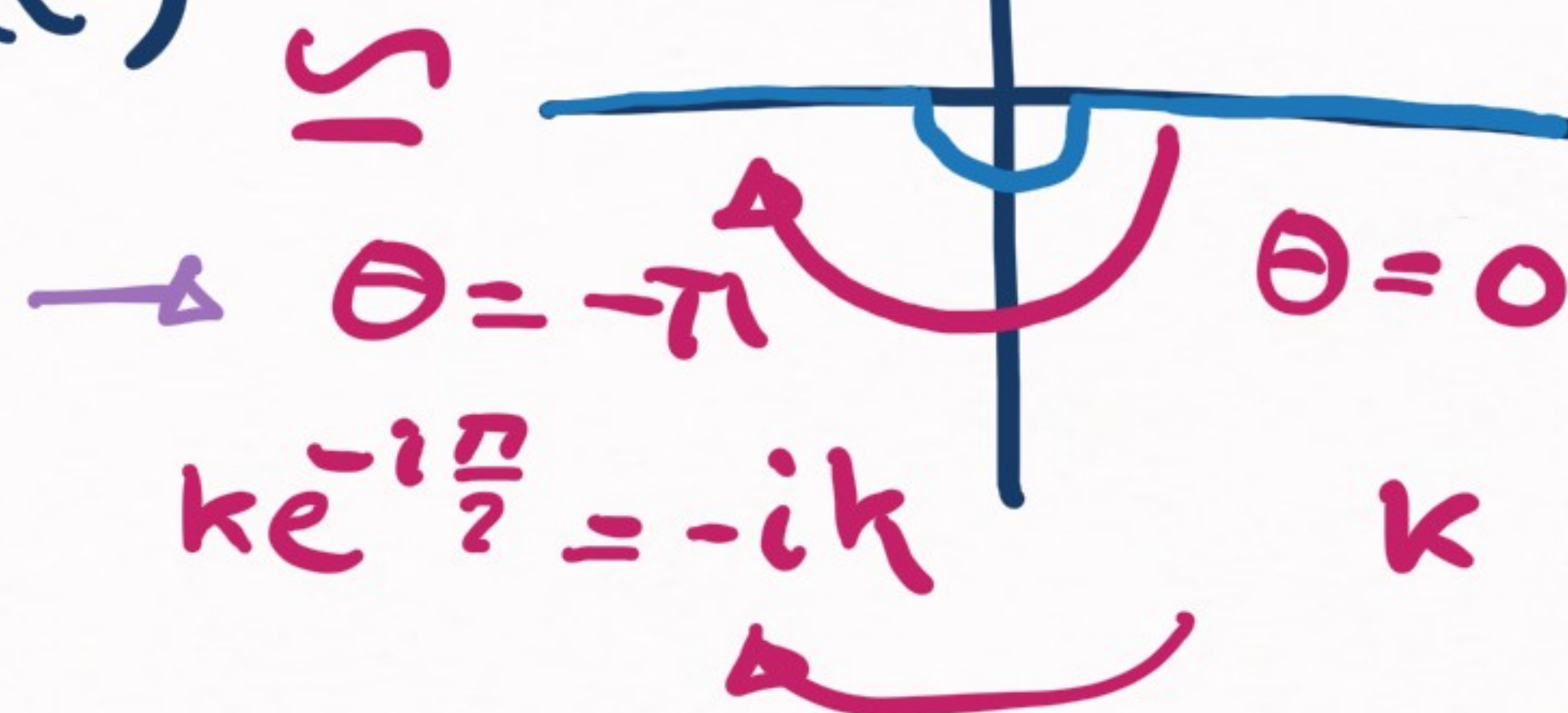
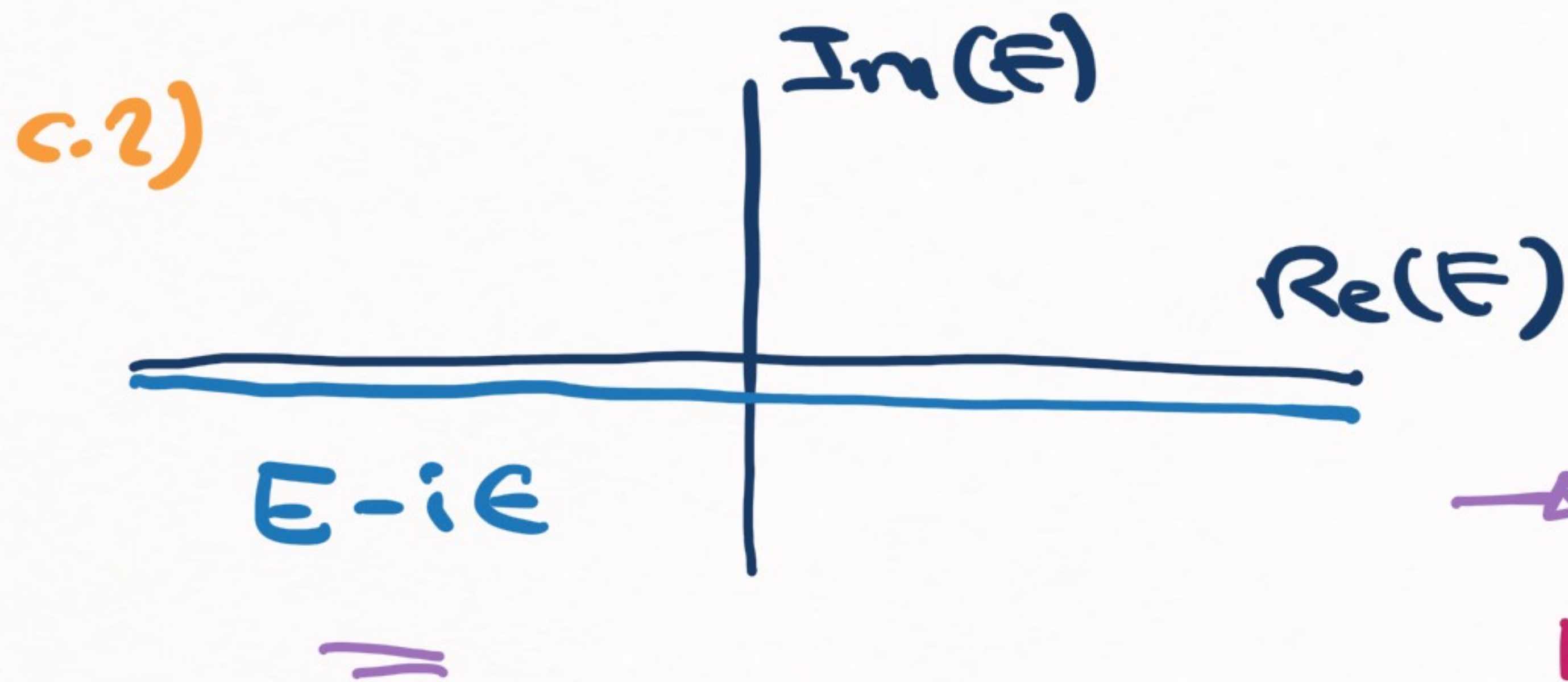
$$\rightarrow E = |E| e^{i\theta}$$

\Downarrow

$$k = \sqrt{2\mu E}$$

\Downarrow

$$k = |k| e^{i\frac{\theta}{2}}$$



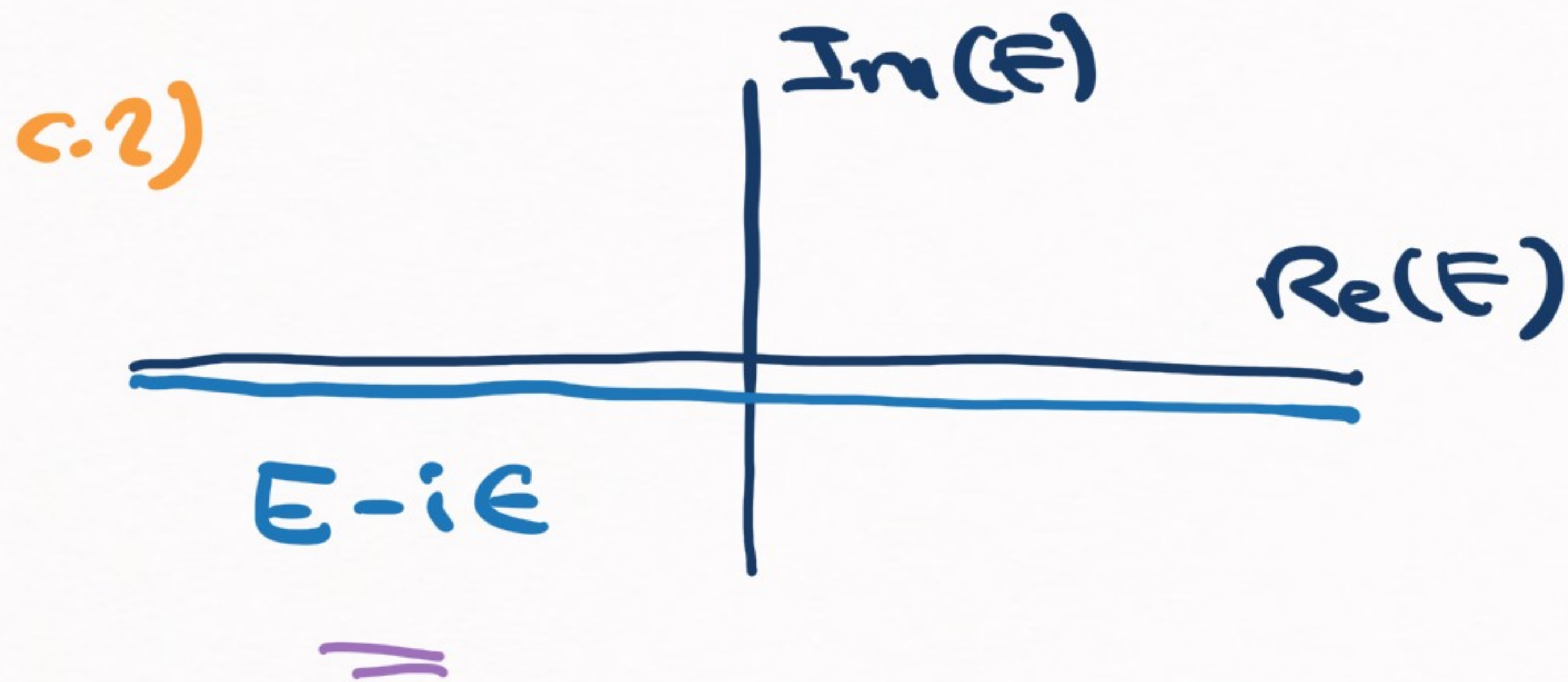
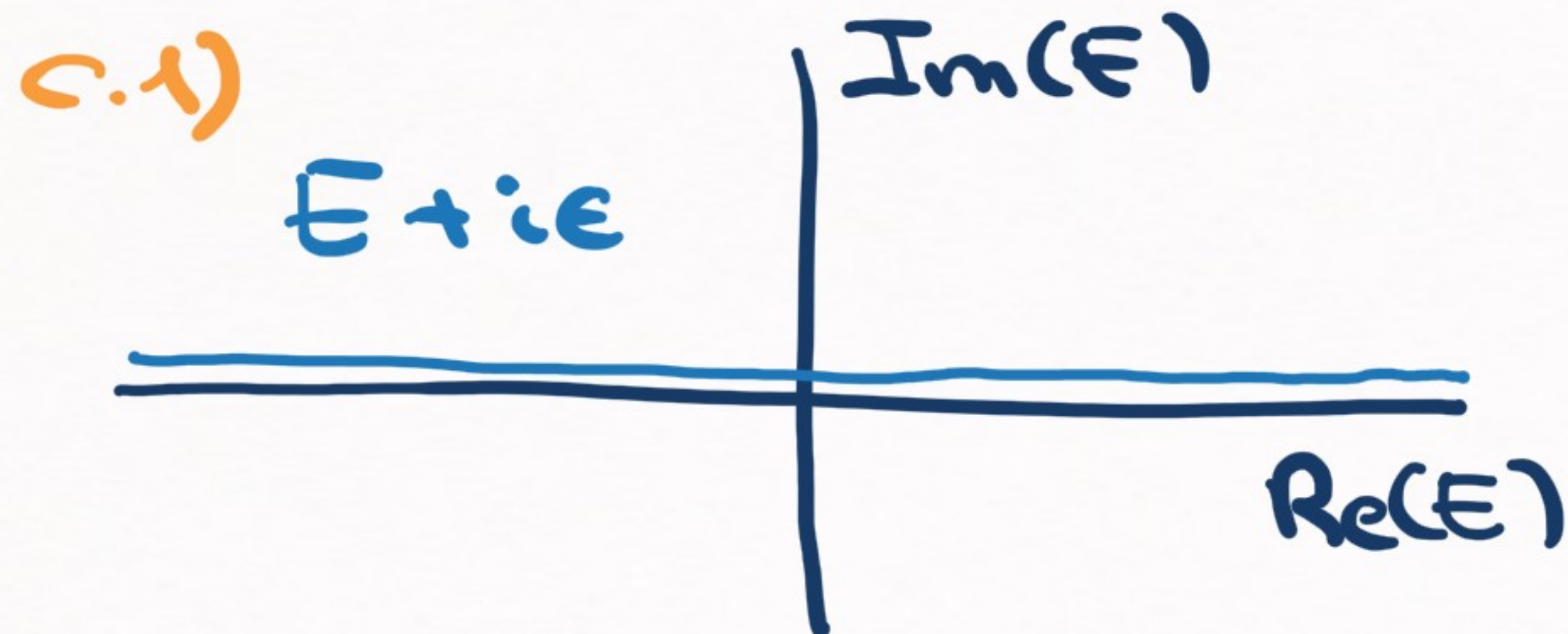
Sign of θ :

$\theta > 0$ for counterclockwise

$\theta < 0$ for clockwise

[T-MATRIX AND CONTACTS] \Rightarrow FROM SCATTERING TO BOUND STATES

c) The $E \pm i\epsilon$ prescriptions:



CAVEAT

We are using " $E \pm i\epsilon$ " in a heuristical way here.

Stricto sensu

$$T(E \pm i\epsilon) = \frac{2\pi}{\mu} \frac{1}{1/a \pm ik}$$

but here we are using the $E \pm i\epsilon$ idea as notation for the two ways of going to $E < 0$

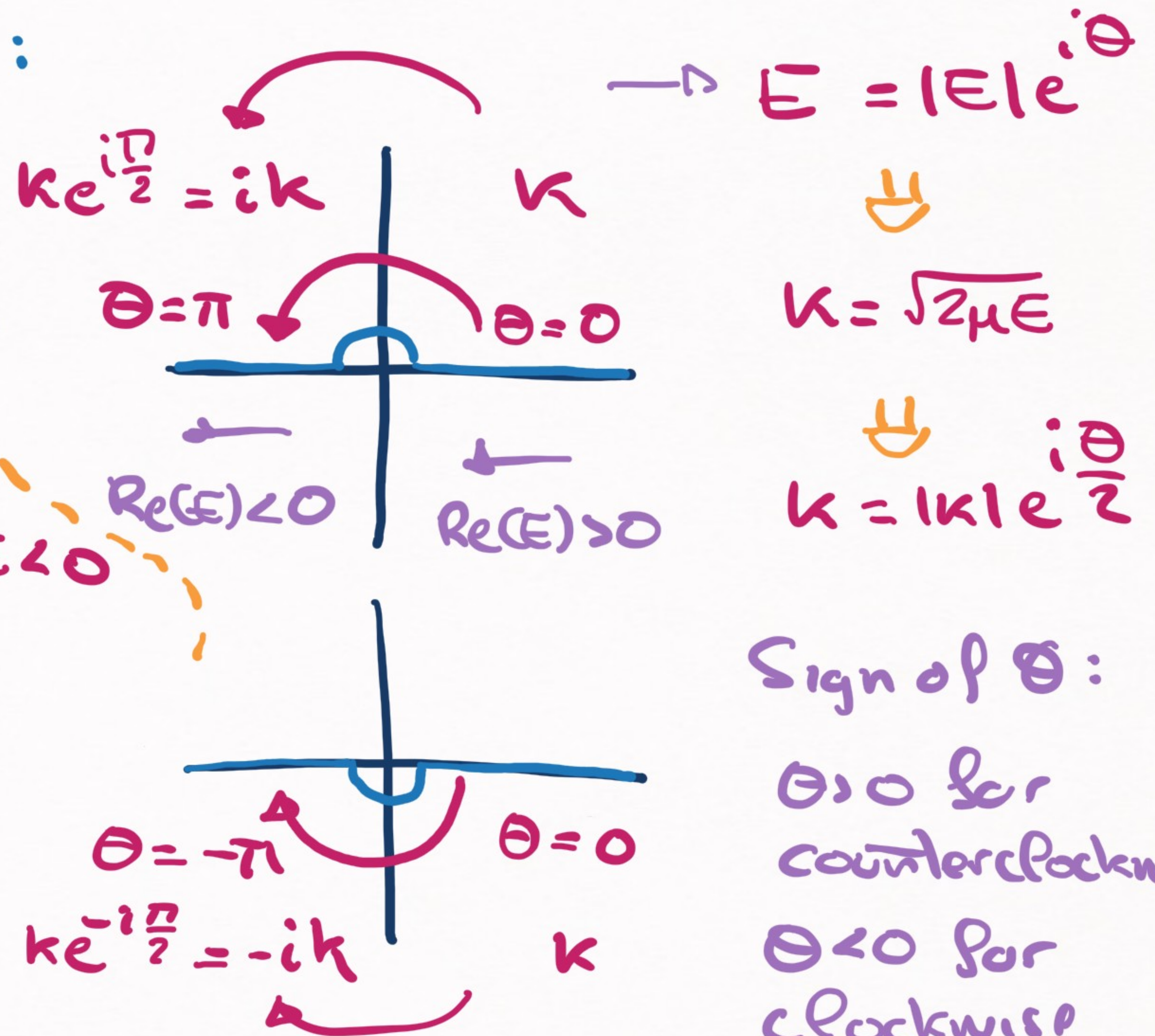
[T-MATRIX AND CONTACTS] \Rightarrow FROM SCATTERING TO BOUND STATES

c) The $E \pm i\epsilon$ prescriptions:

CAVEAT (cont'd)

Actually, this would be the key idea:

for going from $E > 0$ to $E < 0$ there are two possibilities



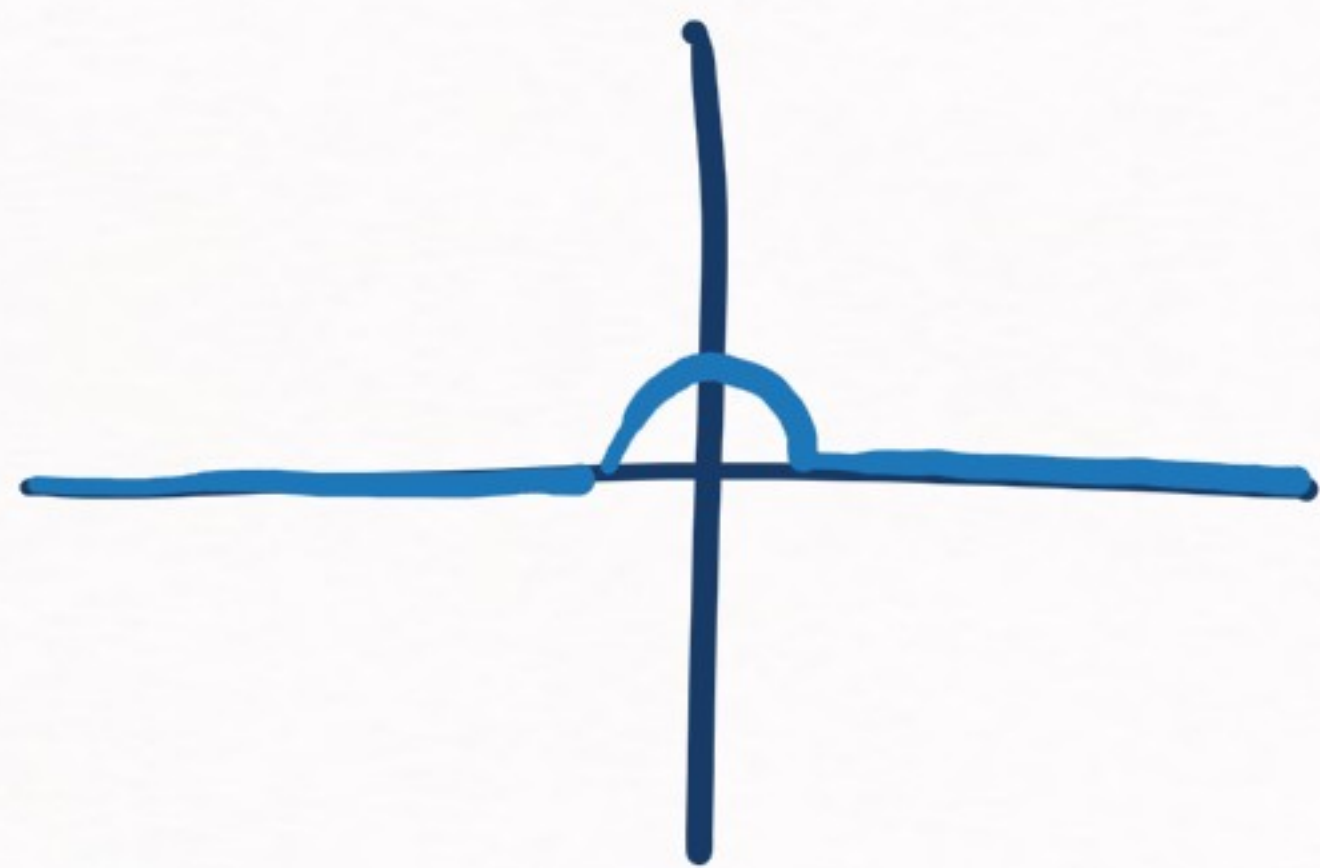
[T-MATRIX AND CONTACTS] \Rightarrow FROM SCATTERING TO BOUND STATES

c) The $\epsilon \pm i\epsilon$ prescription (cont'd)

$$E = \frac{k^2}{2\mu} \quad (E > 0)$$

$$E = -\frac{\gamma^2}{2\mu} \quad (E < 0)$$

c.1)



