NUCLEAR PHYSICS 15

a) THE EFFECTIVE RANGE EXPANSION

→ Useful description of scattering at low energies

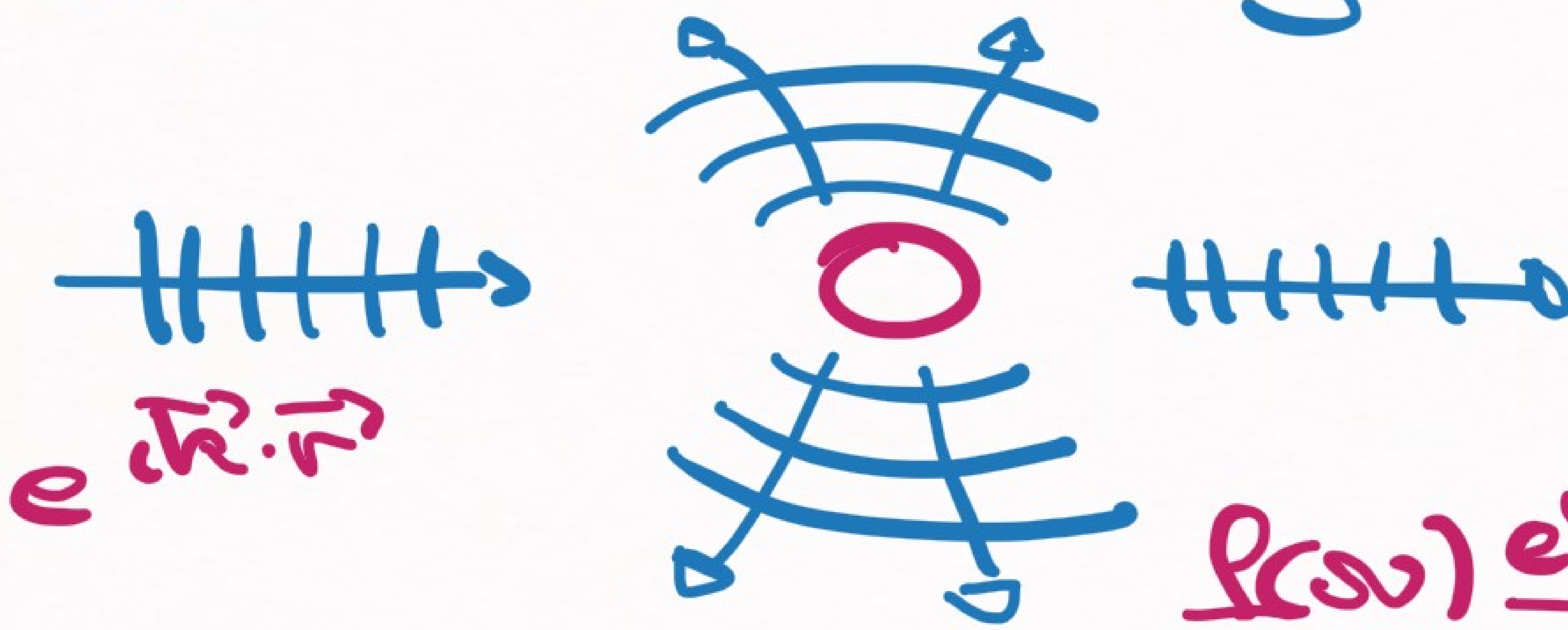
b) FORMAL SCATTERING THEORY

→ Very abstract way of understanding scattering

RECAP I [The cross section in Quantum mechanics]

a) Quantum scattering:

$$\psi_{\vec{k}}(\vec{r}) \xrightarrow[|\vec{r}| \to \infty]{} e^{i\vec{k}\cdot\vec{r}} + \underbrace{f(\Omega)}\frac{e^{ikr}}{r}$$



