NUCLEAR PHYSICS 16

a) FORMAL SCATTERING THEORY (PART II)

⇒ Solutions for contact-range theories

b) BOUND / VIRTUAL STATES & RESONANCES

c) THE ONE BOSON EXCHANGE MODEL

Recap

a) Formal scattering theory $\Rightarrow$ Usual scattering theory, but we use the language of operators acting on a Hilbert space

$$\left.\begin{array}{l} \psi(\vec{r}) \\ \left[-\frac{\vec{\nabla}^2}{2\mu}+V(\vec{r})\right]\psi(\vec{r}) \\ \qquad = E\psi(\vec{r}) \end{array}\right\} \Rightarrow \begin{array}{l} |\psi\rangle \\ (H_0+V)|\psi\rangle \\ \qquad = E|\psi\rangle \end{array}$$

b) This idea also applies to scattering:

$$\left.\psi(\vec{r}) \to e^{i\vec{k}\cdot\vec{r}} + f(\Omega)\frac{e^{ikr}}{r}\right\} \Rightarrow |\psi_{\vec{k}}\rangle = |\vec{k}\rangle + G_0 V|\psi_{\vec{k}}\rangle$$

with $G_0(E) = \dfrac{1}{E-H_0}$

RECAP/

c) The scattering amplitude $f(\Omega)$ is now the matrix element of an operator (T-matrix):

$$f(\Omega) = -\frac{\mu}{2\pi}\langle \vec{k}' | T(E) | \vec{k}\rangle \quad \text{with} \quad T(E) = V + V G_0(E) T(E)$$

(physical scattering: $E + i\epsilon,\ \epsilon \to 0^+$)

d) Bound states are poles in the T-matrix:

$$T(E) \xrightarrow[E \to E_B]{} \frac{V|\psi_B\rangle\langle\psi_B|V}{E - E_B}$$

$(E_B < 0)$

Bound state equation:

$$|\psi_B\rangle = G_0(E) V |\psi_B\rangle$$

[T-MATRIX AND CONTACT-RANGE INTERACTIONS]

a) We begin w/ a reminder: separable interactions

$$\langle \vec{k}' | V | \vec{k} \rangle = g\, f(\vec{k}')\, f(\vec{k}) \Rightarrow \langle \vec{k}' | T(E) | \vec{k} \rangle = \tau(E)\, f(\vec{k}')\, f(\vec{k})$$

$$\tau(E) = \frac{1}{\frac{1}{g} - \int \frac{d^3\vec{q}}{(2\pi)^3} \frac{f^2(\vec{q})}{E - q^2/2\mu}}$$

b) And a second reminder: contact-range potentials

$$V_C(\vec{r}) = C_0\, \delta^{(3)}(\vec{r} - \vec{r}') \xrightarrow{\text{Fourier transform}} V_C(\vec{q}) = C_0$$

$$\langle \vec{k}' | V_C | \vec{k} \rangle = C_0$$

$$\vec{q} = \vec{k} - \vec{k}'$$

[T-MATRIX & CONTACTS]

c) Contacts have a really big problem: $\rho(\vec{k}) = 1$

$$\frac{1}{\tau(E)} = \frac{1}{C_0} - \underbrace{\int \frac{d^3\vec{q}}{(2\pi)^3} \frac{1}{E - \vec{q}^2/2\mu}}$$

$$\int \frac{d^3\vec{q}}{(2\pi)^3} \frac{1}{E - \vec{q}^2/2\mu} \sim -\frac{2\mu}{(2\pi)^3} \int \frac{d^3\vec{q}}{|\vec{q}|^2} \sim -\frac{\mu}{\pi^2} \int \frac{q^2 dq}{q^2} \sim -\frac{\mu}{\pi^2} \int dq$$

$$\int_0^\infty dq \to \infty \Rightarrow \int^\Lambda dq \sim \Lambda$$

There is a linear divergence with this potential

[T-MATRIX AND CONTACTS]

$f(x) \to$ a regulator function

d) But we can regularize the Dirac-delta:

$$\langle \vec{k}' | V_c | \vec{k} \rangle = C_0 \longrightarrow \langle \vec{k}' | V_c | \vec{k} \rangle = C_0 \underline{\underline{f\left(\frac{k'}{\Lambda}\right)}} \underline{\underline{f\left(\frac{k}{\Lambda}\right)}}$$

$f(x)$: d.1) $f(x) \xrightarrow[x \to 0]{} 1$ (at low momenta: $\langle \vec{k}' | V_c | \vec{k} \rangle \to C_0$)

d.2) $f(x) \xrightarrow[x \to \infty]{} 0 \longrightarrow \int \frac{d^3\vec{q}}{(2\pi)^3} \frac{1}{E - q^2/2\mu} \to \int \frac{d^3q}{(2\pi)^3} \frac{f^2(q)}{E - q^2/2m}$

d.3) $\int dx\, f^2(x) < \infty$

$$\sim -\frac{\mu}{\pi^2} \int dq\, f^2(q/\Lambda) < \infty$$

[T-MATRIX AND CONTACTS]

The two-body system collapses

e) For $\Lambda \to \infty$, we can see that this potential is singular

We consider $E = E_B$ (bound state) $E_B = -\frac{\gamma^2}{2\mu}$

$\Rightarrow \left[\frac{1}{T(E=E_B)} = 0 \right]$

$x = q/\Lambda$, $\tilde{\gamma} = \gamma/\Lambda$

$$\int_0^\infty dx\, g^2(x) - \tilde{\gamma}^2 \int_0^\infty \frac{dx\, g^2(x)}{x^2 + \tilde{\gamma}^2} = -\frac{\pi^2}{\mu} \frac{1}{\Lambda C_0} \to 0^+ \text{ (for } C_0<0)$$

$$\int \frac{d^3\vec{q}}{(2\pi)^3} \frac{g^2(q/\Lambda)}{E_B - \frac{q^2}{2\mu}} = \frac{1}{C_0}$$

If $\Lambda \to \infty$, $\tilde{\gamma} \to$ constant

$\Rightarrow$ for C_0 attractive

$\left.\begin{array}{l} \Rightarrow \gamma \propto \Lambda \to \infty \\ E_B \to -\infty \end{array}\right\}$ COLLAPSE!

[T-MATRIX AND CONTACTS]

§) How do we solve the collapse? $\Rightarrow \underbrace{C_0 = C_0(\Lambda)}$

[C_0 depends on Λ
g is chosen to
reproduce E_B / γ]

$$\frac{1}{C_0(\Lambda)} = -\frac{\mu}{\pi^2}\int_0^\infty \frac{q^2\,dq}{\gamma^2+q^2}\, g^2(q/\Lambda)$$

$$= -\frac{\mu}{\pi^2}\left[\Lambda\int_0^\infty g^2(x)\,dx - \gamma\int_0^\infty \frac{dx}{1+x^2}\, g^2(x\cancel{\tfrac{\Lambda}{\gamma}})\right]$$

$F(x)$, $G(x)$ are functions
B_0 is a constant

$$= \frac{\mu}{\pi^2}\left[\gamma F(\gamma/\Lambda) - \Lambda B_0\right]$$

$$= \frac{\mu}{2\pi}\left[\gamma - \Lambda G(\gamma/\Lambda)\right] \rightarrow$$

this reworking of the Eq. requires a bit of thinking, but it's really convenient

[T-MATRIX AND CONTACTS]

a sharp cutoff

⇒ Exactly solvable example: $f(x) = \Theta(1-x)$

$$\frac{1}{C_0(\Lambda)} = -\frac{\mu}{\pi^2}\int_0^\Lambda \frac{q^2\,dq}{q^2+\gamma^2} = -\frac{\mu}{\pi^2}\left[\Lambda - \gamma \operatorname{atan}\frac{\Lambda}{\gamma}\right]$$

$\operatorname{atan} x$
$= \frac{\pi}{2} - \operatorname{acot} x$

$$= \frac{\mu}{2\pi}\left[\gamma - \frac{2}{\pi}\Lambda\left(1 + \frac{\gamma}{\Lambda}\operatorname{acot}\frac{\Lambda}{\gamma}\right)\right]$$

⇒ For $\Lambda \to \infty$,
it simplifies to

$$\frac{1}{C_0(\Lambda)} \xrightarrow[\Lambda\to\infty]{} \frac{\mu}{2\pi}\left(\gamma - \frac{2}{\pi}\Lambda\right)$$

[T-MATRIX AND CONTACTS] $\Rightarrow$ SCATTERING STATES

$\Rightarrow$ Now we solve $f(x) = \Theta(1-x)$ for $E > 0$:

singular point

$$\int \frac{d^3\vec{q}}{(2\pi)^3} \frac{f^2(q/\Lambda)}{E + i\epsilon - q^2/2\mu} = \frac{\mu}{\pi^2} \int_0^\Lambda \frac{q^2 dq}{k^2 + i\epsilon - q^2} \rightarrow \left[\frac{1}{k^2 - q^2} \text{ for } q = k\right]$$

$\Rightarrow$ Trick: the principal value of an integral

$$\mathcal{P}\int_{-\infty}^{+\infty} f(x)\,dx = \lim_{\epsilon \to 0^+} \left[\int_{-\infty}^{-\epsilon} + \int_{-\epsilon}^{+\epsilon} + \int_{+\epsilon}^{\infty} f(x)\,dx\right]$$

$$f(x) = \frac{c_0}{x} \Rightarrow \mathcal{P}\int_{-\infty}^{\infty} f(x)\,dx = 0$$

$\Rightarrow$ Exactly symmetric limit

[T-MATRIX AND CONTACTS] $\Rightarrow$ SCATTERING STATES

$\Rightarrow$ Principal value : $\mathcal{P}\int_{-\infty}^{\infty} \frac{dx}{x} = 0$ (example)

a) Useful formula : $\frac{1}{x \pm i\epsilon} = \mathcal{P}\left(\frac{1}{x}\right) \mp i\pi\delta(x)$

b) If we apply it to our integral :

$$\frac{\mu}{\pi^2}\int_0^{\Lambda} \frac{q^2 dq}{k^2 + i\epsilon - q^2} = \frac{\mu}{\pi^2}\left[\mp i\frac{\pi}{2}k - \Lambda + \frac{k}{2}\log\left|\frac{\Lambda - k}{\Lambda + k}\right|\right]$$

$k = \sqrt{2\mu E}$

please check, using a)
+ b)

[T-MATRIX AND CONTACTS] $\Rightarrow$ SCATTERING STATES

$\Rightarrow$ Now we can calculate $\tau(E)$:

$$\tau(E) = \frac{1}{\frac{1}{C_0(\Lambda)} - \frac{\mu}{\pi}\int_0^{\infty}\frac{q^2\,dq}{k^2+i\epsilon-q^2}} = \frac{1}{\frac{1}{C_0(\Lambda)} + \frac{\mu}{2\pi}\left[ik + \frac{2}{\pi}\Lambda - \frac{k}{\pi}\log\left|\frac{\Lambda-k}{\Lambda+k}\right|\right]}$$

$E+i\epsilon$

$$f(\omega) = -\frac{\mu}{2\pi}\langle\vec{k}'|T(E+i\epsilon)|\vec{k}\rangle$$

match the expressions

$$\hookrightarrow \tau(E) = -\frac{2\pi}{\mu}\frac{1}{k\cot\delta - ik} = -\frac{2\pi}{\mu}\frac{1}{-\frac{2\pi}{\mu}\frac{1}{C_0(\Lambda)} - \frac{2}{\pi}\Lambda + \ldots - ik}$$

[T-MATRIX AND CONTACTS] $\Rightarrow$ SCATTERING STATES

$\Rightarrow$ Matching with $\rho_0(k)$ we arrive at: $k\cot\delta = -\frac{1}{a_0} + \dots$

$$k\cot\delta = -\frac{2\pi}{\mu C_0(\Lambda)} - \frac{2}{\pi}\Lambda + \dots \quad \Rightarrow \quad \underbrace{\frac{1}{C_0(\Lambda)} = \frac{\mu}{2\pi}\left(\frac{1}{a_0} - \frac{2}{\pi}\Lambda\right)}_{\text{Really neat equation}}$$

$\Rightarrow$ For $\Lambda \to \infty$, we obtain:

$$\langle \vec{k}'| T(E+i\epsilon) |\vec{k}\rangle \longrightarrow \frac{2\pi}{\mu} \frac{1}{1/a_0 + ik}$$

try to check this result

[T-MATRIX AND CONTACTS] $\Rightarrow$ FROM SCATTERING TO BOUND STATES

a) For the $\Lambda \to \infty$ limit, we have:

$$\left[\langle \vec{k}' | T(E+i\epsilon) | \vec{k} \rangle = \frac{2\pi}{\mu} \frac{1}{\frac{1}{a} + ik} \quad , k = \sqrt{2\mu E} \right]$$

b) But, where are the bound states here? $\to$ Poles

$$E_B = -B \quad (E_B < 0 \Rightarrow B > 0) \Rightarrow k = \sqrt{2\mu E} = \underbrace{\sqrt{-2\mu B}}$$

If $B = \frac{\gamma^2}{2\mu} \Rightarrow \boxed{k = \pm i\gamma}$ $\leftarrow$ Could be problematic

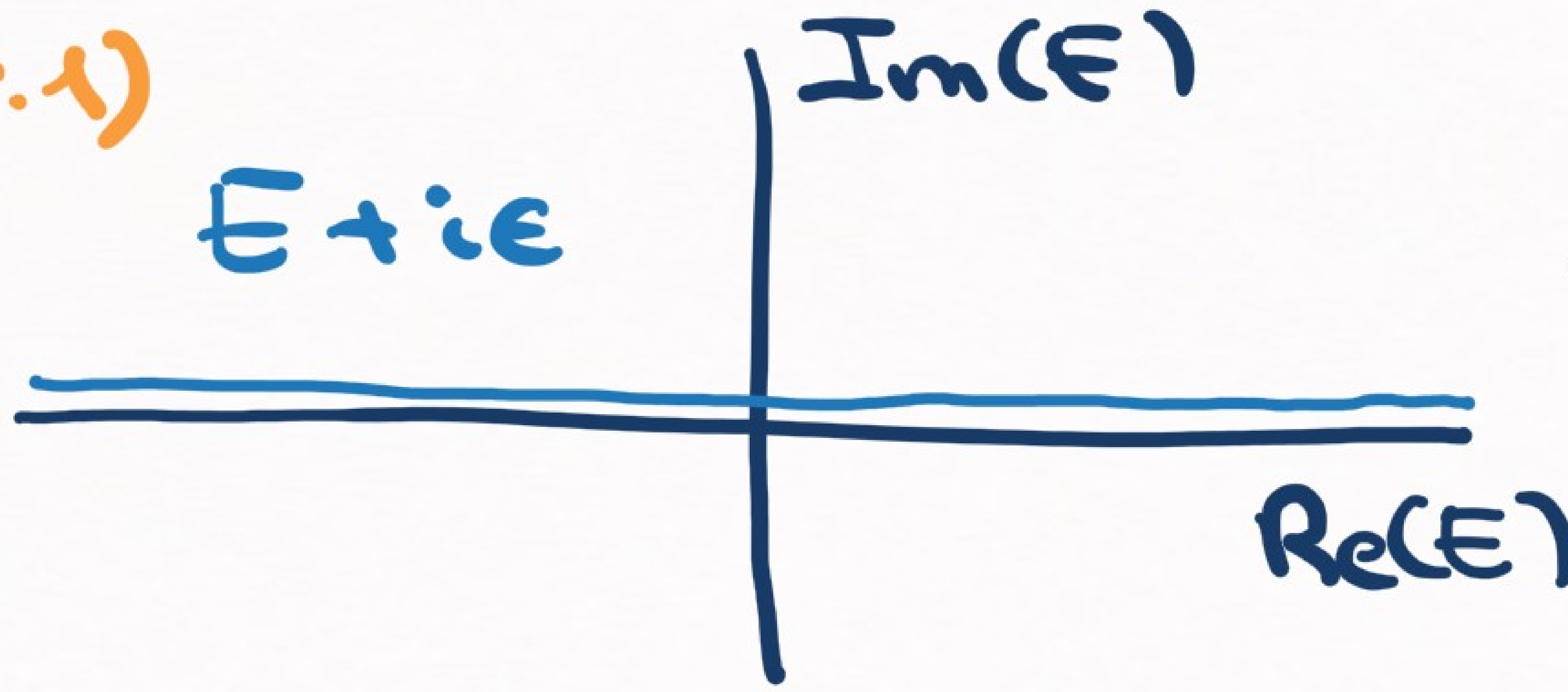
How do we choose the correct root?

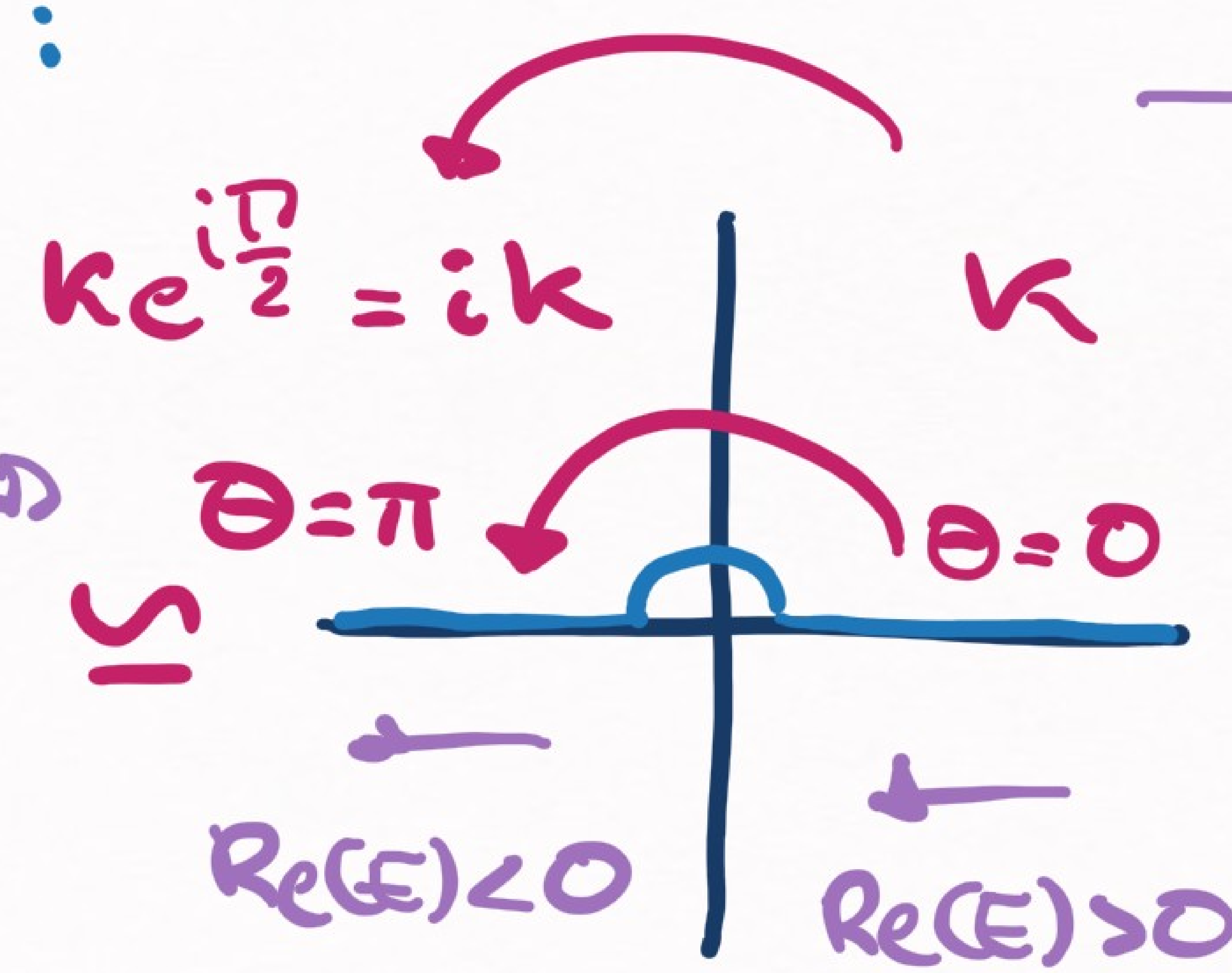
[T-MATRIX AND CONTACTS] $\Rightarrow$ FROM SCATTERING TO BOUND STATES

c) The $E \pm i\epsilon$ prescriptions:

c.1)

$E+i\epsilon$





$\rightarrow E = |E| e^{i\theta}$

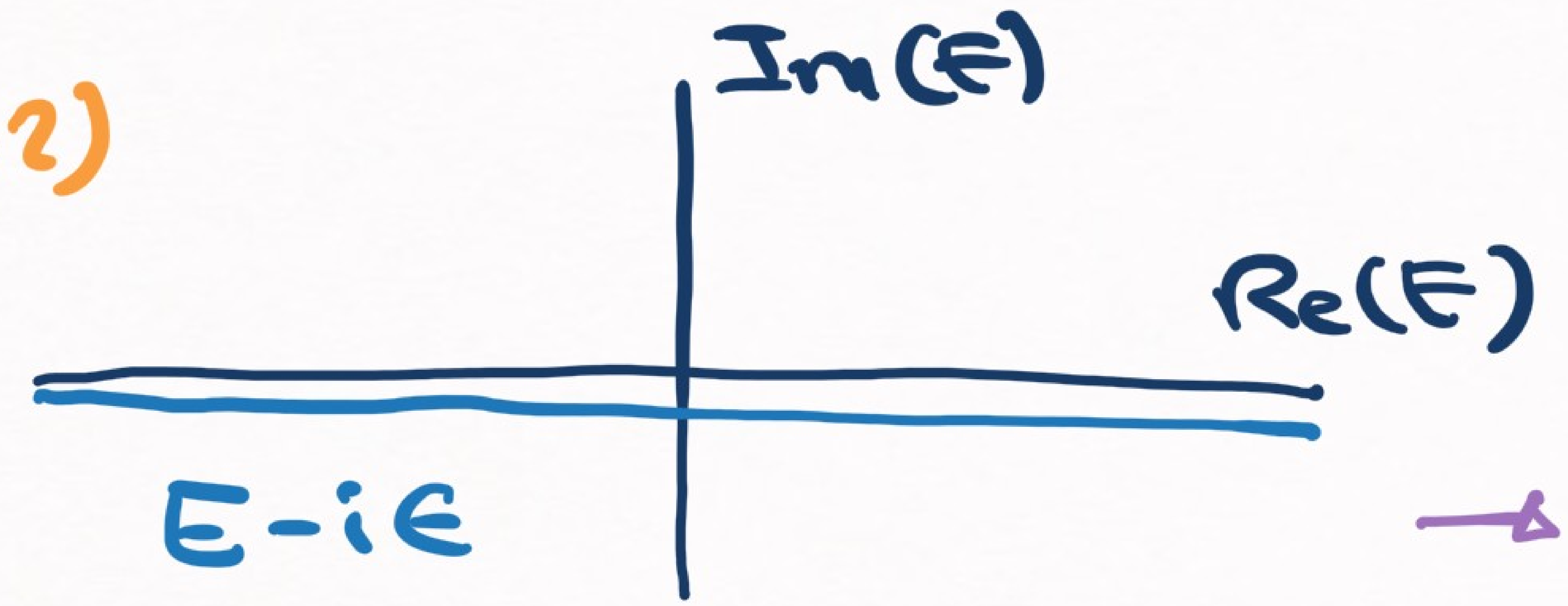
$\Downarrow$

$k = \sqrt{2\mu E}$

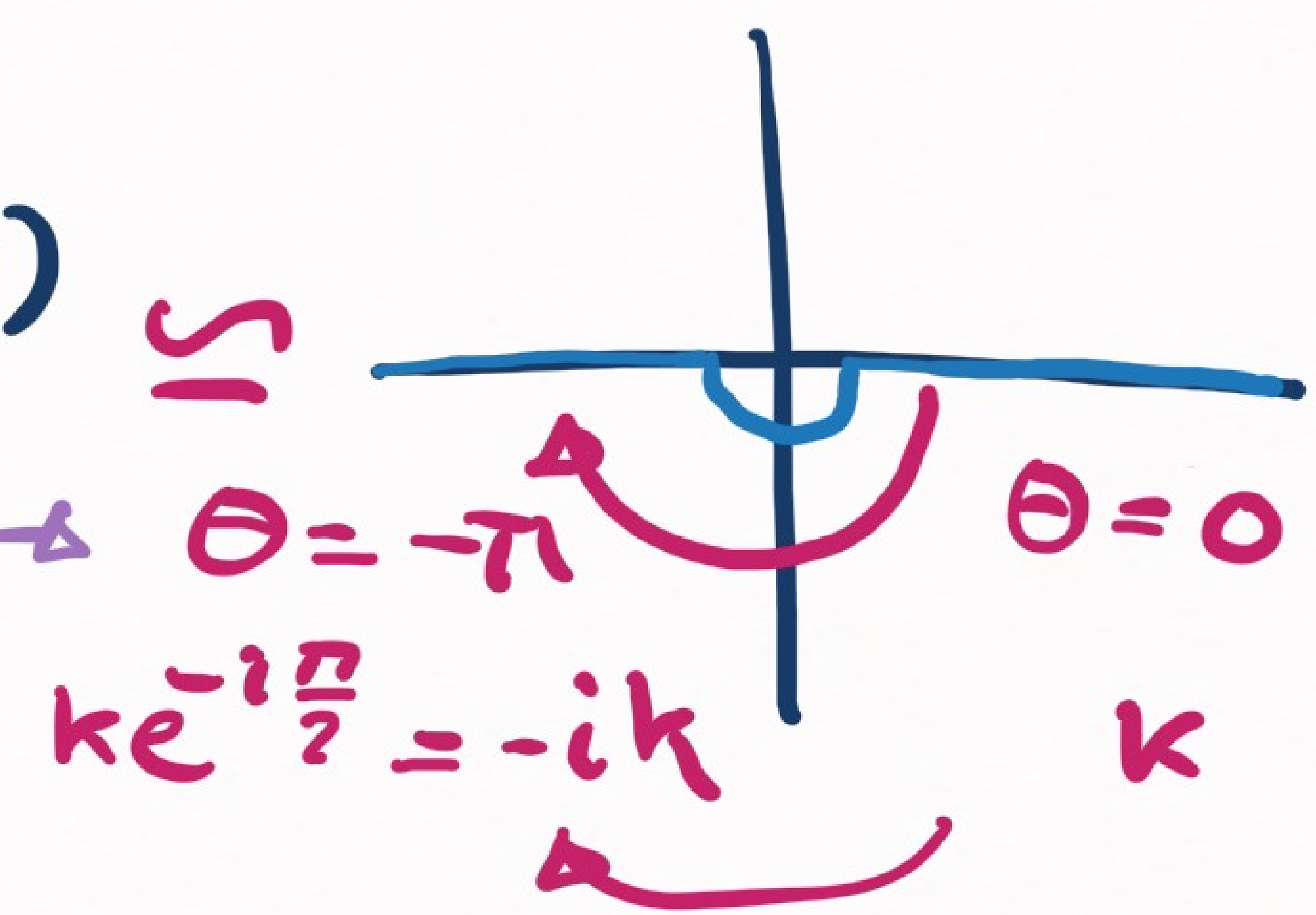
$\Downarrow$

$k = |k| e^{i\frac{\theta}{2}}$

c.2)