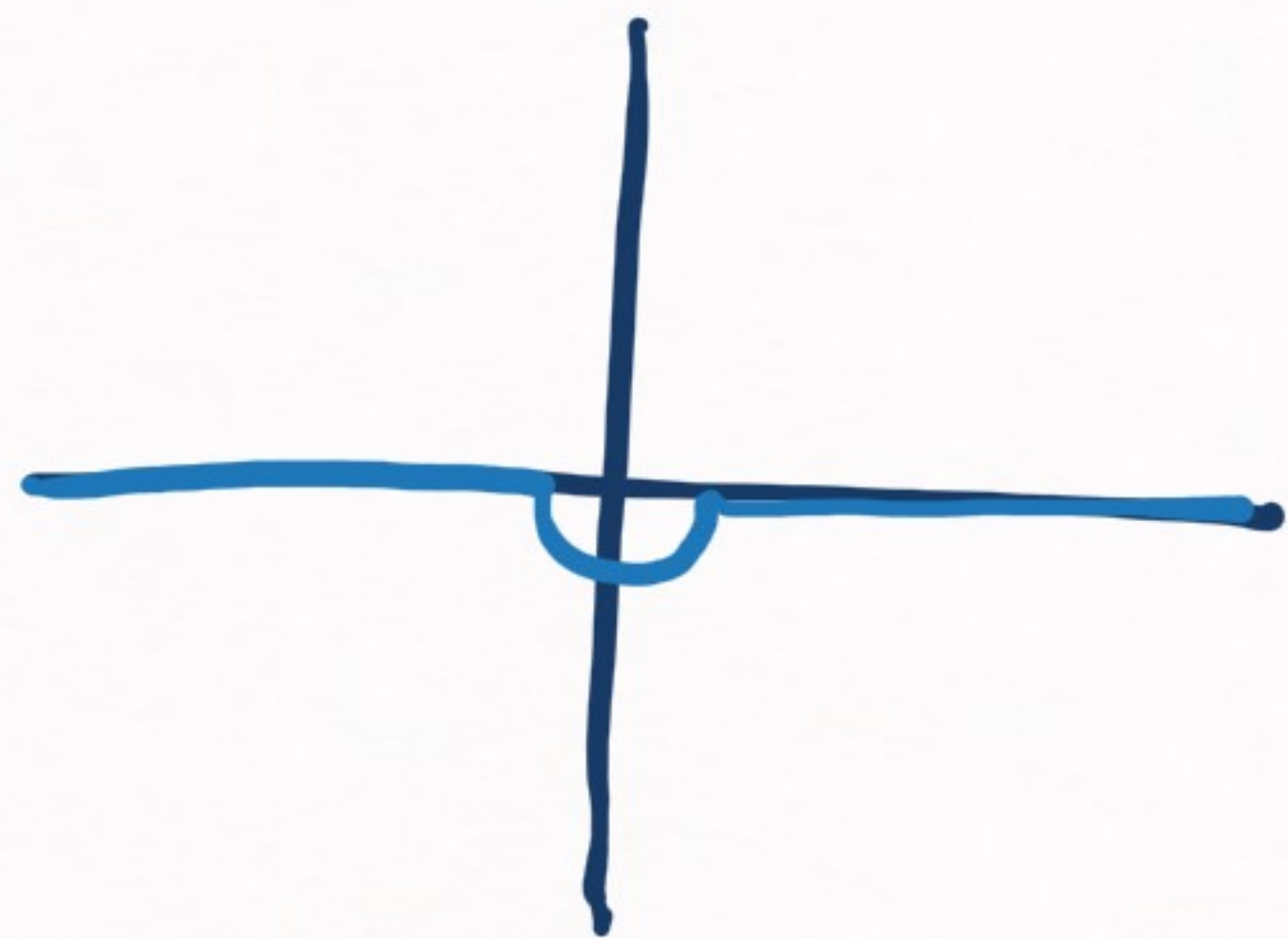
$$k \rightarrow i\gamma$$

$$\frac{1}{\frac{1}{a_0} + ik} \rightarrow \frac{1}{\frac{1}{a_0} - \gamma}$$

$$(E \rightarrow |E|e^{i\pi})$$

c.1) \exists pole at $\gamma = \frac{1}{a_0}$ if $a_0 > 0$

c.2)



$$k \rightarrow -i\gamma$$

$$\frac{1}{\frac{1}{a_0} + ik} \rightarrow \frac{1}{\frac{1}{a_0} + \gamma}$$

$$(E \rightarrow |E|e^{-i\pi})$$

c.2) \exists pole at $\gamma = -\frac{1}{a_0}$ if $a_0 < 0$

[T-MATRIX AND CONTACTS] \Rightarrow FROM SCATTERING
TO BOUND STATES

c) The " $E \pm i\epsilon$ " ($|E|e^{i\theta}$) prescription

c.1) For " $+i\epsilon$ ": $T(E) = -\frac{2\pi}{\mu} \frac{1}{\frac{1}{a_0} + ik} \longrightarrow -\frac{2\pi}{\mu} \frac{1}{\frac{1}{a_0} - \gamma} \checkmark$
($\theta = \pi$)

c.2) For " $-i\epsilon$ ": $T(E) = -\frac{2\pi}{\mu} \frac{1}{\frac{1}{a_0} + ik} \longrightarrow -\frac{2\pi}{\mu} \frac{1}{\frac{1}{a_0} + \gamma} \checkmark$
($\theta = -\pi$)

d) Poles:

$$E = |E|e^{i\pi} \longrightarrow E_B = -\frac{1}{2\mu} \left(\frac{1}{a_0}\right)^2, \gamma = \frac{1}{a_0}$$

$$E = |E|e^{-i\pi} \longrightarrow E_B = -\frac{1}{2\mu} \left(\frac{1}{a_0}\right)^2, \gamma = -\frac{1}{a_0}$$

Which one of these poles corresponds to a bound state?

[T-MATRIX AND CONTACTS] \Rightarrow FROM SCATTERING
TO BOUND STATES

e) Which of these choices
gives us a bound state?

Reminder \Rightarrow Orthogonality of bound & scattering
states

$$\langle \psi_B | \vec{v}_k \rangle \propto \int_0^{\infty} dr e^{-\gamma r} \sin(kr + \delta) = \frac{k \cos \delta + \gamma \sin \delta}{k^2 + \gamma^2}$$

$\stackrel{0}{\parallel}$

If $\langle \psi_B | \vec{v}_k \rangle = 0 \Rightarrow k \cot \delta = -\gamma$ ($k \cot \delta = -\frac{1}{a_0}$)

a bound state (γ) requires $a_0 > 0$

[T-MATRIX AND CONTACTS] \Rightarrow FROM SCATTERING
TO BOUND STATES

e) Which of these choices
gives us a bound state?

The answer is: For $E = |E|e^{i\pi}$, $E_B = -\frac{1}{2\mu} \left(\frac{1}{a_0}\right)^2$, $\gamma = \frac{1}{a_0}$

this gives us the bound state

$$T(E+i\epsilon) = \frac{2\pi}{\mu} \frac{1}{\gamma a + i\epsilon} = \frac{2\pi}{\mu} \frac{\gamma a_0 + \sqrt{-2\mu E}}{\gamma a_0 + 2\mu E}$$

\hookrightarrow pole in $T(E)$

$$\Rightarrow \text{Res } T(E = E_B) = \frac{2\pi}{\mu^2} \frac{1}{a_0} = \frac{2\pi \gamma_B}{\mu^2} \quad \left(\frac{1}{a_0} = \gamma_B\right)$$

[T-MATRIX AND CONTACTS] \Rightarrow FROM SCATTERING TO BOUND STATES

f) Calculating the wave function:

$$\text{Res} T(E = -E_B) = \frac{2\pi}{\mu^2} \left(\frac{1}{a_0} \right) = \frac{2\pi}{\mu^2} \gamma_B \rightarrow \text{residue}$$

$$\text{Res} T = V |\psi_B\rangle \langle \psi_B| V = G_0^{-1}(E = E_B) |\psi_B\rangle \langle \psi_B| G_0^{-1}(E = E_B)$$

$$|\psi_B\rangle = G_0 V |\psi_B\rangle = \left(E_B - \frac{\vec{k}'^2}{2\mu} \right) \psi_B(\vec{k}') \psi_B(\vec{k}) \left(E_B - \frac{\vec{k}^2}{2\mu} \right)$$

$$\langle \vec{k}' | (\dots) | \vec{k} \rangle$$

Our wave function $\Rightarrow \psi_B(\vec{p}) = \pm \frac{\sqrt{2\mu\gamma_B/\mu^2}}{E_B - \frac{\vec{p}^2}{2\mu}} = \mp \frac{\sqrt{8\pi\gamma_B}}{\gamma^2 + \vec{p}^2}$

[T-MATRIX AND CONTACTS] \Rightarrow FROM SCATTERING TO BOUND STATES

f) Calculating the wave function (II)

\Rightarrow The final result: $\psi_B(\vec{r}) = \frac{\sqrt{2\pi\gamma_B}}{\gamma_B^2 + k^2}$

\Rightarrow Something amazing about this result:

Fourier transform $\psi(\vec{r})$

$$\int \frac{d^3\vec{q}}{(2\pi)^3} |\psi_B(\vec{q})|^2 = \int d^3\vec{r} |\psi_B(\vec{r})|^2 = 1$$

$$= \frac{1}{\sqrt{4\pi}} \frac{\sqrt{2\gamma_B} e^{-\gamma_B r}}{r}$$

\Rightarrow The residue of $T(E)$ even gives us the correct normalization. Wave function for a pure contact-range theory

[T-MATRIX AND CONTACTS] \Rightarrow THE OTHER SIDE OF THE COMPLEX PLANE 

\Rightarrow But, if $\omega_0 < 0$,

what happens with the pole in $E = |E|e^{-i\pi}$?

$E = |E|e^{i\pi}$, $E = |E|e^{-i\pi}$
 two prescriptions

$$T(E) = -\frac{2\pi}{\mu} \frac{1}{\frac{1}{\omega_0} + i\kappa} \rightarrow -\frac{2\pi}{\mu} \frac{1}{\frac{1}{\omega_0} + \gamma}$$

\Rightarrow We acknowledge the existence of this second type of pole

We call it a virtual state pole

$$\gamma = -\frac{1}{\omega_0}$$

is a pole
now

[T-MATRIX AND CONTACTS] \Rightarrow THE OTHER SIDE
OF THE COMPLEX PLANE

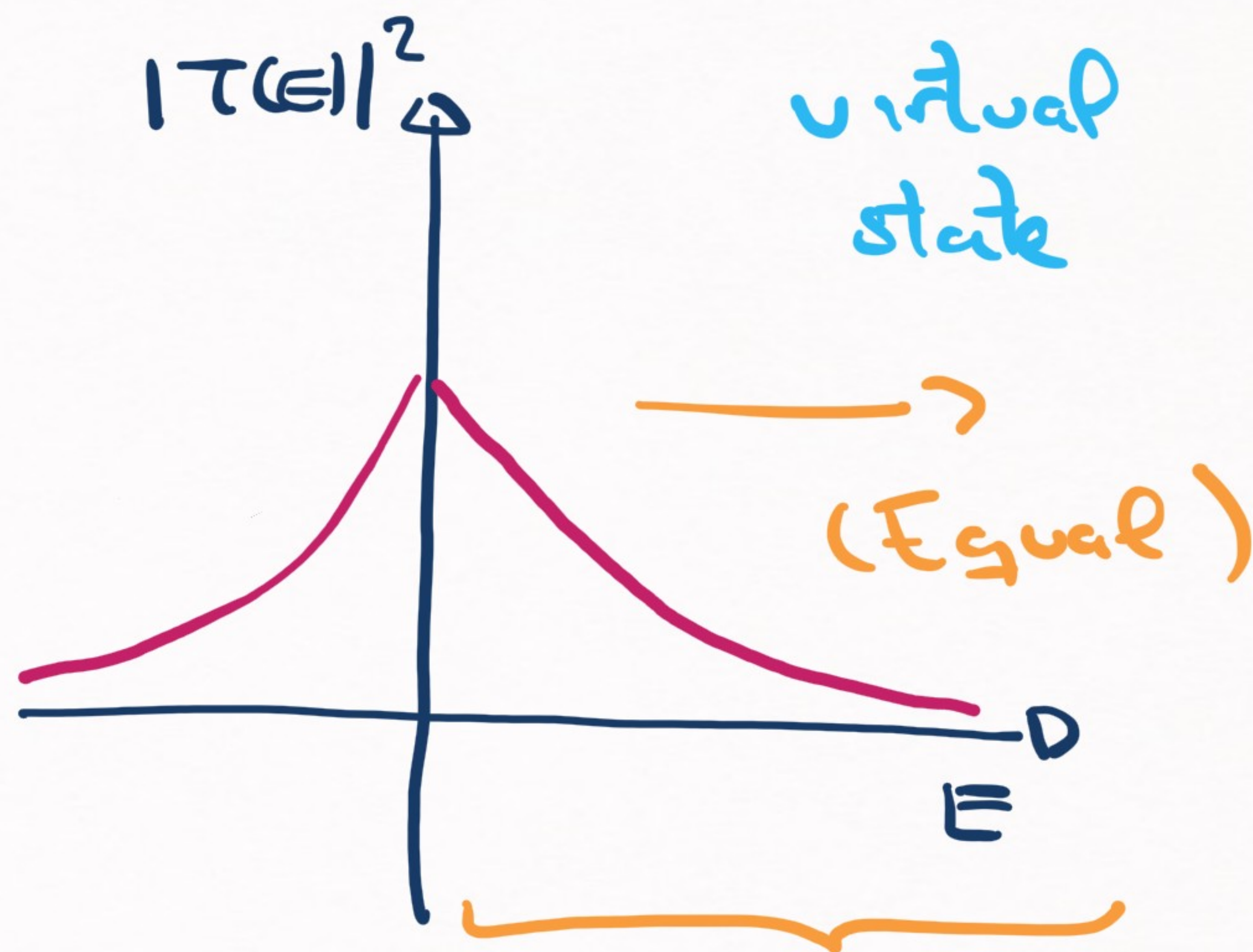
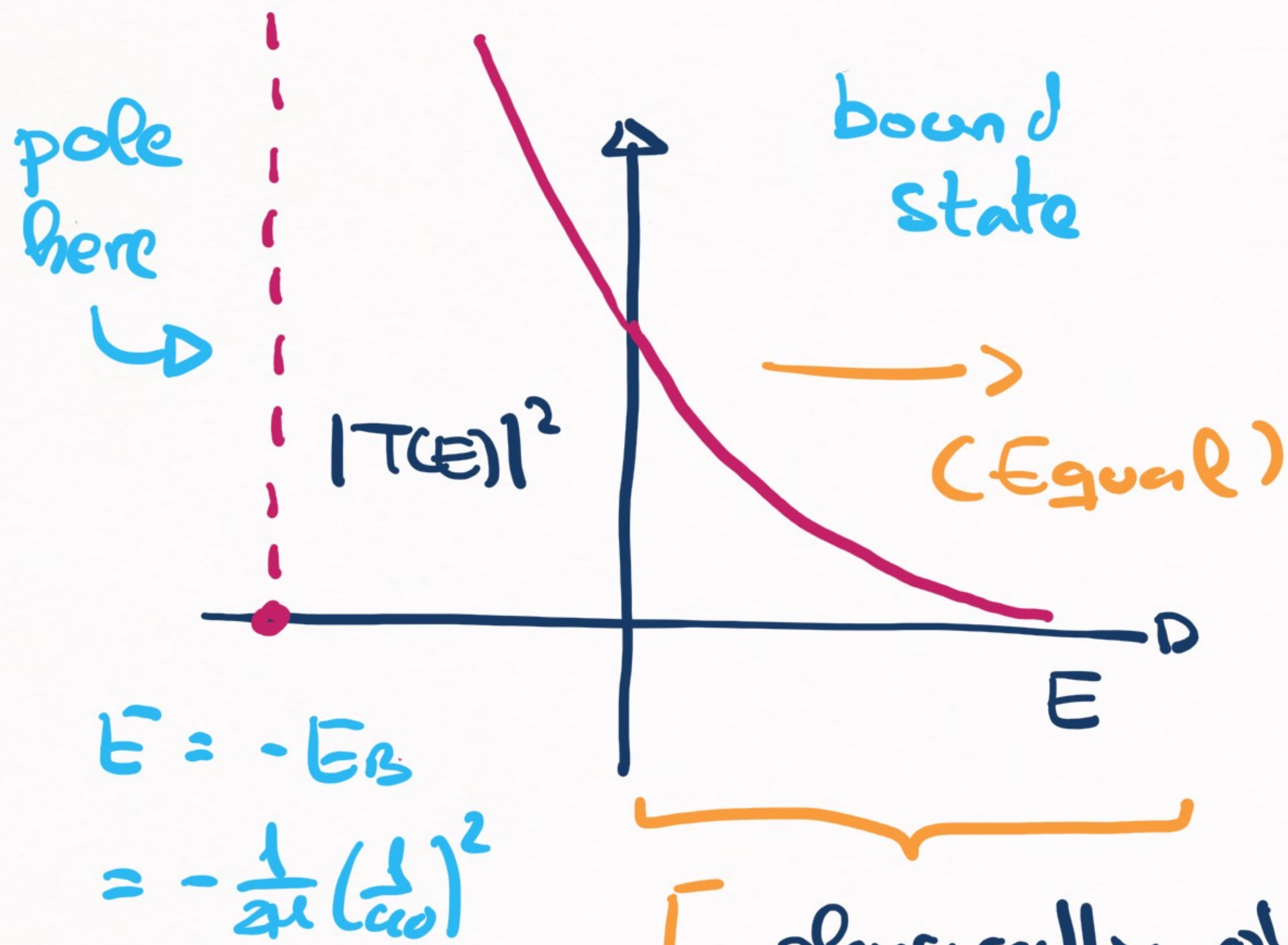
\Rightarrow But, a virtual state pole,
is it less of a pole than a bound state pole?

It depends.... even though there is no two-body,
physical state that corresponds
to this pole ...

\downarrow
If we look
at low energy
scattering \rightarrow $\left[\sigma \xrightarrow{k \rightarrow 0} 4\pi |a_0|^2 \right] \rightarrow$ the sign of
 a_0 does
not matter

[T-MATRIX AND CONTACTS] \Rightarrow THE OTHER SIDE OF THE COMPLEX PLANE

\Rightarrow Bound vs virtual state:



[physically observable energies in scattering]

[T-MATRIX AND CONTACTS] \Rightarrow THE OTHER SIDE
OF THE COMPLEX PLANE

a) Bound state solution:

$$\frac{e^{ikr}}{r} \rightarrow \frac{e^{-\gamma r}}{r} \quad (k \rightarrow +i\gamma)$$

This is a mathematical consequence

b) Virtual state solution:

$$\frac{e^{ikr}}{r} \rightarrow \frac{e^{+\gamma r}}{r} \quad (k \rightarrow -i\gamma)$$

of $E = \frac{k^2}{2\mu}$ being valid for both signs of k :

$$\left[E = \frac{(\pm k)^2}{2\mu} \right]$$

[T-MATRIX AND CONTACTS] \Rightarrow THE OTHER SIDE
OF THE COMPLEX PLANE

c) Or we can directly begin with:

$$\psi_{\vec{r}}(\vec{r}) \rightarrow 4\pi \sum_{lme} i^l \frac{S_l(k) e^{ikr} - e^{-ikr}}{2ir} \sum_{m'} Y_{lm}(\hat{r}) Y_{l'm'}(\hat{r})$$

$$\Rightarrow S_l(k) = e^{2i\delta_l(k)} = 1 + 2ik P_l(k) = 1 - ik \frac{\mu}{\hbar} t_l(k)$$

$\Rightarrow S_l(k)$ has also a pole for $k = +i\gamma$ / $-i\gamma$
(bound / virtual)

$$k = +i\gamma \Rightarrow \psi_k(\vec{r}) \rightarrow (\infty) \frac{e^{-\gamma r}}{r}$$

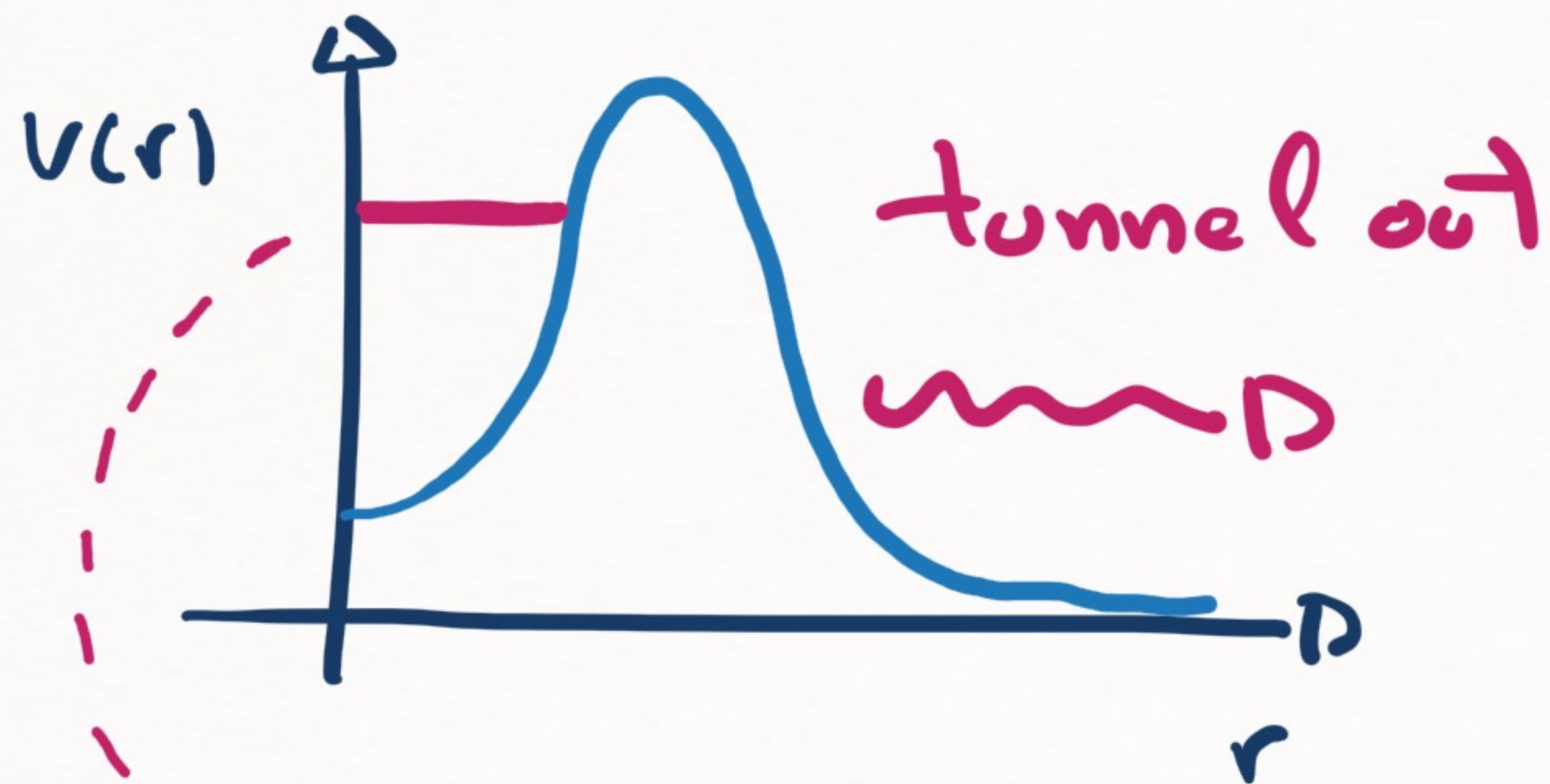
$$k = -i\gamma \Rightarrow \psi_k(\vec{r}) \rightarrow (\infty) \frac{e^{+\gamma r}}{r}$$