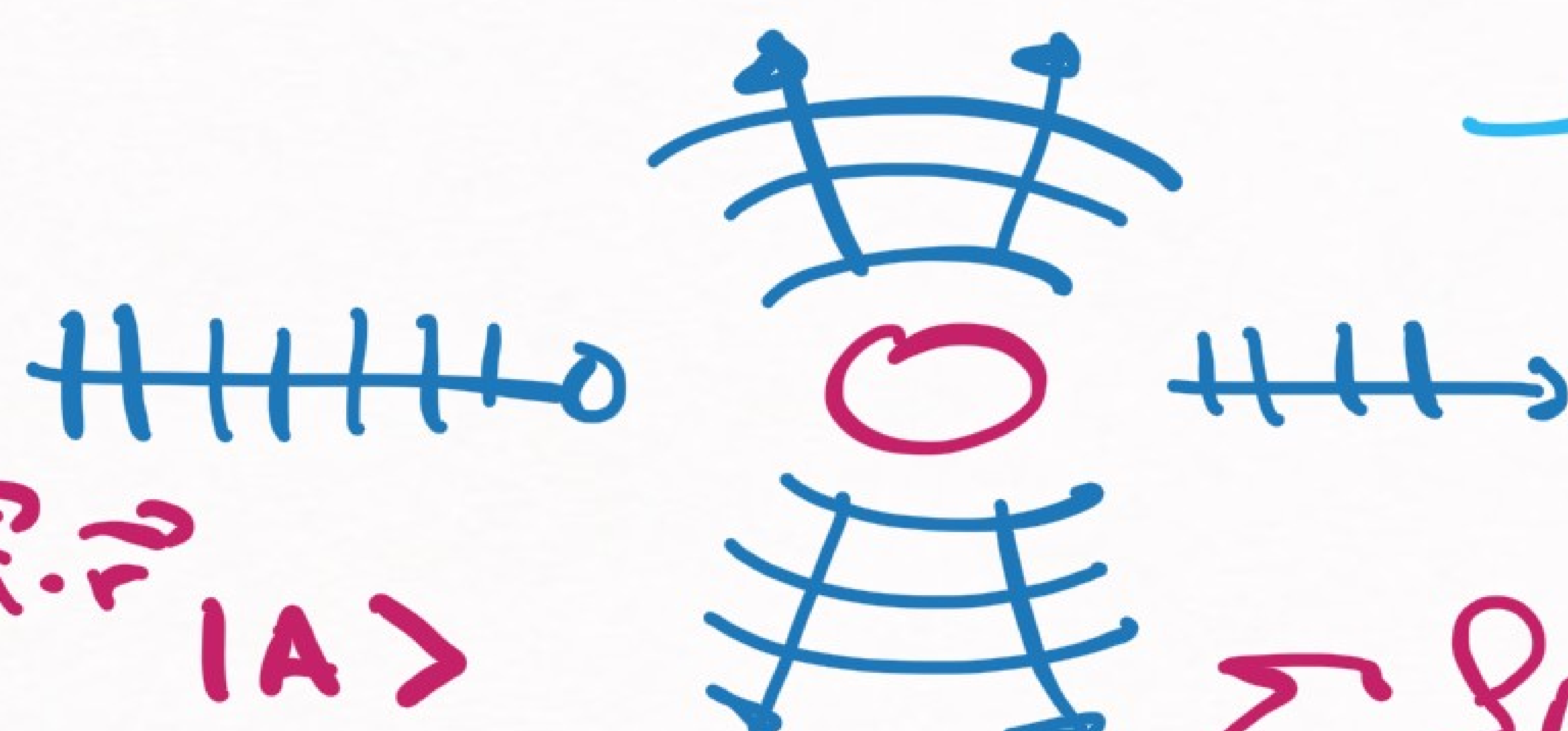
$e^{i\vec{k}\cdot\vec{r}}$

$f(\Omega)\frac{e^{ikr}}{r}$

$$\left[\frac{d\sigma}{d\Omega} = |f(\Omega)|^2\right]$$

b) Quantum scattering w/additional quantum numbers

$$\rightarrow \psi_{\vec{k}}(\vec{r})|A\rangle \xrightarrow[|\vec{r}| \to \infty]{} e^{i\vec{k}\cdot\vec{r}}|A\rangle + \sum_{AB} f_{AB}\frac{e^{ikr}}{r}|B\rangle$$



$e^{i\vec{k}\cdot\vec{r}}|A\rangle$

$\sum_B f_{AB}(\Omega)\frac{e^{ikr}}{r}|B\rangle$

$$\left[\frac{d\sigma}{d\Omega}(A\to B) = |f_{AB}(\Omega)|^2\right]$$

RECAP

[Case b) can be applied to spin!]

c) Quantum scattering w/ spin:

$|A\rangle = |s_1 m_1\rangle |s_2 m_2\rangle \qquad |B\rangle = |s_1 m_1'\rangle |s_2 m_2'\rangle$

c.1) Differential + polarized cross section

$$\frac{d\sigma}{d\Omega}(m_1 m_2 \to m_1' m_2') = |f_{m_1 m_2 m_1' m_2'}(\Omega)|^2$$

c.2) Differential + unpolarized cross section

$$\frac{d\sigma}{d\Omega} = \frac{1}{(2s_1+1)}\frac{1}{(2s_2+1)} \sum_{\substack{m_1 m_1' \\ m_2 m_2'}} |f_{m_1 m_2 m_1' m_2'}(\Omega)|^2$$