$E-i\epsilon$



Sign of θ:

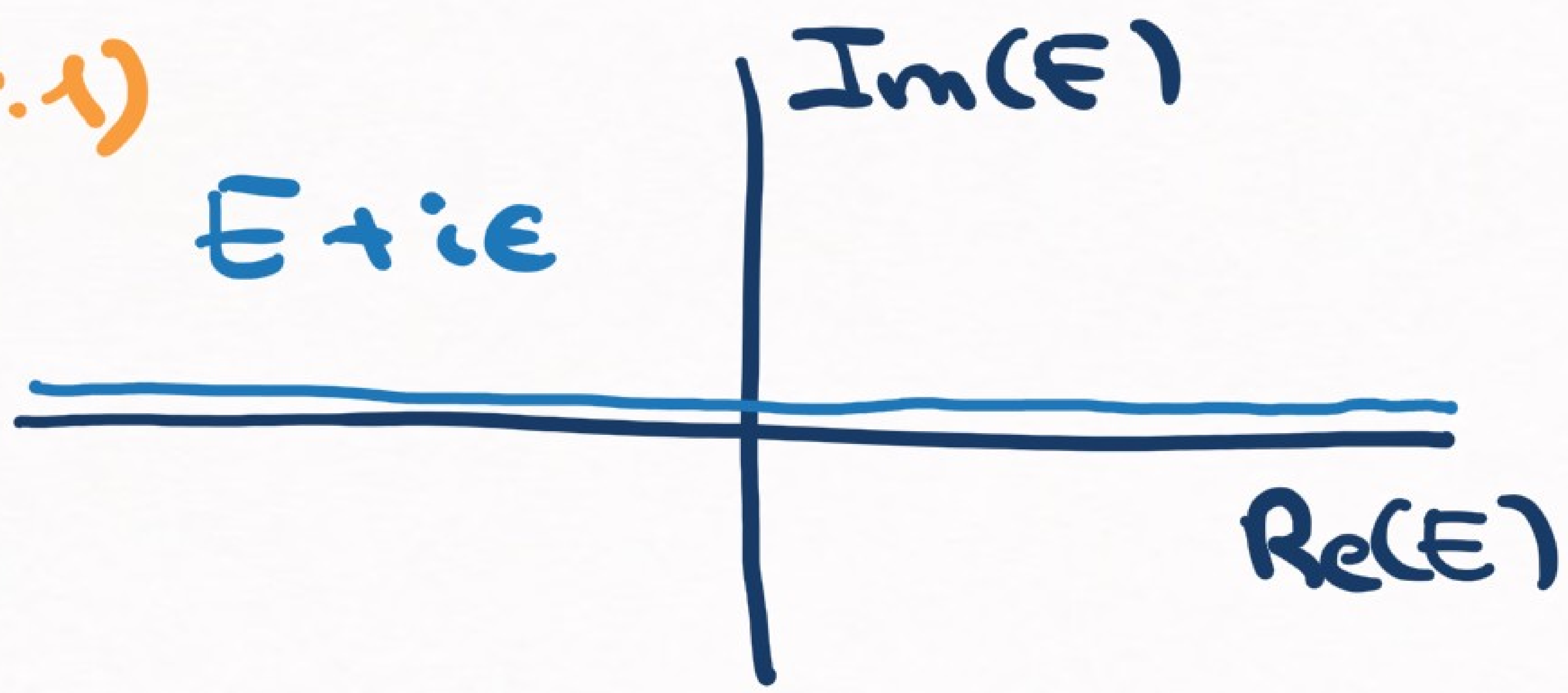
$\theta>0$ for counterclockwise

$\theta<0$ for clockwise

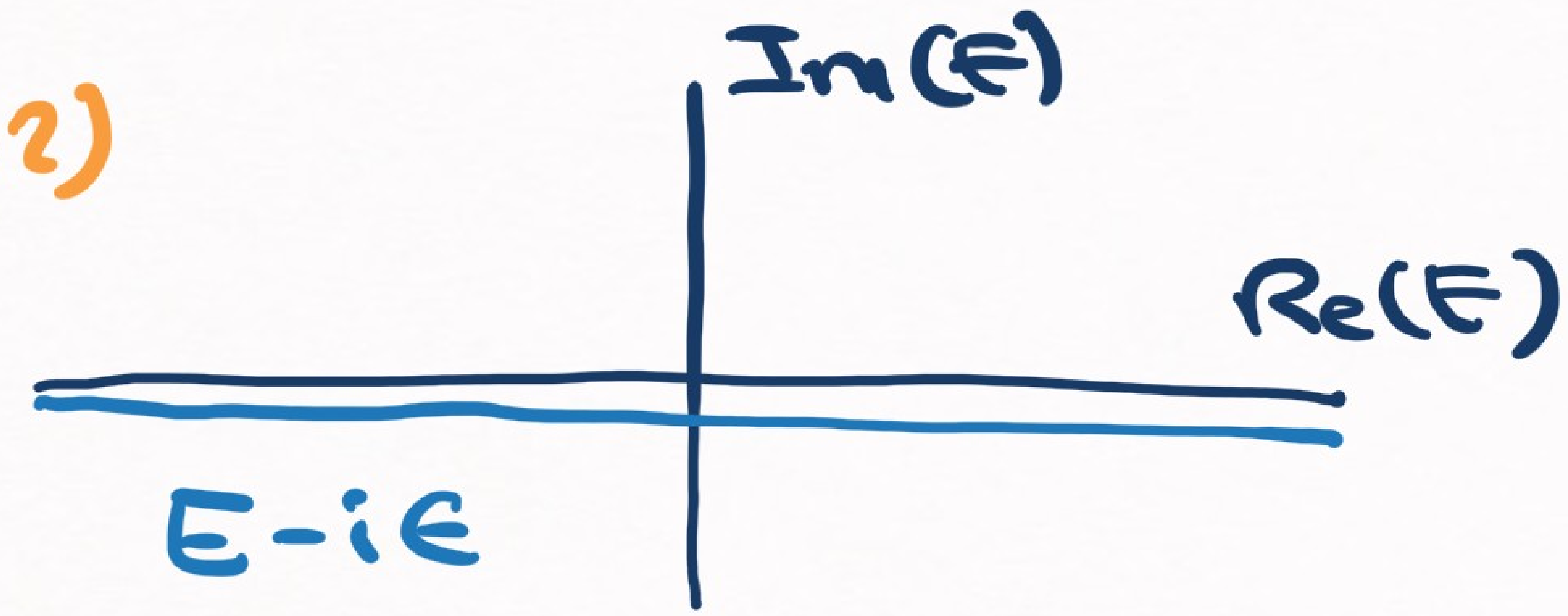
[T-MATRIX AND CONTACTS] ⟹ FROM SCATTERING TO BOUND STATES

c) The $E \pm i\epsilon$ prescriptions:

c.1)



c.2)



CAVEAT

We are using "$E \pm i\epsilon$" in a heuristical way here.

Stricto sensu

$$T(E \pm i\epsilon) = \frac{2\pi}{\mu} \frac{1}{1/a \pm ik}$$

but here we are using the $E \pm i\epsilon$ idea as notation for the two ways of going to $E<0$

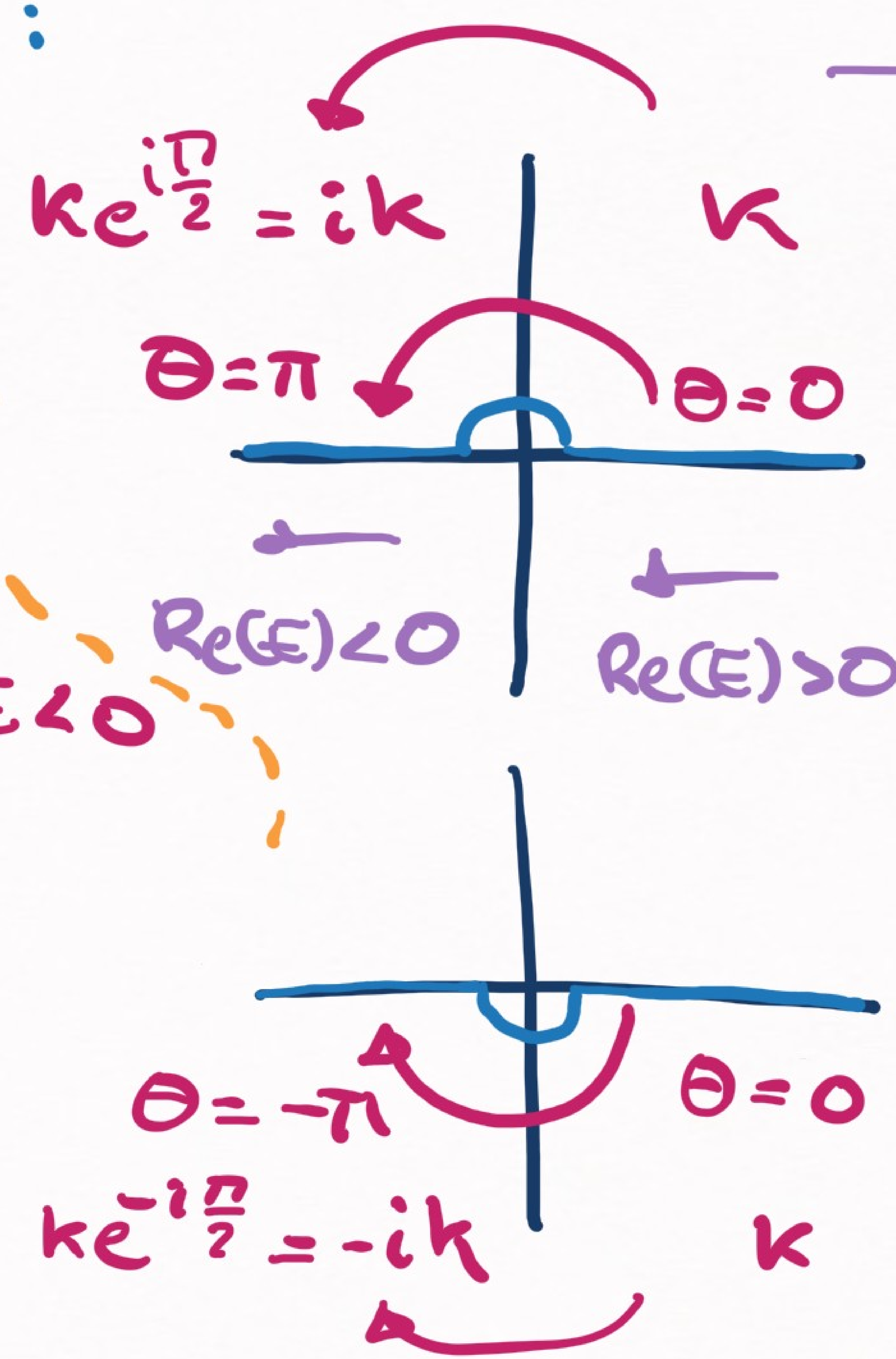
[T-MATRIX AND CONTACTS] ⟹ FROM SCATTERING TO BOUND STATES

c) The $E \pm i\epsilon$ prescriptions:

$\boxed{\text{CAVEAT (cont'd)}}$

Actually, this would be the key idea:

for going from $E>0$ to $E<0$ there are two possibilities



$\longrightarrow E = |E| e^{i\theta}$

$\Downarrow$

$k = \sqrt{2\mu E}$

$\Downarrow$

$k = |k| e^{i\frac{\theta}{2}}$

Sign of θ:

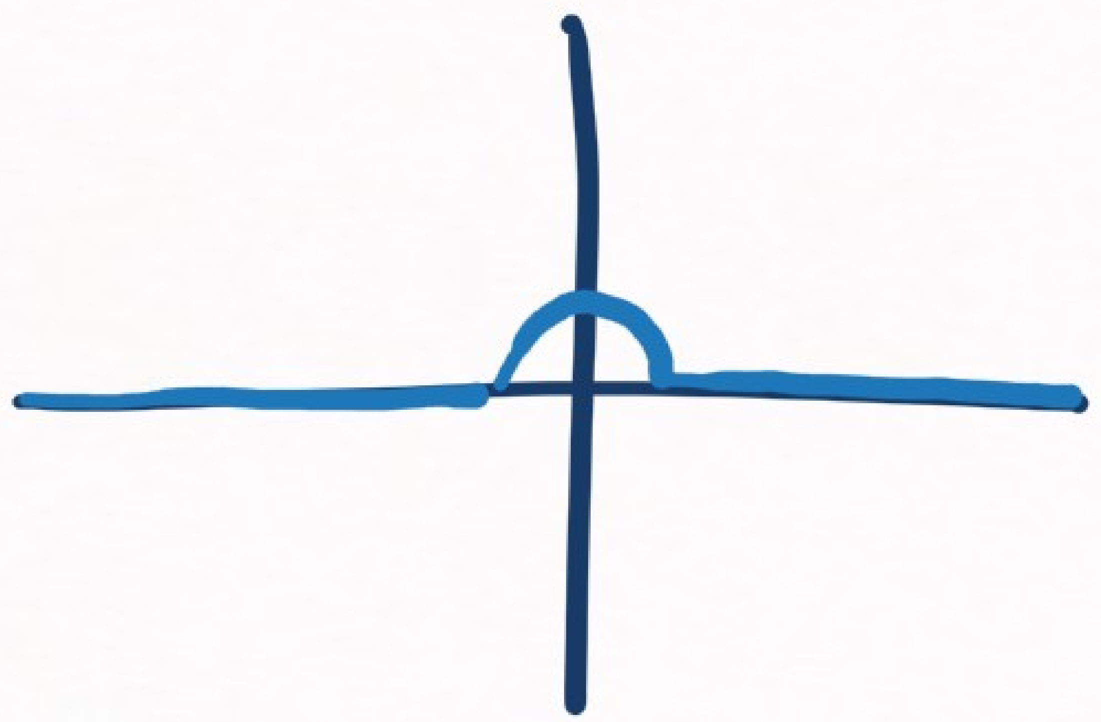
$\theta > 0$ for counterclockwise

$\theta < 0$ for clockwise

[T-MATRIX AND CONTACTS] $\Rightarrow$ FROM SCATTERING TO BOUND STATES

c) The $E \pm i\epsilon$ prescription (cont'd)

c.1)

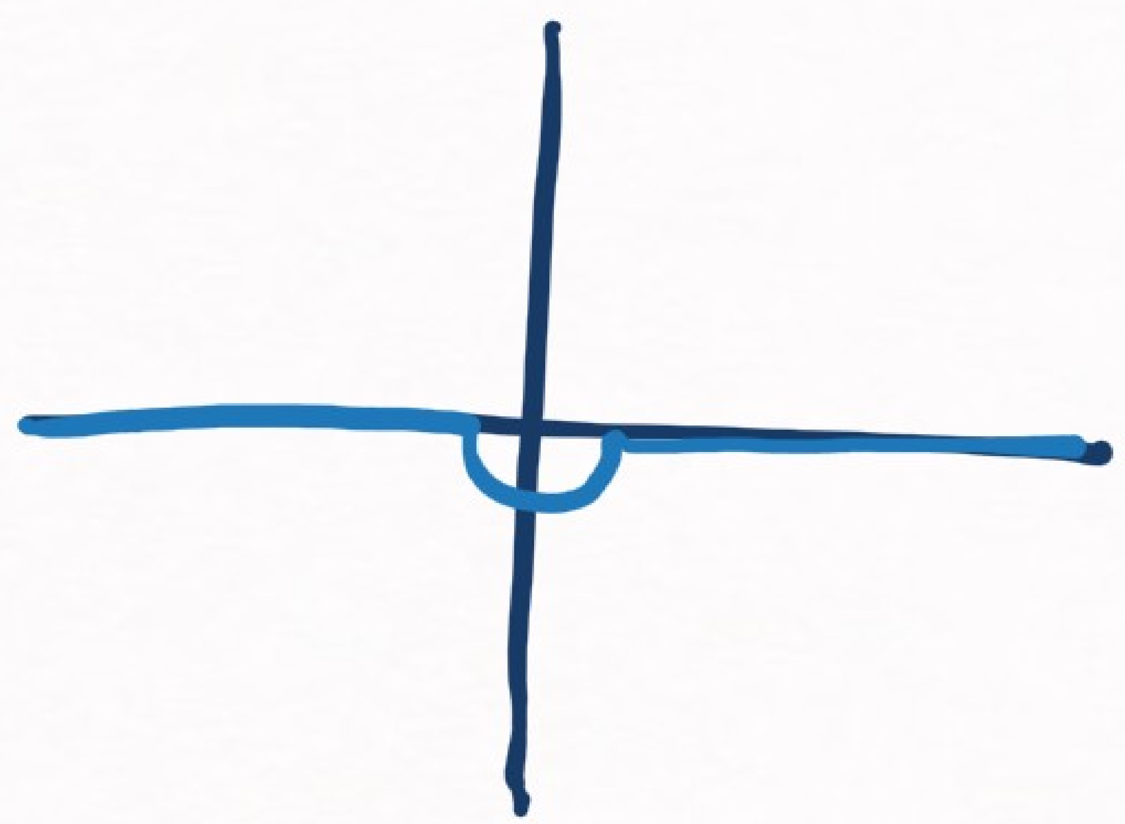


$$k \to i\gamma$$

$$\frac{1}{1/a_0 + ik} \longrightarrow \frac{1}{1/a_0 - \gamma}$$

$$(E \to |E| e^{i\pi})$$

c.2)



$$k \to -i\gamma$$

$$\frac{1}{1/a_0 + ik} \longrightarrow \frac{1}{1/a_0 + \gamma}$$

$$(E \to |E| e^{-i\pi})$$

$$E = \frac{k^2}{2\mu} \quad (E>0) \qquad E = -\frac{\gamma^2}{2\mu} \quad (E<0)$$

c.1) $\exists$ pole at $\gamma = 1/a_0$ if $a_0 > 0$

c.2) $\exists$ pole at $\gamma = -1/a_0$ if $a_0 < 0$

[T-MATRIX AND CONTACTS] $\Rightarrow$ FROM SCATTERING TO BOUND STATES

c) The "$E \pm i\epsilon$" ($|E|e^{i\theta}$) prescription

c.1) For "$+i\epsilon$": $T(E) = -\frac{2\pi}{\mu}\frac{1}{1/a_0 + ik} \longrightarrow -\frac{2\pi}{\mu}\frac{1}{1/a_0 - \gamma}$ ✓
($\theta = \pi$)

c.2) For "$-i\epsilon$": $T(E) = -\frac{2\pi}{\mu}\frac{1}{1/a_0 + ik} \longrightarrow -\frac{2\pi}{\mu}\frac{1}{1/a_0 + \gamma}$ ✓
($\theta = -\pi$)

d) Poles:

$$E = |E|e^{i\pi} \longrightarrow E_B = -\frac{1}{2\mu}\left(\frac{1}{a_0}\right)^2, \ \gamma = 1/a_0$$

$$E = |E|e^{-i\pi} \longrightarrow E_B = -\frac{1}{2\mu}\left(\frac{1}{a_0}\right)^2, \ \gamma = -1/a_0$$

Which one of these poles corresponds to a bound state?

[T-MATRIX AND CONTACTS] $\Rightarrow$ FROM SCATTERING TO BOUND STATES

e) Which of these choices gives us a bound state?

$\boxed{\text{Reminder}}$ $\Rightarrow$ Orthogonality of bound & scattering states

$$\langle \psi_B | \vec{r}_- \rangle \propto \int^{\infty} dr\, e^{-\gamma r} \sin(kr+\delta) = \frac{k\cos\delta + \gamma \sin\delta}{k^2+\gamma^2}$$

$$\langle \psi_B | \vec{r}_- \rangle = 0$$

If $\langle \psi_B | \vec{r}_- \rangle = 0 \Rightarrow k\cot\delta = -\gamma \quad \left(k\cot\delta = -\frac{1}{a_0}\right)$

a bound state (γ) requires $a_0 > 0$

[T-MATRIX AND CONTACTS] $\Rightarrow$ FROM SCATTERING TO BOUND STATES

e) Which of these choices gives us a bound state?

The answer is: for $E = |E|e^{i\pi}$, $E_B = -\frac{1}{2\mu}\left(\frac{1}{a_0}\right)^2$, $\gamma = \frac{1}{a_0}$

this gives us the bound state
$\hookrightarrow$ pole in $T(E)$

$$T(E+i\epsilon) = \frac{2\pi}{\mu}\frac{1}{1/a + ik} = \frac{2\pi}{\mu}\frac{1/a_0 + \sqrt{-2\mu E}}{1/a_0^2 + 2\mu E}$$

$$\Rightarrow \operatorname{Res} T(E = E_B) = \frac{2\pi}{\mu^2}\frac{1}{a_0} = \frac{2\pi\gamma_B}{\mu^2} \quad \left(\frac{1}{a_0} = \gamma_B\right)$$

[T-MATRIX AND CONTACTS] $\Rightarrow$ FROM SCATTERING TO BOUND STATES

f) Calculating the wave function:

$$\mathrm{Res}\,T(E=-E_B) = \frac{2\pi}{\mu^2}\left(\frac{1}{a_0}\right) = \frac{2\pi}{\mu^2}\gamma_B \rightarrow \text{residue}$$

$$\mathrm{Res}\,T = V|\psi_B\rangle\langle\psi_B|V = G_0^{-1}(E=E_B)|\psi_B\rangle\langle\psi_B|G_0^{-1}(E=E_B)$$

$$|\psi_B\rangle = G_0 V|\psi_B\rangle$$

$$= \left(E_B - \frac{\vec{k}'^2}{2\mu}\right)\psi_B(\vec{k}')\,\psi_B(\vec{k})\left(E_B - \frac{\vec{k}^2}{2\mu}\right)$$

$$\langle\vec{k}'|(\dots)|\vec{k}\rangle$$

Our wave function $\Rightarrow$ $$\psi_B(\vec{p}) = \pm\frac{\sqrt{2\mu\gamma_B/\mu^2}}{E_B - \frac{\vec{p}^2}{2\mu}} = \mp\frac{\sqrt{8\pi\gamma_B}}{\gamma^2+\vec{p}^2}$$

[T-MATRIX AND CONTACTS] ⇒ FROM SCATTERING TO BOUND STATES

f) Calculating the wave function (II)

⇒ The final result: $\psi_B(\vec{k}) = \dfrac{\sqrt{8\pi\gamma_B}}{\gamma_B^2 + k^2}$ $\xrightarrow{\text{Fourier transform}}$ $\psi(\vec{r})$

$$= \frac{1}{\sqrt{4\pi}} \frac{\sqrt{2\gamma_B}\, e^{-\gamma_B r}}{r}$$

⇒ Something amazing about this result:

$$\int \frac{d^3\vec{q}}{(2\pi)^3} |\psi_B(\vec{q})|^2 = \int d^3\vec{r}\, |\psi_B(\vec{r})|^2 = 1$$

⇒ The residue of $T(E)$ even gives us the correct normalization

Wave function for a pure contact-range theory

[T-MATRIX AND CONTACTS] $\Rightarrow$ THE OTHER SIDE OF THE COMPLEX PLANE

$\Rightarrow$ But, if $a_0 < 0$,

what happens with the pole in $E = |E| e^{-i\pi}$?

$\underbrace{E = |E| e^{i\pi}, \quad \boxed{E = |E| e^{-i\pi}}}_{\text{two prescriptions}}$ $\Rightarrow$ $T(E) = -\frac{2\pi}{\mu} \frac{1}{1/a_0 + ik} \rightarrow -\frac{2\pi}{\mu} \frac{1}{\frac{1}{a_0} + \gamma}$

$\Rightarrow$ We acknowledge the existence of this second type of pole

We call it a virtual state pole

$\gamma = -\frac{1}{a_0}$ is a pole now

[T-MATRIX AND CONTACTS] ⟹ THE OTHER SIDE OF THE COMPLEX PLANE

⟹ But, a virtual state pole, is it less of a pole than a bound state pole?