[RESONANCES] ③

the state will decay:

a) Consider this potential:

$$[P(t) = e^{-t/\tau}]$$



} = D

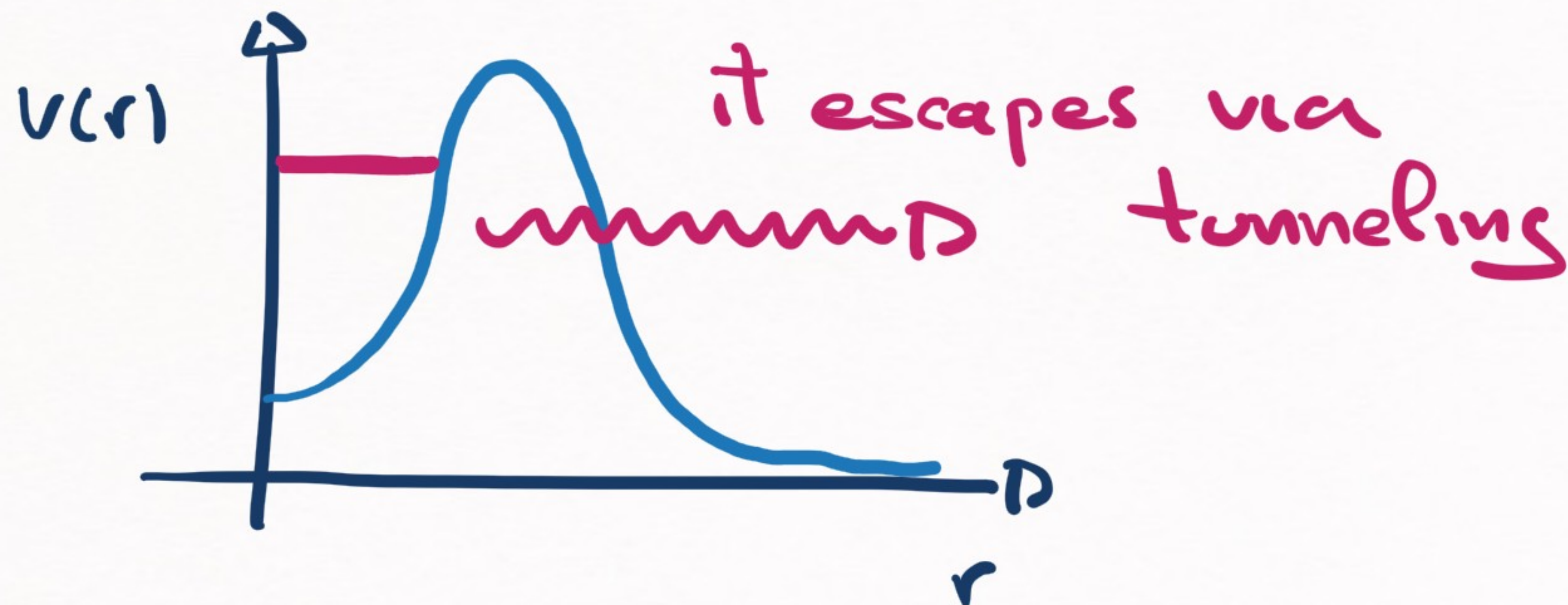
probability of finding the two particles in the classical bound state configuration

This "bound state" is possible in classical mechanics, but not in quantum mechanics

⇒ they tunnel out

[RESONANCES] ②

a) Consider this potential:



⇒ The QM version of this state will have:

$$\Rightarrow \left[\begin{array}{l} \text{Im}(E_3) < 0 \\ \text{or} \quad \tau = \frac{2}{-\text{Im}(E)} = \frac{1}{\Gamma} \end{array} \right]$$

Probability of survival:

$$P(t) \propto e^{-t/\tau}$$

For a true bound state we have the time dependence:

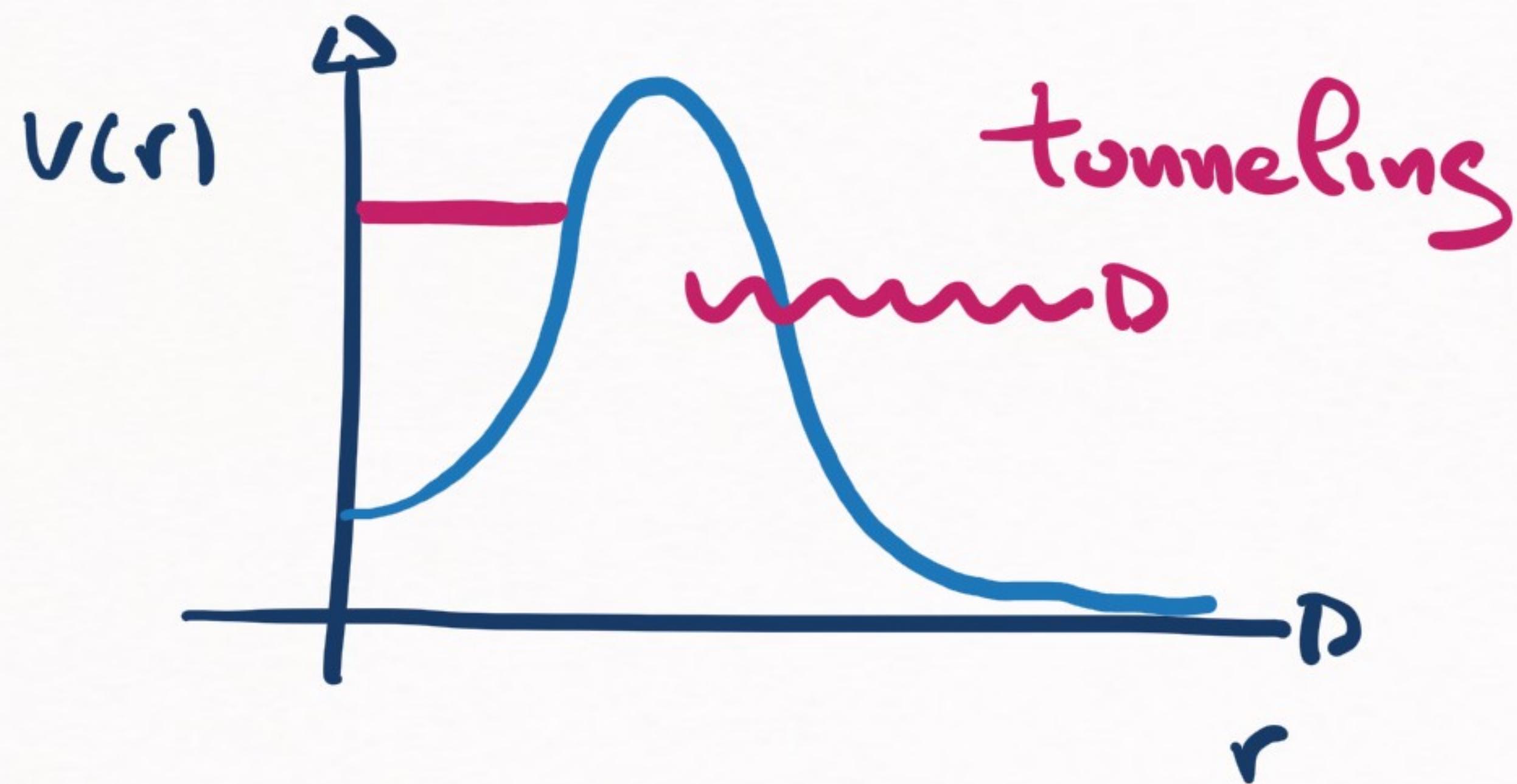
$$\psi(\vec{r}, t) = \psi(\vec{r}) e^{-iE_0 t}$$

if we allow complex E_3 :

$$\psi(\vec{r}, t) = \psi(\vec{r}) e^{-i\text{Re}(E_3)t - \text{Im}(E_3)t}$$

[RESONANCES] ③

a) Consider this potential \Rightarrow In QM, this classical state will have complex energy



$$E = E_r - i \frac{\Gamma}{2} \quad (\Gamma > 0, \tau = \frac{1}{\Gamma})$$

(also $E_r > 0$ in this example)

\Rightarrow If a resonance is also represented by a pole in $P(E)/T(E)$, we have

④ \rightarrow

$$P(E) \propto \frac{1}{E - (E_r - i \frac{\Gamma}{2})}$$

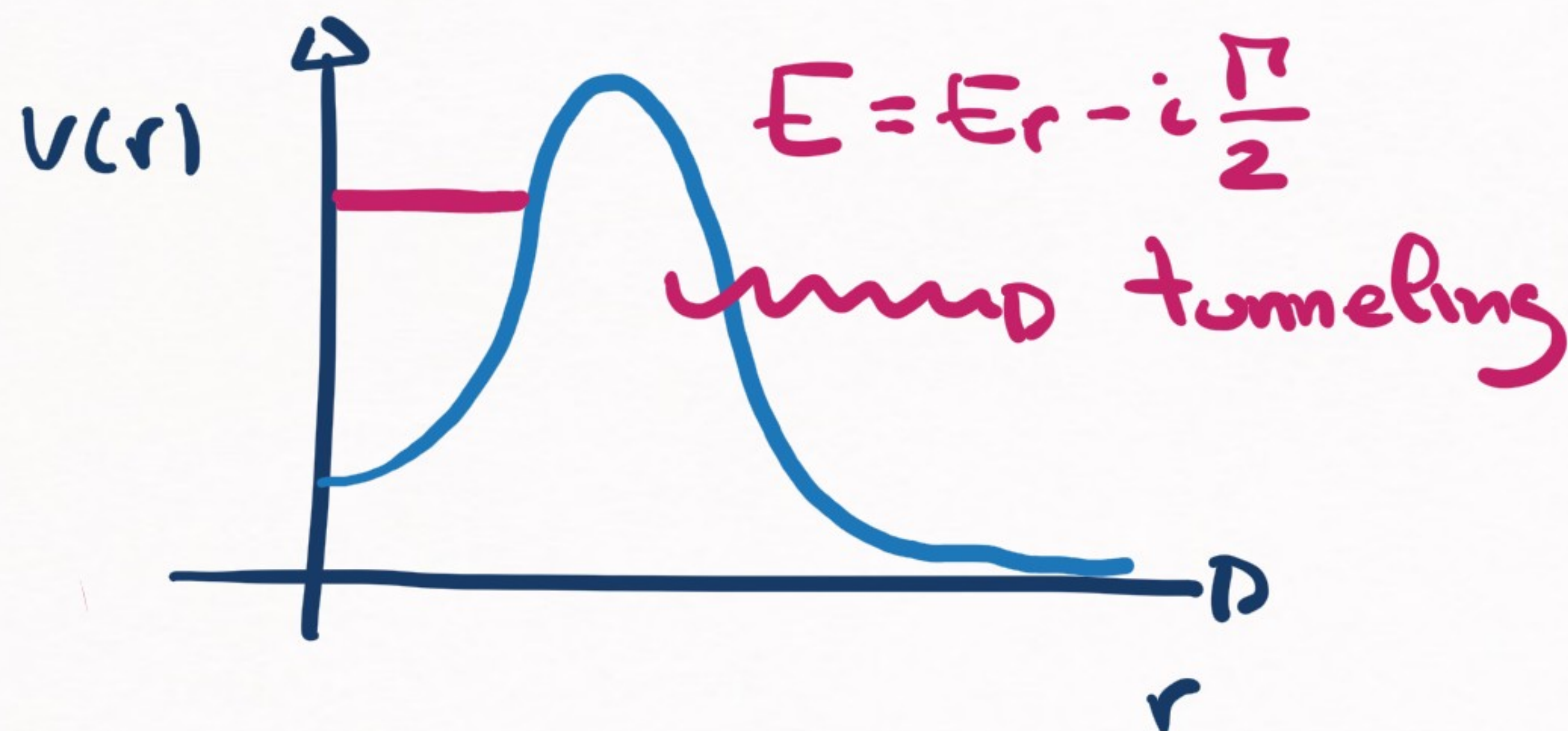
\leftarrow (match) \rightarrow

$$P(E) = \frac{1}{k \cot \delta - i k}$$

\downarrow
④

[RESONANCES] (4)

a) Consider this potential $= D$



a.1) $P(E) \propto \frac{1}{E - (E_r - i\frac{\Gamma}{2})}$ (close to $E = E_r$)

a.2) $P(E) = \frac{1}{\kappa \cot \delta - ik}$



a.3) $P(E) = -\frac{1}{2\kappa} \frac{\Gamma}{E - (E_r - i\frac{\Gamma}{2})}$

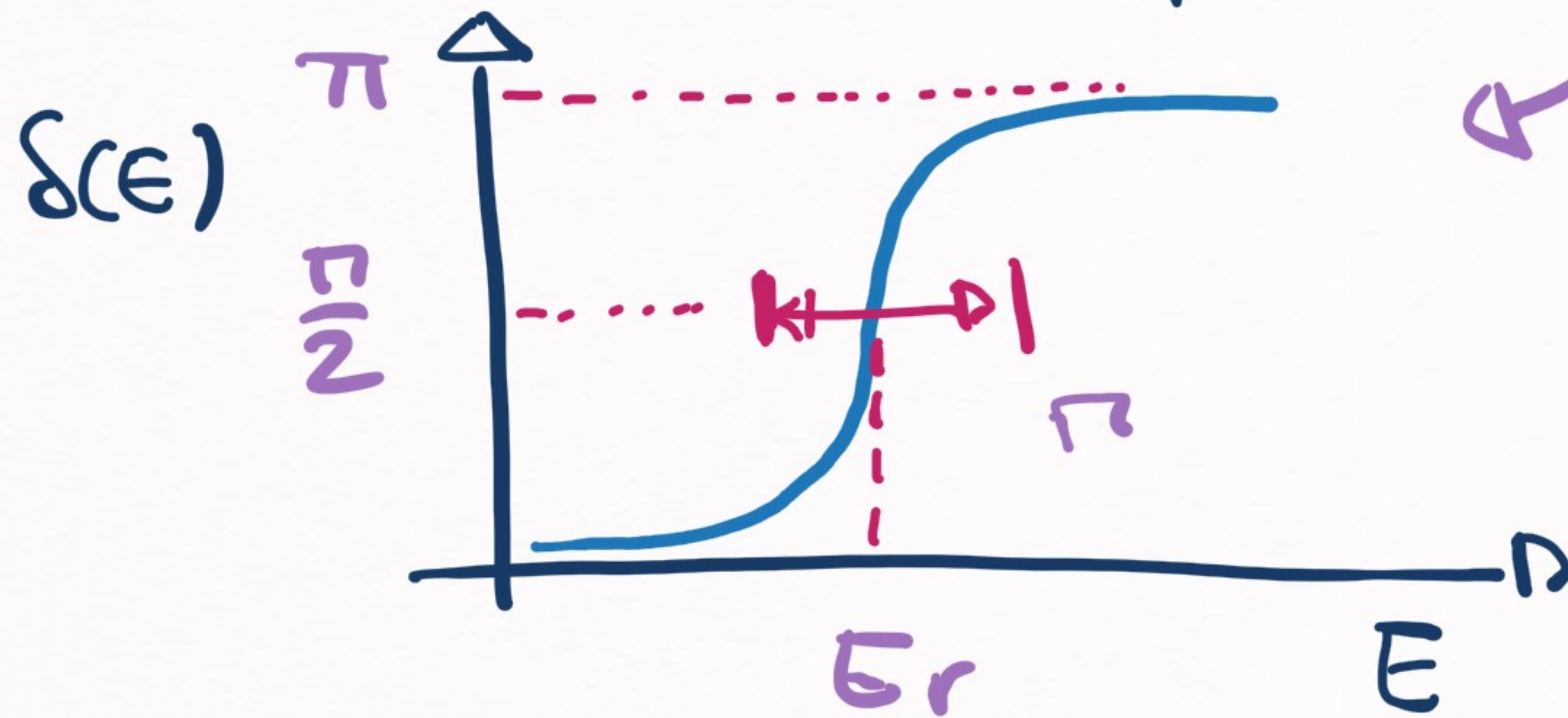
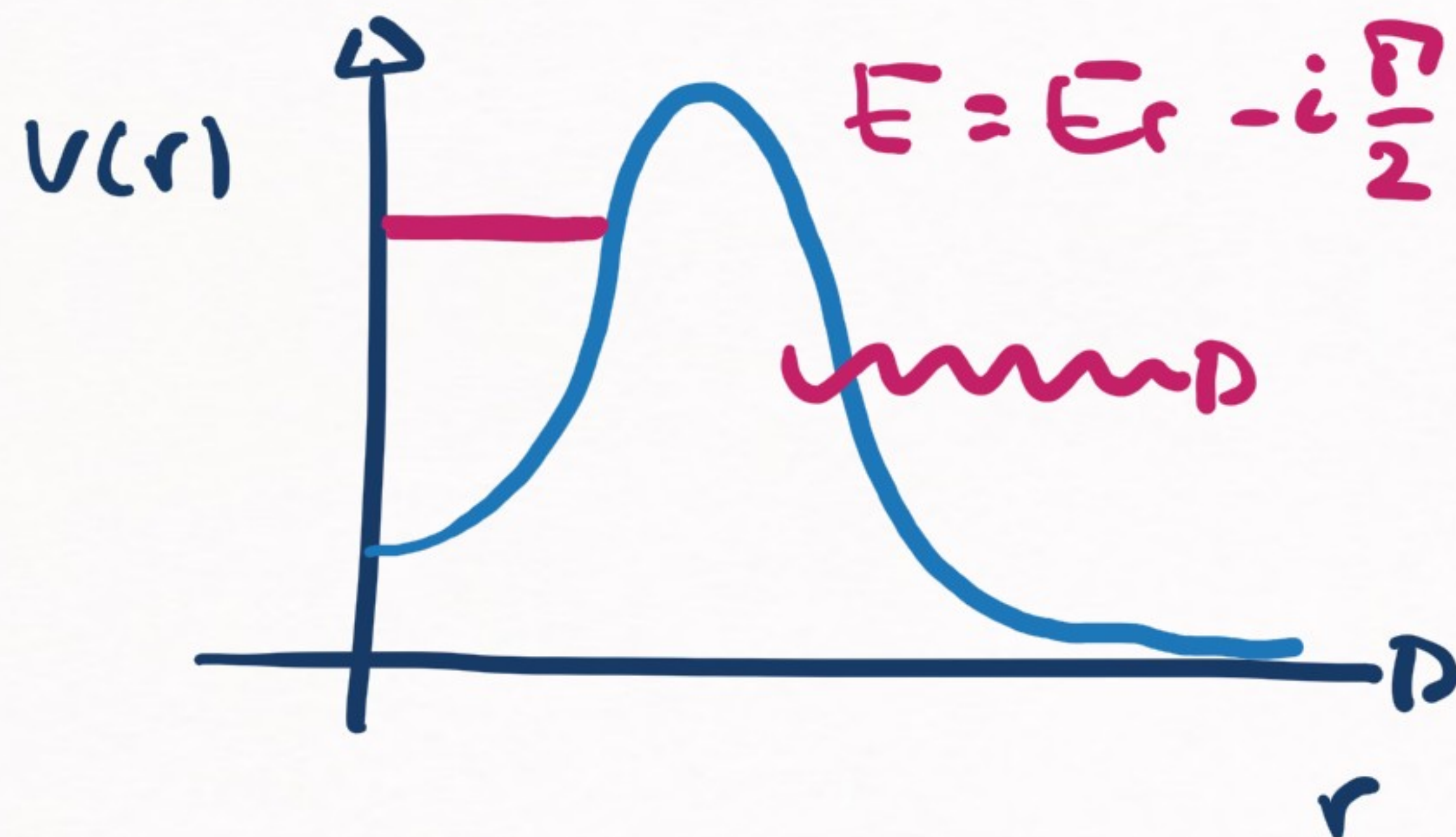
(most simple solution to a.1 & a.2)

a.3) $\Rightarrow \delta(E) = -\text{atan} \left(\frac{\Gamma}{2} \frac{1}{E - E_r} \right)$

try to plot it

[RESONANCES] ⑤

a) Consider this potential



$$\Rightarrow f(E) = -\frac{1}{2k} \frac{\Gamma}{E - (E_r - i\frac{\Gamma}{2})}$$

$$\delta(E) = -\text{atan} \left(\frac{\Gamma}{2} \frac{1}{E - E_r} \right)$$

Caveat \rightarrow

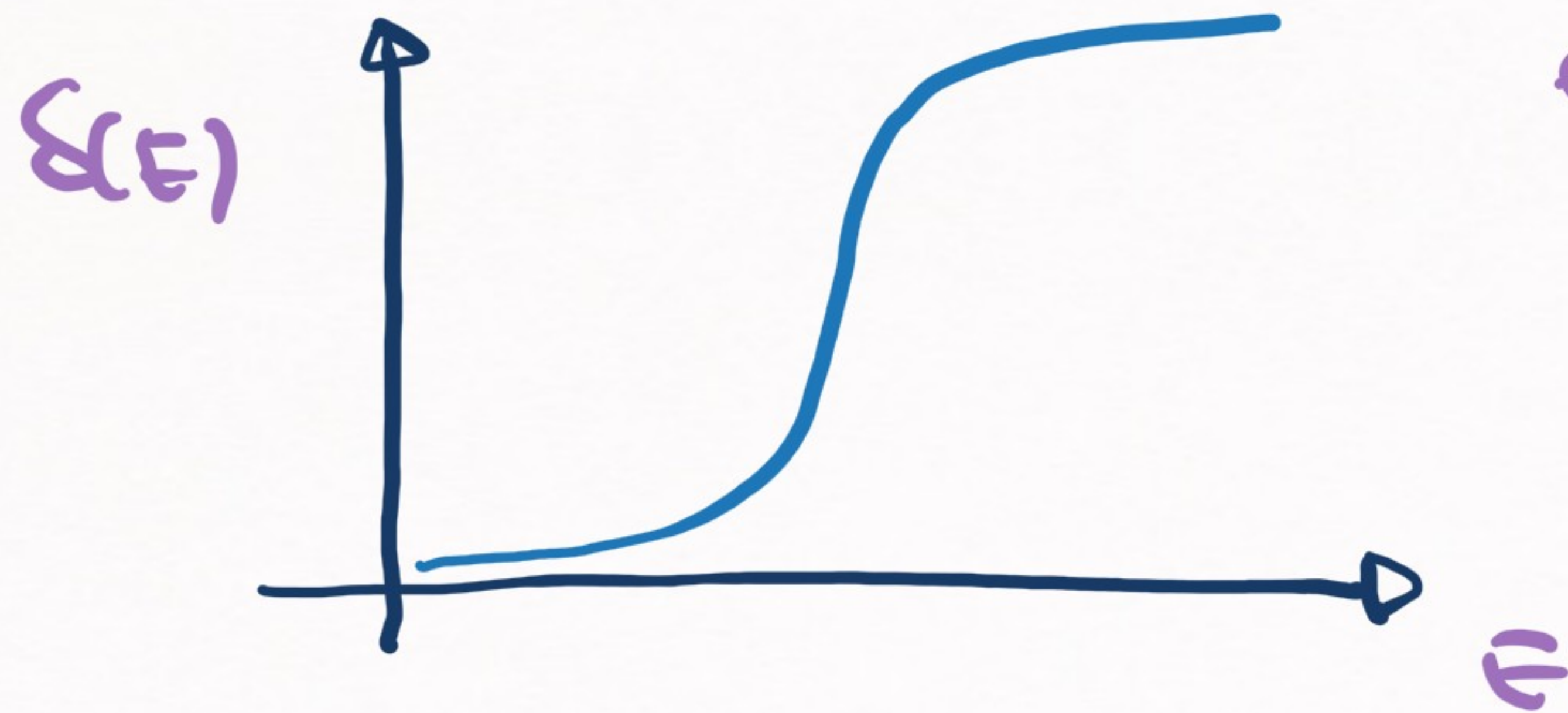
this is only the most easy solution to the previous conditions