RECAP

d) Scattering w/ spin + $[H, \vec{S}^2] = 0$

$$f_{m_1 m_2 m_1' m_2'} \longrightarrow f^S_{m_s m_s'}$$

$$\left(|s_1 m_1\rangle |s_2 m_2\rangle = \sum |s\, m_s\rangle \langle s_1 m_1 s_2 m_2\rangle \right)$$

d.1) If we also have $[H, S_z] = 0 \Rightarrow f^S_{m_s m_s'} = f^S \delta_{m_s m_s'}$

$$\frac{d\sigma}{d\Omega} = \frac{1}{(2s_1+1)} \frac{1}{(2s_2+2)} \sum_S (2S+1) |f^S(\Omega)|^2$$

Example: neutron-proton scattering $(s_1 = 1/2,\ s_2 = 1/2)$

$$\frac{d\sigma}{d\Omega} = \frac{1}{4} |f_s(\Omega)|^2 + \frac{3}{4} |f_t(\Omega)|^2$$ [singlet $(s=0)$ + triplet $(s=1)$]

RECAP !

d) Scattering w/ spin + $[H, \vec{S}^2] = 0$

d.2) If we additionally have $[H, \vec{L}^2] = 0$

$$\Rightarrow \quad f^S(\Omega) = \sum_\ell (2\ell+1) f^S_\ell(\Omega)\, P_\ell(\cos\theta)$$

$+$

$$\sigma = \frac{1}{(2s_1+1)} \frac{1}{(2s_2+1)} \sum_S (2S+1) \frac{4\pi}{k^2} \sum_\ell (2\ell+1) \sin^2 \delta_\ell^S(k)$$

RECAP |

e) Scattering w/ spin + $[H, \vec{L}^2] \neq 0 \Rightarrow$ [Mixing of orbital angular momenta]

$\Rightarrow$ Usually is because of tensor forces:

$$V(\vec{r}) = V_C(\vec{r}) + V_S(\vec{r})\,\vec{\sigma}_1\cdot\vec{\sigma}_2 + V_T(\vec{r})\underbrace{\left(3\,\vec{\sigma}_1\cdot\hat{r}\,\vec{\sigma}_2\cdot\hat{r} - \vec{\sigma}_1\cdot\vec{\sigma}_2\right)}_{\text{This is the tensor operator (mixes } \ell, \ell')} + \dots$$

e.1) Without tensor forces:

$$\psi_{\vec{k}}(\vec{r}) = 4\pi \sum_{\ell m_\ell} Y^*_{\ell m}(\hat{k})\, \psi_{\ell m_\ell}(\vec{r})$$

$$\psi_{\ell m_\ell}(\vec{r}) = \frac{u_\ell(r)}{r} Y_{\ell m_\ell}(\hat{r}), \quad \frac{u_\ell(r)}{r} \xrightarrow[|\vec{r}|\to\infty]{} \frac{S_\ell(k) h_\ell^{(+)}(kr) - h_\ell^{(-)}(kr)}{2i}$$

RECAP

e) Scattering w/ spin + $[H, \vec{L}^2] \neq 0$

e.2) With tensor forces ← expansion in jm

$$\psi_{\vec{k}}(\vec{r}) = 4\pi \sum_{jm} \sum_{\ell=j-1}^{j+1} Z_{jm}^{\ell s m_s^*}(\hat{k})\, \psi_{jm}^{\ell}(\vec{r})$$

(before it was in ℓm)

$$= 4\pi \sum_{jm} \Big[Z_{jm}^{j-1\, s m_s^*}(\hat{k})\, \psi_{jm}^{j-1}(\vec{r}) + Z_{jm}^{j s m_s^*}(\hat{k})\, \psi_{jm}^{j}(\vec{r}) + Z_{jm}^{j+1\, s m_s^*}\, \psi_{jm}^{j+1}(\vec{r}) \Big]$$

$\boxed{\ell = j}$

$$\psi_{jm}^{j}(\vec{r}) = \frac{v(r)}{r}\, \mathcal{Y}_{jm}^{j}(\hat{r})$$

$$\frac{v(r)}{r} \to \frac{S_j(k)\, h_j^{(+)}(kr) - h_j^{(-)}(kr)}{2i}$$

Same as before

RECAP

e) Scattering w/ spin + $[H, \vec{L}^2] \neq 0$

e.2) With tensor forces (cont'd)

$\boxed{\ell = j \pm 1}$

$$f_{jm}^{\ell = j\pm 1}(\vec{r}) = \frac{u(r)}{r}\, y_{jm}^{\ell=j-1}(\hat{r}) + \frac{w(r)}{r}\, y_{jm}^{\ell=j+1}(\hat{r})$$

$$\frac{1}{r}\begin{pmatrix} u \\ w \end{pmatrix} \xrightarrow[r\to\infty]{} \frac{1}{2i}\left[\begin{pmatrix} e_{j-1}^{(+)} & 0 \\ 0 & e_{j+1}^{(+)} \end{pmatrix} S - \begin{pmatrix} e_{j-1}^{(-)} & 0 \\ 0 & e_{j-1}^{(-)} \end{pmatrix}\right]\begin{pmatrix} a_{j-1} \\ a_{j+1} \end{pmatrix}$$