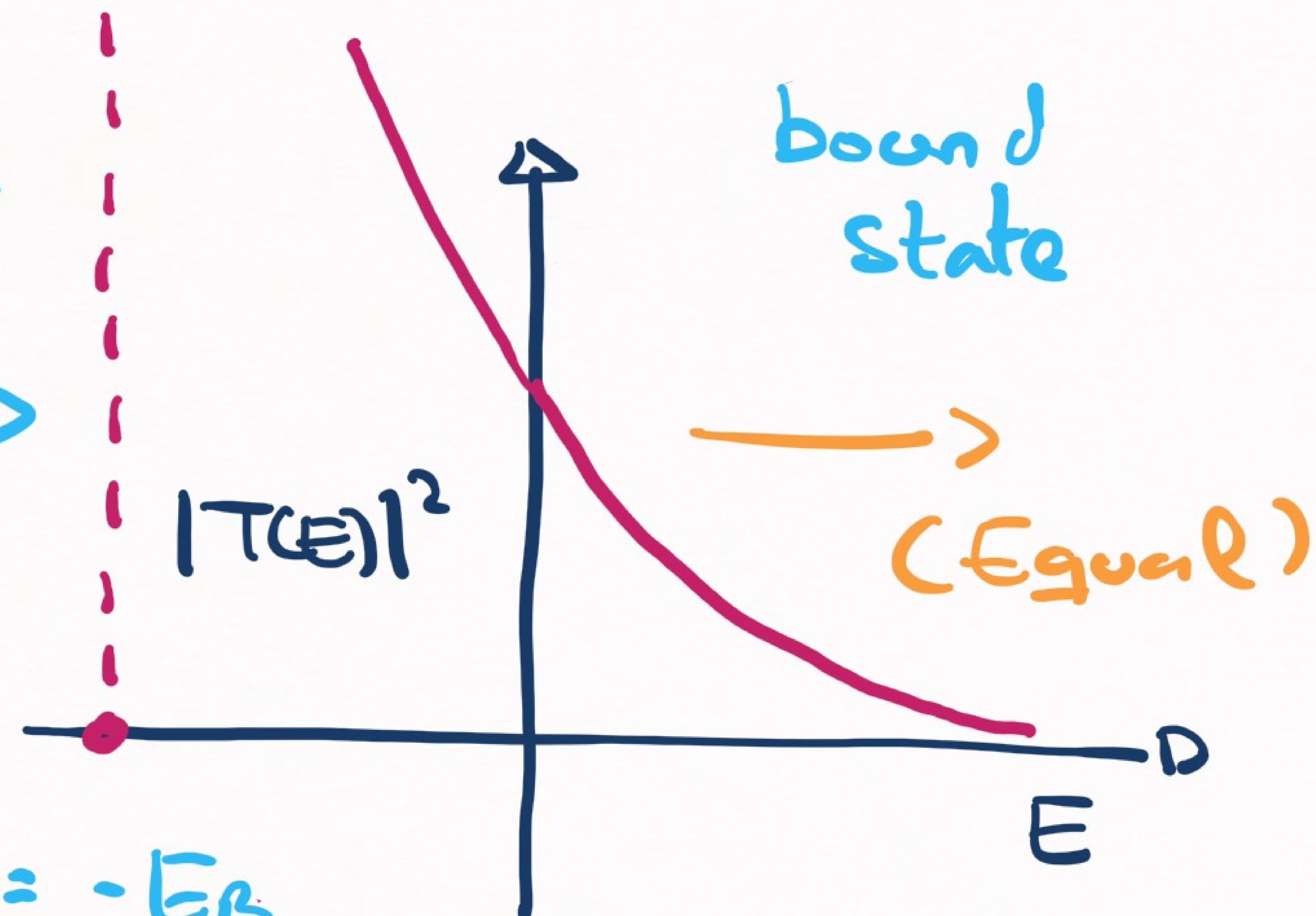
It depends.... even though there is no two-body physical state that corresponds to this pole...

If we look at low energy scattering → $\left[\sigma \xrightarrow[k \to 0]{} 4\pi |a_0|^2\right]$ → the sign of a_0 does not matter

[T-MATRIX AND CONTACTS] ⇒ THE OTHER SIDE OF THE COMPLEX PLANE

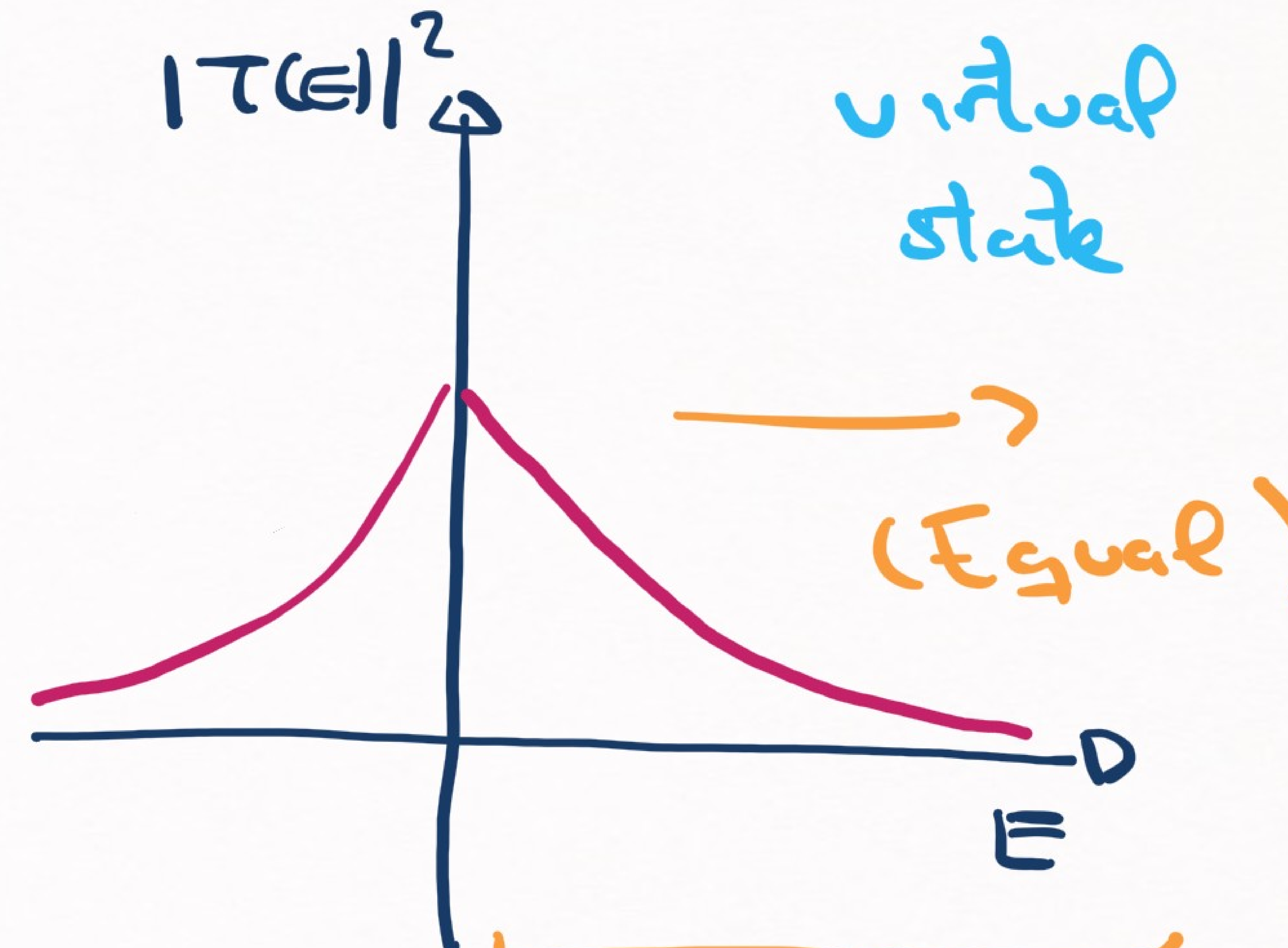
⇒ Bound vs virtual state:

pole here ↪



$E = -E_B$

$= -\frac{1}{2\mu}\left(\frac{1}{a_0}\right)^2$



[physically observable energies in scatterings]

[T-MATRIX AND CONTACTS] $\Rightarrow$ THE OTHER SIDE OF THE COMPLEX PLANE

a) Bound state solution:

$$\frac{e^{ikr}}{r} \longrightarrow \frac{e^{-\gamma r}}{r} \quad (k \rightarrow +i\gamma)$$

b) Virtual state solution:

$$\frac{e^{ikr}}{r} \longrightarrow \frac{e^{+\gamma r}}{r} \quad (k \rightarrow -i\gamma)$$

This is a mathematical consequence of $E = \frac{k^2}{2\mu}$ being valid for both signs of k:

$$\left[E = \frac{(\pm k)^2}{2\mu} \right]$$

[T-MATRIX AND CONTACTS] $\Rightarrow$ THE OTHER SIDE OF THE COMPLEX PLANE

c) Or we can directly begin with:

$$\psi_{\vec{k}}(\vec{r}) \to 4\pi \sum_{\ell m} i^{\ell} \frac{S_\ell(k)e^{ikr} - e^{-ikr}}{2ir} Y^*_{\ell m}(\hat{k}) Y_{\ell m}(\hat{r})$$

$$\Rightarrow S_\ell(k) = e^{2i\delta_\ell(k)} = 1 + 2ik f_\ell(k) = 1 - ik\frac{\mu}{\pi} t_\ell(k)$$

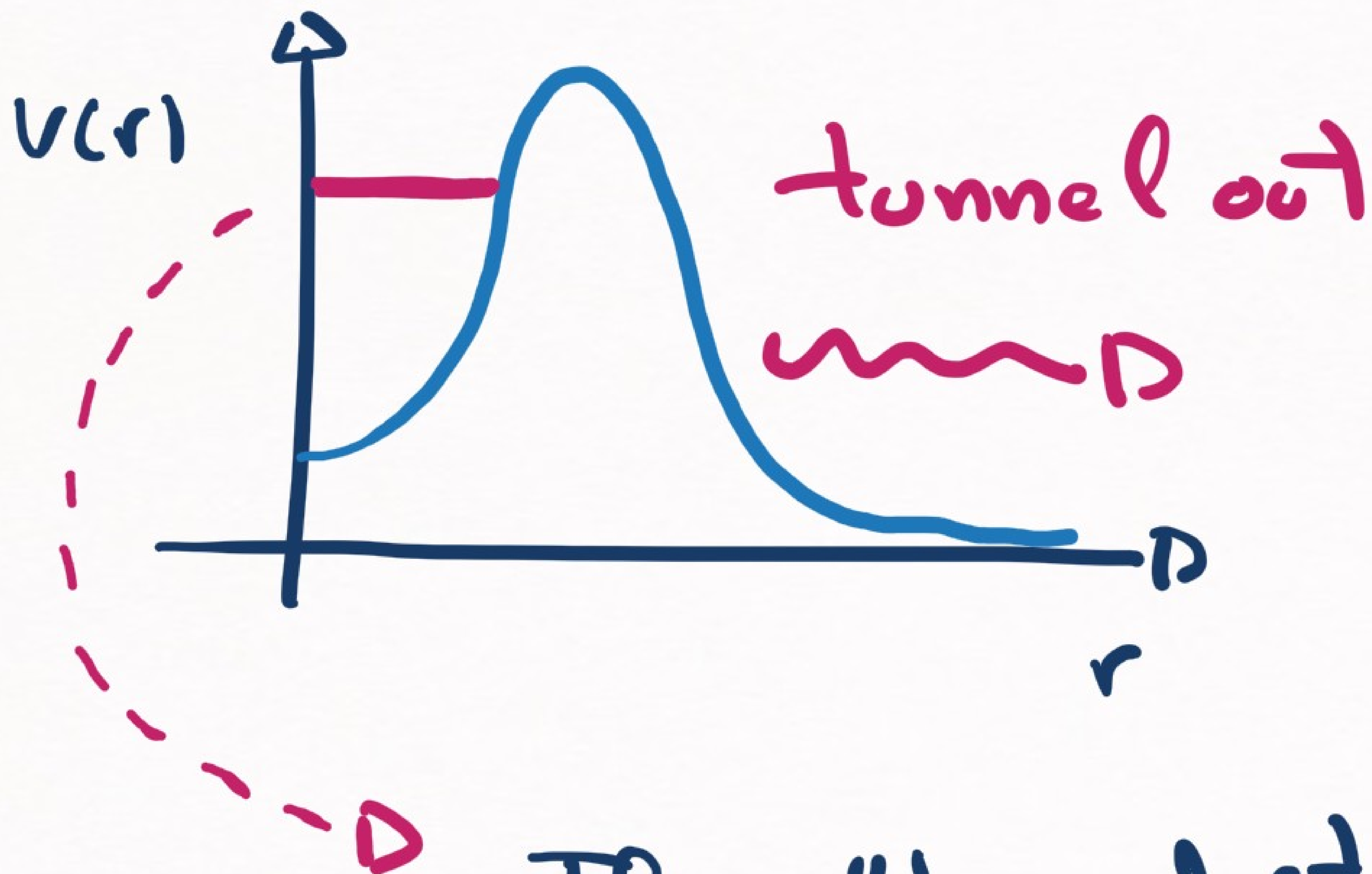
$\Rightarrow$ $S_\ell(k)$ has also a pole for $k = +i\gamma$ / $-i\gamma$ (bound / virtual)

$k = +i\gamma \Rightarrow \psi_k(\vec{r}) \to (\infty)\, \frac{e^{-\gamma r}}{r}$

$k = -i\gamma \Rightarrow \psi_k(\vec{r}) \to (\infty)\, \frac{e^{+\gamma r}}{r}$

[RESONANCES] 3

a) Consider this potential:



This "bound state" is possible in classical mechanics, but not in quantum mechanics

$\Rightarrow$ they tunnel out

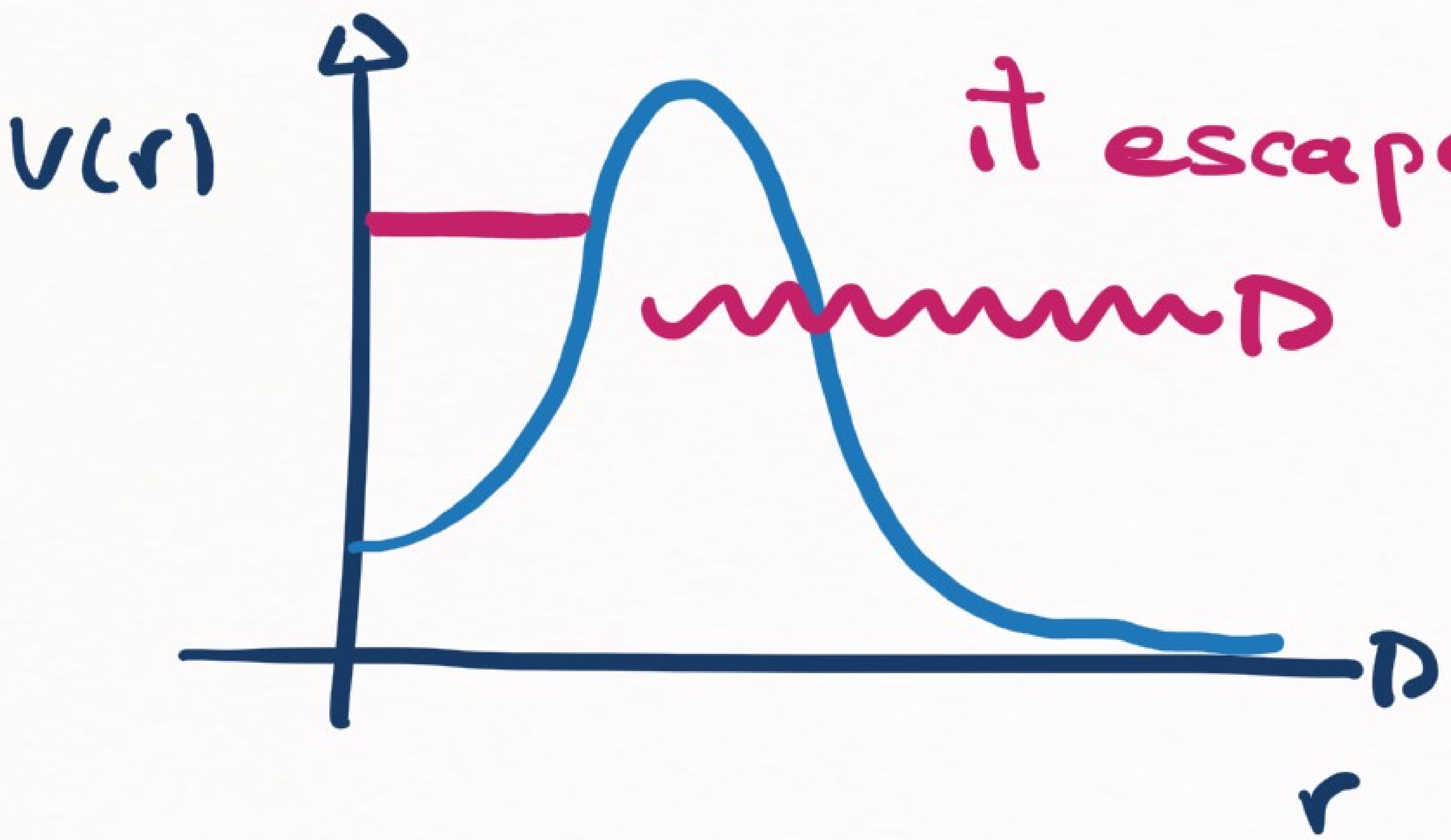
the state will decay:

$$\left[\ P(t) = e^{-t/\tau}\ \right]$$

$\Rightarrow$ probability of finding the two particles in the classical bound state configuration

[RESONANCES] 2

a) Consider this potential:



it escapes via tunneling

$\Rightarrow$ The QM version of this state will have:

$$\Rightarrow \left[\begin{array}{l} \mathrm{Im}(E_B) < 0 \\ \text{or} \quad \tau = \dfrac{2}{\mathrm{Im}(E)} = \dfrac{1}{\Gamma} \end{array}\right]$$

Probability of survival:

$$P(t) \propto e^{-t/\tau}$$

For a true bound state we have the time dependence:

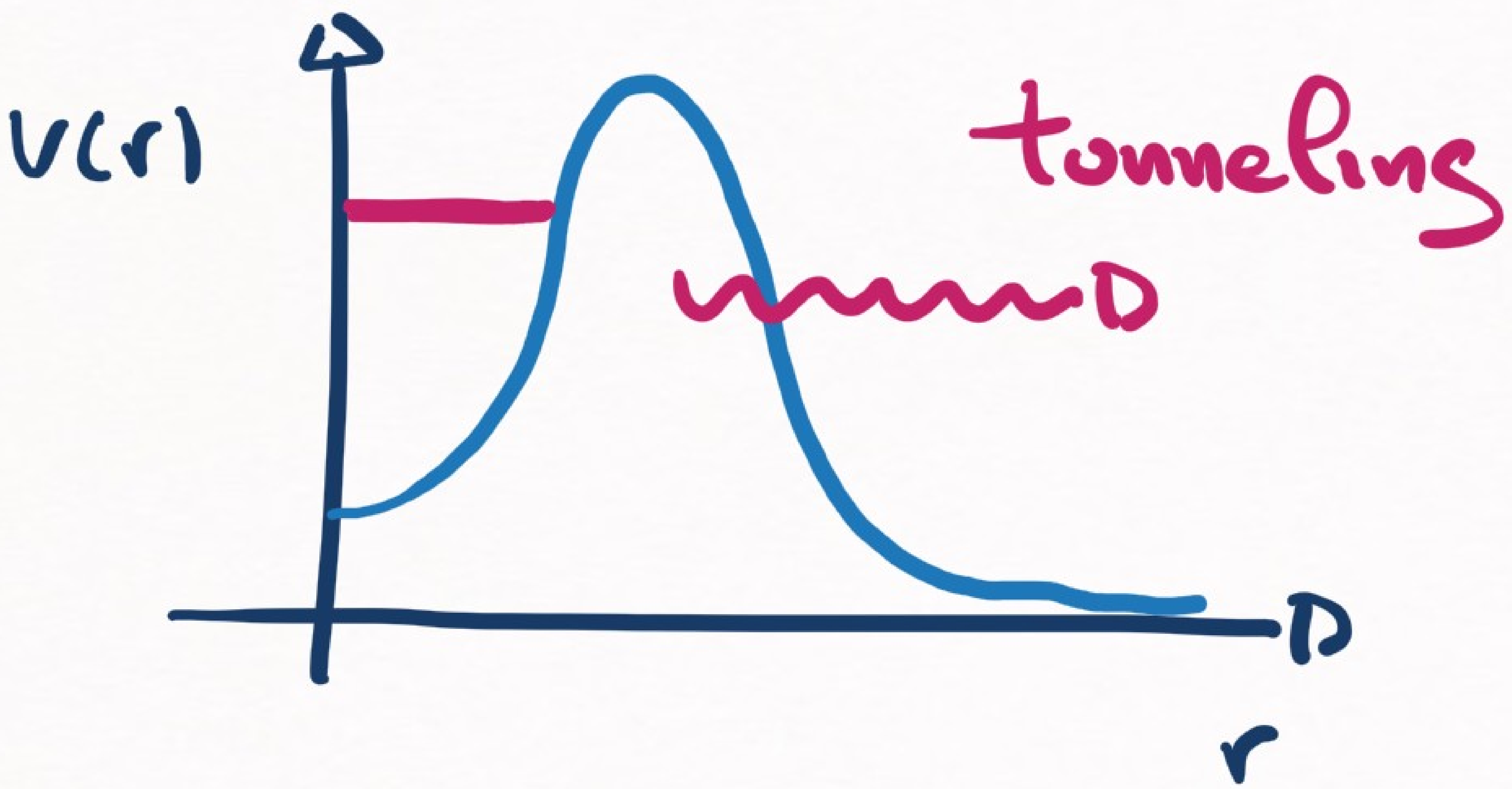
$$\psi(\vec{r},t) = \psi(\vec{r})\, e^{-iE_B t}$$

if we allow complex E_B:

$$\psi(\vec{r},t) = \psi(\vec{r})\, e^{-i\mathrm{Re}(E_B)t + \mathrm{Im}(E_B)t}$$

[RESONANCES] 3

a) Consider this potential $\Rightarrow$ In QM, this classical state will have complex energy



$$E = E_r - i\frac{\Gamma}{2} \qquad \left(\Gamma > 0,\ \tau = \frac{1}{\Gamma}\right)$$

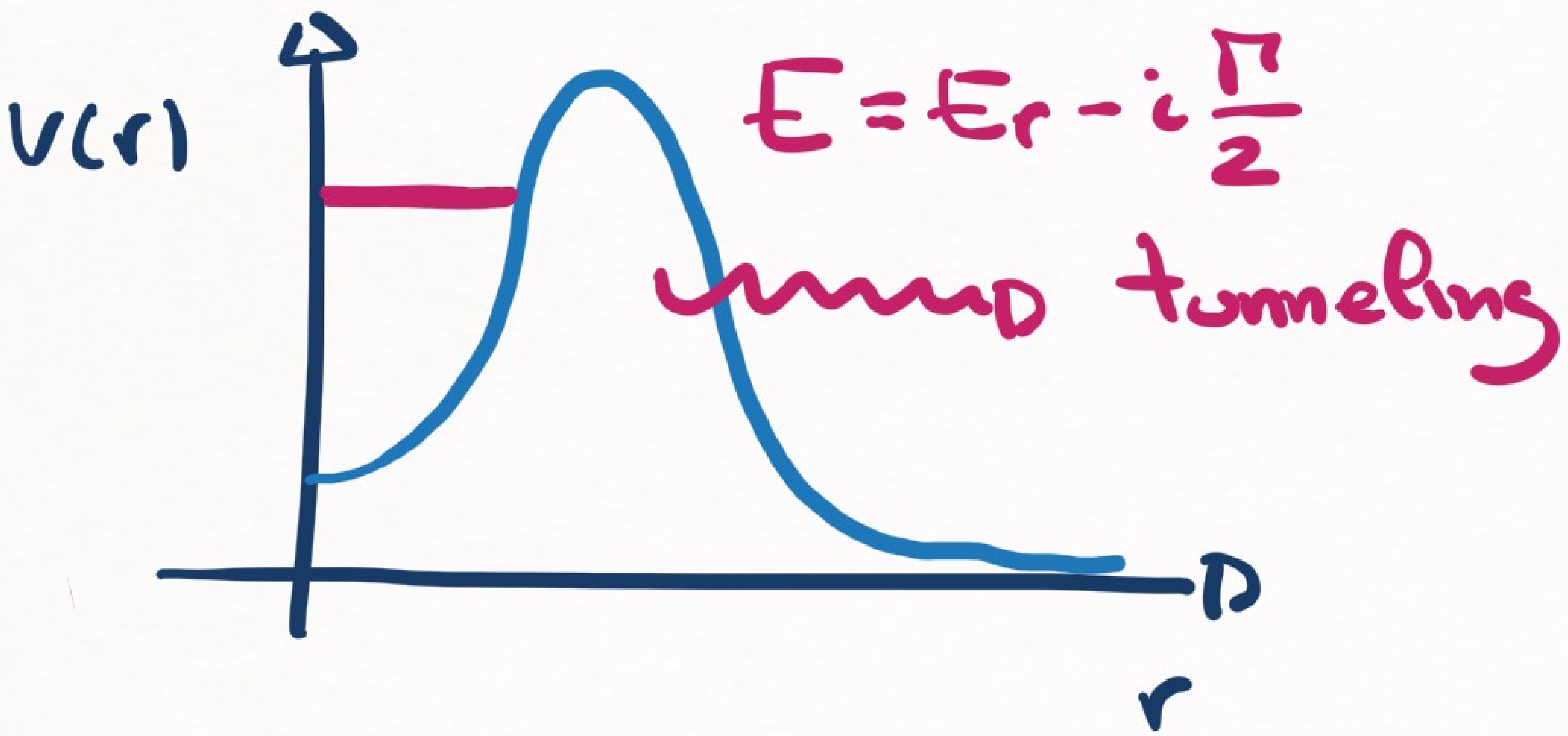
(also $E_r > 0$ in this example)

$\Rightarrow$ If a resonance is also represented by a pole in $f(E)/T(E)$, we have

$$f(E) \propto \frac{1}{E - \left(E_r - i\frac{\Gamma}{2}\right)} \quad \longleftarrow \text{(match)} \longrightarrow \quad f(E) = \frac{1}{k\cot\delta - ik}$$

[RESONANCES] (4)

a) Consider this potential $\Rightarrow$



a.1) $f(E) \propto \dfrac{1}{E-(E_r - i\frac{\Gamma}{2})}$ (close to $E=E_r$)