\Rightarrow

Resonance translates into a sudden increase of $\delta(E)$ of π

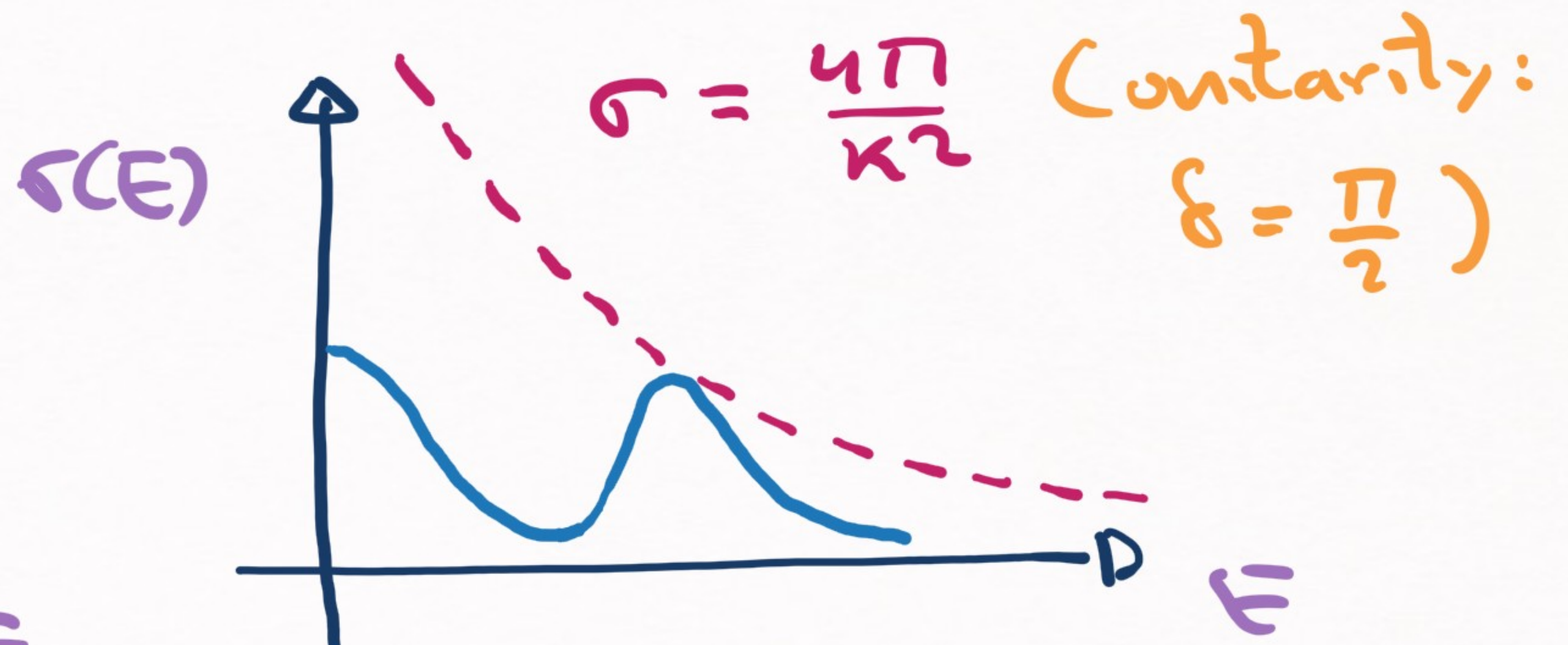
[RESONANCES] (6)

⇒ Phase shifts and cross section:



$$\delta(E) = -\text{atan}\left(\frac{\Gamma}{2} \frac{1}{E - E_r}\right)$$

Jumps from 0 to π quickly



↳ Looks like a peak

$$\left[\sigma(E) \xrightarrow{E \rightarrow E_r} \frac{4\pi}{k^2} \right]$$

(unitarity $\hat{=}$ max value of $\sigma(E)$)

[RESONANCES] (7)

⇒ A more general form can be obtained from $S_e(k)$:

1) $S(E)$ is a phase for $E \in \mathbb{R}, E > 0$ $|S(E)| = 1, S(E) = e^{2i\delta(E)}$

2) $S(E)$ has a pole for $E = E_r - i\frac{\Gamma}{2}$

$$\Rightarrow S(E) = e^{2i\delta_{nr}(E)} \frac{E - (E_r + i\frac{\Gamma}{2})}{E - (E_r - i\frac{\Gamma}{2})}$$

poles 1) 2)

[RESONANCES] (3)

⇒ A more general form can be obtained from $S_e(E)$:

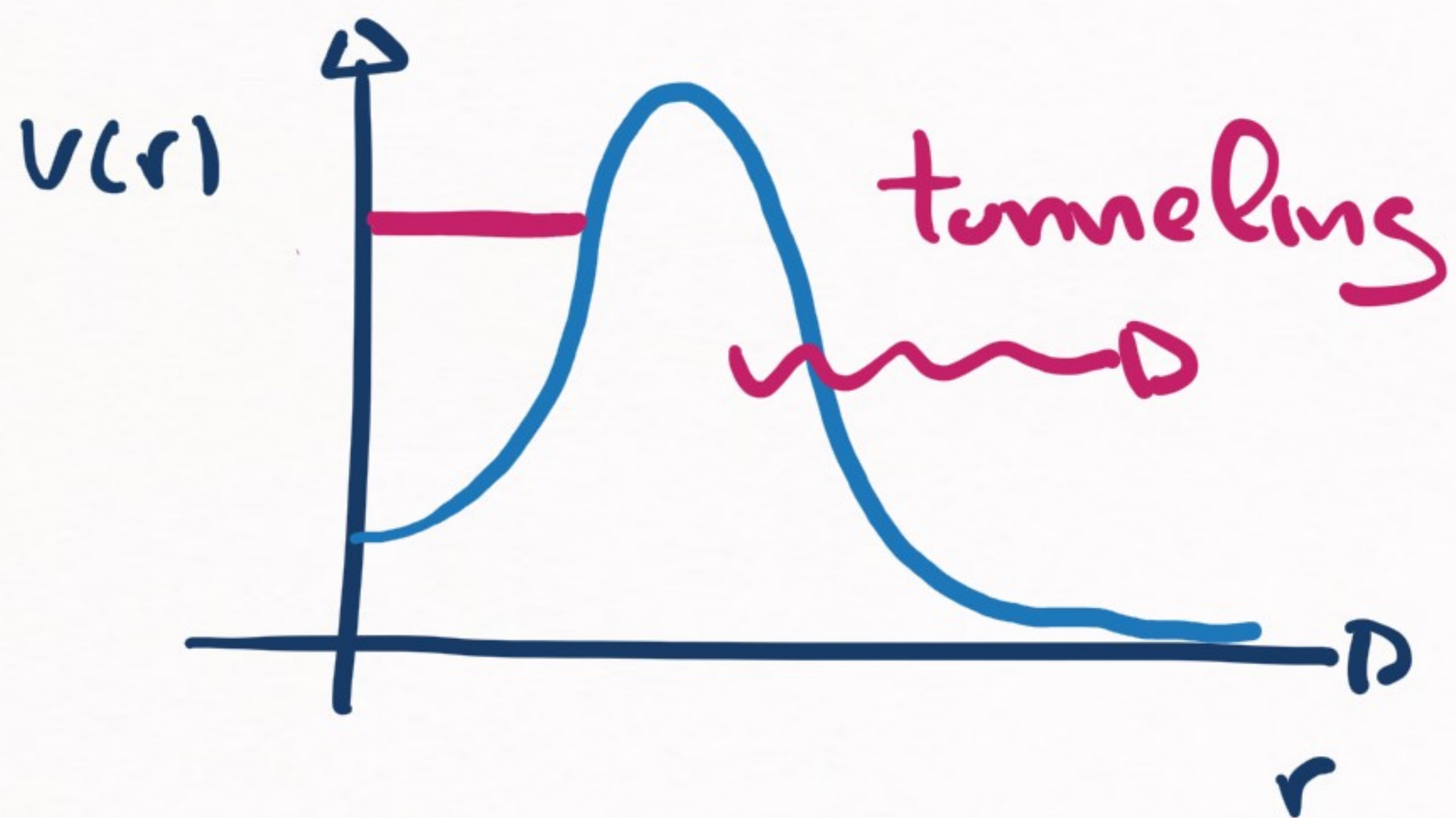
$$\left[S(E) = e^{2i\delta_{nr}(E)} \frac{E - (E_r + i\Gamma/2)}{E - (E_r - i\Gamma/2)} \right]$$

$$\begin{aligned} \delta(E) &= \frac{1}{2i} \log S(E) = \delta_{nr}(E) - \text{atan} \left(\frac{\Gamma}{2} \frac{1}{E - E_r} \right) \\ &= \underbrace{\delta_{nr}(E)} + \delta_r(E) \end{aligned}$$

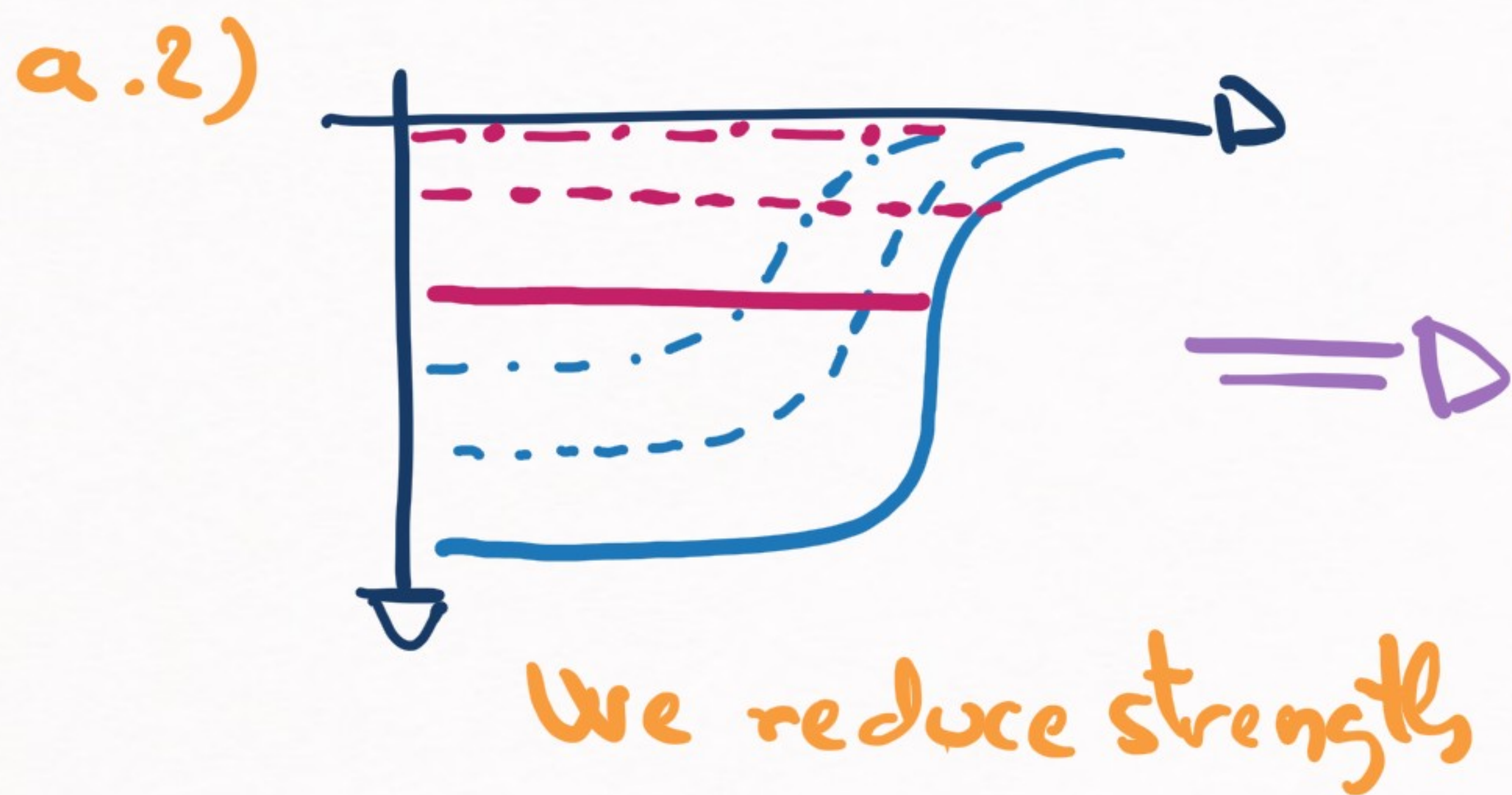
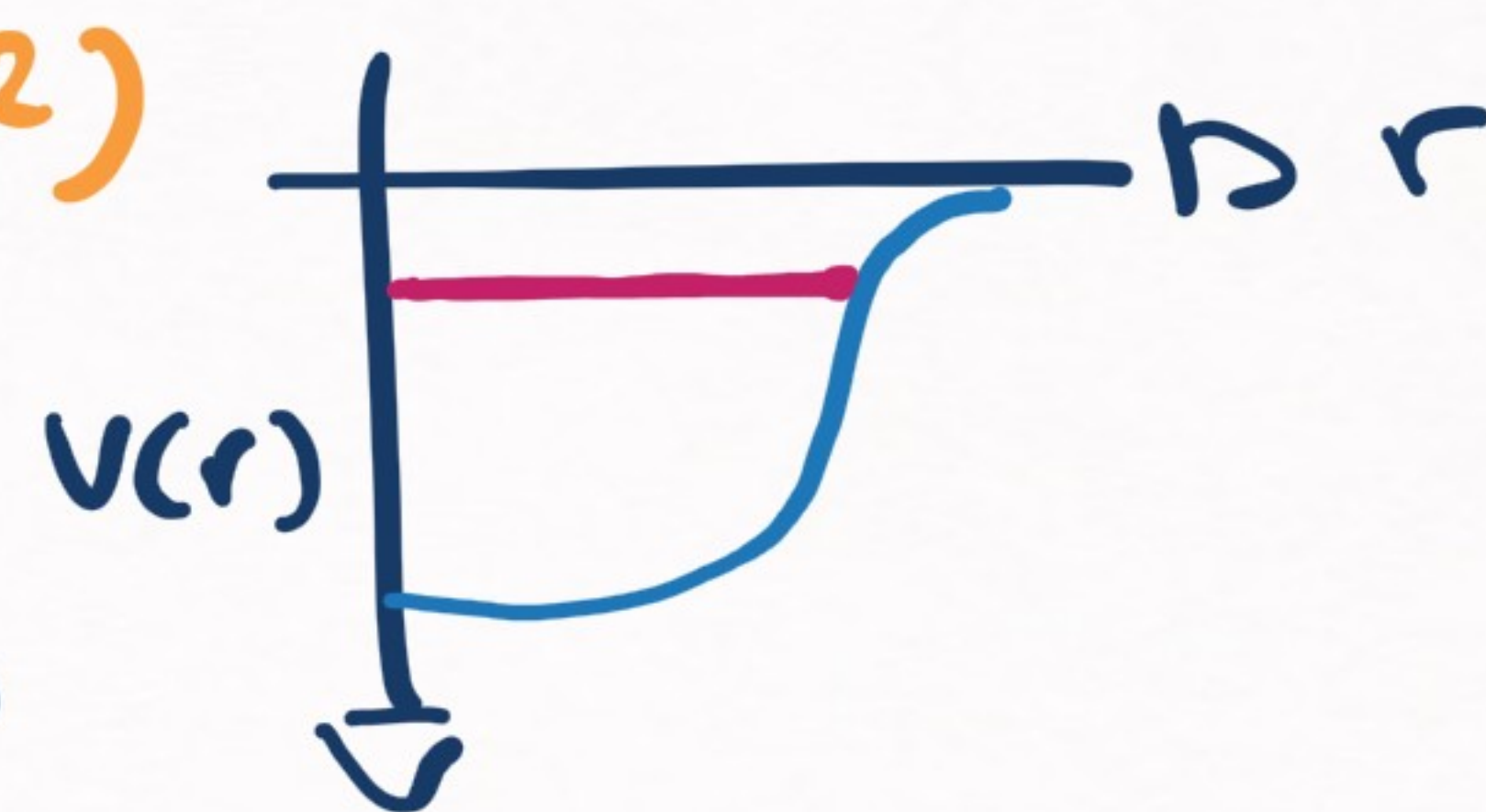
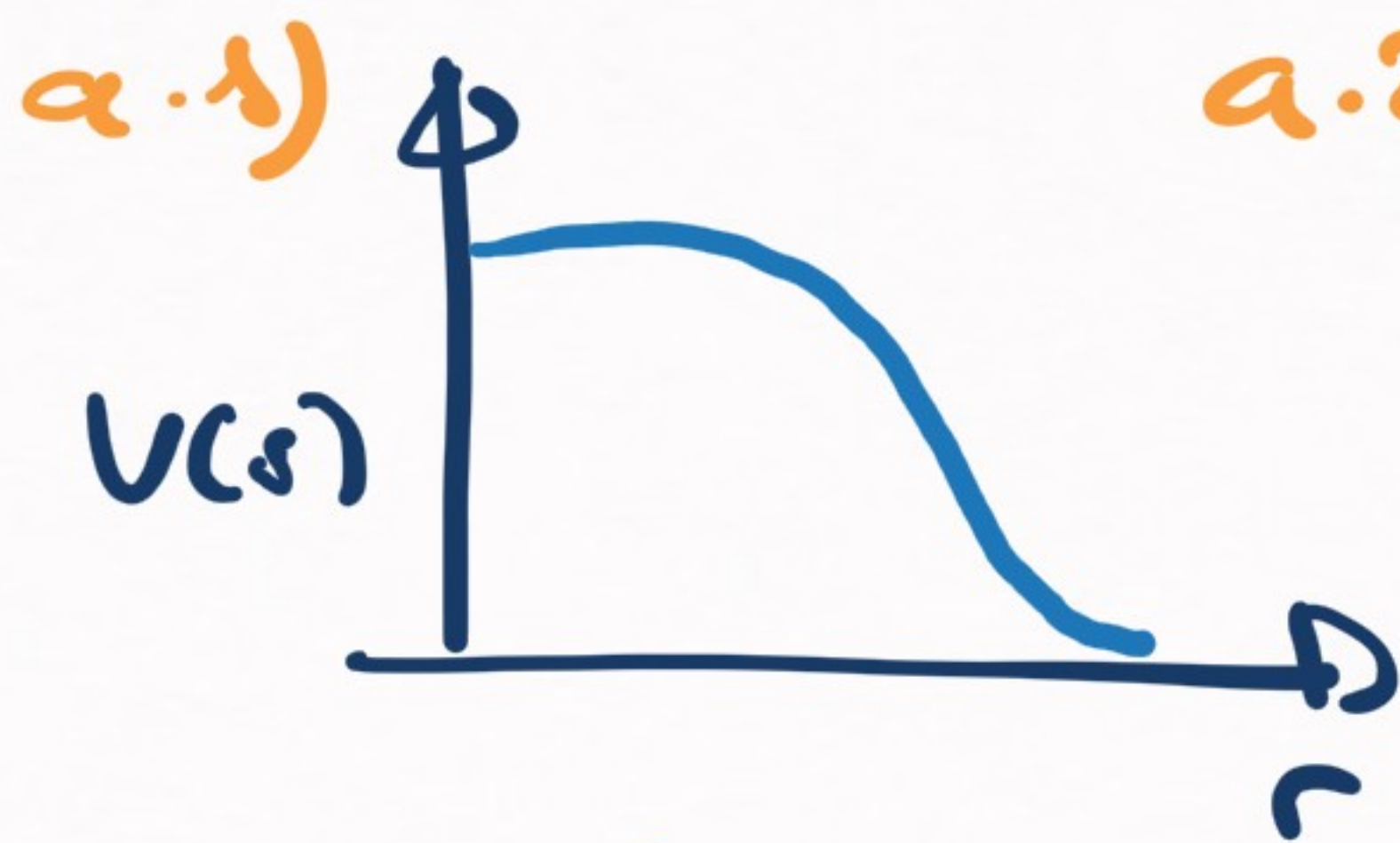
non-resonant + resonant contribution

[RESONANCES] (a)

=> But, when do we find this type of potential?

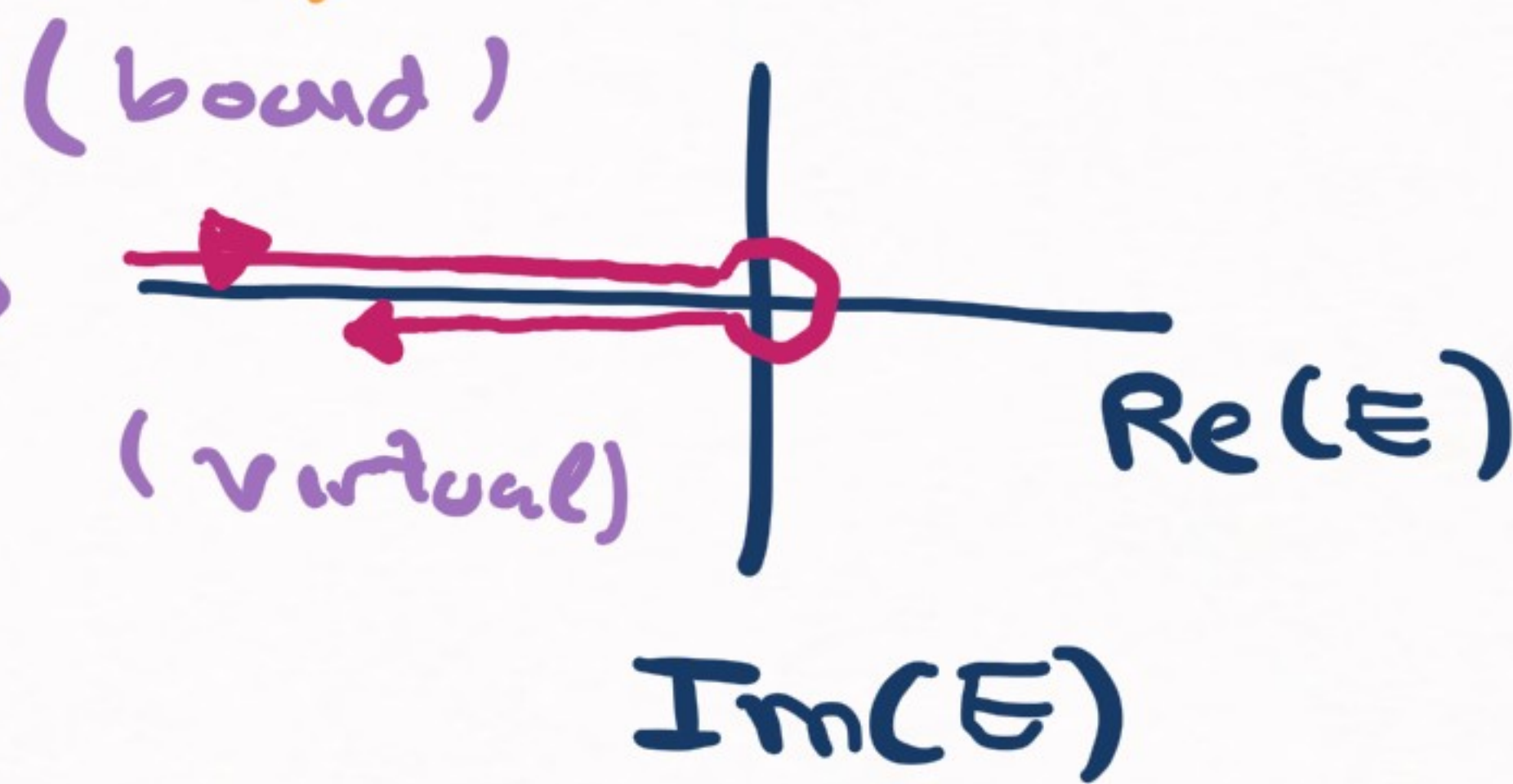


a) Most usually we find:



Repulsive

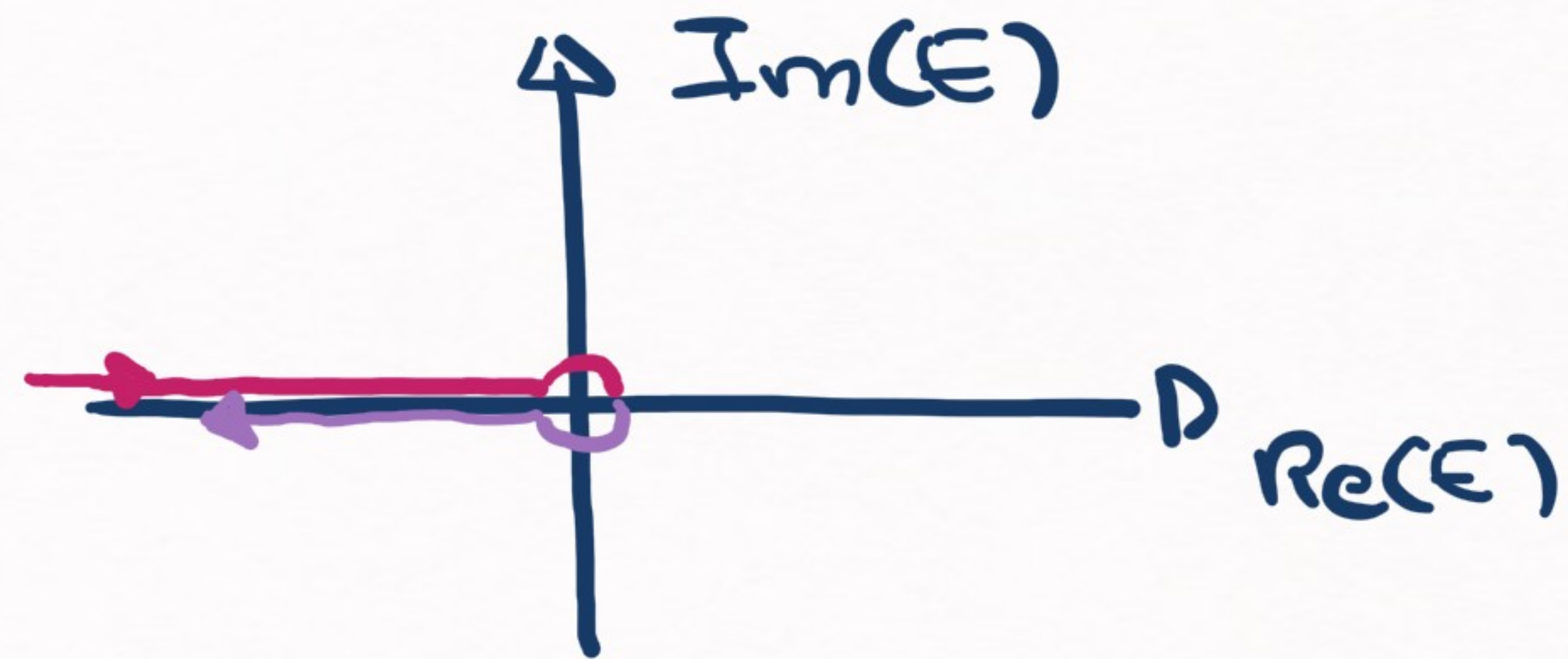
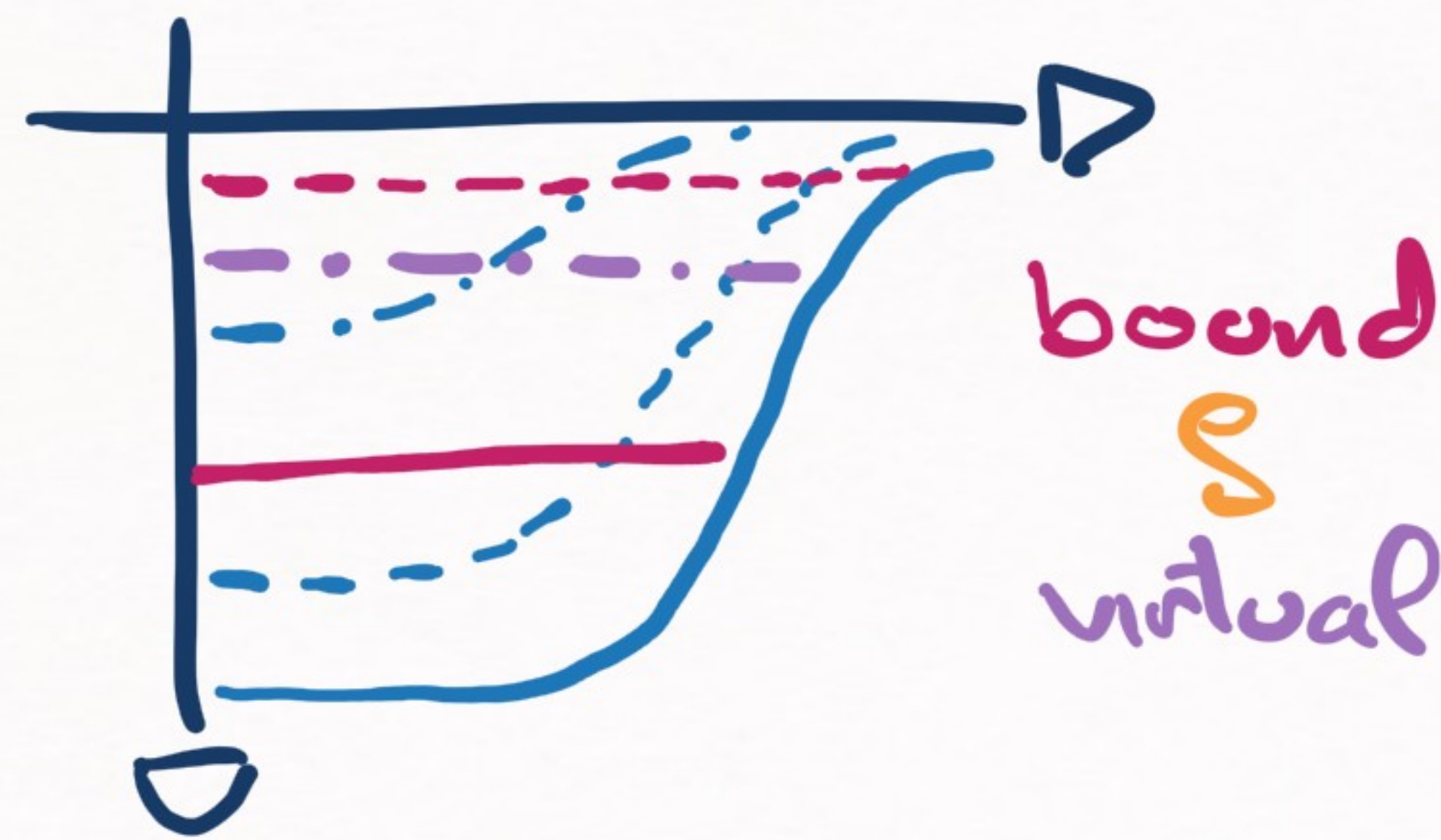
Attractive



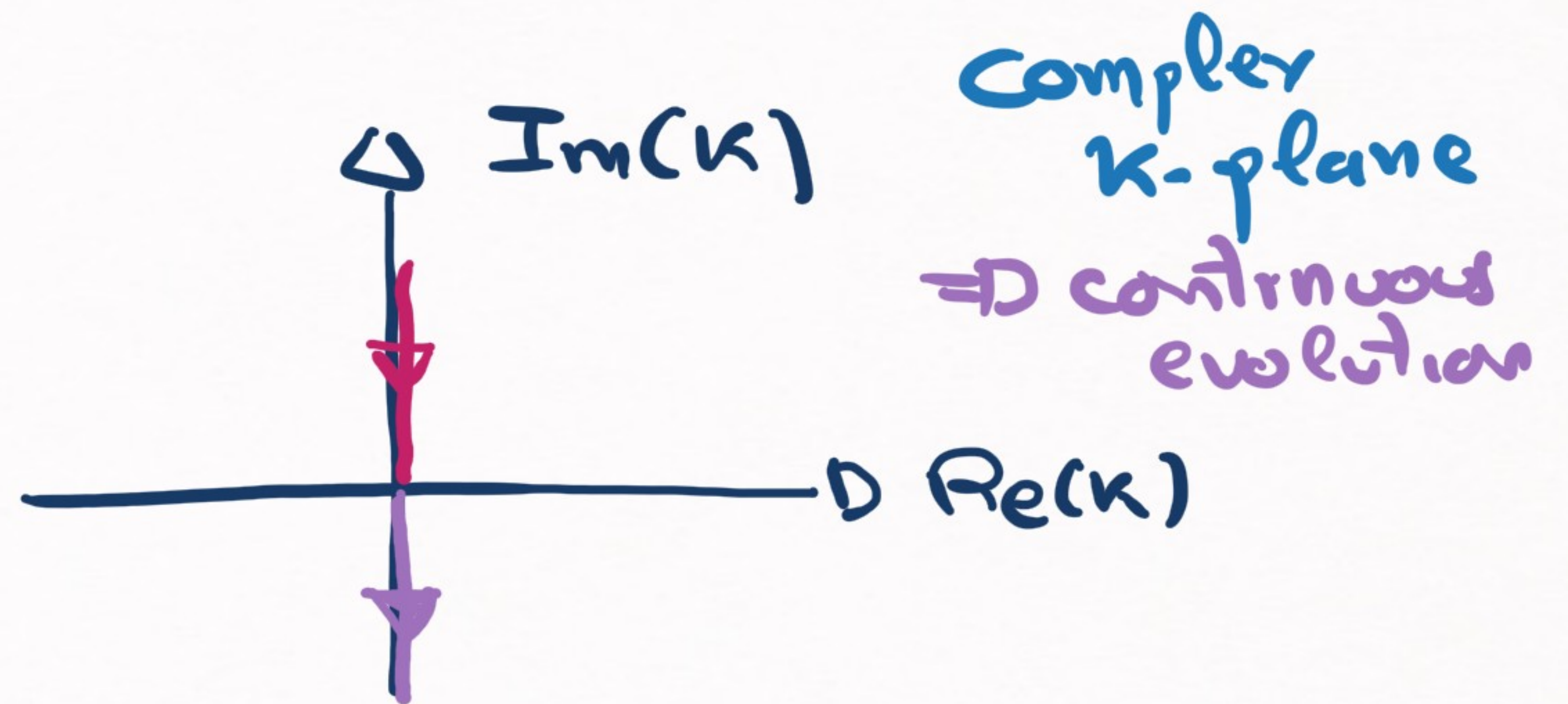
=> { We go from a bound state to a virtual state

[RESONANCES] (10) (+ S-wave)

a.2) Attractive potentials: bound & virtual states

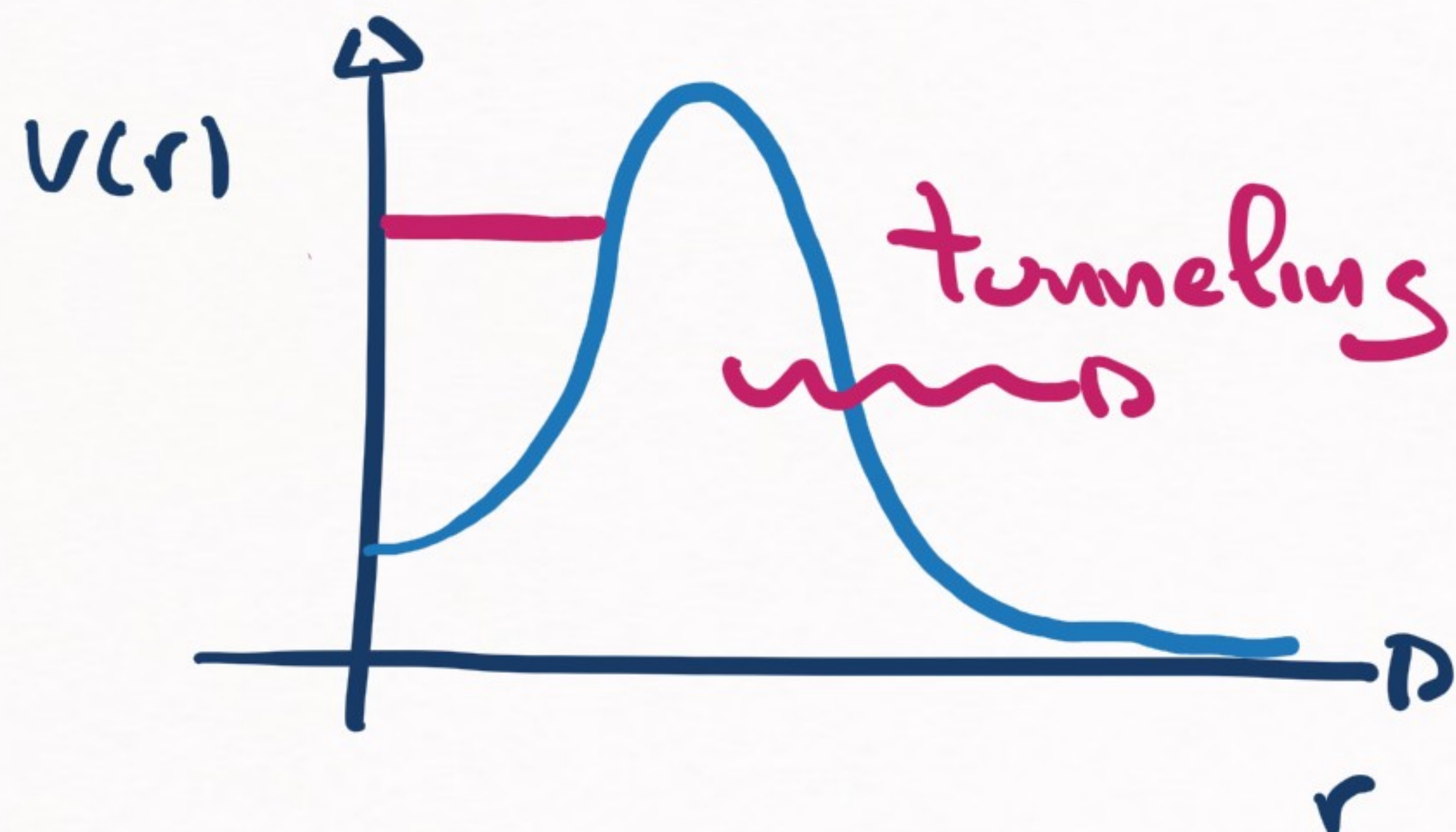


Again, we are changing the strength of the potential!



[RESONANCES] (11)

=> But, when do we find this type of potential?



b) The effective potential for $l \neq 0$:

$$\left[V_{\text{eff}}(r) = V(r) + \frac{l(l+1)}{r^2} \right]$$

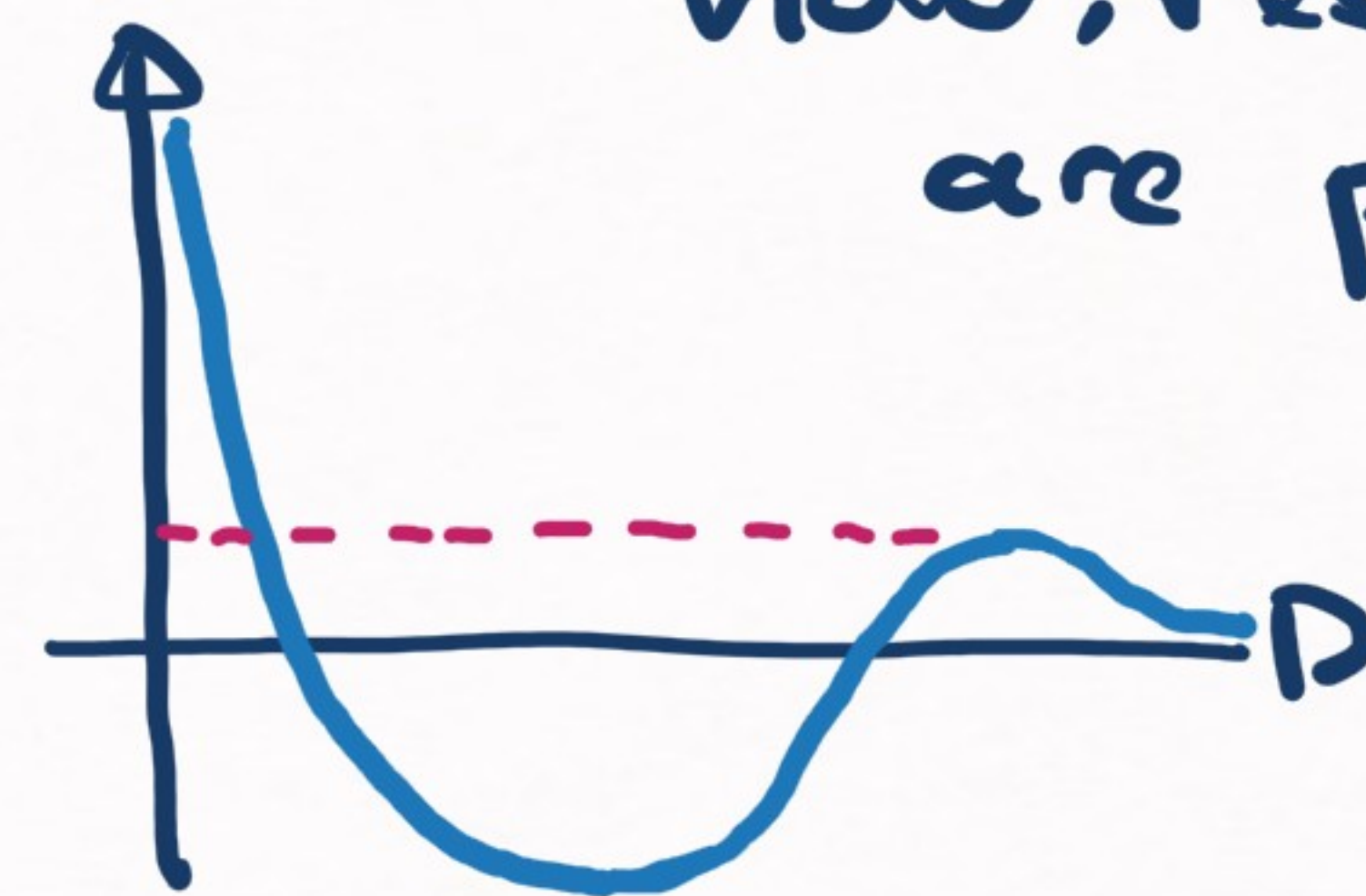


$V(r)$
potential



centrifugal
barrier

=

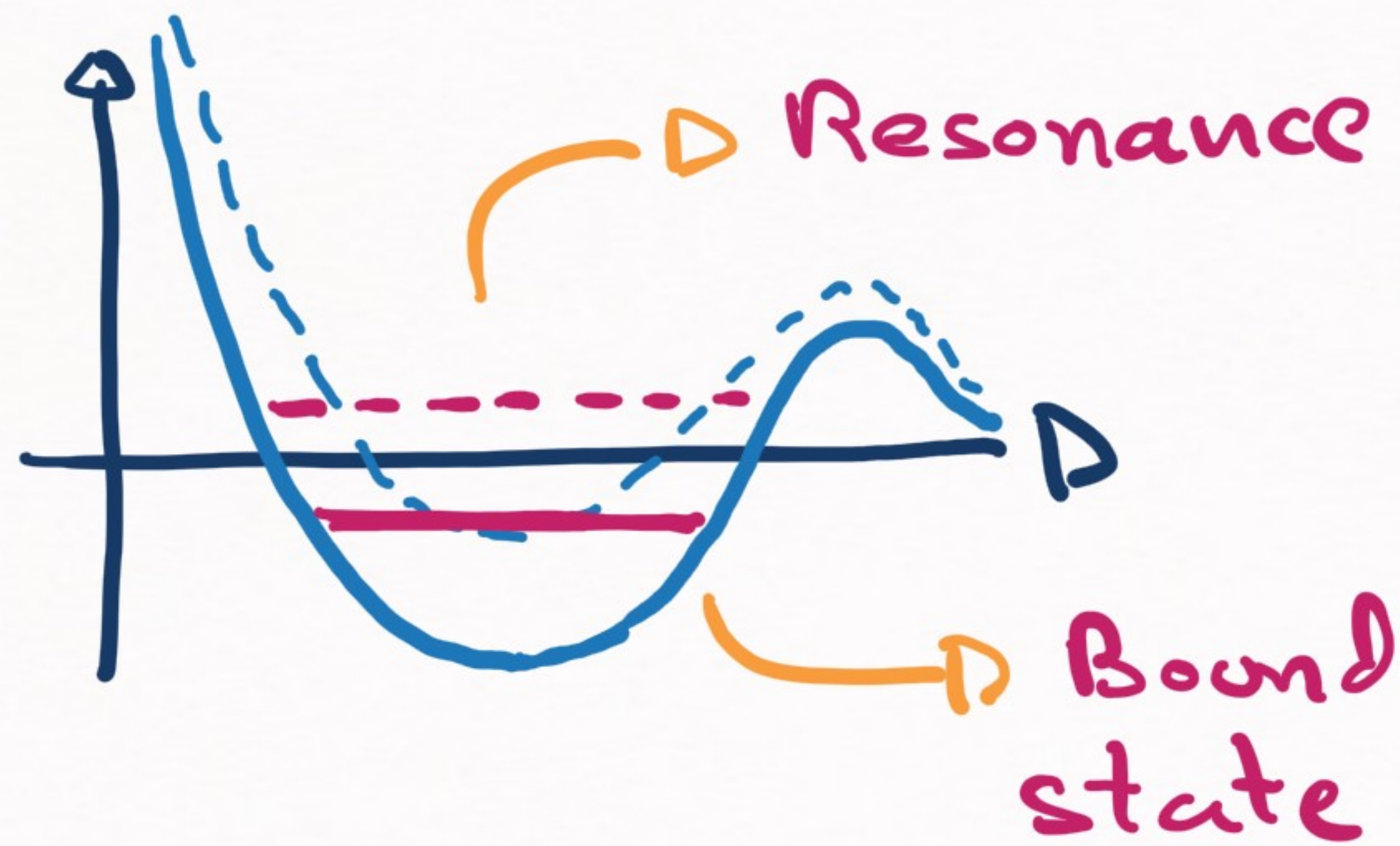


now, resonances
are possible

↙

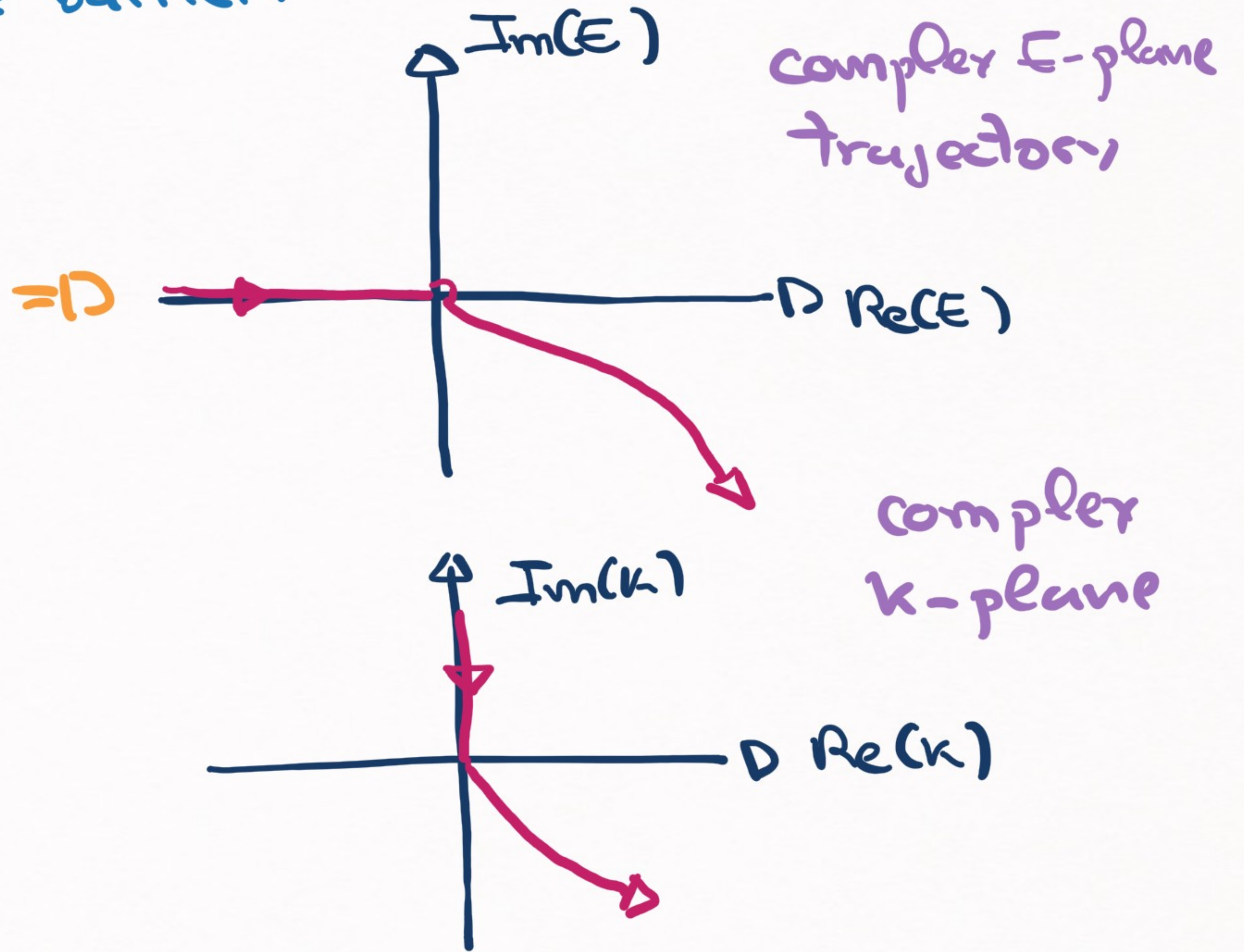
[RESONANCES] (12) ($l \neq 0$)

b) Potential + centrifugal barrier:



The effect of reducing the potential =

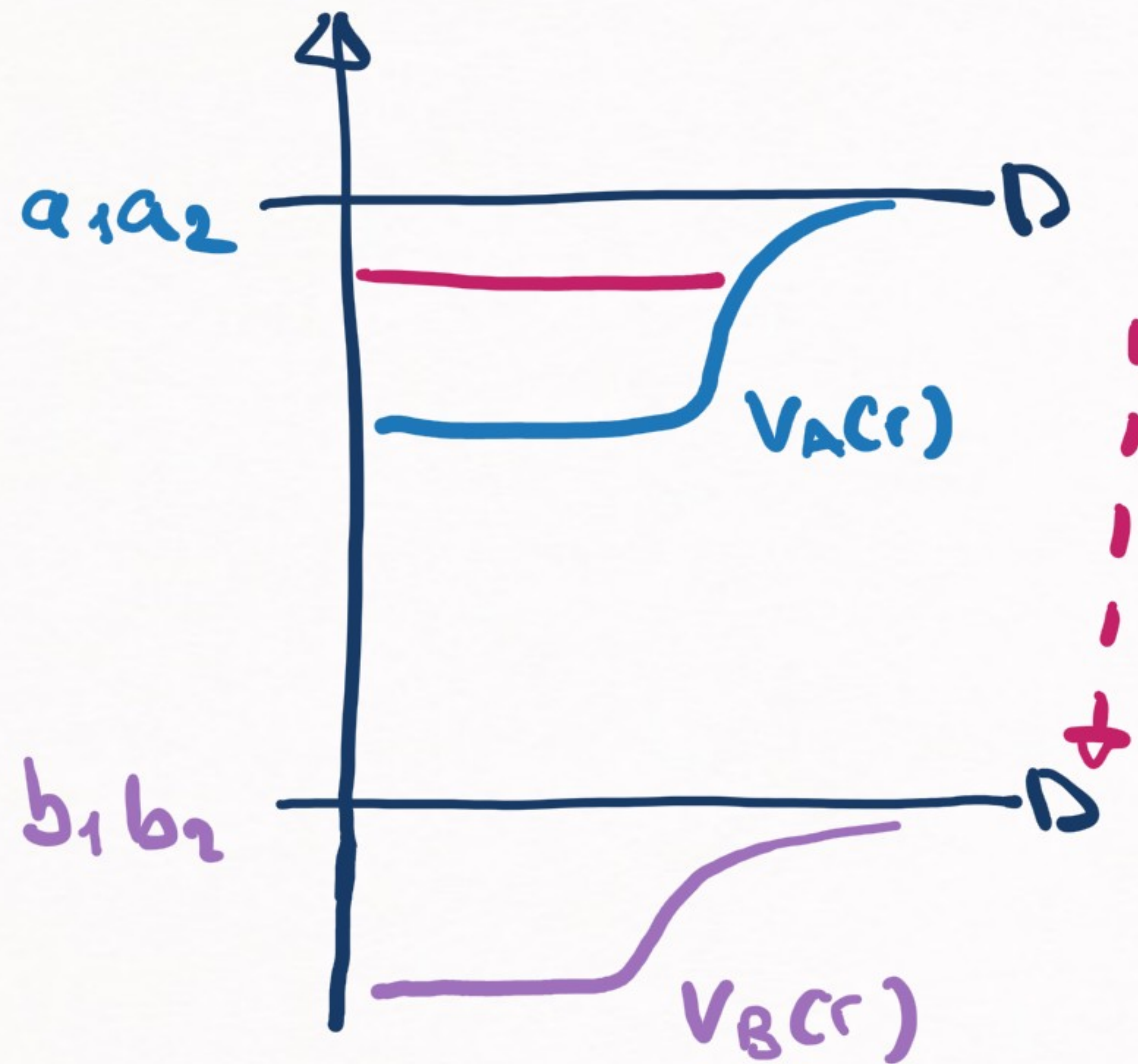
→ direction when the potential weakens



[RESONANCES] (13)

c) Multichannel problem: 

[Scattering from $a_1 a_2$ to $b_1 b_2$]

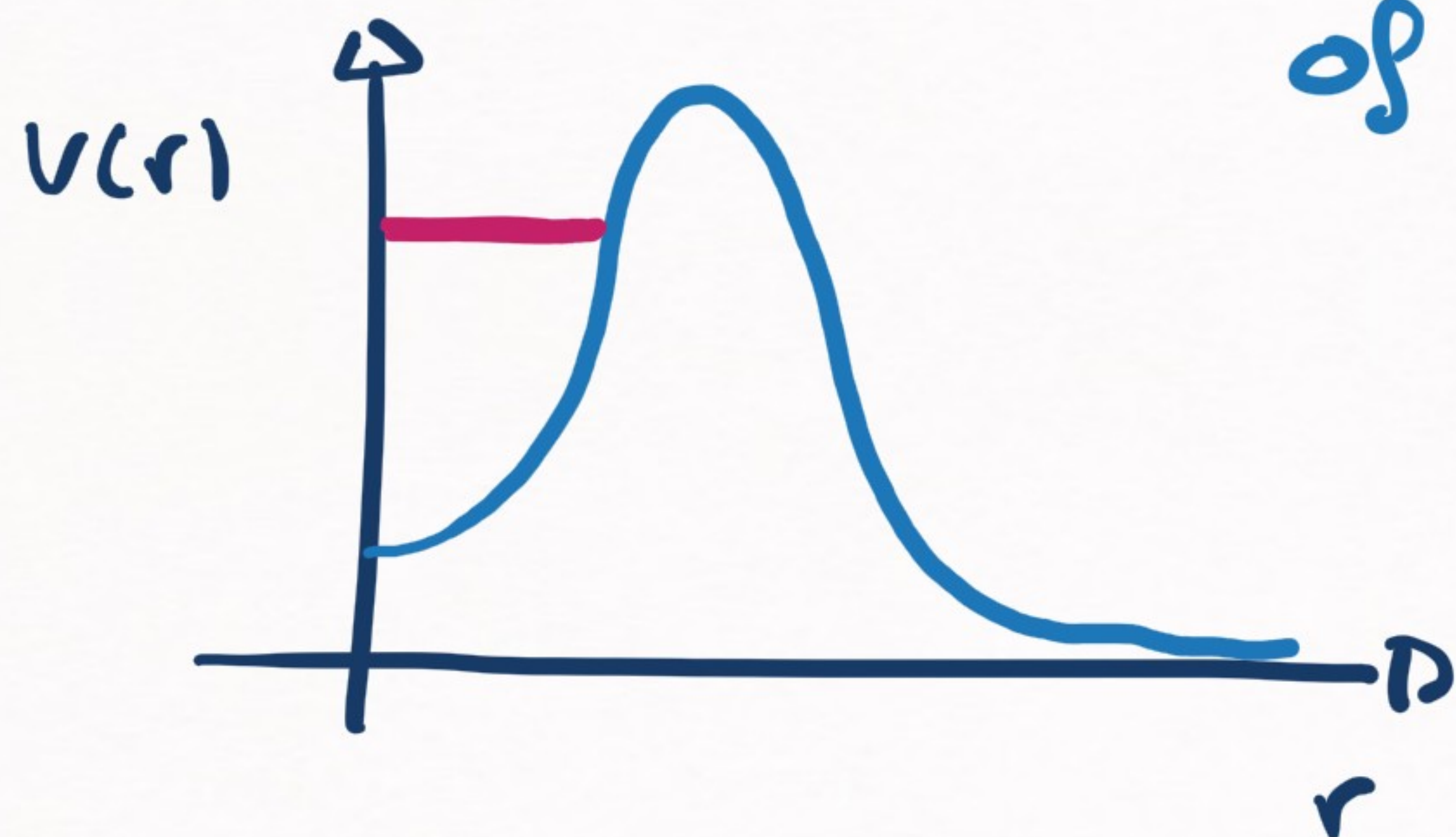


$(a_1 a_2)$
↓
 $b_1 + b_2$

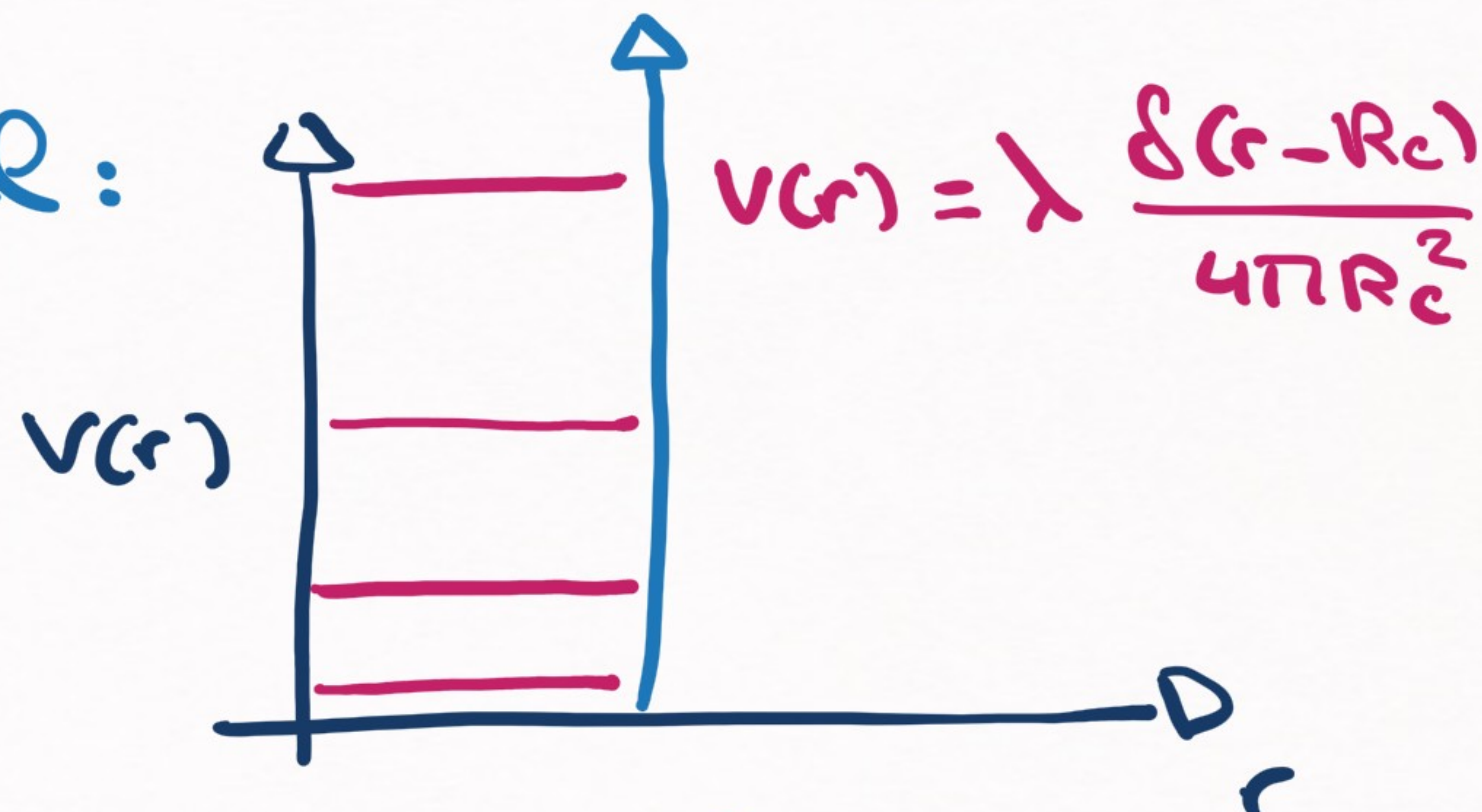
inelastic scattering:
the outgoing particles are different than the incoming particles (advanced topic)

[RESONANCES] (14)

⇒ There is a simplified version of this potential:



⇒



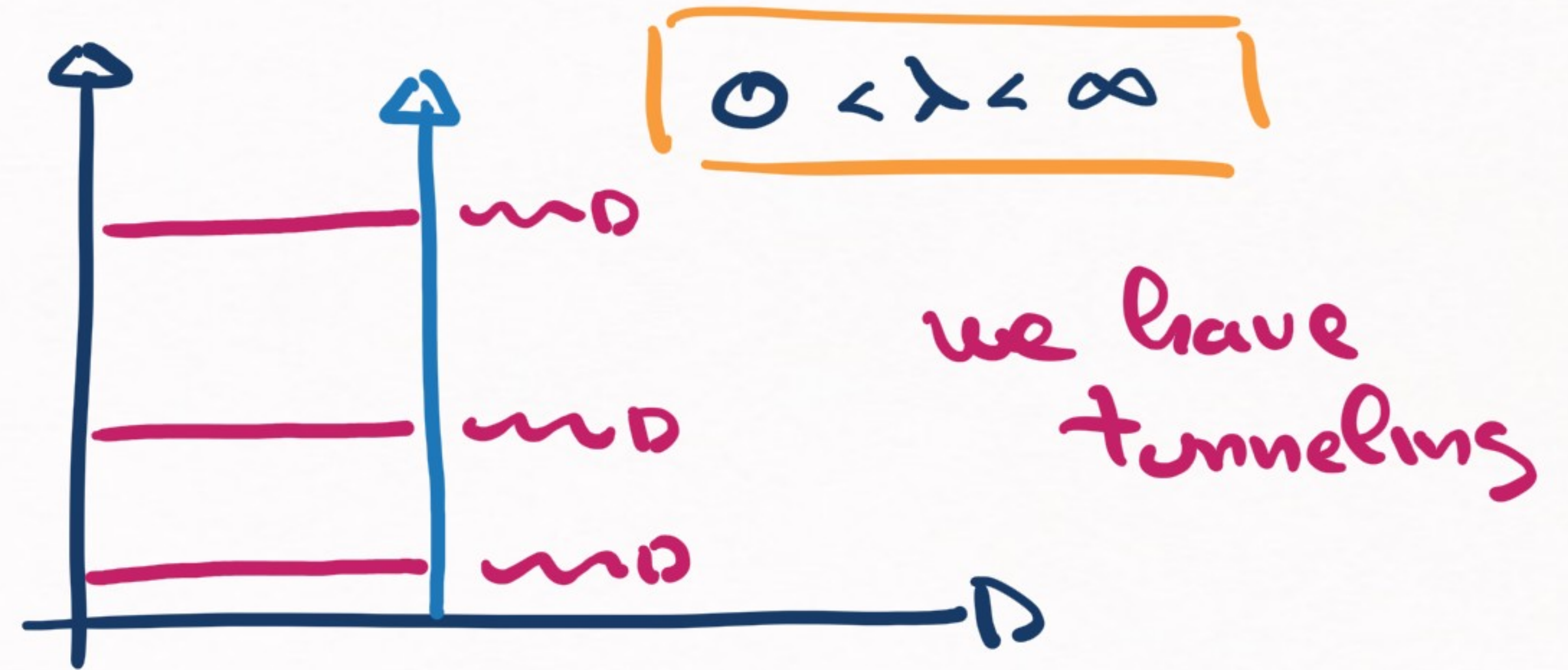
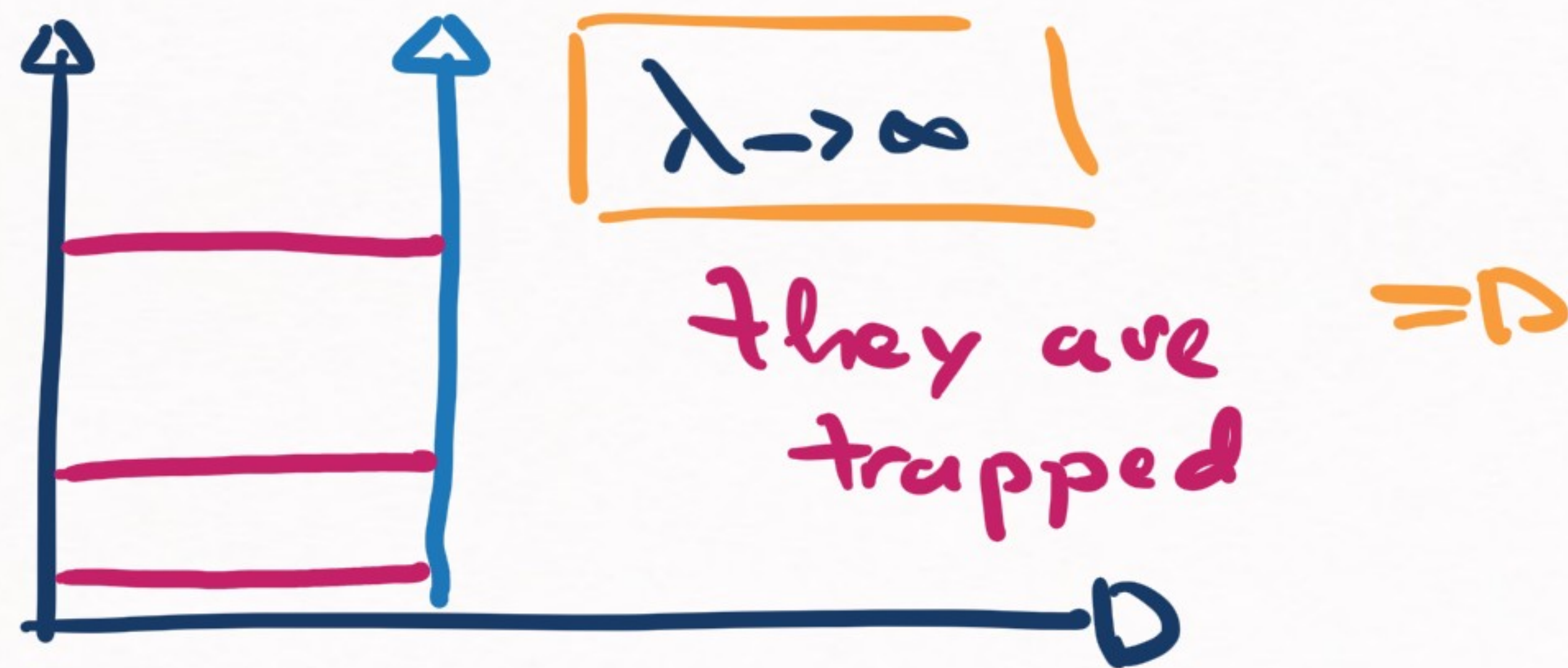
Repulsive δ -shell