$+$

a.2) $f(E) = \dfrac{1}{k\cot\delta - ik}$

$\downarrow$

a.3) $f(E) = -\dfrac{1}{2k}\,\dfrac{\Gamma}{E-(E_r - i\frac{\Gamma}{2})}$

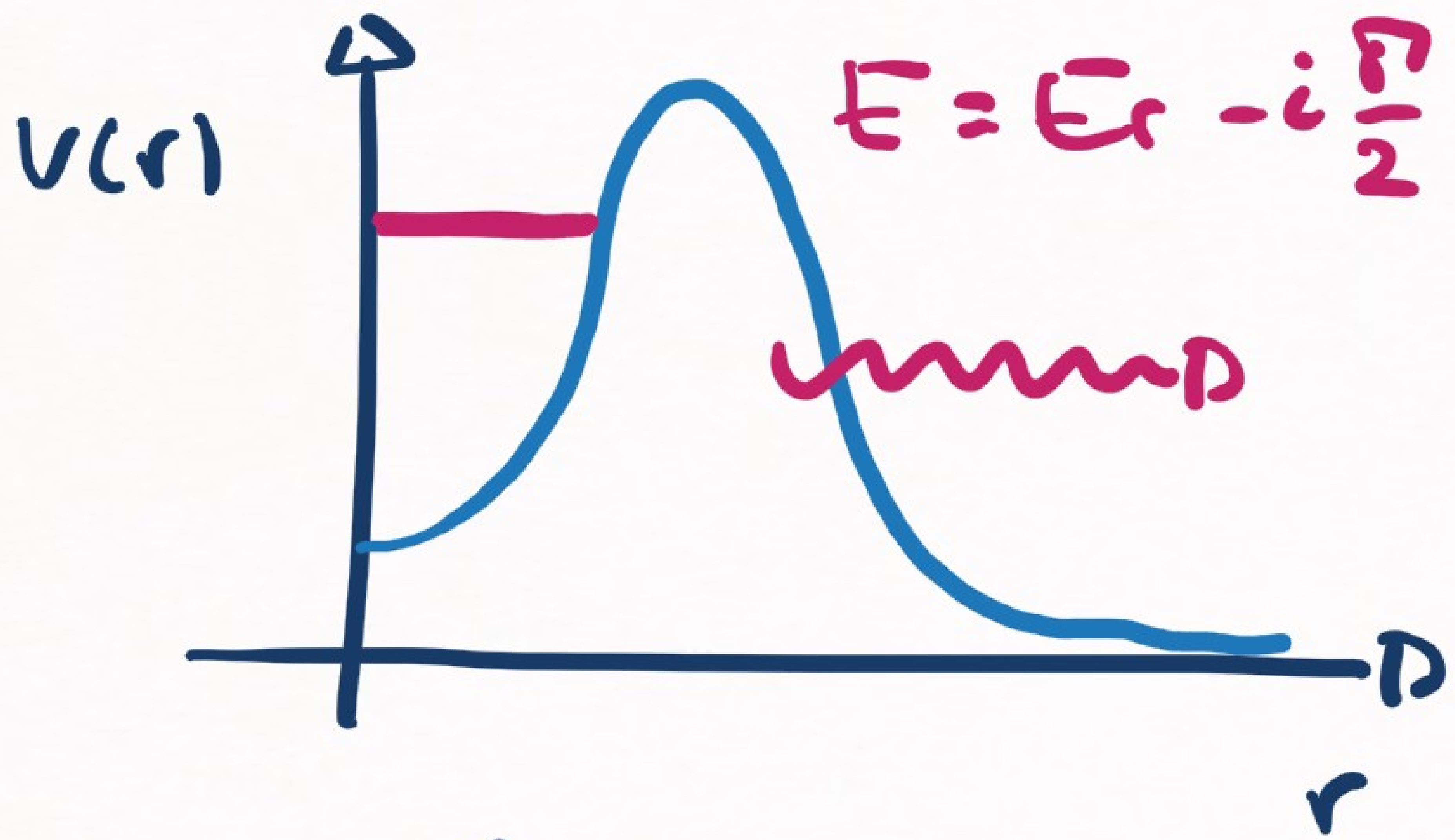
(most simple solution to a.1 & a.2)

a.3) $\Rightarrow$ $\delta(E) = -\operatorname{atan}\left(\dfrac{\Gamma}{2}\,\dfrac{1}{E-E_r}\right)$

try to plot it

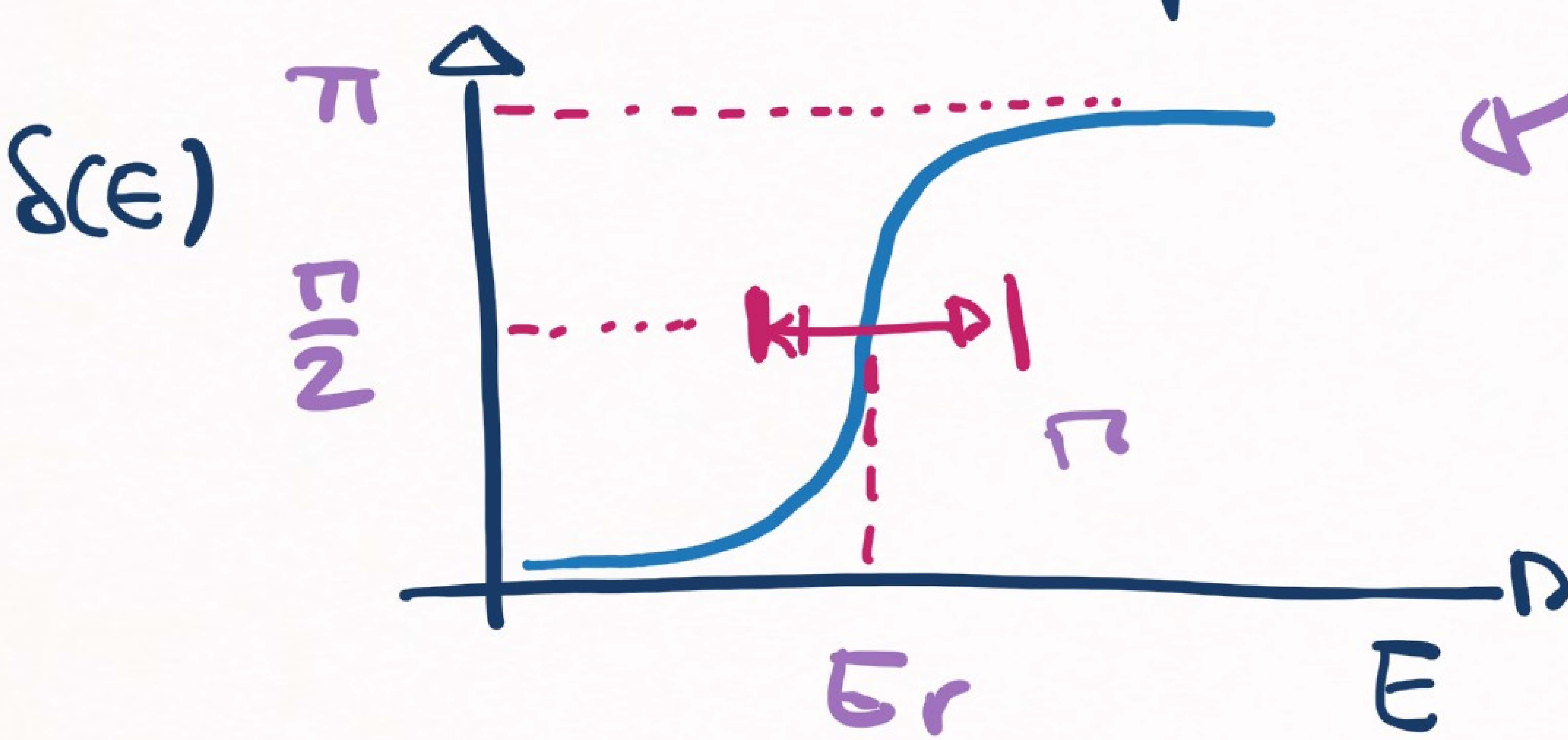
[RESONANCES] 5

a) Consider this potential $\Rightarrow$ $f(E) = -\frac{1}{2k}\,\frac{\Gamma}{E-(E_r - i\frac{\Gamma}{2})}$



$\delta(E) = -\operatorname{atan}\left(\frac{\Gamma}{2}\,\frac{1}{E-E_r}\right)$

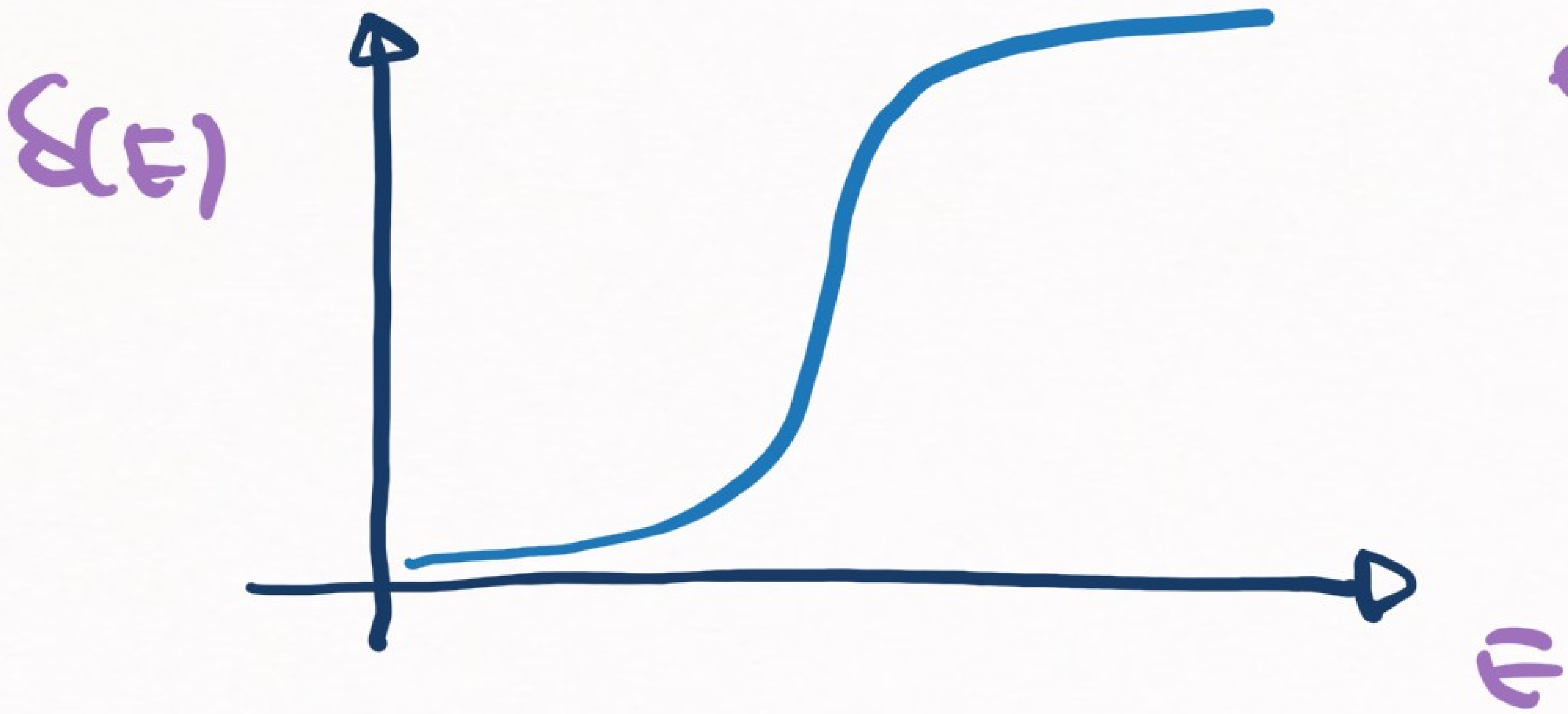
$\boxed{\text{Caveat}}$ $\rightarrow$ this is only the most easy solution to the previous conditions

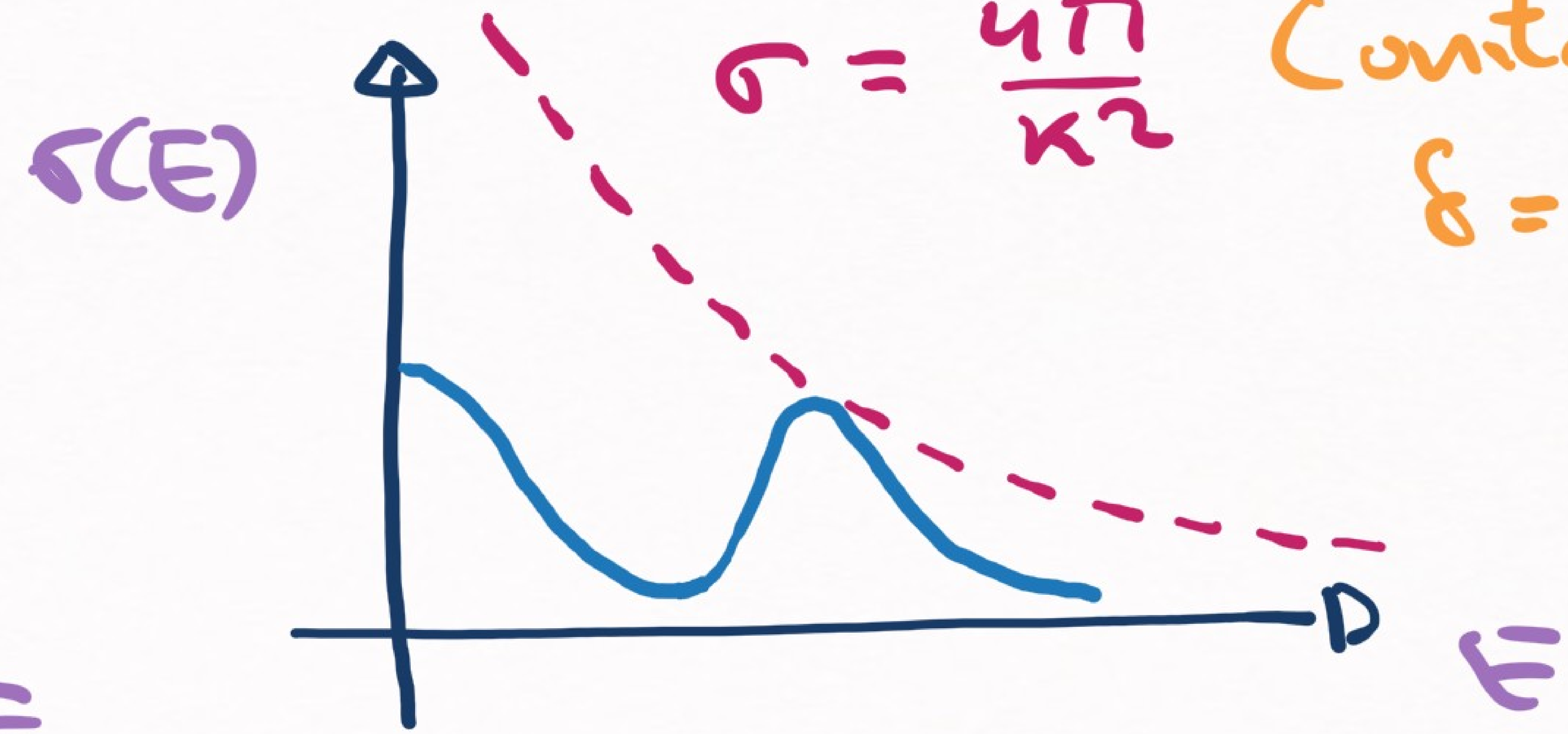


$\Rightarrow$ Resonance translates into a sudden increase of $\delta(E)$ of π

[RESONANCES] 6

⇒ Phase shifts and cross section:





$\sigma = \frac{4\pi}{k^2}$ (unitarity: $\delta = \frac{\pi}{2}$)

$$\delta(E) = -\mathrm{atan}\left(\frac{\Gamma}{2}\frac{1}{E-E_r}\right)$$

jumps from 0 to π quickly

↳ looks like a peak

$$\left[\ \sigma(E) \xrightarrow[E \to E_r]{} \frac{4\pi}{k^2}\ \right]$$

(unitarity ≃ max value of $\sigma(E)$)

[RESONANCES] 7

⇒ A more general form can be obtained from $S_\ell(k)$:

1) $S(E)$ is a phase for $E \in \mathbb{R}, E > 0$ $\quad |S(E)| = 1, \; S(E) = e^{2i\delta(E)}$

2) $S(E)$ has a pole for $E = E_r - i\frac{\Gamma}{2}$

$$\Rightarrow \quad S(E) = e^{2i\delta_{nr}(E)} \, \frac{E - (E_r + i\frac{\Gamma}{2})}{E - (E_r - i\frac{\Gamma}{2})}$$

fulfills 1) & 2)

[RESONANCES] 8

⇒ A more general form can be obtained from $S_\ell(E)$:

$$\left[\; S(E) = e^{2i\delta_{nr}(E)} \frac{E-(E_r+i\Gamma/2)}{E-(E_r-i\Gamma/2)} \;\right]$$

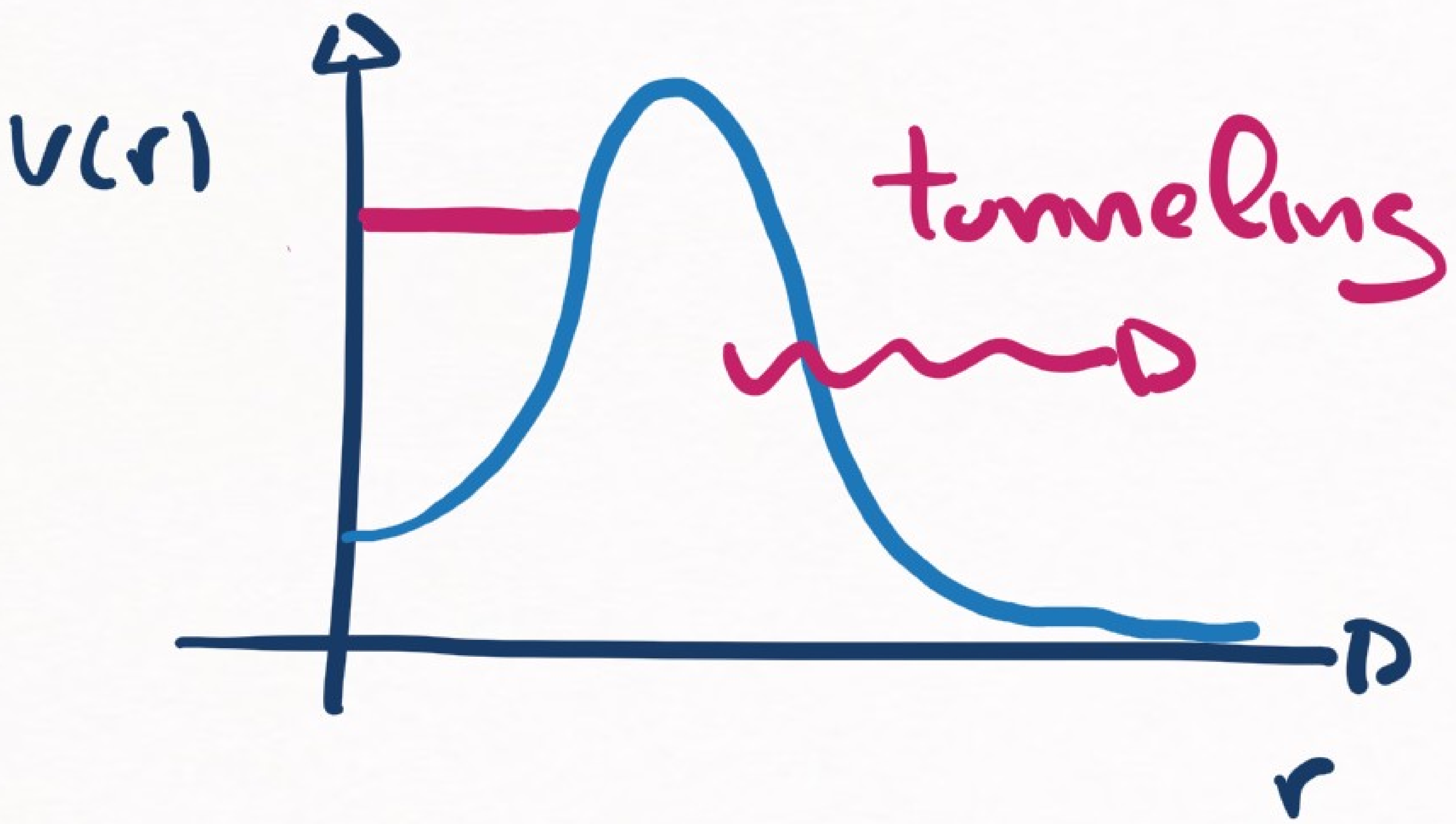
$$\delta(E) = \frac{1}{2i}\log S(E) = \delta_{nr}(E) - \operatorname{atan}\left(\frac{\Gamma}{2}\frac{1}{E-E_r}\right)$$

$$= \underbrace{\delta_{nr}(E) + \delta_r(E)}$$

non-resonant + resonant contribution

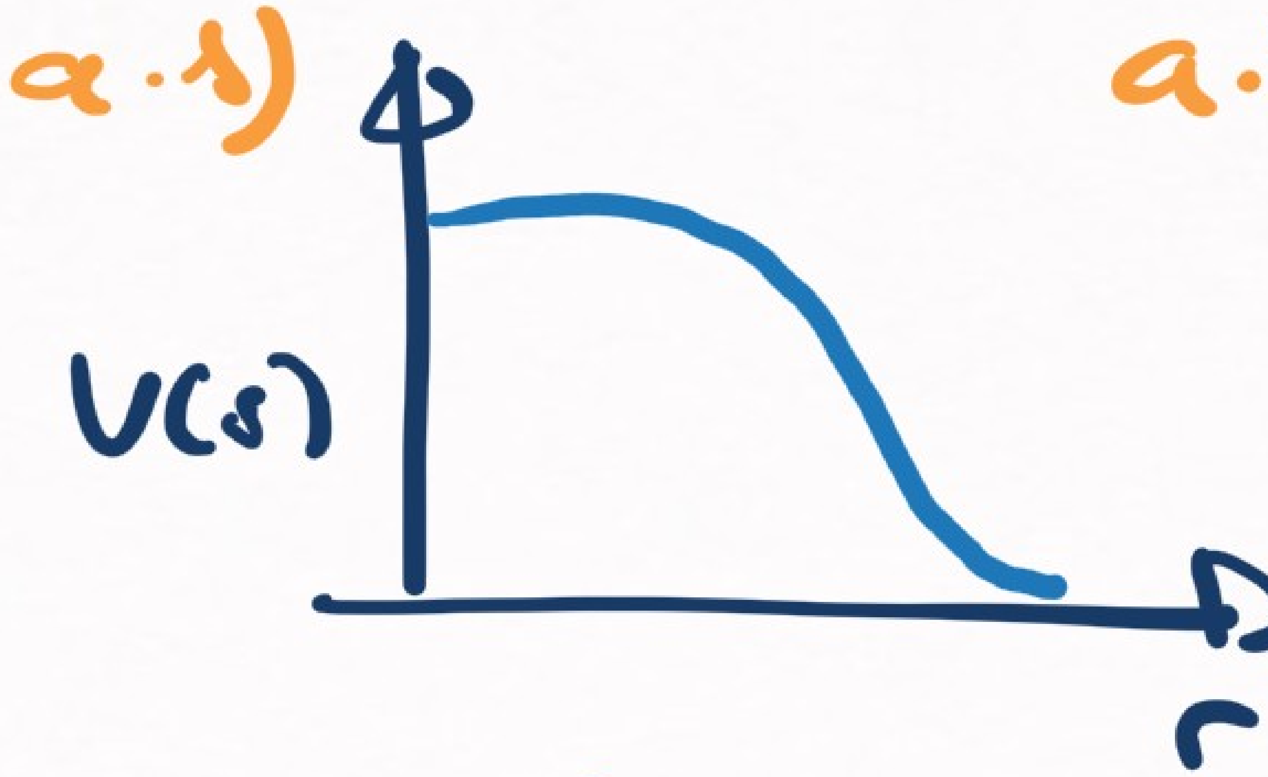
[RESONANCES] (9)

$\Rightarrow$ But, when do we find this type of potential?



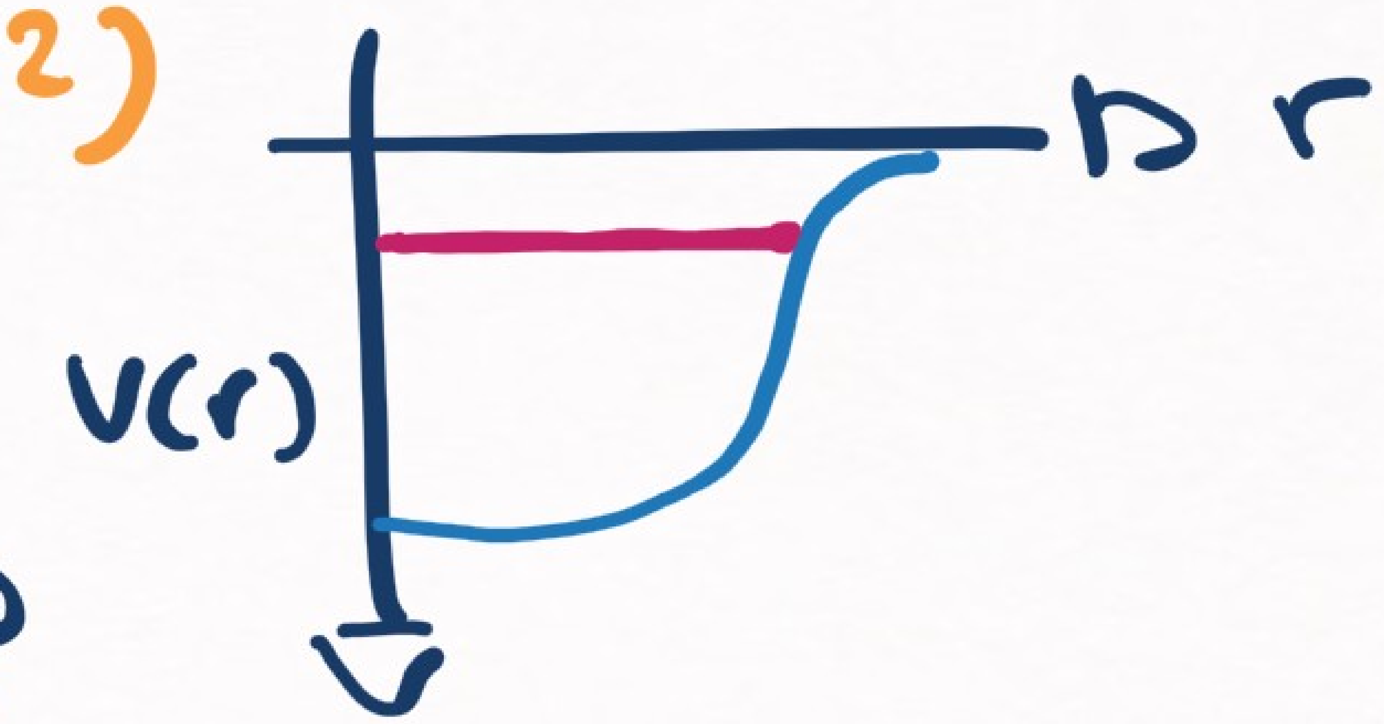
a) Most usually we find:

a.1)



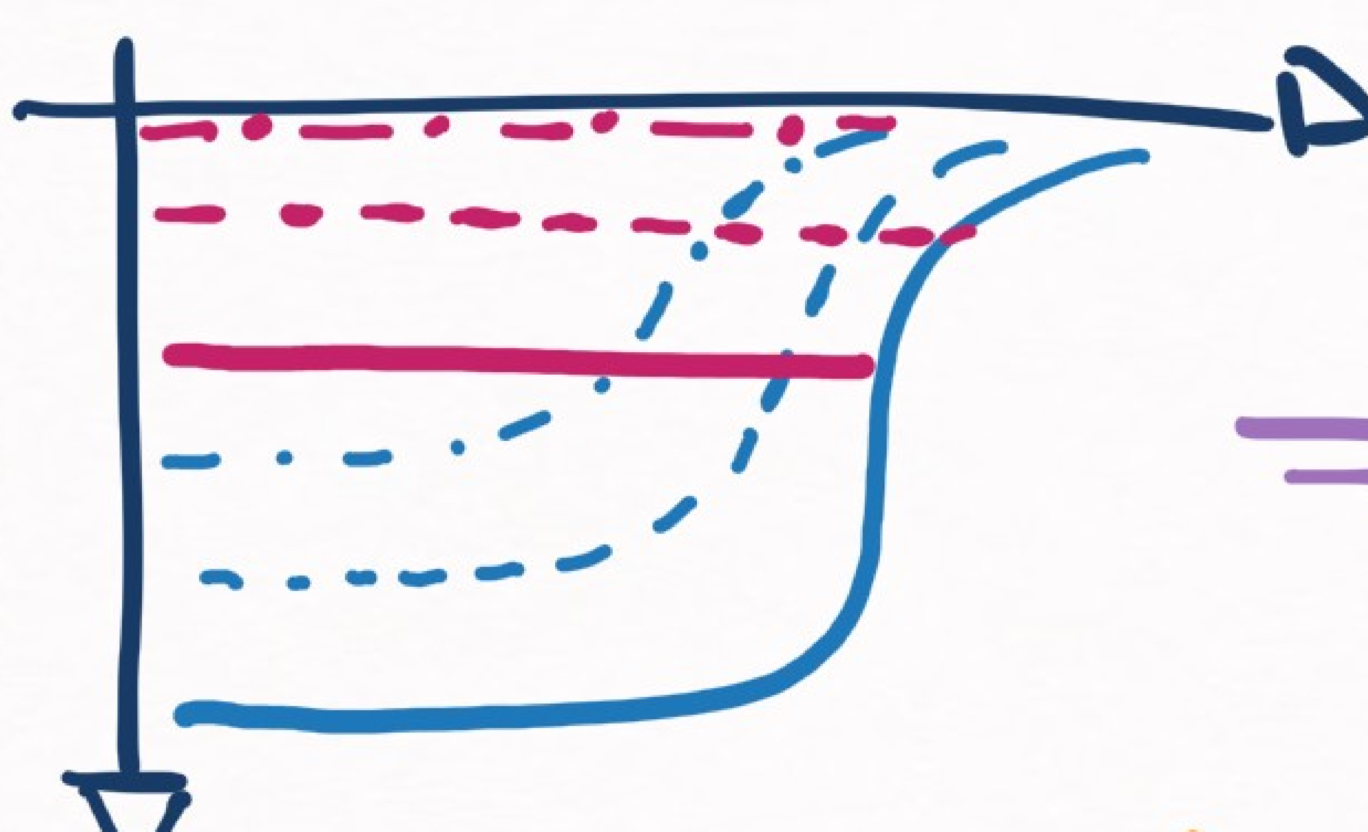
Repulsive

a.2)



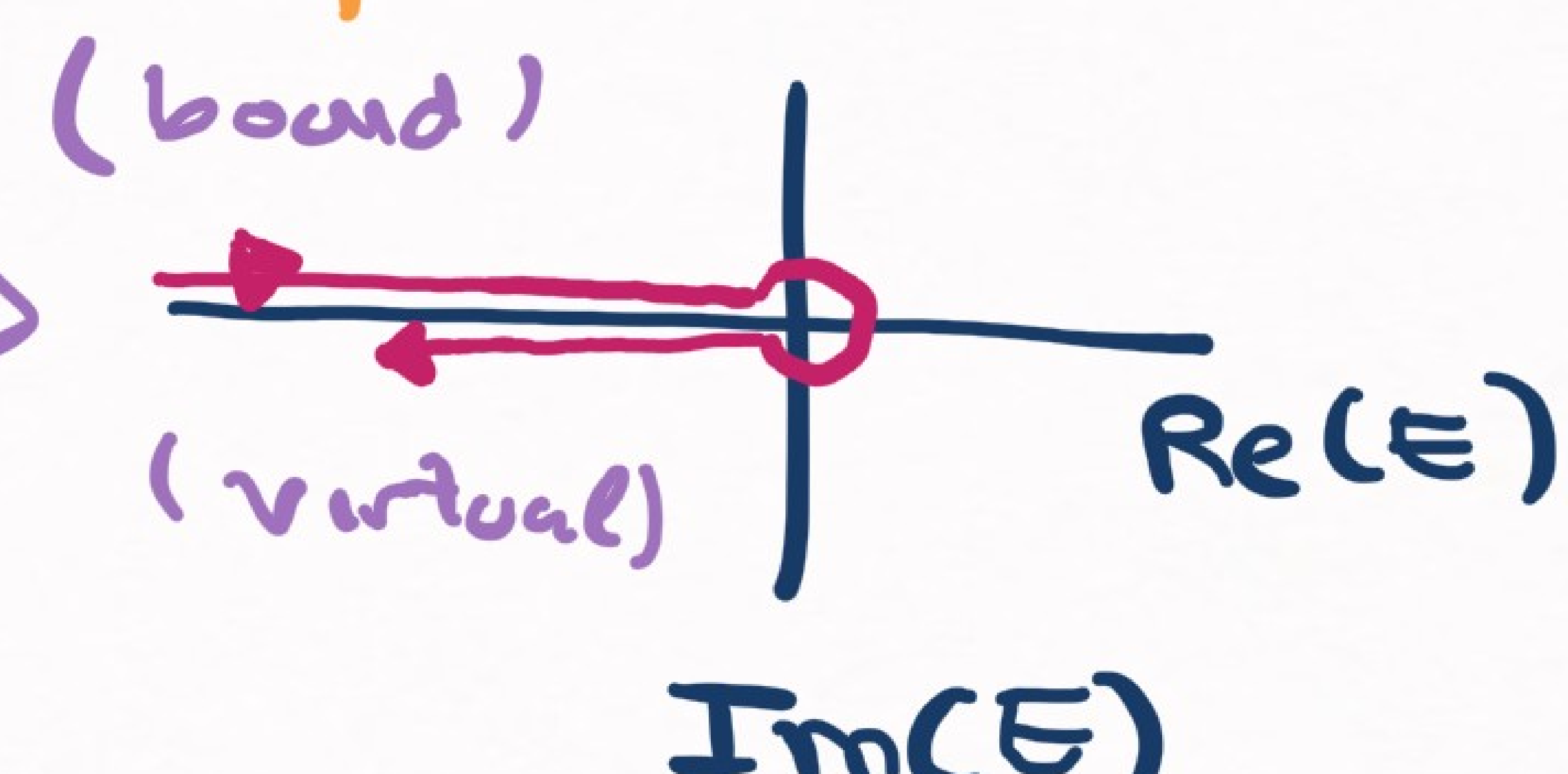
Attractive

a.2)



We reduce strength

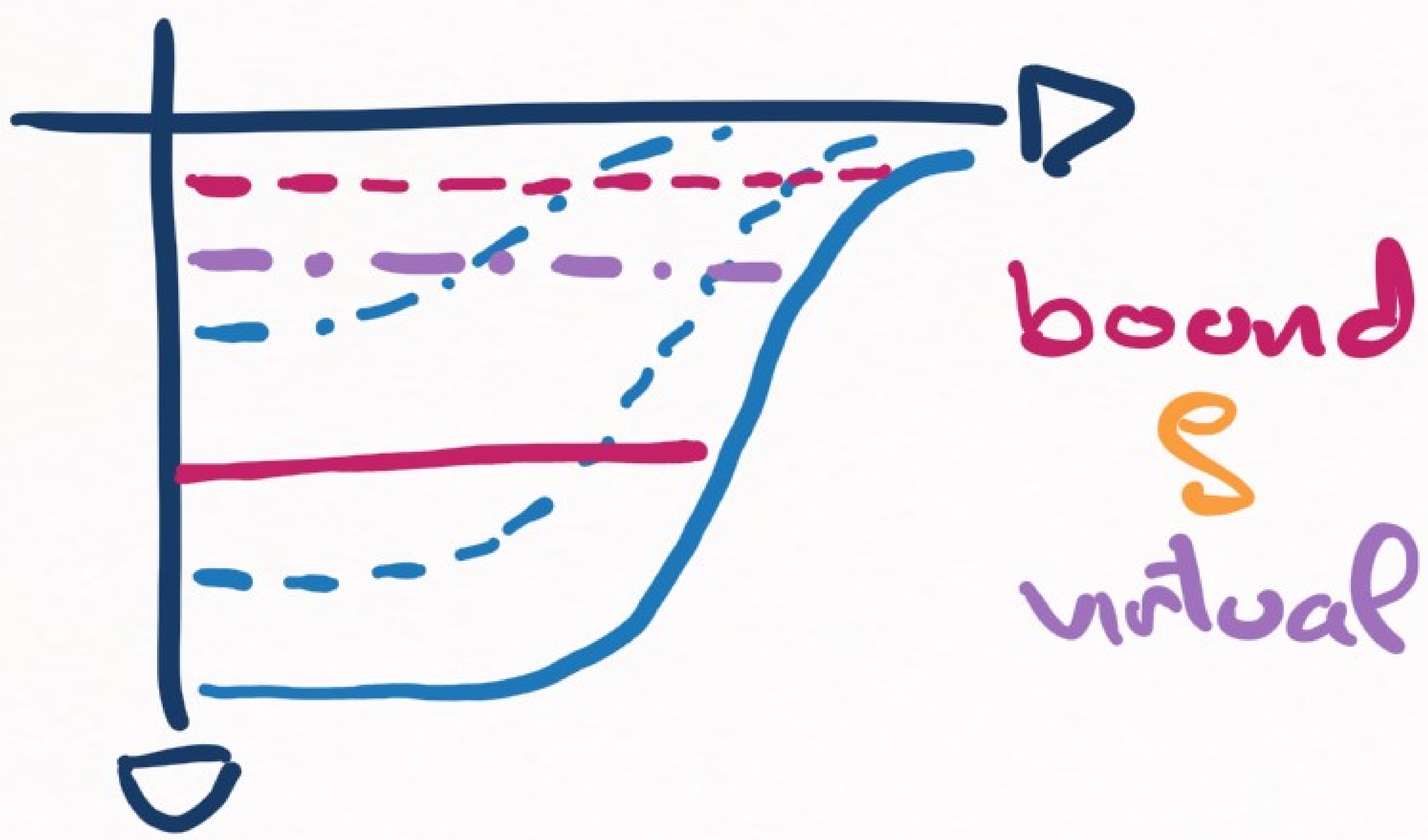
$\Rightarrow$



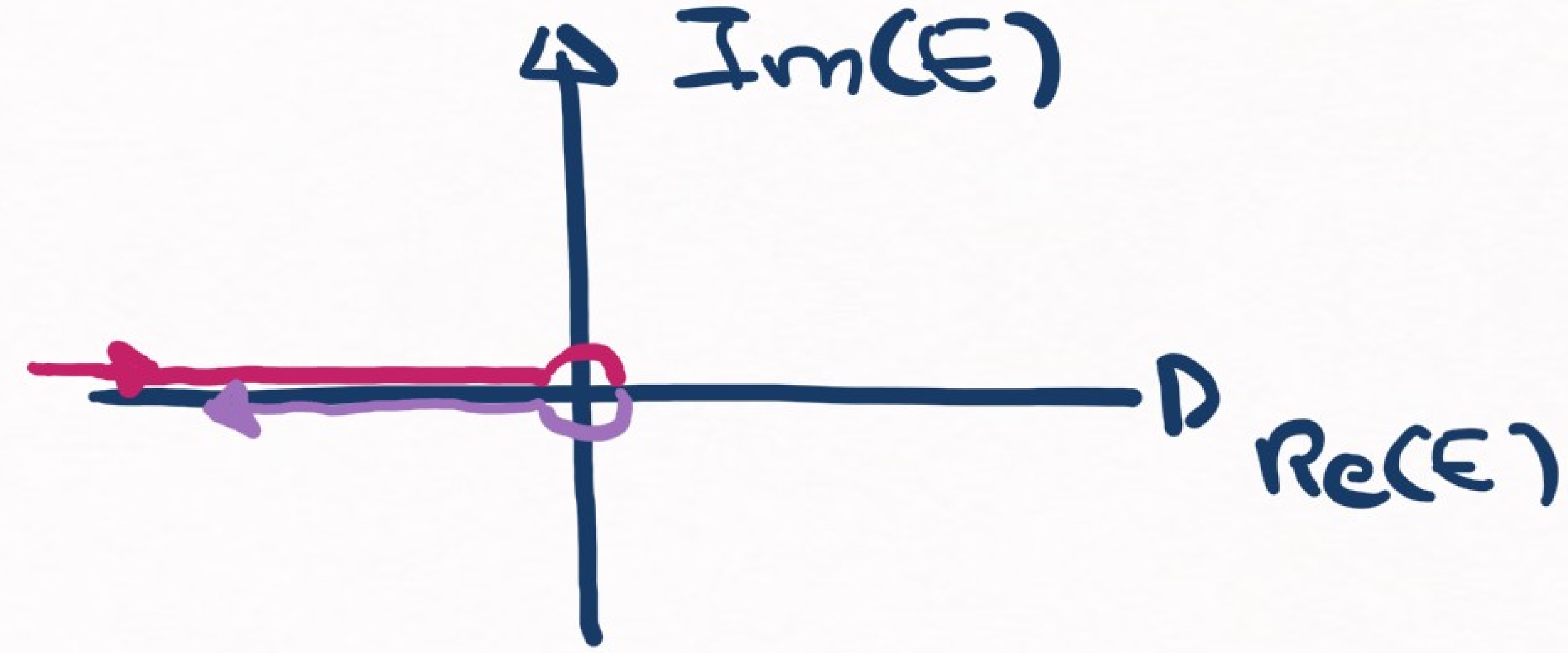
$\Rightarrow$ { We go from a bound state to a virtual state

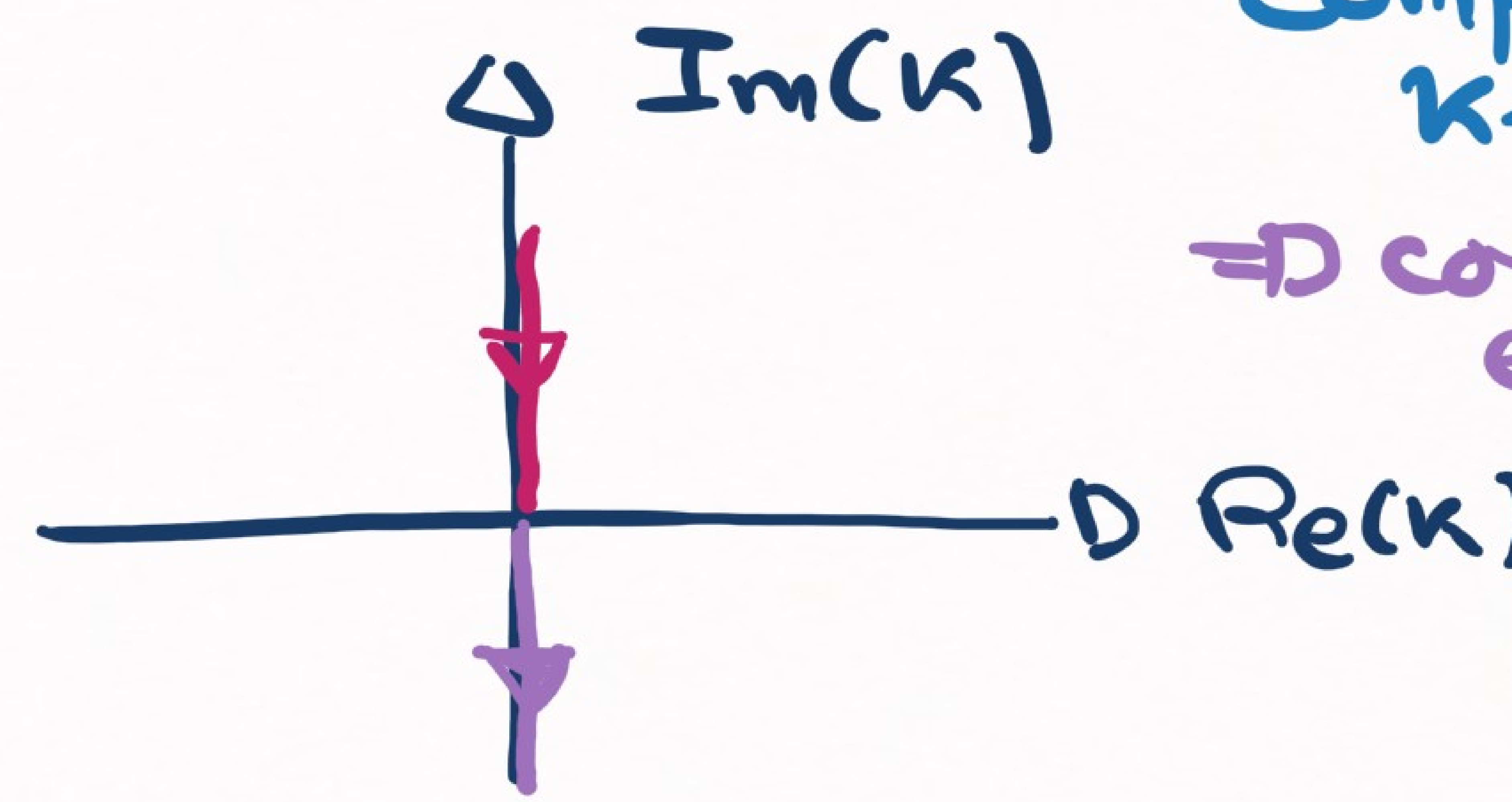
[RESONANCES] 10 (+ S-wave)

a.2) Attractive potentials: bound & virtual states



⇒





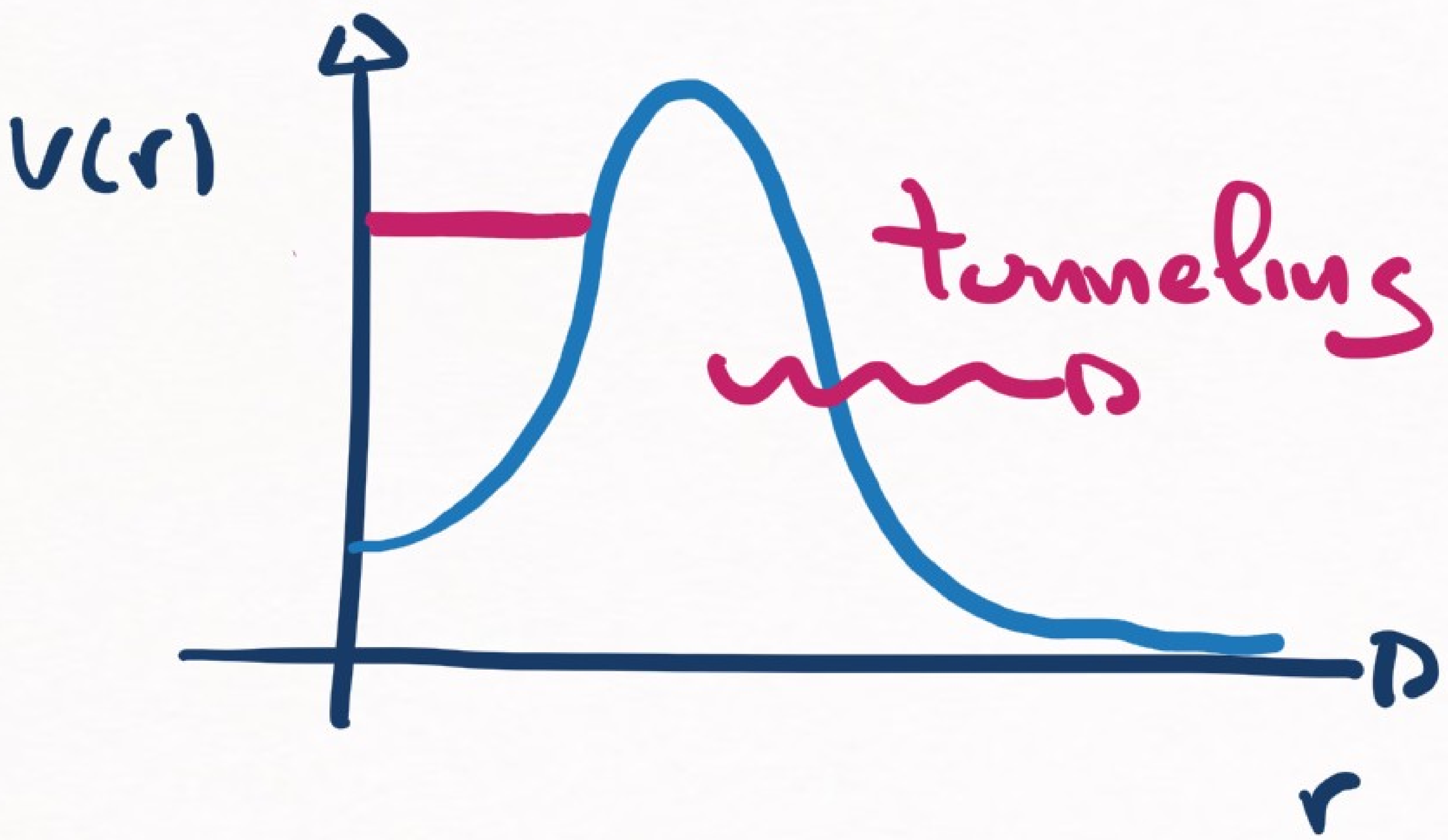
complex k-plane

⇒ continuous evolution

Again, we are changing the strength of the potential!