$\lambda \rightarrow \infty$ ⇒ Bound states
at $\sin(kR_c) = 0$ ($V(R_c) = 0$)

⇒ $kR_c = n\pi$, $n = 1, 2, 3, \dots$

[RESONANCES] (SS)

⇒ Repulsive δ -shell: good model to understand resonances



$$E = \frac{1}{2\mu} \left(n \frac{\pi}{R_c} \right)^2$$

$n = 1, 2, 3, \dots$

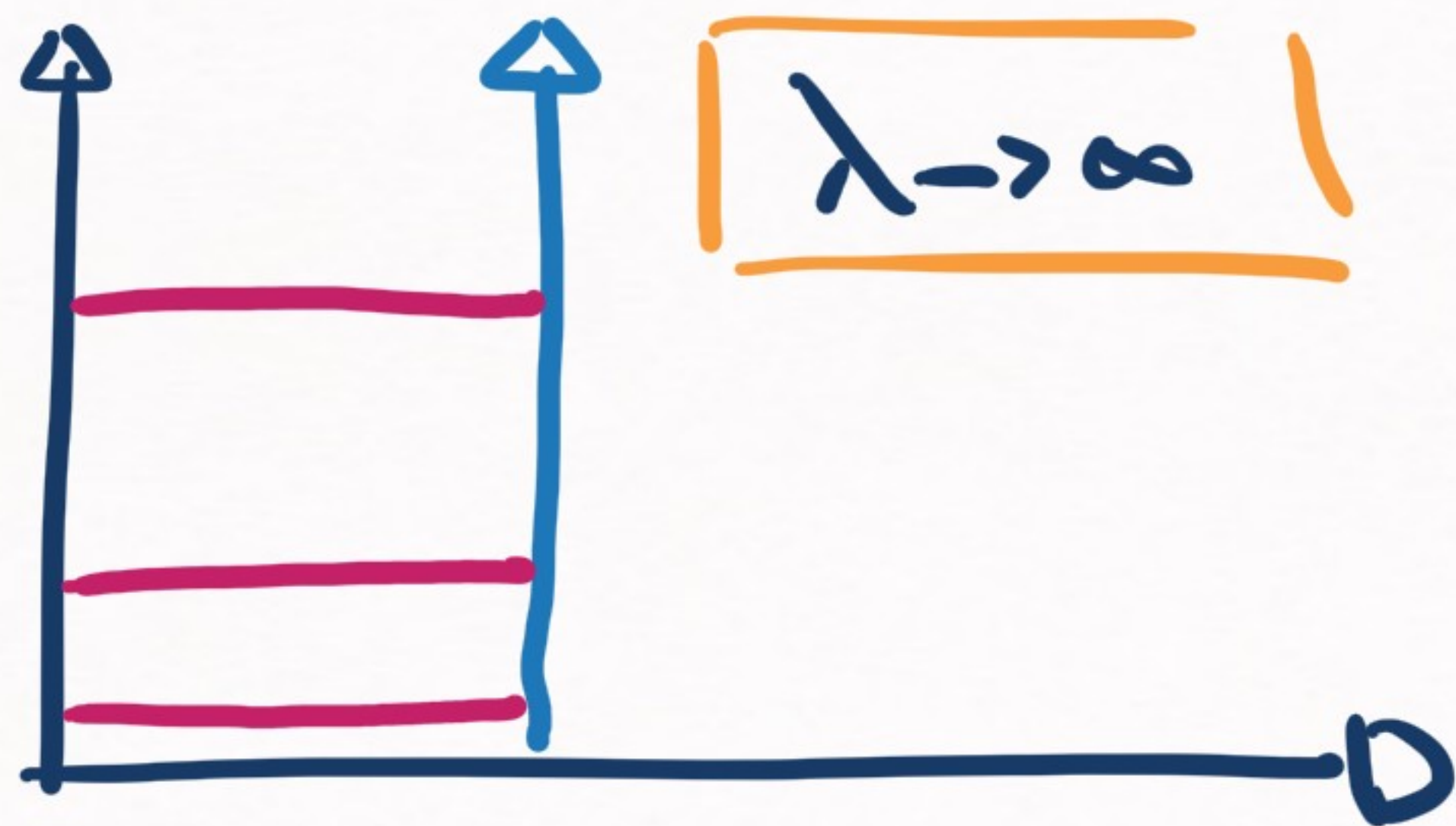
not difficult to calculate

$$E = \frac{1}{2\mu} \left(n \frac{\pi}{R_c} \right)^2 + \delta E_r - i \frac{\Gamma}{2}$$

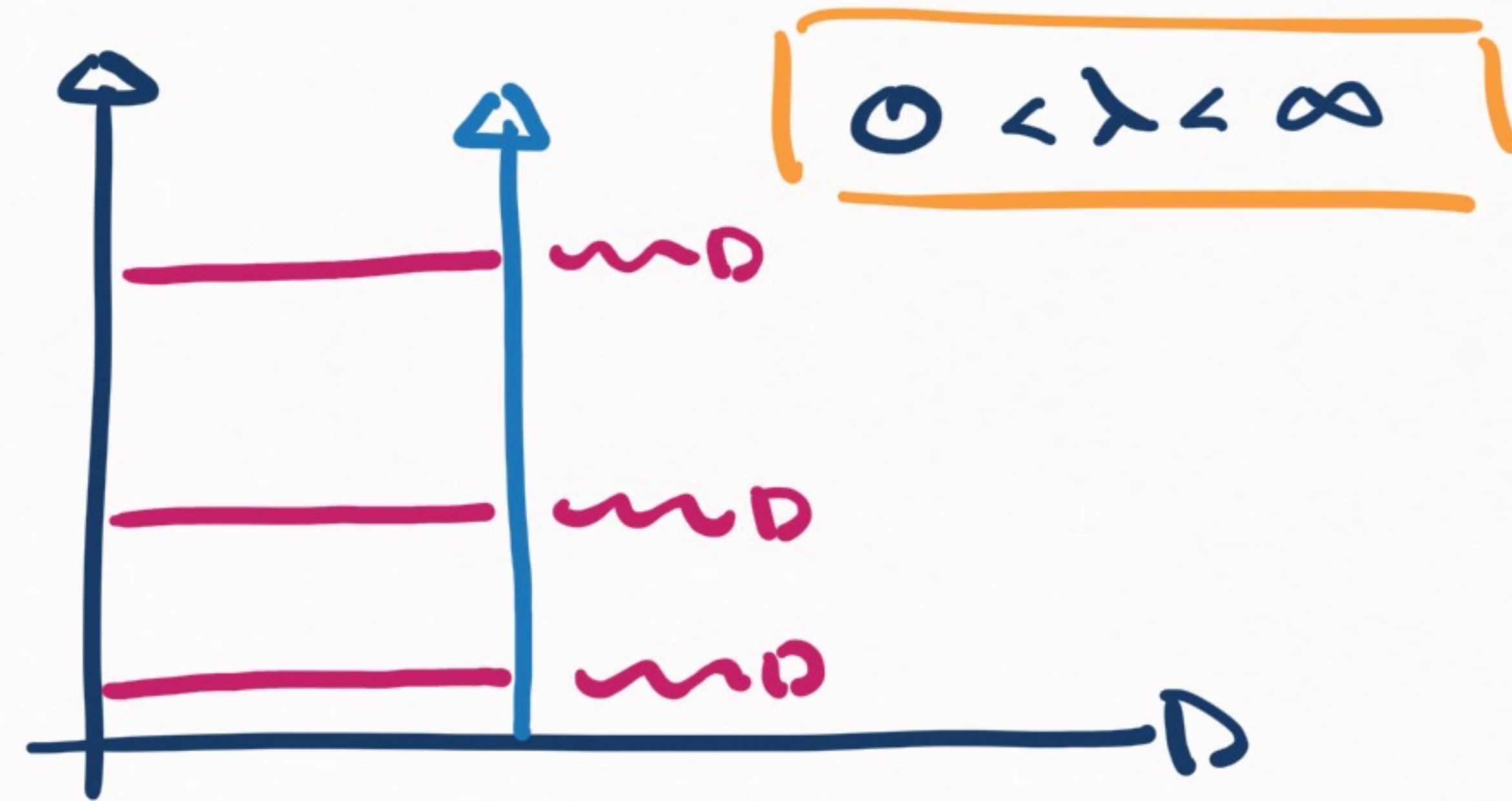
$$\Gamma \sim \mathcal{O}\left(\frac{1}{\lambda}\right) \quad (\text{if } \lambda \text{ is large enough})$$

[RESONANCES] (16)

⇒ Repulsive δ -shell: good model to understand resonances



⇒



⇒ Nice, but how do you calculate the widths?

$$\begin{aligned}
 u(r > R_c) &= e^{ikr} - \frac{e^{-ikr}}{S(k)} \\
 &= e^{ikr}
 \end{aligned}$$

Resonances:

$$\begin{aligned}
 S(E \rightarrow E_r) &\rightarrow \infty \\
 \frac{1}{S(E \rightarrow E_r)} &\rightarrow 0
 \end{aligned}$$



[BOUND, VIRTUAL STATES AND RESONANCES]

1) The wave function interpretation: ($l=0$ for simplicity)

$$u_0(r) \xrightarrow{r \rightarrow \infty} \underline{S(k) e^{ikr} - e^{-ikr}} \quad \text{or} \quad e^{ikr} - \underline{\frac{1}{S(k)} e^{-ikr}}$$

(normalization in this case is arbitrary)

1.a) Pole in the scattering amplitude:

$$\left. \begin{array}{l} S(k) \rightarrow \infty \\ \text{or} \\ \frac{1}{S(k)} \rightarrow 0 \end{array} \right\} \Rightarrow u_0(r) \xrightarrow{r \rightarrow \infty} e^{ikr} \quad (k \text{ will be complex})$$

[BOUND, VIRTUAL STATES AND RESONANCES]

1) The wave function interpretation (continuation):

1.b) Bound state $k = i\gamma$ ($\text{Im}(k) > 0$)

$$\Rightarrow u(r) \xrightarrow{r \rightarrow \infty} e^{-\gamma r}$$

1.c) Virtual state $k = -i\gamma$ ($\text{Im}(k) < 0$)

$$\Rightarrow u(r) \xrightarrow{r \rightarrow \infty} e^{+\gamma r}$$

1.d) Resonance $k = \text{Re}(k) + i\text{Im}(k)$ ($\text{Im}(k) < 0$)

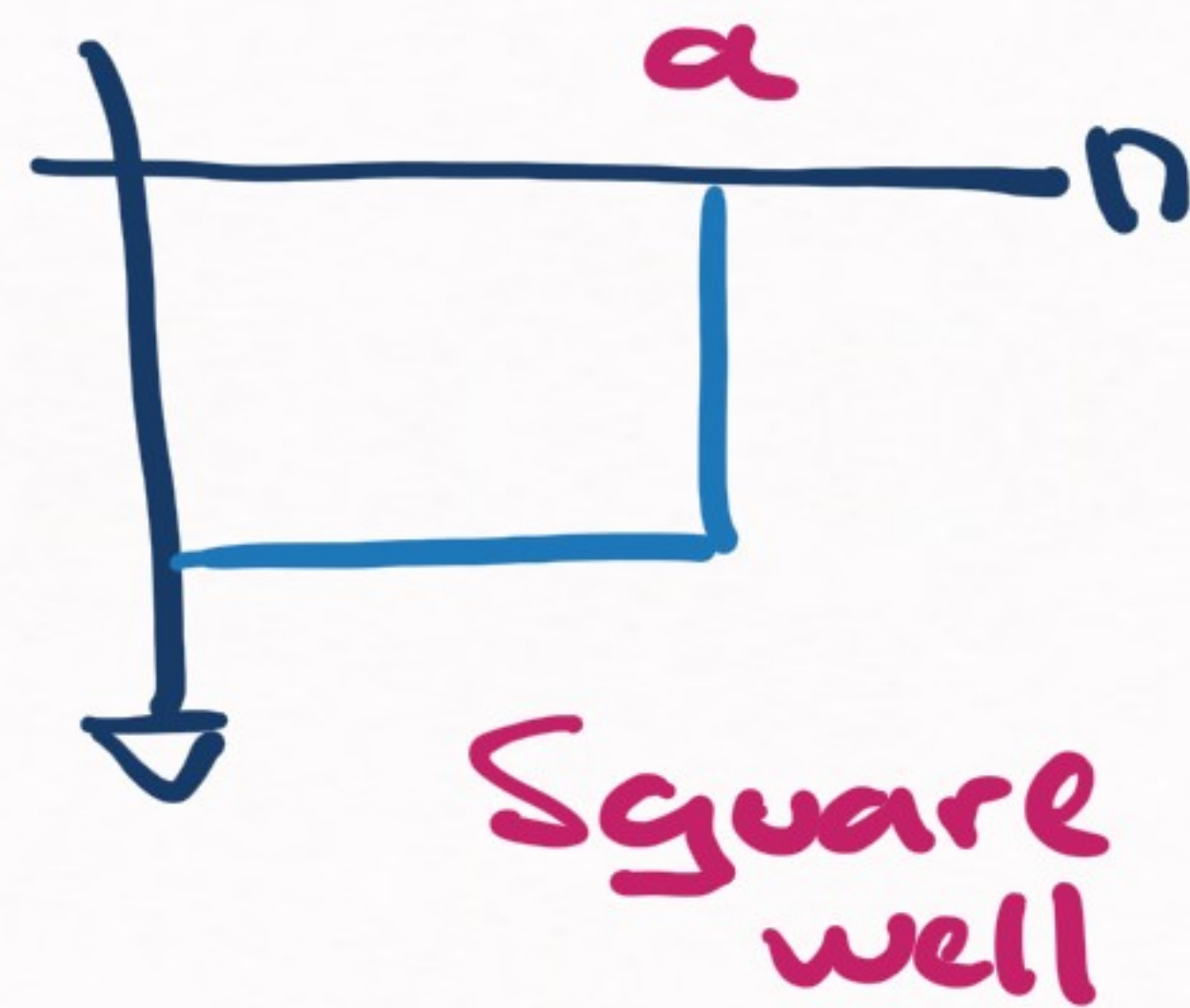
$$\Rightarrow u(r) \xrightarrow{r \rightarrow \infty} e^{ikr}$$

[BOUND, VIRTUAL STATES AND RESONANCES]

⇒ Examples:

1.a) & 1.b)

$$\kappa = \sqrt{2\mu(V_0 + E)} \quad (E < 0)$$



1.a) $u(r > a) = A_S e^{-\gamma_B r}$

$$\kappa \cot(\kappa a) = -\gamma_B$$

1.b) $u(r > a) = A_S e^{+\gamma_V r}$

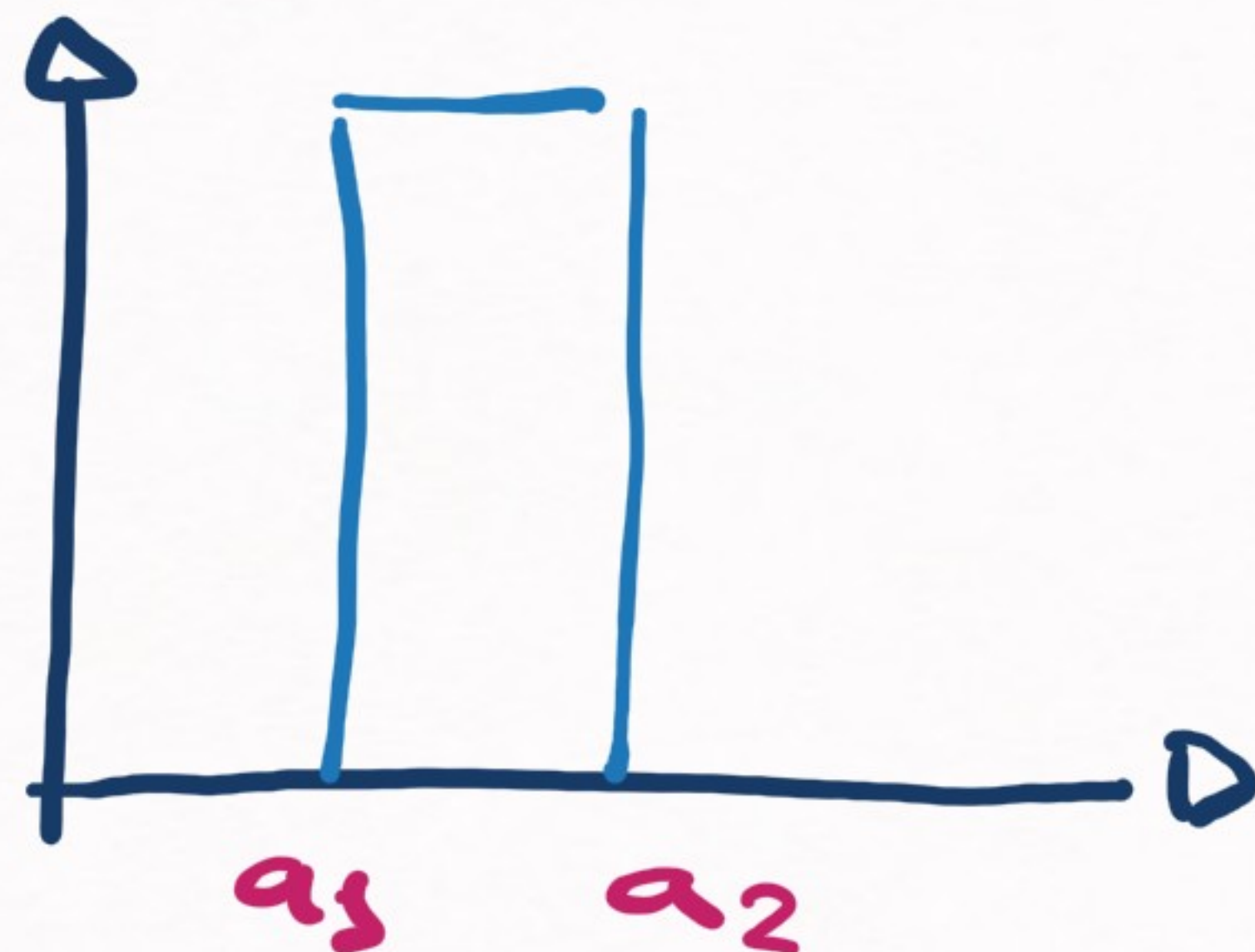
$$\kappa \cot(\kappa a) = +\gamma_V$$

1.a) & 1.b)

⇒ the same condition: $\kappa \cot(\kappa a) = -\gamma$

$\gamma > 0$
bound
 $\gamma < 0$
virtual

1.c)



1.c) $u(r > a) = A_S e^{i\kappa r}$

⇒ the condition will be rather involved

[BOUND, VIRTUAL STATES AND RESONANCES]

2) The "pole of the scattering amplitude" interpretation

$$T(E) \rightarrow \frac{1}{E - E_{\text{pole}}} \Rightarrow \text{we try to rewrite } T(E) \text{ in terms of } k = \sqrt{2\mu E},$$

pole \leftrightarrow $[k_p = \sqrt{2\mu E_p}]$

2.a) Bound state: $\text{Re}(k_p) = 0, \text{Im}(k_p) > 0$

2.b) Virtual state: $\text{Re}(k_p) = 0, \text{Im}(k_p) < 0$

2.c) Resonance: $\text{Re}(k_p) \neq 0, \text{Im}(k_p) < 0$

} pattern
↖

[BOUND, VIRTUAL STATES AND RESONANCES]

\Rightarrow The two Riemann sheets: $E = \frac{\hbar^2 k^2}{2\mu}$, E, k are complex

R I) $\text{Im}(k) > 0 \Rightarrow$ first (I) Riemann sheet

R II) $\text{Im}(k) < 0 \Rightarrow$ second (II) Riemann sheet

\downarrow
For each E ,
you have
two k 's

Bound states \Rightarrow sheet I (physical sheet)

Virtual states & Resonances \Rightarrow sheet II
(unphysical sheets)

[BOUND, VIRTUAL STATES AND RESONANCES]

⇒ Examples (of poles of the scattering amplitude)

2.a) & 2.b) $\langle \vec{p}' | T(E+i\epsilon) | \vec{p} \rangle = \frac{2\pi}{\mu} \frac{1}{\frac{1}{a_0} + ik} \rightarrow$ slides
M - 2S
(check again)

2.c) $\langle \vec{p}' | T(E+i\epsilon) | \vec{p} \rangle = - \frac{2\pi}{\mu} \frac{1}{-\frac{1}{a_0} + \frac{1}{2}r_0k^2 - ik}$

For $r_0 < 0$ (+ certain conditions for a_0)

⇒ Possible to have resonances

RECAP

1) Contact-range theories \Rightarrow they are the perfect example to understand how the T-matrix works

$$\left[\begin{aligned} \langle \vec{p}' | T(E+i\epsilon) | \vec{p} \rangle \\ = \frac{2\pi}{\mu} \frac{1}{\frac{1}{a_0} + ik} \end{aligned} \right]$$

+
Renormalization

2) We learnt that bound states are poles of the T-matrix
 \Rightarrow but there are more types of poles!

bound states, virtual states and resonances

RECAP

sheet: 3) Their interpretation:

I { 3.a) Bound states \Rightarrow We know them well

3.b) Virtual states \Rightarrow Same effects as a bound state for low energy scattering

But \nexists a physical state below threshold $\Leftarrow (\sigma \rightarrow 4\pi|a_0|^2 \text{ or } \frac{4\pi}{181^2})$

{ 3.c) Resonances \Rightarrow A bit like bound states, but they can decay

4) Riemann sheets \Rightarrow (I) : physical sheet $\text{Im}(k) > 0$

(II) : unphysical sheet

$\text{Im}(k) < 0$

[THE ONE BOSON EXCHANGE (OBE) MODEL]

⇒ Wanted (dead or alive):
a quantitatively successful
description of nuclear forces



OBE was the first of
such descriptions



A description
of nuclear
forces

Let's begin with...