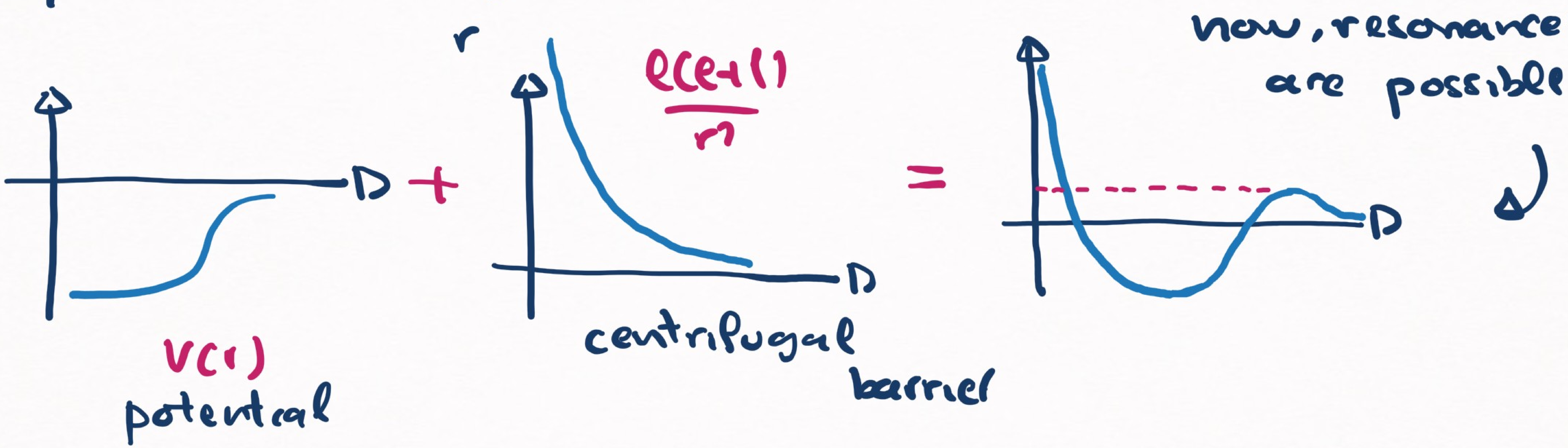
[RESONANCES] 11

⇒ But, when do we find this type of potential?



b) The effective potential for $\ell \neq 0$:

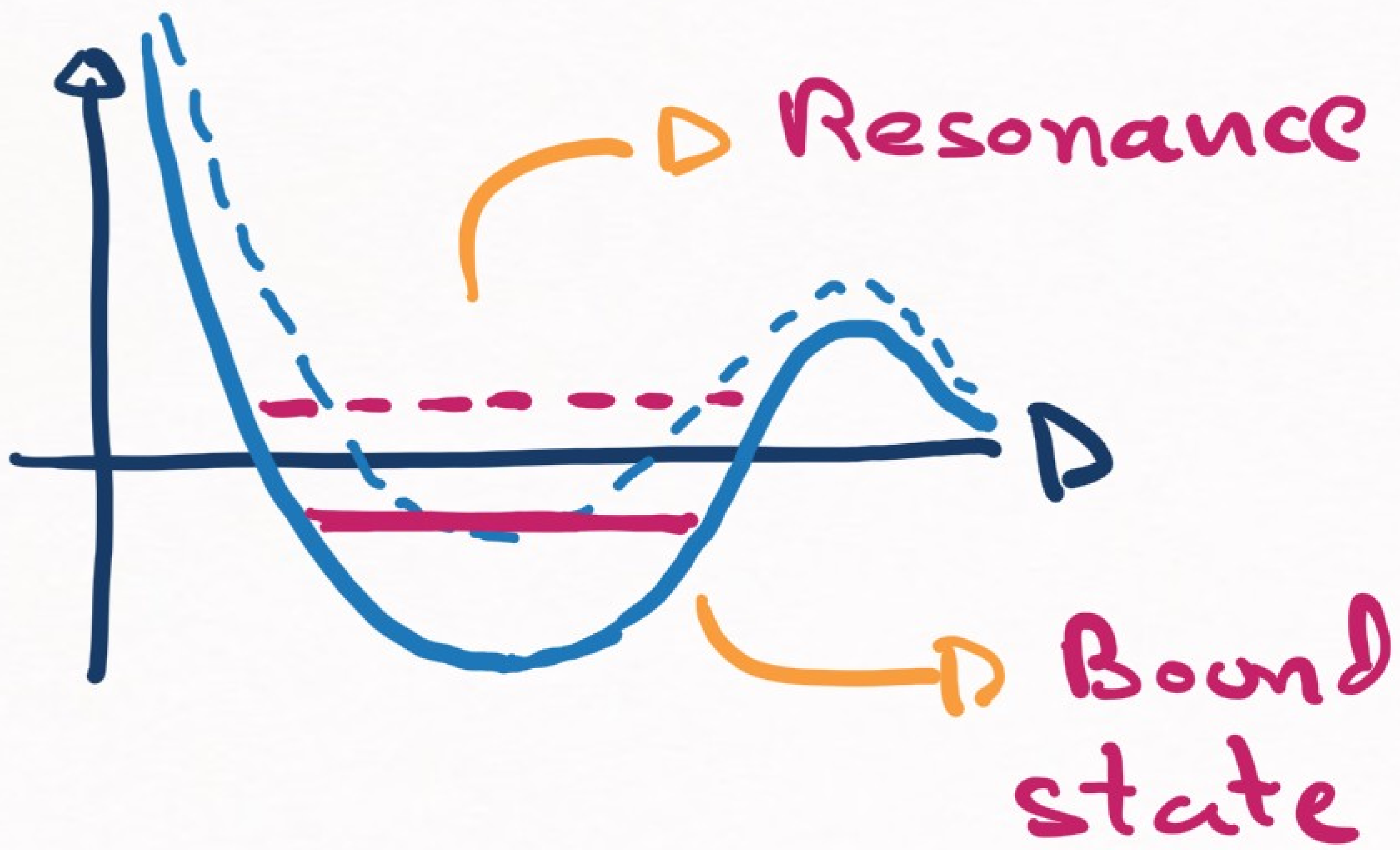
$$\left[V_{eff}(r) = V(r) + \frac{\ell(\ell+1)}{r^2} \right]$$

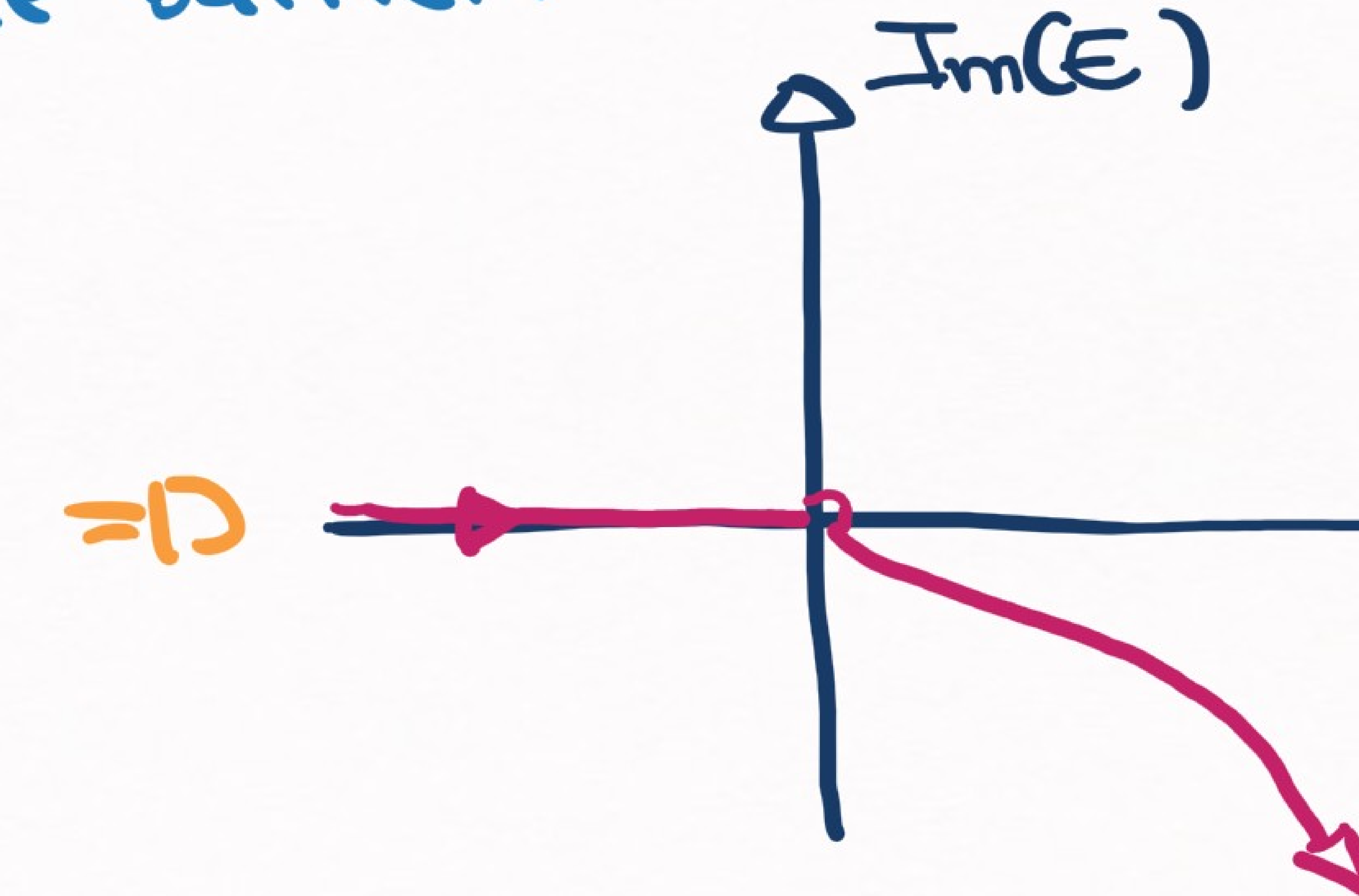


[RESONANCES] 12 ($\ell \neq 0$)

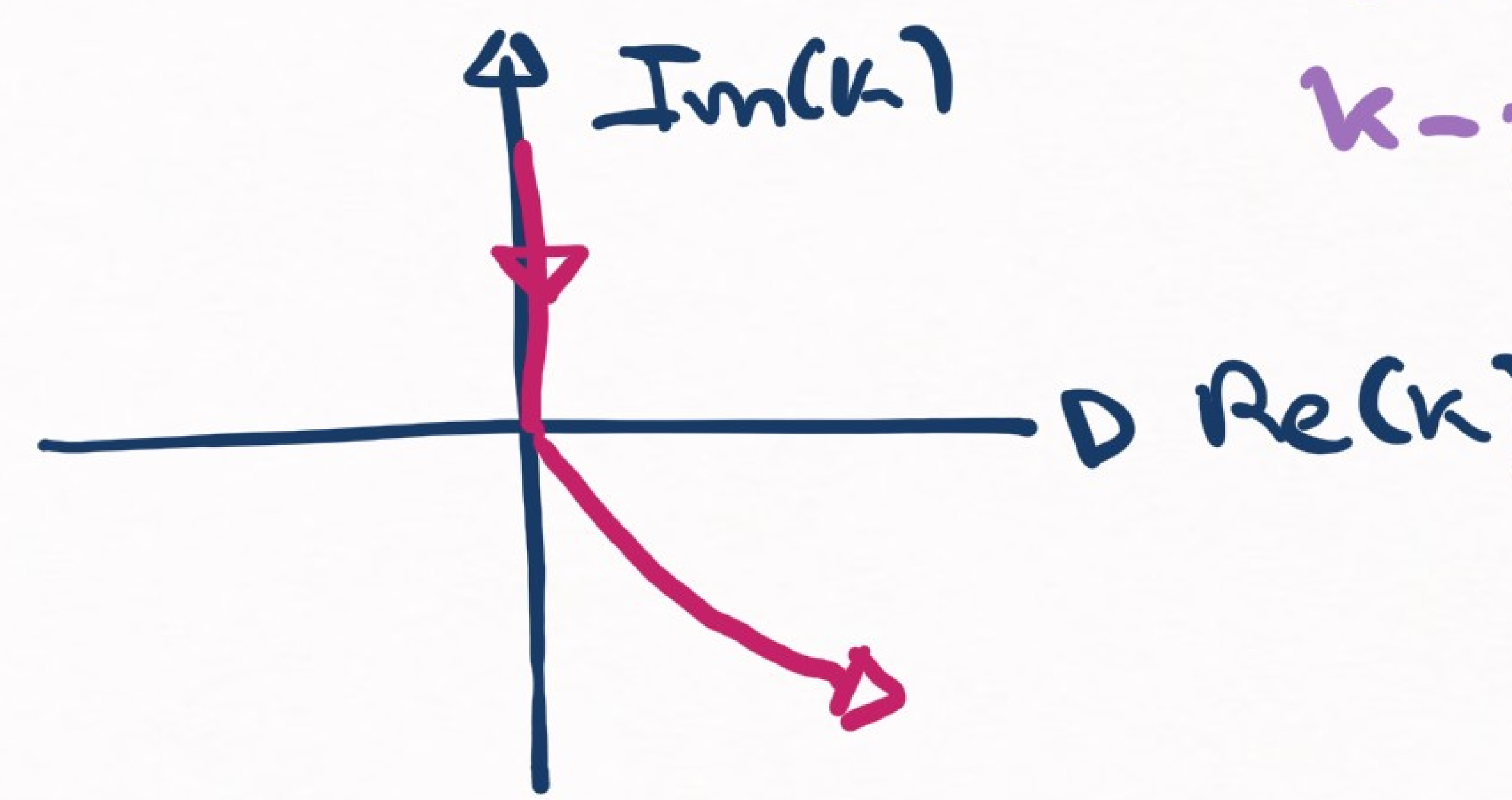
→ direction when the potential weakens

b) Potential + centrifugal barrier:





complex E-plane trajectory

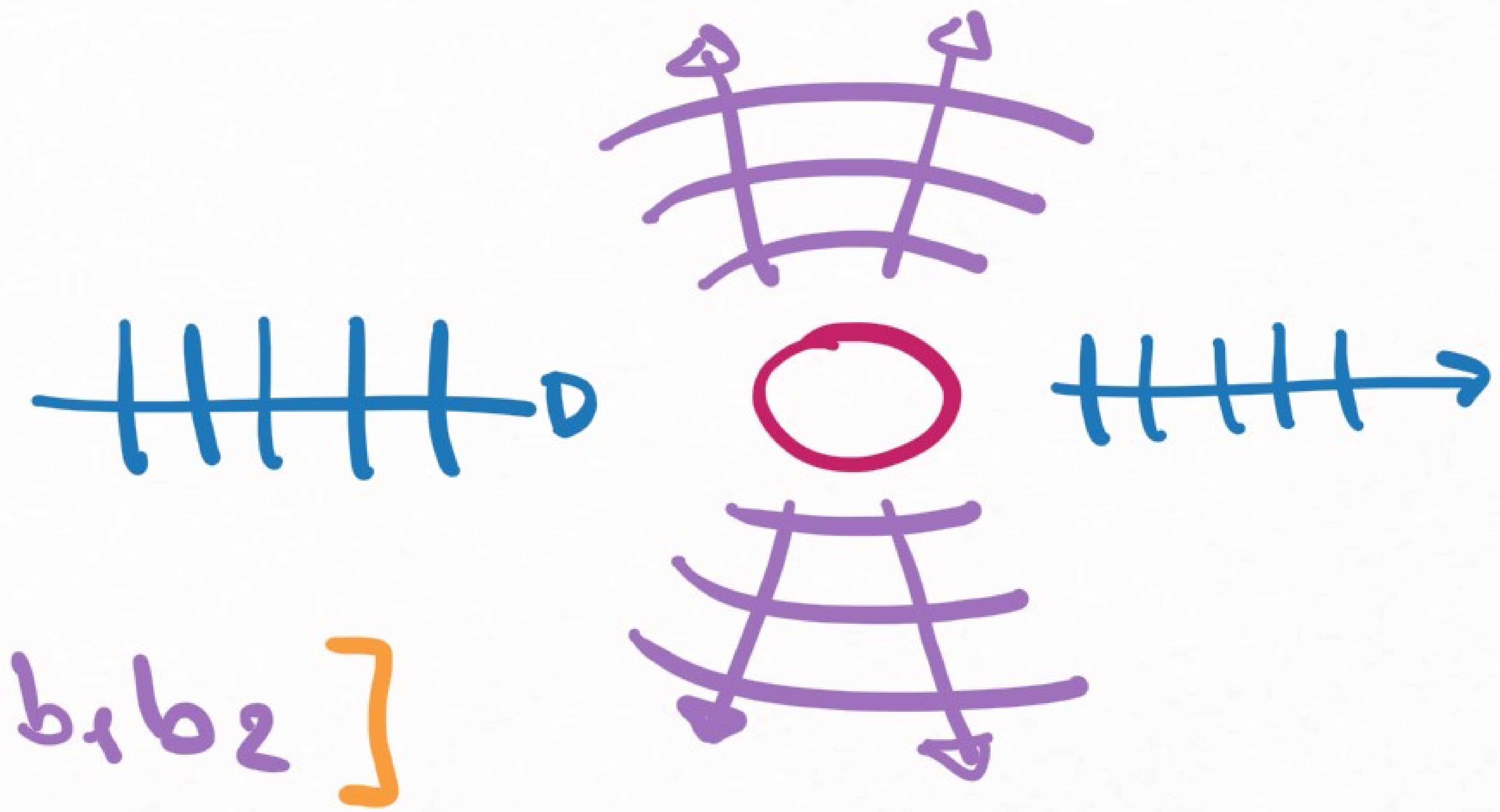


complex k-plane

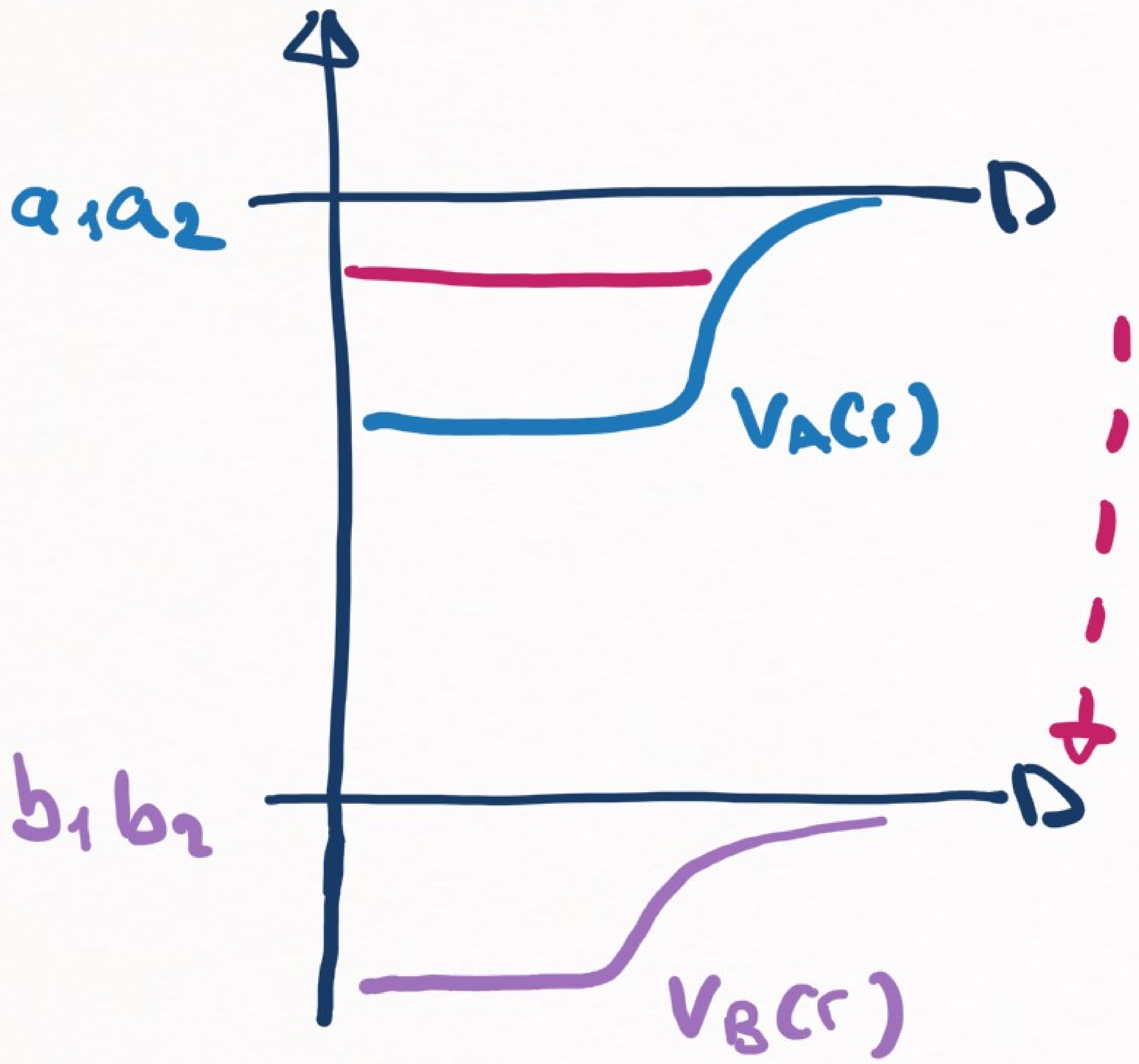
The effect of reducing the potential

[RESONANCES] 13

c) Multichannel problem:



[Scattering from $\underline{\underline{a_1 a_2}}$ to $\underline{\underline{b_1 b_2}}$]



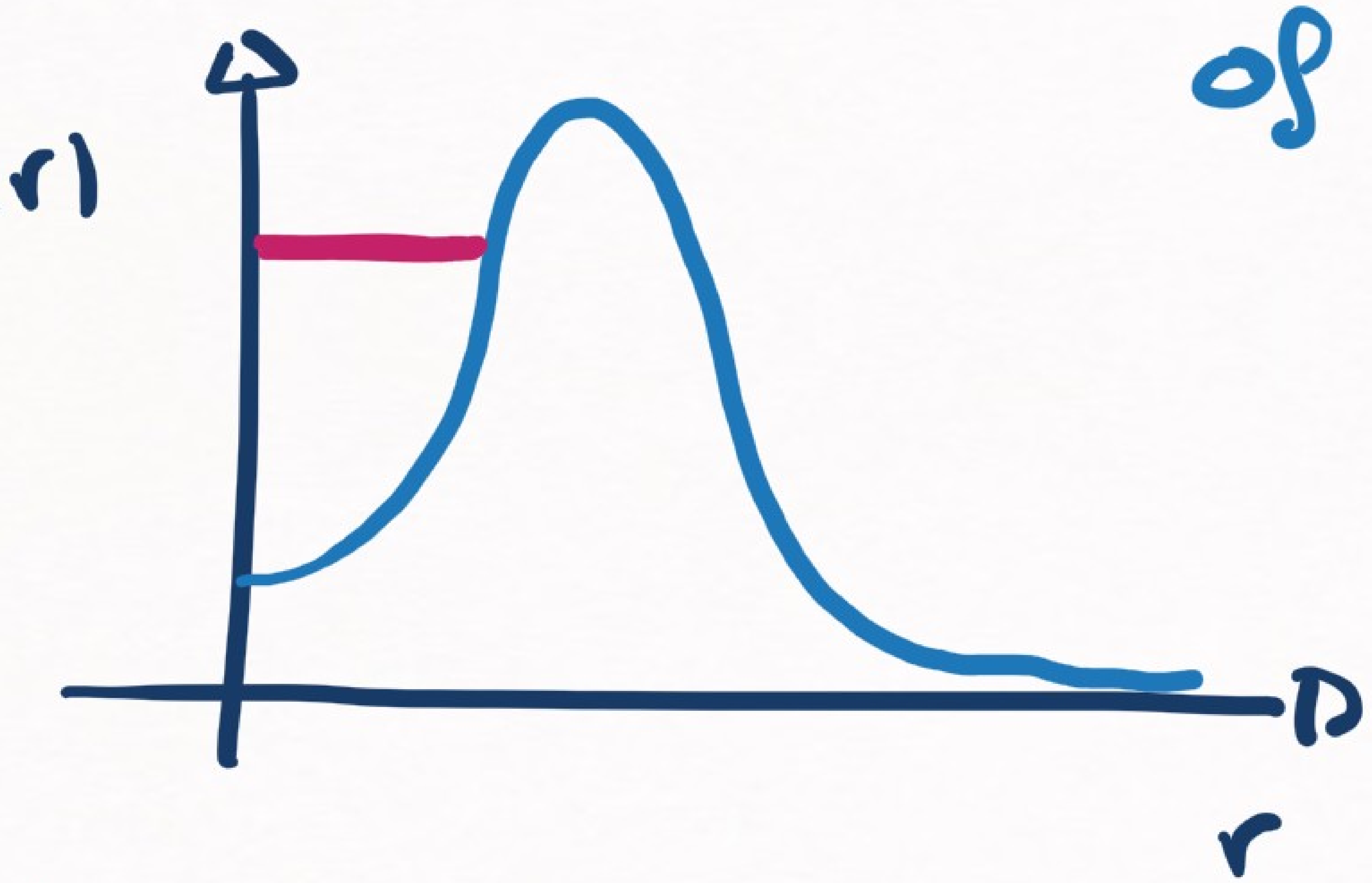
$(a_1 a_2)$
$\downarrow$
$\underline{\underline{b_1 + b_2}}$

inelastic scatterings:

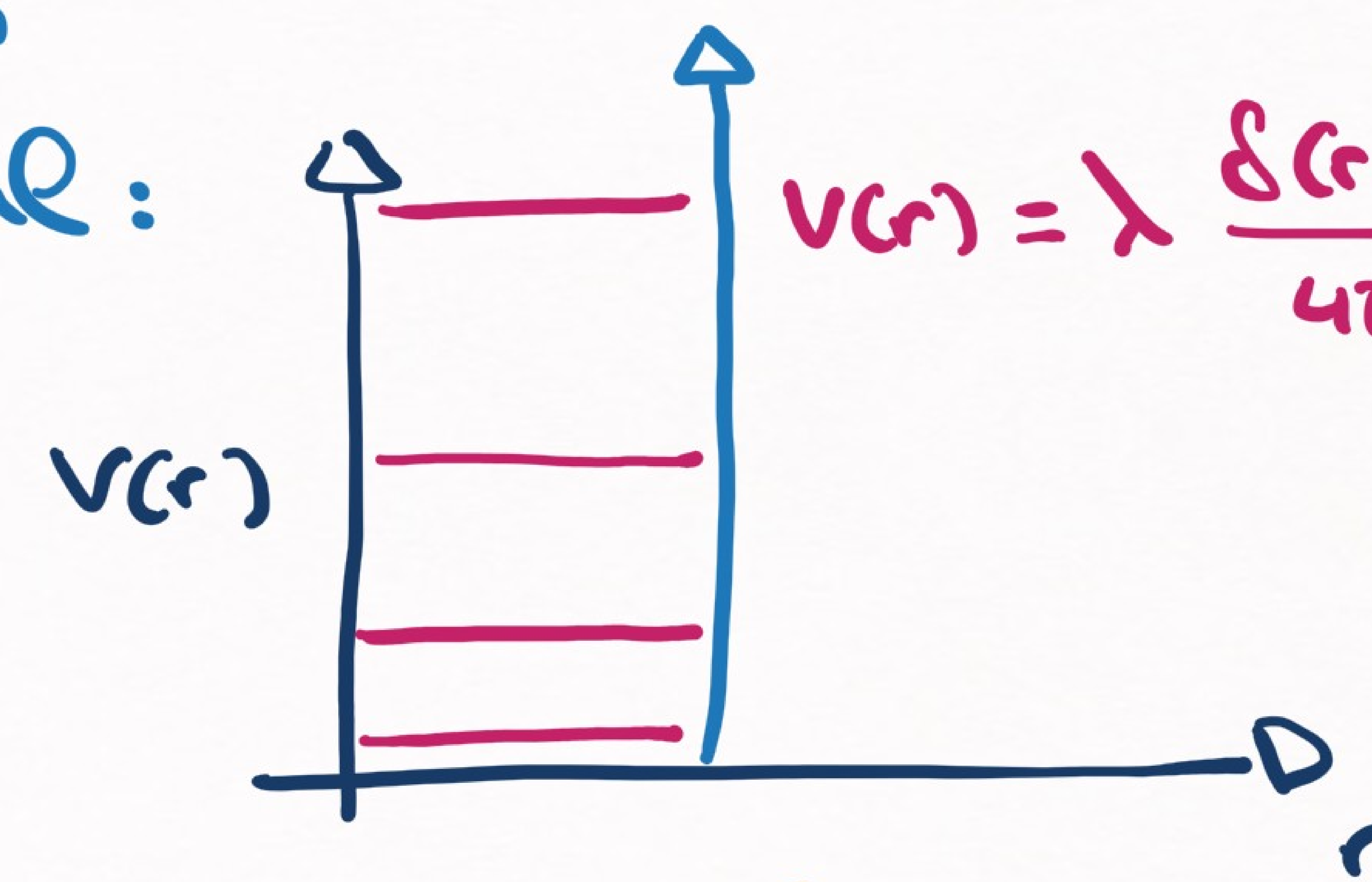
the outgoing particles are different than the incoming particles (advanced topic)

[RESONANCES] 14

$\Rightarrow$ There is a simplified version of this potential:



$\Rightarrow$



$$V(r) = \lambda \frac{\delta(r-R_c)}{4\pi R_c^2}$$

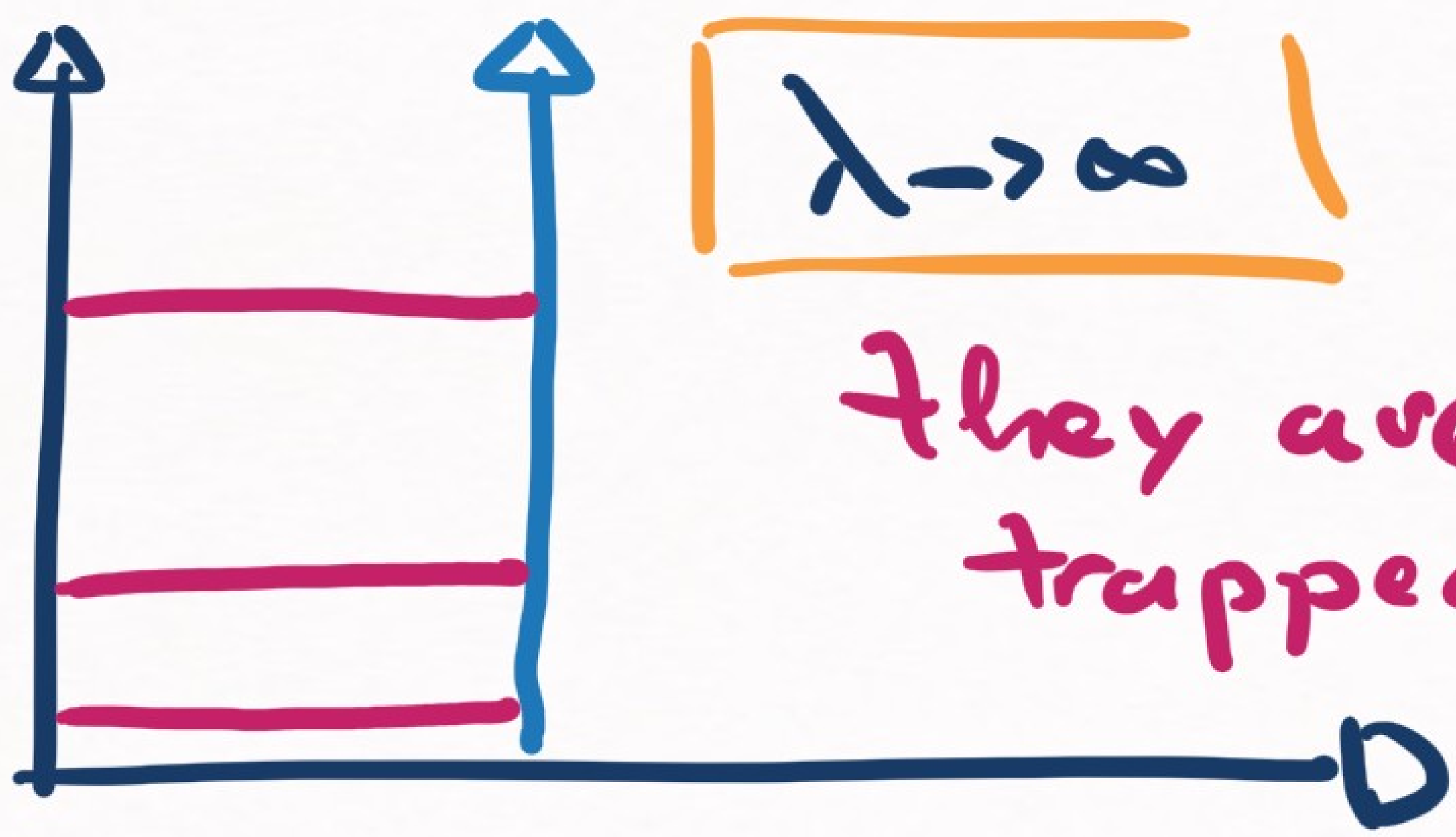
Repulsive δ-shell

$\boxed{\lambda \to \infty}$ $\Rightarrow$ Bound states at $\sin(kR_c) = 0$ $\quad (u(R_c) = 0)$

$\Rightarrow$ $kR_c = n\pi \quad , n = 1, 2, 3 \ldots$

[RESONANCES] 35

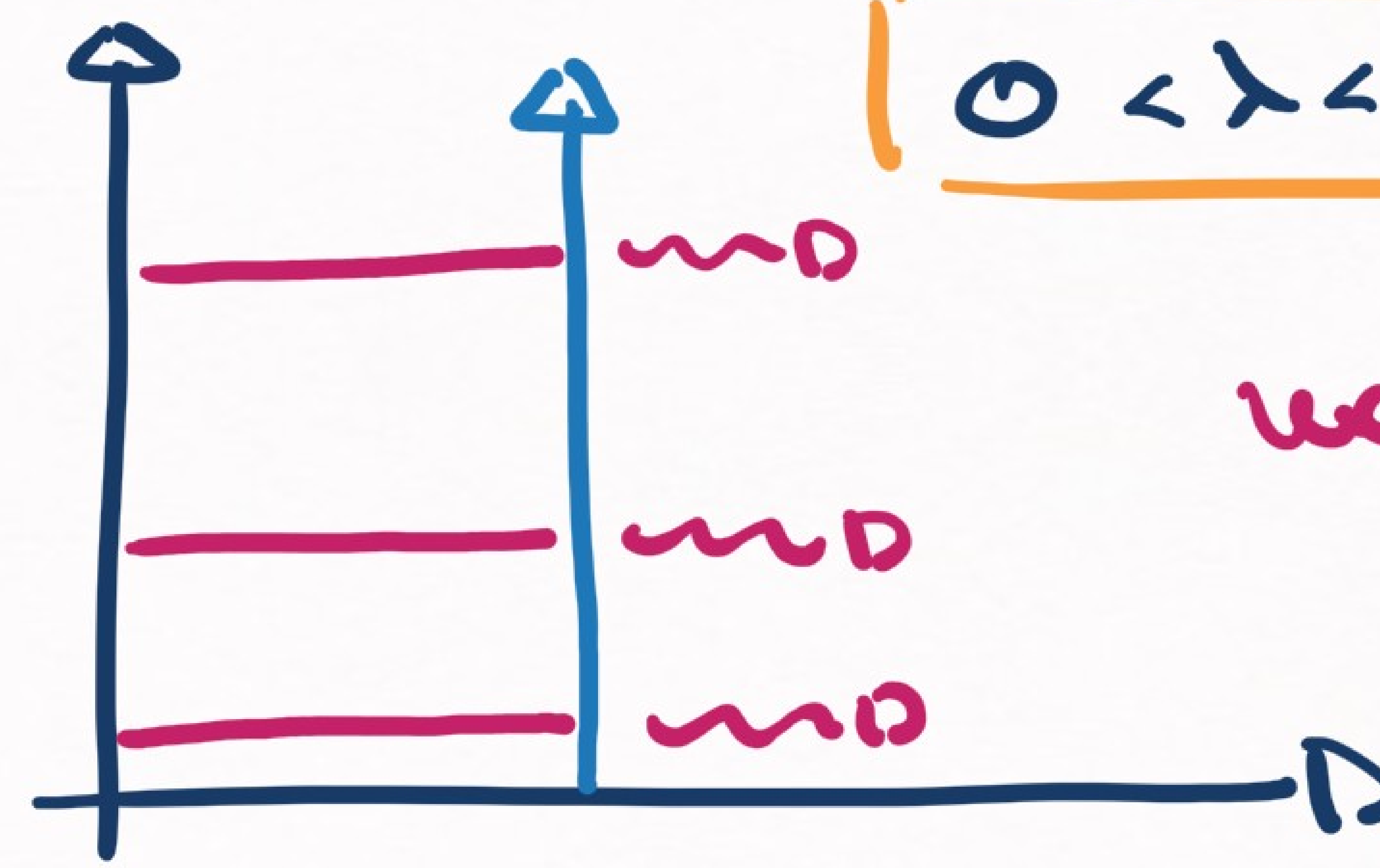
⇒ Repulsive δ-shell: good model to understand resonances



$\lambda \to \infty$

they are trapped

⇒



$0 < \lambda < \infty$

we have tunneling

$$E = \frac{1}{2\mu}\left(n\frac{\pi}{R_c}\right)^2$$

$n = 1, 2, 3, \ldots$

⇒

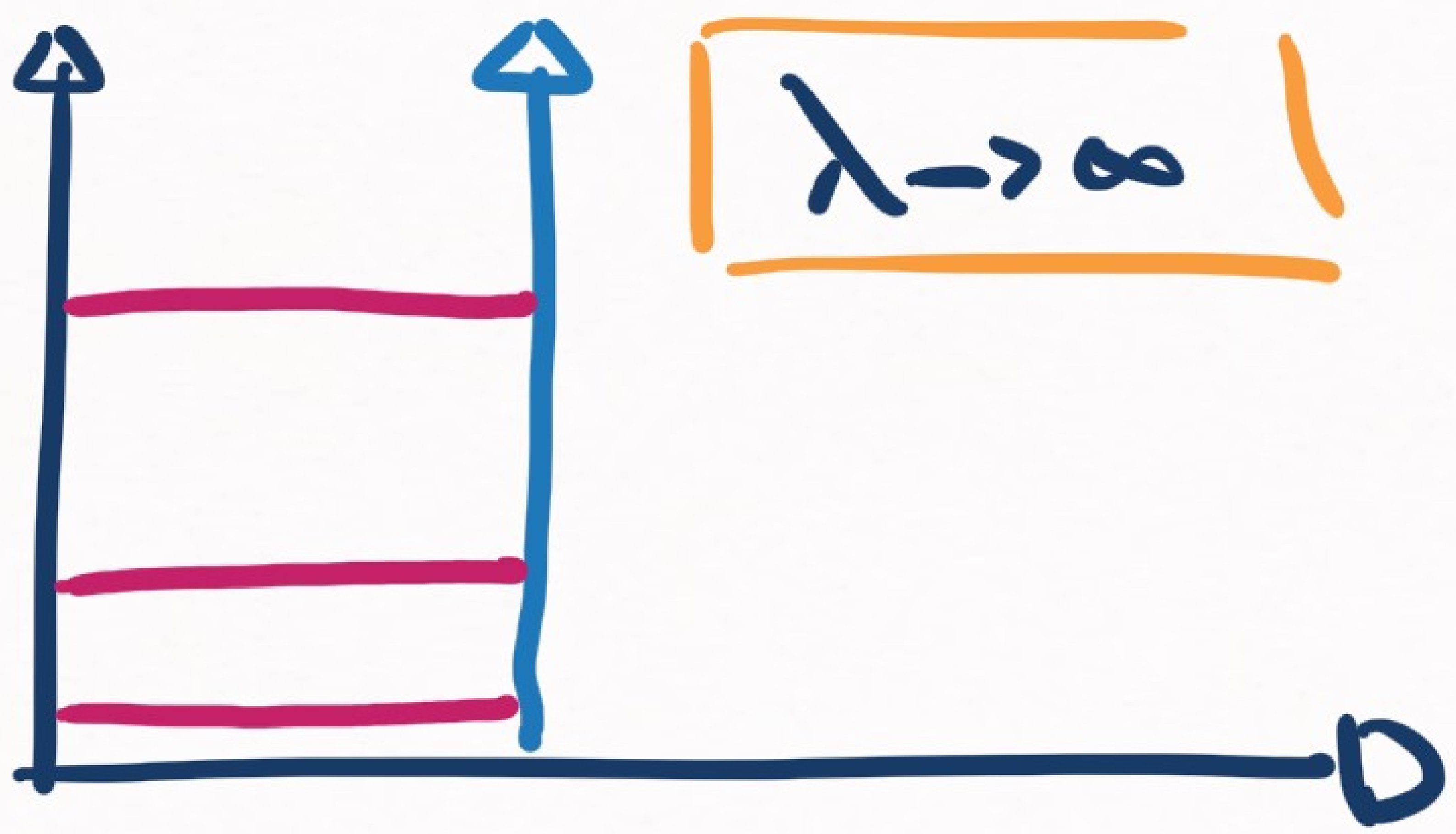
$$E = \frac{1}{2\mu}\left(n\frac{\pi}{R_c}\right)^2 + \delta E_r - i\frac{\Gamma}{2}$$

$\Gamma \sim \mathcal{O}\left(\frac{1}{\lambda}\right)$ (if λ is large enough)

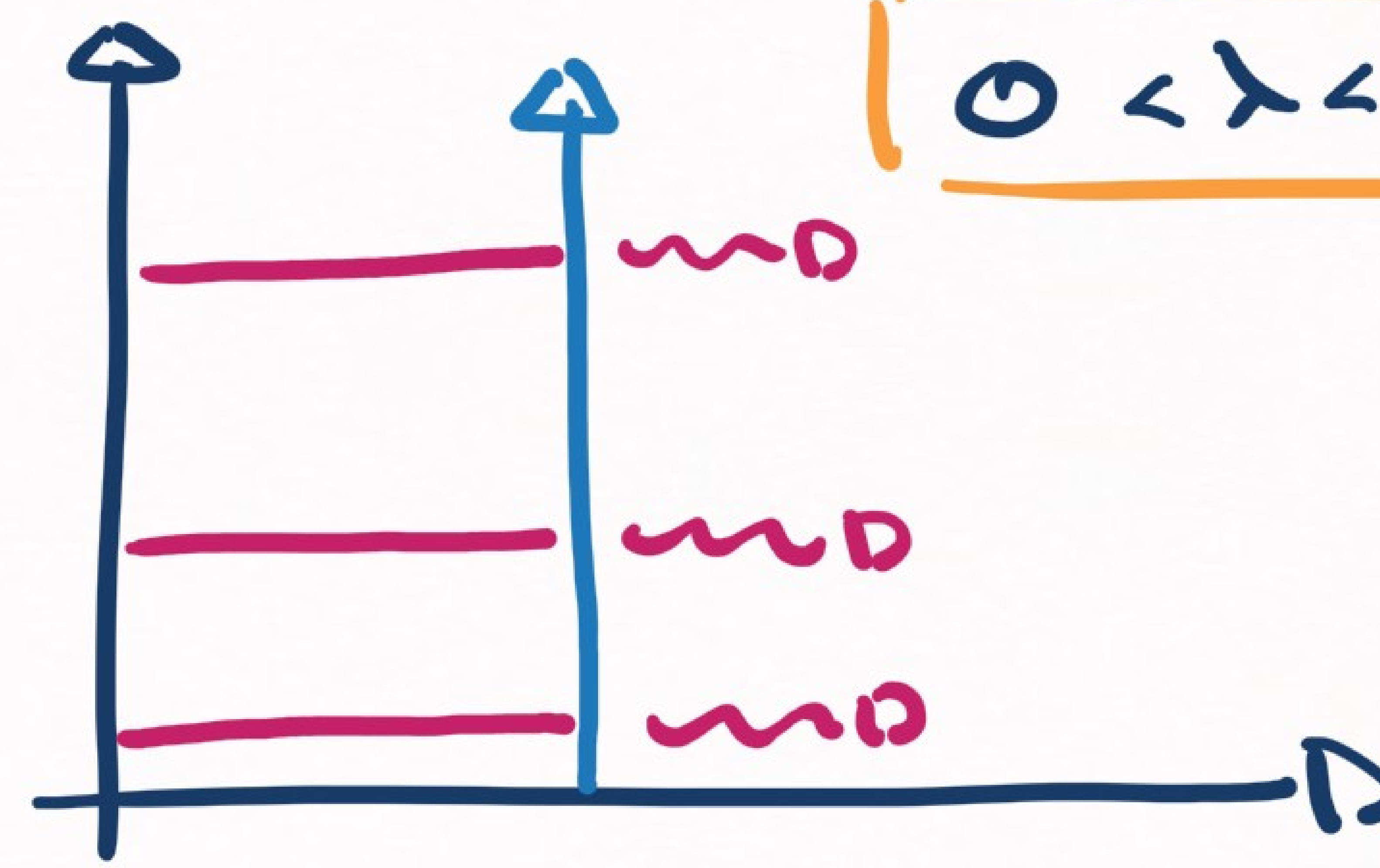
not difficult to calculate

[RESONANCES] 16

⇒ Repulsive δ-shell: good model to understand resonances



⇒



⇒ Nice, but how do you calculate the widths?

$$u(r > R_c) = e^{ikr} - \frac{e^{-ikr}}{S(k)}$$

$$= e^{ikr}$$

Resonances: $S(E \to E_r) \to \infty$

$$\frac{1}{S(E \to E_r)} \to 0$$

[BOUND, VIRTUAL STATES AND RESONANCES]

1) The wave function interpretation: ($\ell = 0$ for simplicity)

$$u_0(r) \xrightarrow[r\to\infty]{} \underline{\underline{S(k)}}\, e^{ikr} - e^{-ikr} \quad \text{or} \quad e^{ikr} - \underline{\underline{\frac{1}{S(k)}}}\, e^{-ikr}$$

(normalization in this case is arbitrary)

1.a) Pole in the scattering amplitude:

$$\left.\begin{array}{c} S(k) \to \infty \\ \text{or} \\ \dfrac{1}{S(k)} \to 0 \end{array}\right\} \Rightarrow u_0(r) \xrightarrow[r\to\infty]{} e^{ikr}$$

(k will be complex)

[BOUND, VIRTUAL STATES AND RESONANCES]

1) The wave function interpretation (continuation):

1.b) Bound state $k = i\gamma$ $(\mathrm{Im}(k) > 0)$

$$\Rightarrow u(r) \xrightarrow[r\to\infty]{} e^{-\gamma r}$$

1.c) Virtual state $k = -i\gamma$ $(\mathrm{Im}(k) < 0)$

$$\Rightarrow u(r) \xrightarrow[r\to\infty]{} e^{+\gamma r}$$

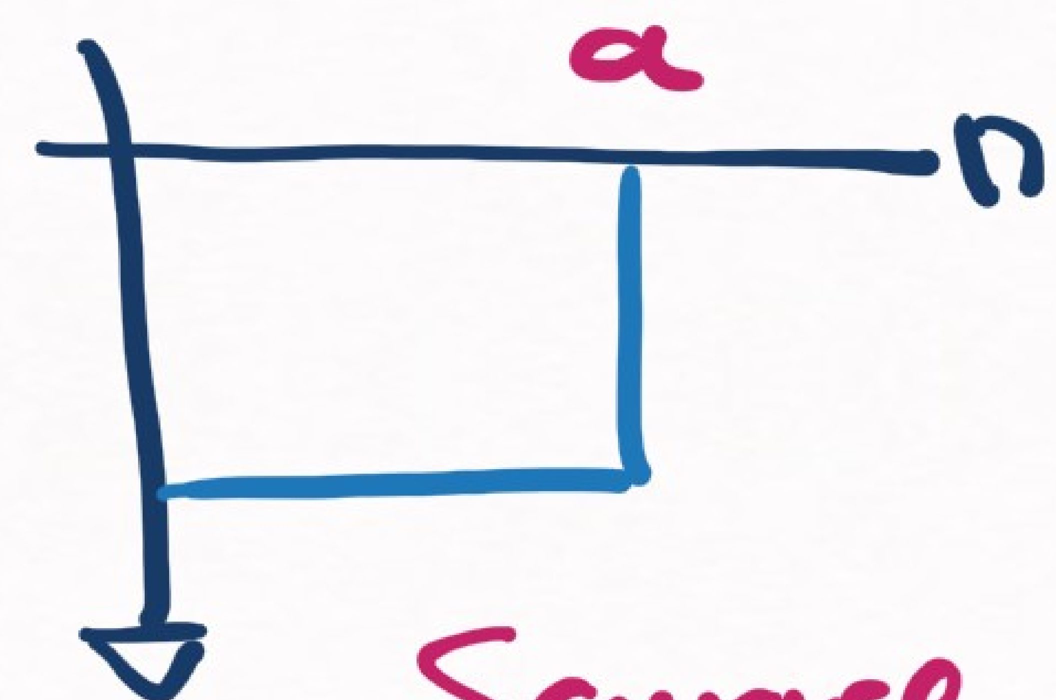
1.d) Resonance $k = \mathrm{Re}(k) + i\,\mathrm{Im}(k)$ $(\mathrm{Im}(k) < 0)$

$$\Rightarrow u(r) \xrightarrow[r\to\infty]{} e^{ikr}$$

[BOUND, VIRTUAL STATES AND RESONANCES]

$\Rightarrow$ Examples:

1.a) & 1.b)



Square well

$k = \sqrt{2\mu(V_0+E)}$
$(E<0)$

1.a) $u(r>a) = A_s e^{-\gamma_B r}$

$k\cot(ka) = -\gamma_B$

1.b) $u(r>a) = A_s e^{+\gamma_V r}$

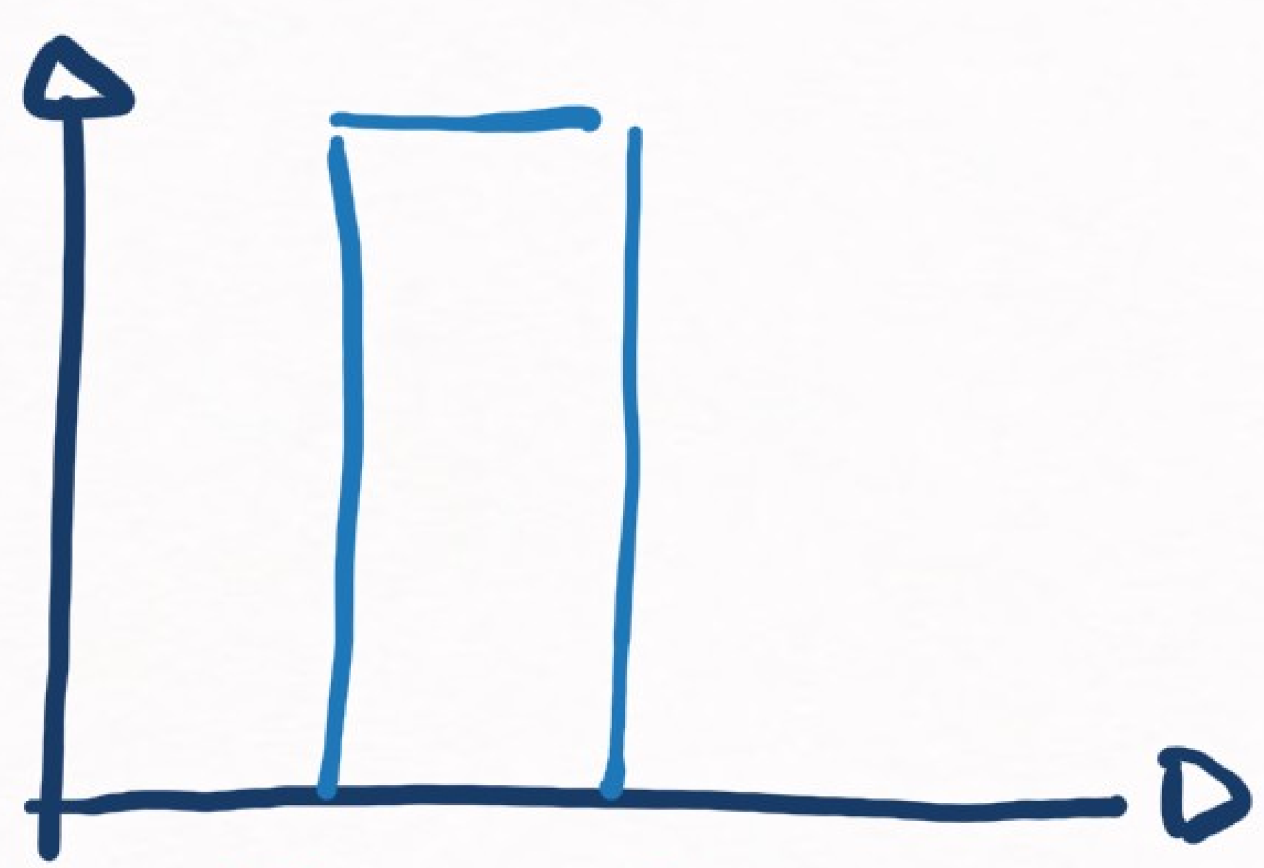
$k\cot(ka) = +\gamma_V$

$\boxed{1.a) \,\&\, 1.b)}$ $\Rightarrow$ the same condition: $k\cot(ka) = -\gamma$

$\gamma > 0$ bound

$\gamma < 0$ virtual

1.c)



1.c) $u(r>a) = A_s e^{ikr}$

$\Rightarrow$ the condition will be rather involved

[BOUND, VIRTUAL STATES AND RESONANCES]

2) The "pole of the scattering amplitude" interpretation

$T(E) \rightarrow \frac{1}{E - E_{pole}}$ $\Rightarrow$ we try to rewrite $T(E)$ in terms of $k = \sqrt{2\mu E}$,

pole $\leftarrow$ $[k_p = \sqrt{2\mu E_p}]$

2.a) Bound state : $Re(k_p) = 0$, $Im(k_p) > 0$

2.b) Virtual state : $Re(k_p) = 0$, $Im(k_p) < 0$

2.c) Resonance : $Re(k_p) \neq 0$, $Im(k_p) < 0$

pattern

[BOUND, VIRTUAL STATES AND RESONANCES]

$\Rightarrow$ The two Riemann sheets: $E = \frac{k^2}{2\mu}$, E, k are complex

$\Rightarrow$ for each E, you have two k's

RI) $\mathrm{Im}(k) > 0 \Rightarrow$ first (I) Riemann sheet

RII) $\mathrm{Im}(k) < 0 \Rightarrow$ second (II) Riemann sheet

Bound states $\Rightarrow$ sheet I (physical sheet)

Virtual states & Resonances $\Rightarrow$ sheet II (unphysical sheets)

[BOUND, VIRTUAL STATES AND RESONANCES]

$\Rightarrow$ Examples (of poles of the scattering amplitude)

2.a) & 2.b) $\langle \vec{p}\,' | T(E+i\epsilon) | \vec{p} \rangle = \frac{2\pi}{\mu} \frac{1}{\frac{1}{a_0} + ik}$ $\rightarrow$ slides

H - 2S

(check again)

2.c) $\langle \vec{p}\,' | T(E+i\epsilon) | \vec{p} \rangle = -\frac{2\pi}{\mu} \frac{1}{\underbrace{-\frac{1}{a_0} + \frac{1}{2} r_0 k^2 - ik}}$

For $r_0 < 0$ (+ certain conditions for a_0)

$\Rightarrow$ Possible to have resonances

RECAP !