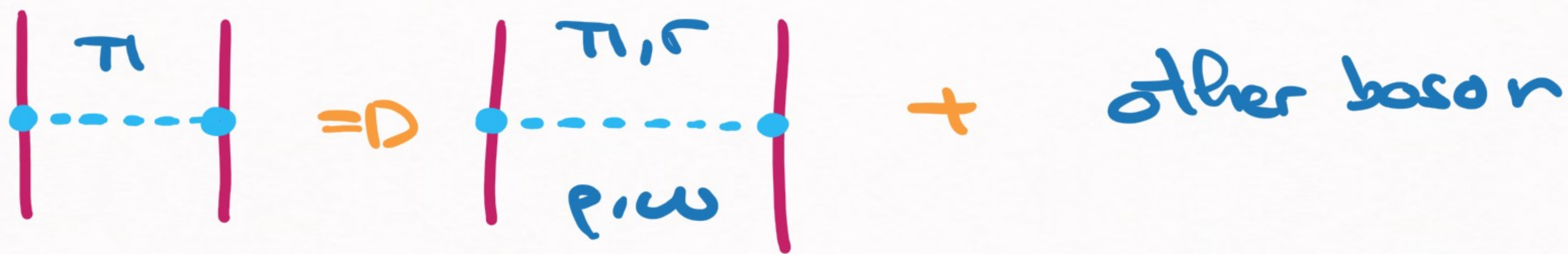
[What is the origin of the nuclear force?]

Historical overview:

- 0) Yukawa's idea \Rightarrow Exchange of a meson as explanation
- 1) Before QCD era:
 - 1.a) multi-pion theories (SO₃) \Rightarrow Failed
 - 1.b) one boson exchange model
- 2) Post QCD era: \rightarrow nucleons are not fundamental
 - 2.a) quark-model inspired explanation
 - 2.b) effective field theory formulations

ONE MODEL

→) Extension of the original idea by Yukawa:



2) Relatively simple and intuitive description

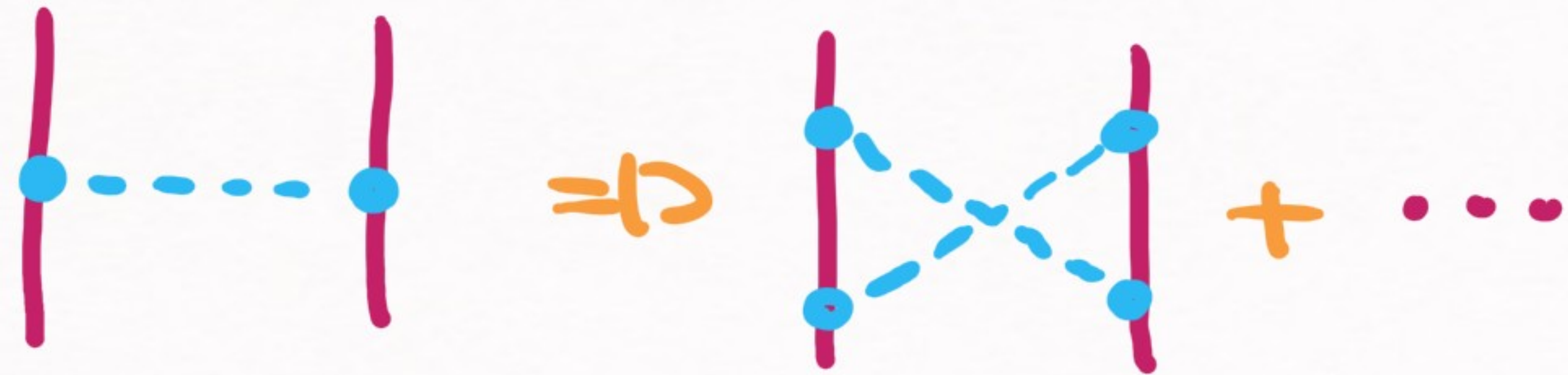
3) First quantitatively successful description
of the nuclear force

$$\chi^2/d.o.f \approx 1$$

for the description of the NN scattering data
(2000 data)

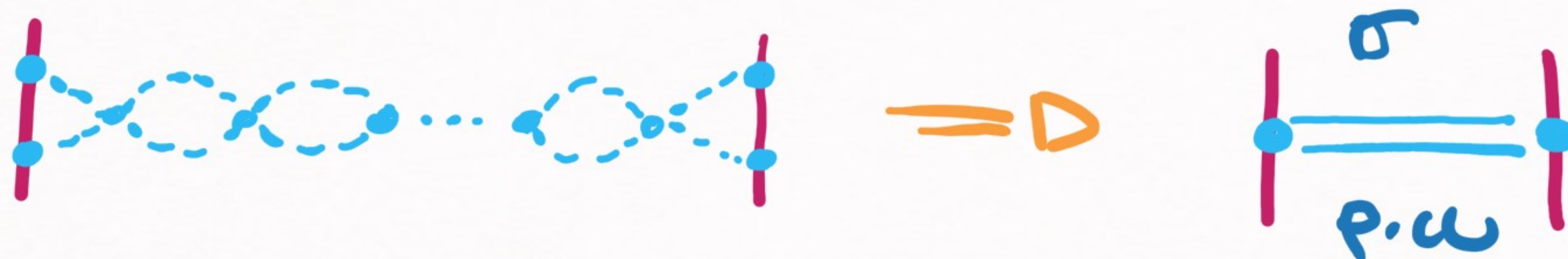
OBE MODEL \Rightarrow MOTIVATION: [Failure of the old pion theories]

1) Direct extension of Yukawa's idea: more pions!



2) But it did not work \Rightarrow why? Because back then no chiral symmetry, no renormalization

3) Idea: multi-pion exchanges \simeq resonance exchange



σ, ρ appear as resonances in $\pi\pi$ scattering

OBE MODEL \Rightarrow Each boson has a job within this model

a) The pion: $J^P = 0^-$, $I = 1$, $m_\pi \approx 140$ MeV

It explains the deuteron quadrupole moment

b) The sigma: $J^P = 0^+$, $I = 0$, $m_\sigma \approx 500$ MeV

provides a strong mid-range attraction

c) The rho: $J^P = 1^-$, $I = 1$, $m_\rho \approx 770$ MeV

counter the excessive tensor force from the pion

d) The omega: $J^P = 1^-$, $I = 0$, $m_\omega \approx 780$ MeV

provides short-range repulsion

OBE MODEL \Rightarrow Each boson has a job within this model

Table 1: The four most important mesons and the main characteristics of their contributions to components of the nuclear force.

Meson	Central	Spin-Spin	Tensor	Spin-Orbit
$\pi(138)$	---	weak, long-ranged	strong , long-ranged	---
$\sigma(500)$	strong, attractive , intermediate-ranged	---	---	moderate, intermediate-ranged
$\omega(782)$	strong, repulsive , short-ranged	---	---	strong , short-ranged, coherent with σ
$\rho(770)$	---	weak, short-ranged, coherent with π	moderate, short-ranged, opposite to π	---

Relativistic correctors

From scholarpedia's review by R. Machleidt

[THE OBE POTENTIAL]

=> Really simple: $V_{NN}(\vec{q}) = \sum_{\mu} V_{\mu}(\vec{q}) = V_{\pi} + V_{\sigma} + V_{\rho} + V_{\omega}$

$$\pi) V_{\pi}(\vec{q}) = -\frac{g_{\pi NN}^2}{4M_N^2} \frac{\vec{\sigma}_1 \cdot \vec{q} \vec{\sigma}_2 \cdot \vec{q}}{q^2 + m_{\pi}^2}$$

$$\sigma) V_{\sigma}(\vec{q}) = -\frac{g_{\sigma}^2}{q^2 + m_{\sigma}^2}$$

$$\omega) V_{\omega}(\vec{q}) = \frac{g_{\omega}^2}{q^2 + m_{\omega}^2} - \frac{(f_{\omega} + g_{\omega})^2}{4M_N^2} \frac{(\vec{\sigma}_1 \cdot \vec{q})(\vec{\sigma}_2 \cdot \vec{q})}{q^2 + m_{\omega}^2}$$

$$\rho) V_{\rho}(\vec{q}) = \vec{\tau}_1 \cdot \vec{\tau}_2 \left[\frac{g_{\rho}^2}{q^2 + m_{\rho}^2} - \frac{(f_{\rho} + g_{\rho})^2}{4M_N^2} \frac{(\vec{\sigma}_1 \cdot \vec{q})(\vec{\sigma}_2 \cdot \vec{q})}{q^2 + m_{\rho}^2} \right]$$

[OBE MODEL]

\Rightarrow Usual simplifications:

$$f_p \gg g_e \Rightarrow g_e \approx 0$$

$$f_w \ll g_w \Rightarrow f_w \approx 0$$

[THE OBE POTENTIAL]

=> Really simple: $V_{NN}(\vec{r}) = \sum_{\mu} V_{\mu}(\vec{r}) = V_{\pi} + V_{\sigma} + V_{\rho} + V_{\omega}$

$$\pi) \quad V_{\pi}(\vec{r}) = \frac{1}{3} \frac{g_{\pi NN}^2}{4M_N^2} \vec{\tau}_1 \cdot \vec{\tau}_2 \left[\vec{\sigma}_1 \cdot \vec{\sigma}_2 m_{\pi}^3 W_{\gamma}(m_{\pi} r) + S_{12}(\vec{r}) m_{\pi}^3 W_{\gamma}(m_{\pi} r) \right]$$

$$\sigma) \quad V_{\sigma}(\vec{r}) = -g_{\sigma}^2 m_{\sigma} W_{\gamma}(m_{\sigma} r)$$

$$\omega) \quad V_{\omega}(\vec{r}) = g_{\omega}^2 m_{\omega} W_{\gamma}(m_{\omega} r)$$

$$+ \frac{(\rho_{\omega} + S_{\omega})^2}{4M_N^2} \left[\frac{2}{3} \vec{\sigma}_1 \cdot \vec{\sigma}_2 m_{\omega}^3 W_{\gamma}(m_{\omega} r) \right.$$

$$\left. - \frac{1}{3} S_{12}(\vec{r}) m_{\omega}^3 W_{\gamma}(m_{\omega} r) \right]$$

$\rho)$ like ω , but including $\vec{\tau}_1 \cdot \vec{\tau}_2$

[THE OBE POTENTIAL]

=> Really simple: $V_{NN}(\vec{r}^2) = \sum_{\mu} V_{\mu}(\vec{r}^2) = V_{\pi} + V_{\sigma} + V_{p} + V_{\omega}$

Two Functions: a) $W_{\gamma}(x) = \frac{e^{-x}}{4\pi x}$

b) $W_{\pi}(x) = \frac{e^{-x}}{4\pi x} \left(1 + \frac{3}{x} + \frac{3}{x^2} \right)$

→

=>

[THE OBE POTENTIAL]

=> Really simple, but we have singular interactions:

$$V_{\text{OBE}}(\vec{r}) \propto \frac{1}{r^3} \text{ for } r \rightarrow 0$$

->

Proposed
solution

[FORM FACTORS]

=> What is a form factor?

$$\underbrace{V_{\mu}(\vec{q})} \longrightarrow V_{\mu}(\vec{q}) \underbrace{F_{\mu}^2(\vec{q}; \Lambda)}$$

=> Most usual choice: $F_{\mu}(\vec{q}; \Lambda) = \left(\frac{\Lambda^2 - m^2}{\Lambda^2 + \vec{q}^2} \right)^{\rho}$

[FORM FACTORS]

⇒ Example: monopolar form factor $F_{\mu}(\vec{q}, \Lambda) = \frac{\Lambda^2 - m^2}{\Lambda^2 + \vec{q}^2}$

$V_{\mu}(\vec{q}; \Lambda) = V_{\mu}(\vec{q}) F_{\mu}^2(\vec{q}, \Lambda)$ equivalent to:

a) $W_Y(x) \rightarrow W_Y(x, \lambda) = W_Y(x) - \lambda W_Y(\lambda x) - \frac{(\lambda^2 - 1)}{2x} \frac{e^{-\lambda x}}{4\pi}$

b) $W_T(x) \rightarrow W_T(x, \lambda) = W_T(x) - \lambda^3 W_T(\lambda x)$

with $\lambda = \frac{\Lambda}{m}$ $- \frac{(\lambda^2 - 1)}{2\lambda} \lambda^2 \left(1 + \frac{1}{\lambda x} \right) \frac{e^{-\lambda x}}{4\pi}$

[OBE PARAMETERS]

The Meson Theory of Nuclear Forces and Nuclear Structure

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139.180.195.74

☆ ↓ <

Physics, Spring 2023

We will compile the lecture notes of the course, the exercise sheets and the slides of this year are under development (actually, they are always under development and you should come here from time to time for updates. Currently we have no update):

[Introduction](#)
[Nucleon system](#)
[Nucleon structure](#)

Materials: [SU\(3\) Clebsch-Gordan coefficients](#), [ChPT Reviews](#), [the OBE model](#).

Table A.3: Non-relativistic configuration space OBEP.

	m_α (MeV)	Potential A		Potential B	
		$g_\alpha^2/4\pi$	Λ_α (GeV)	$g_\alpha^2/4\pi$	Λ_α (GeV)
π	138.03	14.9	1.3	14.9	2.0
η	548.8	2	1.5	0	-
ρ	769	1.2	1.2	1.7	1.1
ω	782.6	25	1.4	28	1.3
δ	983	2.742	2.0	6.729	2.0
σ^a	550 (700-710)	8.7171 (17.6205)	2.0 (2.0)	8.8322 (16.0707)	1.4 (2.0)
$-\epsilon_d$ (MeV)		2.2246		2.2246	
P_D (%)		4.75		5.53	
Q_d (fm ²)		0.274 ^b		0.279 ^b	
μ_d (μ_N)		0.8527 ^b		0.8483 ^b	
A_S (fm ^{-1/2})		0.8865		0.8860	
D/S		0.0259		0.0263	
a_{np} (fm)		-23.75		-23.75	
r_{np} (fm)		2.69		2.70	
a_t (fm)		5.425		5.423	
$\tau_t = \rho(0, 0)$ (fm)		1.762		1.758	

Given are the meson, deuteron, and low energy parameters.

For notation and other information see Table 4.1 and 4.2.

It is always used: $f_\rho/g_\rho = 6.1$, $f_\omega/g_\omega = 0.0$, and $n_\alpha = 1$.

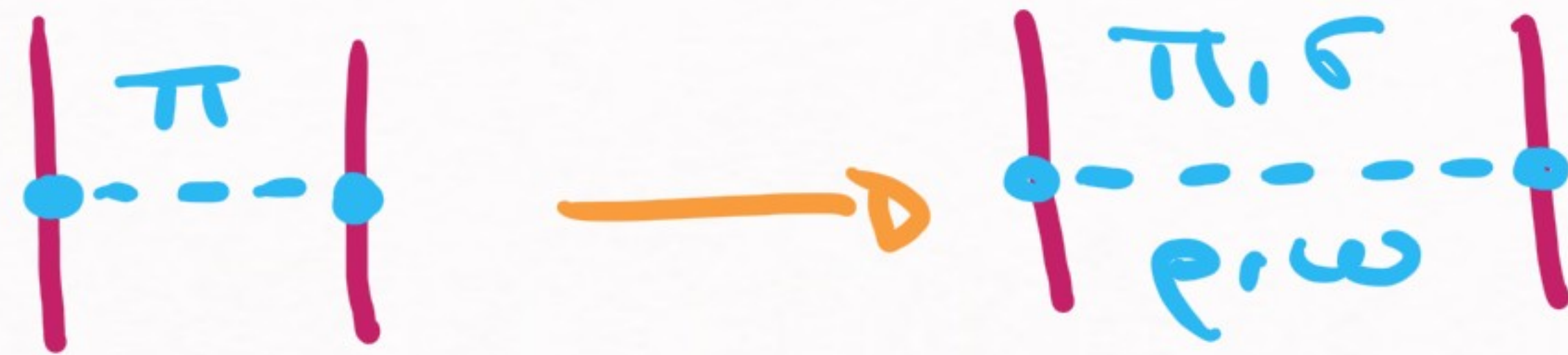
^aThe σ parameters given in brackets apply to the $T = 0$

NN potential. Potential A uses 710 and B uses 700 MeV for the σ mass.

^b Meson-exchange current contributions not included.

RECAP

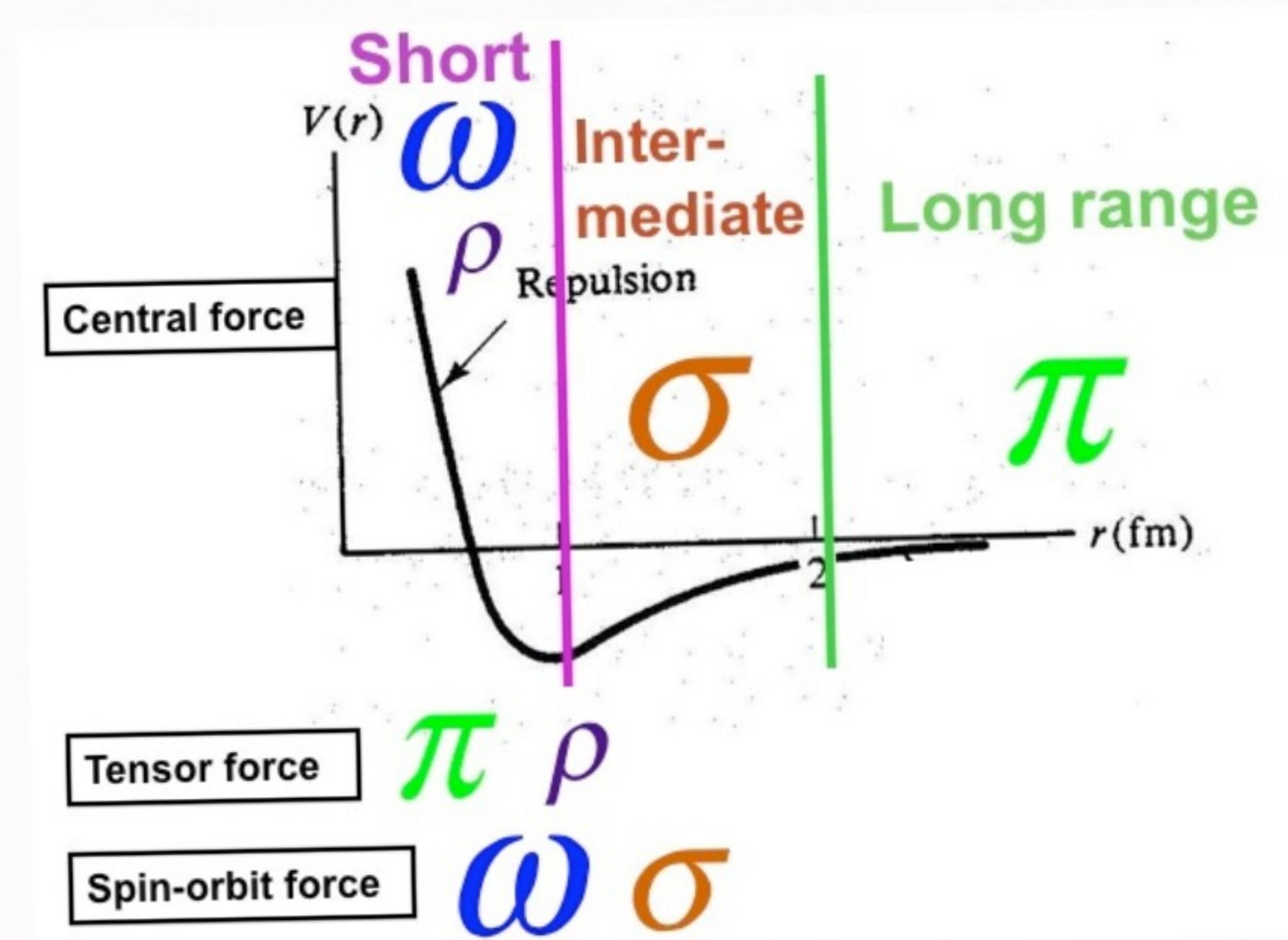
1) The OBE model \Rightarrow Natural extension of Yukawa's idea



2) Each meson has a job:

3) We obtain singular potentials at $r \rightarrow 0$:

We include form factors to regularize the potential



→ We finish this part next week

See you the next day

Tuesday / Friday

15:50

↵