1) Contact-range theories ⟹ they are the perfect example to understand how the T-matrix works + Renormalization

$$\langle \vec{p}\,'|T(E+i\epsilon)|\vec{p}\rangle = \frac{2\pi}{\mu}\frac{1}{1/a_0 + ik}$$

2) We learnt that bound states are poles of the T-matrix

⟹ but there are more types of poles !

bound states, virtual states and resonances

RECAP

sheet:

3) Their interpretation:

I { 3.a) Bound states ⟹ We know them well

{ 3.b) Virtual states ⟹ Same effects as a bound state for low energy scattering

II But $\nexists$ a physical state below threshold ⟸ ($\sigma \to 4\pi |a_0|^2$ or $\frac{4\pi}{|\gamma|^2}$)

{ 3.c) Resonances ⟹ A bit like bound states, but they can decay

4) Riemann sheets ⟹ (I): physical sheet $\mathrm{Im}(k) > 0$

(II): unphysical sheet $\mathrm{Im}(k) < 0$

[THE ONE BOSON EXCHANGE (OBE) MODEL]

⇒ Wanted (dead or alive):
a quantitatively successful
description of nuclear forces

=

⇓

OBE was the first of
such descriptions

A description
of nuclear
forces

Let's begin with...

[What is the origin of the nuclear force?]

Historical overview:

0) Yukawa's idea $\Rightarrow$ Exchange of a meson as explanation

1) Before QCD era:

1.a) multi-pion theories (50's) $\Rightarrow$ failed

1.b) one boson exchange model

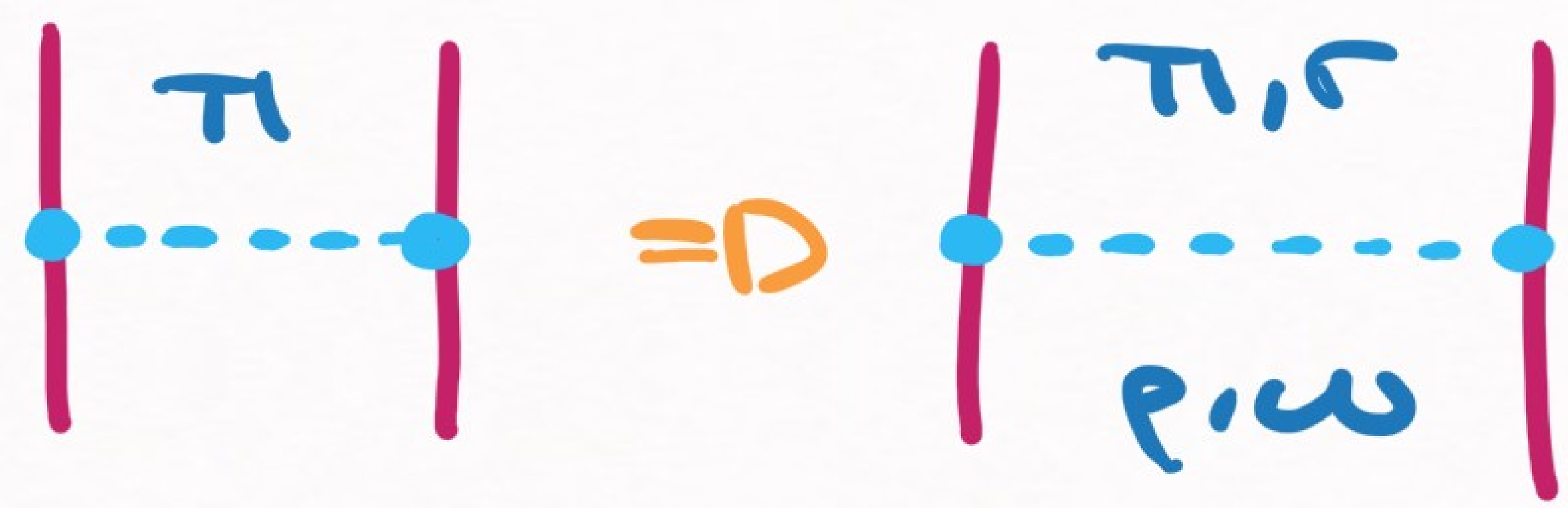
2) Post QCD era: $\rightarrow$ nucleons are not fundamental

2.a) quark-model inspired explanation

2.b) effective field theory formulations

OBE MODEL

1) Extension of the original idea by Yukawa:



+ other bosons

2) Relatively simple and intuitive description

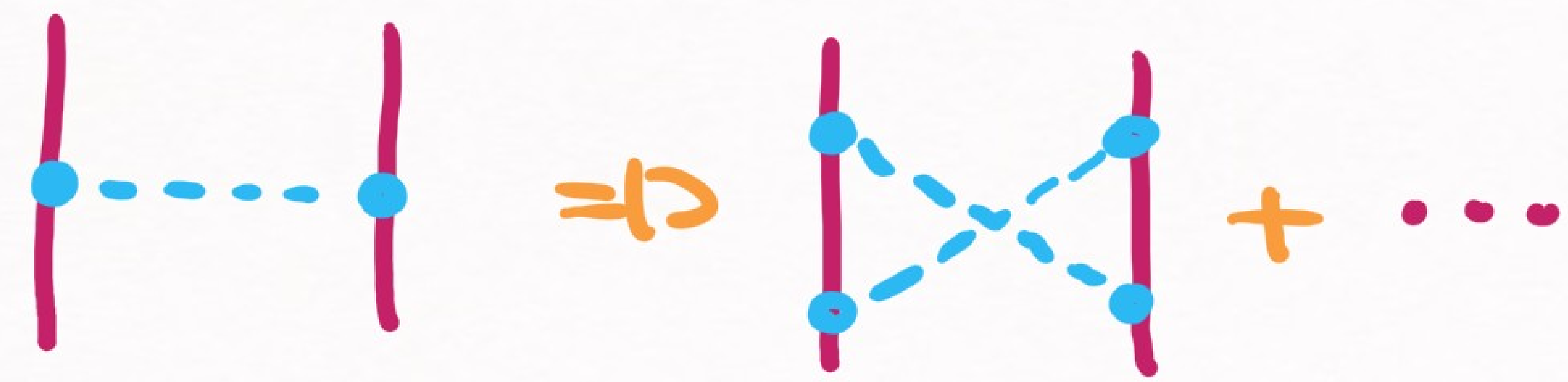
3) First quantitatively successful description of the nuclear force

$\chi^2/\mathrm{d.o.f} \simeq 1$

for the description of the NN scattering data (3000 data)

$\boxed{\text{OBE MODEL}}$ $\Rightarrow$ MOTIVATION: [Failure of the old pion theories]

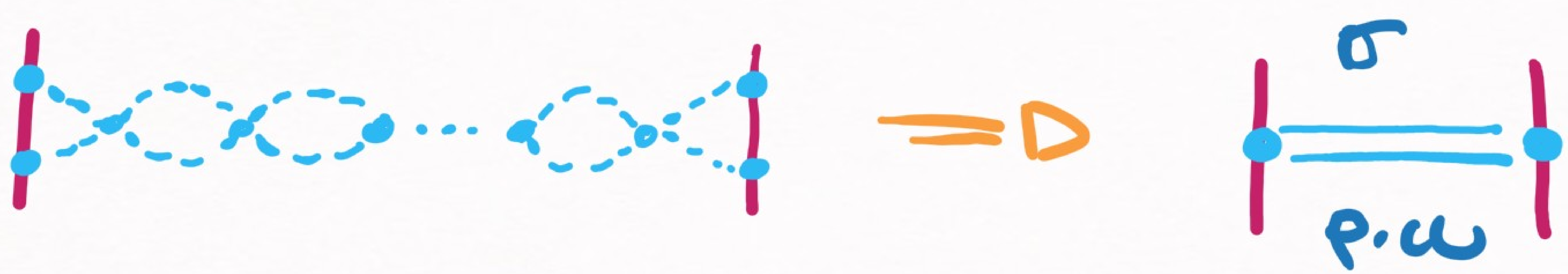
1) Direct extension of Yukawa's idea: more pions!



2) But it did not work $\Rightarrow$ why?

Because back then
no chiral symmetry
no renormalization

3) Idea: multi-pion exchanges $\simeq$ resonance exchange



σ, ρ appear as resonances in $\pi\pi$ scattering

$\boxed{\text{OBE MODEL}}$ ⇒ Each boson has a job within this model

a) The pion : $J^P = 0^-$, $I = 1$, $m_\pi \simeq 140$ MeV

It explains the deuteron quadrupole moment

b) The sigma: $J^P = 0^+$, $I = 0$, $m_\sigma \sim 500$ MeV

provides a strong mid-range attraction

c) The rho: $J^P = 1^-$, $I = 1$, $m_\rho \simeq 770$ MeV

counter the excessive tensor force from the pion

d) The omega: $J^P = 1^-$, $I = 0$, $m_\omega \simeq 780$ MeV

provides short-range repulsion

$\boxed{\text{OBE MODEL}}$ $\Rightarrow$ Each boson has a job within this model

Table 1: The four most important mesons and the main characteristics of their contributions to components of the nuclear force.

Meson	Central	Spin-Spin	Tensor	Spin-Orbit
$\pi(138)$	---	weak, long-ranged	**strong**, long-ranged	---
$\sigma(500)$	**strong, attractive**, intermediate-ranged	---	---	moderate, intermediate-ranged
$\omega(782)$	**strong, repulsive**, short-ranged	---	---	**strong**, short-ranged, coherent with σ
$\rho(770)$	---	weak, short-ranged, coherent with π	moderate, short-ranged, **opposite to** π	---

Relativistic corrections

From scholarpedia's review by R. Machleidt

[THE OBE POTENTIAL]

$\Rightarrow$ Really simple: $V_{NN}(\vec{q}) = \sum_{M} V_M(\vec{q}) = V_\pi + V_\sigma + V_\rho + V_\omega$

(sum of contributions)

π) $V_\pi(\vec{q}) = -\frac{g_{\pi NN}^2}{4M_N^2} \frac{\vec{\sigma}_1\cdot\vec{q}\,\vec{\sigma}_2\cdot\vec{q}}{\vec{q}^2 + m_\pi^2}$

σ) $V_\sigma(\vec{q}) = -\frac{g_\sigma^2}{\vec{q}^2 + m_\sigma^2}$

ω) $V_\omega(\vec{q}) = \frac{g_\omega^2}{\vec{q}^2 + m_\omega^2} - \frac{(f_\omega + g_\omega)^2}{4M_N^2} \frac{(\vec{\sigma}_1\wedge\vec{q})\cdot(\vec{\sigma}_2\wedge\vec{q})}{\vec{q}^2 + m_\omega^2}$

ρ) $V_\rho(\vec{q}) = \vec{\tau}_1\cdot\vec{\tau}_2 \left[\frac{g_\rho^2}{\vec{q}^2 + m_\rho^2} - \frac{(f_\rho + g_\rho)^2}{4M_N^2} \frac{(\vec{\sigma}_1\wedge\vec{q})\cdot(\vec{\sigma}_2\wedge\vec{q})}{\vec{q}^2 + m_\rho^2} \right]$

[OBE MODEL]

⇒ Usual simplifications:

$$f_\rho \gg g_\rho \Rightarrow g_\rho \simeq 0$$

$$f_\omega \ll g_\omega \Rightarrow f_\omega \simeq 0$$

Practical observation

the reason is this simplification

Sometimes you will see V_ρ potentials without the central part ...

[THE OBE POTENTIAL] $\longrightarrow$ r-space

$\Rightarrow$ Really simple: $V_{NN}(\vec{r}) = \sum_{\mu} V_\mu(\vec{r}) = V_\pi + V_\sigma + V_\rho + V_\omega$

$\pi)$ $V_\pi(\vec{r}) = \frac{1}{3}\frac{g_{\pi NN}^2}{4M_N^2}\vec{\tau}_1\cdot\vec{\tau}_2\left[\vec{\sigma}_1\cdot\vec{\sigma}_2\, m_\pi^3 W_Y(m_\pi r) + S_{12}(\hat{r})\, m_\pi^3 W_T(m_\pi r)\right]$

$\sigma)$ $V_\sigma(\vec{r}) = -g_\sigma^2 m_\sigma W_Y(m_\sigma r)$

$\omega)$ $V_\omega(\vec{r}) = g_\omega^2 m_\omega W_Y(m_\omega r) + \frac{(f_\omega + g_\omega)^2}{4M_N^2}\left[\frac{2}{3}\vec{\sigma}_1\cdot\vec{\sigma}_2\, m_\omega^3 W_Y(m_\omega r) - \frac{1}{3}S_{12}(\hat{r})\, m_\omega^3 W_T(m_\omega r)\right]$

$\rho)$ like ω, but including $\vec{\tau}_1\cdot\vec{\tau}_2$

[THE OBE POTENTIAL]

$\Rightarrow$ Really simple: $V_{NN}(\vec{r}) = \sum_{\mu} V_{\mu}(\vec{r}) = V_{\pi} + V_{\sigma} + V_{\rho} + V_{\omega}$

Two functions: a) $W_Y(x) = \dfrac{e^{-x}}{4\pi x}$

b) $W_T(x) = \dfrac{e^{-x}}{4\pi x}\left(1 + \dfrac{3}{x} + \dfrac{3}{x^2}\right)$

$\rightarrow$ $\underline{\underline{W_T(x)}}$ always appears with the tensor force $(S_{12}(\vec{r}))$

$$W_T(x) \xrightarrow[x \to 0]{} \frac{1}{x^3}$$

$\Rightarrow$ Problems: $V_{\pi} \underset{r \to 0}{\sim} \dfrac{S_{12}(\vec{r})}{r^3}$, $V_{\rho,\omega} \underset{r \to 0}{\sim} \dfrac{S_{12}(\vec{r})}{r^3}$

[THE OBE POTENTIAL]

$\Rightarrow$ Really simple, but we have singular interactions:

$$V_{OBE}(\vec{r}) \propto \frac{1}{r^3} \text{ for } r \to 0$$

$\Rightarrow$ At that time, renormalization was not understood

$\boxed{\text{Proposed solution}}$ $\Rightarrow$ finite-size of the hadrons will regulate the potentials at $r \to 0$

$\Rightarrow$ inclusion of form factors

[FORM FACTORS]

⇒ What is a form factor? → function that regularizes the potential at $|\vec{q}| \to \infty$ (or $|\vec{r}| \to 0$)

$$\underbrace{V_M(\vec{q})}_{\text{meson-exchange potential (for point-like hadrons)}} \longrightarrow V_M(\vec{q})\,\underbrace{F_M^2(\vec{q},\Lambda)}_{\text{this function represents the finite size of hadrons}}$$

⇒ Most usual choice: $F_M(\vec{q};\Lambda) = \left(\dfrac{\Lambda^2 - m^2}{\Lambda^2 + \vec{q}^{\,2}}\right)^{\alpha}$

multipolar form factor

[FORM FACTORS] $\rightarrow [\alpha = 1]$

$\Rightarrow$ Example : monopolar form factor $F_M(\vec{q},\Lambda) = \dfrac{\Lambda^2 - m^2}{\Lambda^2 + \vec{q}^{\,2}}$

$$V_M(\vec{q};\Lambda) = V_M(\vec{q})\,F_M^2(\vec{q},\Lambda)$$ equivalent to:

a) $$W_Y(x) \longrightarrow W_Y(x,\lambda) = W_Y(x) - \lambda W_Y(\lambda x) - \frac{(\lambda^2-1)}{2x}\frac{e^{-\lambda x}}{4\pi}$$

b) $$W_T(x) \longrightarrow W_T(x,\lambda) = W_T(x) - \lambda^3 W_T(\lambda x) - \frac{(\lambda^2-1)}{2\lambda}\lambda^2\left(1+\frac{1}{\lambda x}\right)\frac{e^{-\lambda x}}{4\pi}$$

with $\lambda = \dfrac{\Lambda}{m}$

$\rightarrow$ now the potentials are finite $(x \rightarrow 0)$

[OBE PARAMETERS]

The Meson Theory of Nuclear Forces and Nuclear Structure

R. Machleidt*
Department of Physics, University of California, Los Angeles, CA 90024, U.S.A.
and
Los Alamos National Laboratory, MS H850, Los Alamos, NM 87545, U.S.A.

Physics, Spring 2023

we will compile the lecture notes of the course, the exercise sheets and the s

es of this year are under development (actually, they are always under dev and you should come here from time to time for updates. Currently we ha update):

roduction
icleon system
ucture

erials: SU(3) Clebsch-Gordan coefficients, ChPT Reviews, the OBE model.

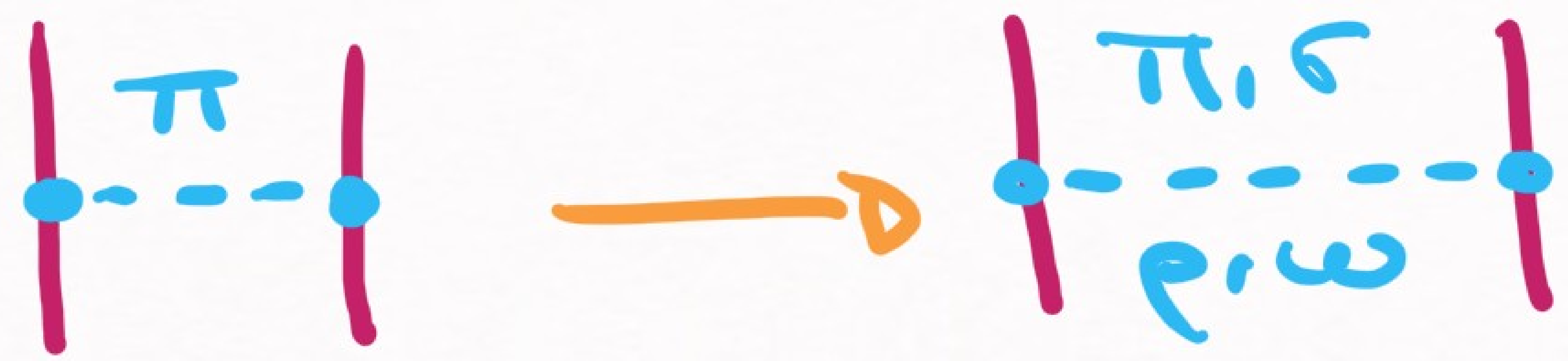
Table A.3: Non-relativistic configuration space OBEP.

	m_α (MeV)	*Potential A* $g_\alpha^2/4\pi$	Λ_α(GeV)	*Potential B* $g_\alpha^2/4\pi$	Λ_α(GeV)
π	138.03	14.9	1.3	14.9	2.0
η	548.8	2	1.5	0	–
ρ	769	1.2	1.2	1.7	1.1
ω	782.6	25	1.4	28	1.3
δ	983	2.742	2.0	6.729	2.0
σ^a	550	8.7171	2.0	8.8322	1.4
	(700-710)	(17.6205)	(2.0)	(16.0707)	(2.0)
$-\epsilon_d$ (MeV)		2.2246		2.2246	
P_D (%)		4.75		5.53	
Q_d (fm^2)		0.274 ^{b}		0.279 ^{b}	
μ_d (μ_N)		0.8527 ^{b}		0.8483 ^{b}	
A_S (fm$^{-1/2}$)		0.8865		0.8860	
D/S		0.0259		0.0263	
a_{np} (fm)		-23.75		-23.75	
r_{np} (fm)		2.69		2.70	
a_t (fm)		5.425		5.423	
$r_t = \rho(0,0)$ (fm)		1.762		1.758	

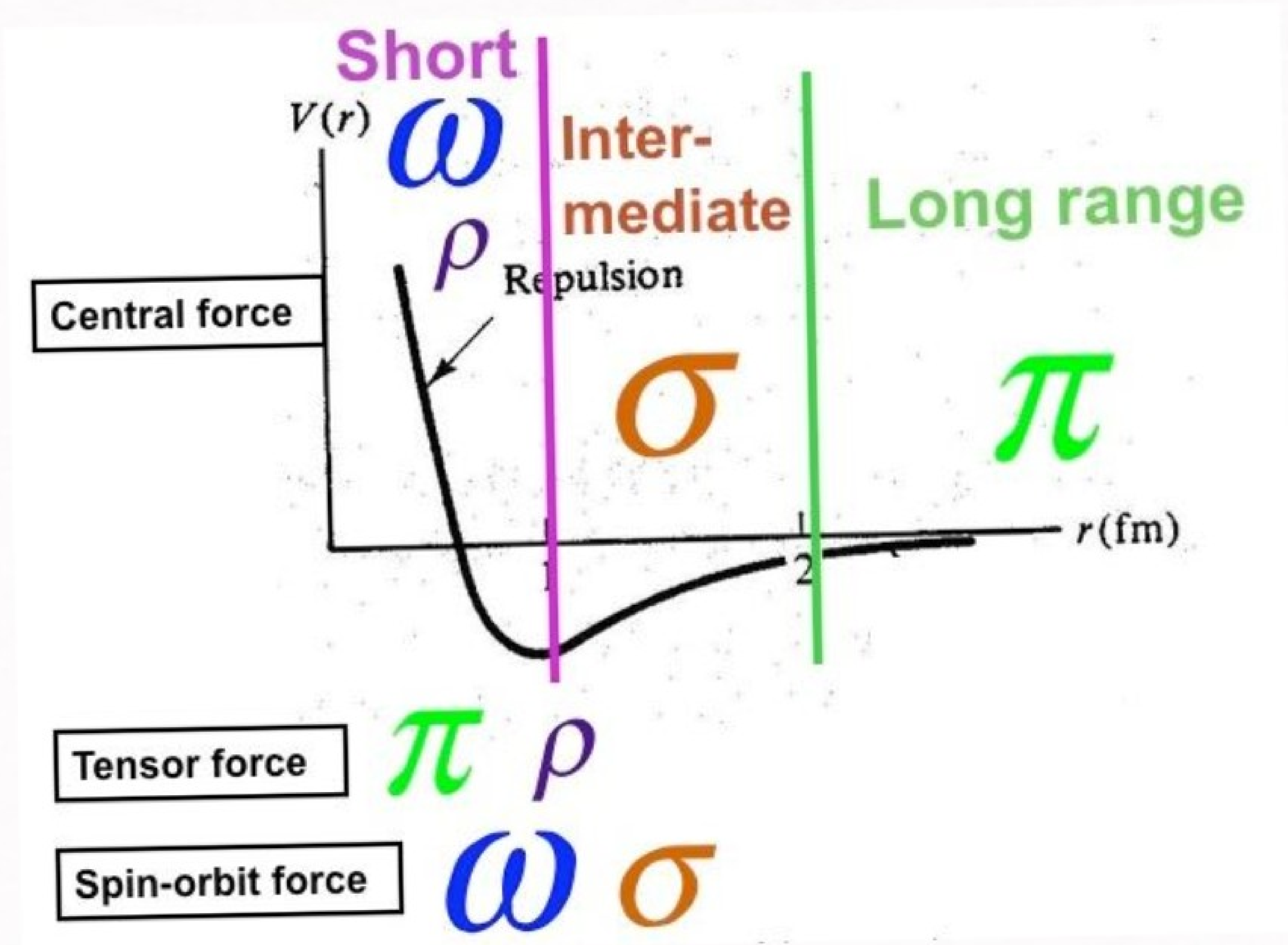
Given are the meson, deuteron, and low energy parameters.
For notation and other information see Table 4.1 and 4.2.
It is always used: $f_\rho/g_\rho = 6.1$, $f_\omega/g_\omega = 0.0$, and $n_\alpha = 1$.
^{a}The σ parameters given in brackets apply to the $T = 0$ NN potential. Potential A uses 710 and B uses 700 MeV for the σ mass.
^{b} Meson-exchange current contributions not included.

RECAP!

1) The OBE model $\Rightarrow$ Natural extension of Yukawa's idea



2) Each meson has a job:



3) We obtain singular potentials at $r \to 0$:

We include form factors to regularize the potential

See you the next day

tuesday / friday

15: 50