$S \to$ [S-matrix]

(here it is a 2x2 matrix)

$$\Rightarrow \begin{bmatrix} \ell = j-1, & \begin{pmatrix} a_{j-1} \\ a_{j+1} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \\ \ell = j+1, & \begin{pmatrix} a_{j-1} \\ a_{j+1} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{bmatrix}$$

RECAP

§) Definition of the phase shifts $\Rightarrow$ not unique

S-matrix + tensor forces $\Rightarrow$ 2×2 matrix

§.1) Eigen phase shifts : (more intuitive)

$$S = \begin{pmatrix} \cos\epsilon & -\sin\epsilon \\ \sin\epsilon & \cos\epsilon \end{pmatrix} \begin{pmatrix} e^{2i\delta_1} & 0 \\ 0 & e^{2i\delta_2} \end{pmatrix} \begin{pmatrix} \cos\epsilon & \sin\epsilon \\ -\sin\epsilon & \cos\epsilon \end{pmatrix}$$

§.2) Nuclear bar phase shifts : (most used)

$$S = \begin{pmatrix} e^{i\bar{\delta}_1} & 0 \\ 0 & e^{i\bar{\delta}_2} \end{pmatrix} \begin{pmatrix} \cos 2\bar{\epsilon} & i\sin 2\bar{\epsilon} \\ i\sin 2\bar{\epsilon} & \cos 2\bar{\epsilon} \end{pmatrix} \begin{pmatrix} e^{i\bar{\delta}_1} & 0 \\ 0 & e^{i\bar{\delta}_2} \end{pmatrix}$$

RECAP!

g) From eigen to bar:

$$\delta_1 + \delta_2 = \bar{\delta}_1 + \bar{\delta}_2$$

$$\sin(\delta_1 - \delta_2) = \frac{\sin 2\bar{\epsilon}}{\sin 2\epsilon}$$

$$\sin(\bar{\delta}_1 - \bar{\delta}_2) = \frac{\tan 2\bar{\epsilon}}{\tan 2\epsilon}$$

(If you calculate NN phase shifts, you will end up using these formulas a lot)

[THE EFFECTIVE RANGE EXPANSION (ERE)] ①

⇒ Starting point: $\delta_0(k) \xrightarrow[k \to 0]{} \underbrace{-a_0 k}_{\text{scattering length}} + \underbrace{\mathcal{O}(k^3)}_{\text{we would like to know how these corrections look like}}$

$\sigma \xrightarrow[k \to 0]{} 4\pi |a_0|^2$

More convenient ⇒ $k \cot \delta_0(k) = -\frac{1}{a_0} + \mathcal{O}(k^2)$

$$f_0(k) = \frac{1}{k \cot \delta_0(k) - ik}$$

[THE DEVELOPMENT OF ERE] ②

⇒ The characters:

1) Schwinger → a really complicated derivation

2) Landau & Smorodinsky → they derive the ERE for proton-proton (which includes Coulomb)

↳ the complicated case

3) Bethe → usual derivation

His strong point: translating physical problems into ordinary differential equations (e.g. ERE, G-matrix)

$\boxed{\text{ERE}}$ ③

a) The actual expansion:

$$\left[k\cot\delta_0(k) = -a_0 + \frac{1}{2}r_0k^2 + \sum_{n=2}^{\infty} v_n k^{2n} \right]$$

b) Range of validity:

If we have $V(r) \to \frac{e^{-mr}}{r^n}$ + (shorter range components)

then $k < \frac{m}{2}$ → ERE will be convergent

[DERIVATION OF THE ERE] (4) (Derivation by Bethe)

$\Rightarrow$ We will use a new trick: the Wronskian identity

Eq.(1) : $-u_k'' + 2\mu V(r) u_k(r) = k^2 u_k(r)$

$$u_k(r) \xrightarrow[r\to\infty]{} \frac{\sin(kr+\delta)}{\sin\delta}$$

} scattering at momentum k

Eq.(2) : $-u_0'' + 2\mu V(r) u_0(r) = 0$

$$u_0(r) \xrightarrow[r\to\infty]{} 1 - \frac{r}{a_0}$$

} Scattering at zero momentum

[DERIVATION OF THE ERE] (5)

⇒ Building the Wronskian identity:

$$[\text{Eq.}(1)] \times u_0(r) - [\text{Eq.}(2)] \times u_k(r)$$

$$\Downarrow$$

(1)

$$-(u_k'' u_0 - u_k u_0'') = k^2 u_k(r)\, u_0(r)$$

⇒ Notice the exact derivative:

$$(u_k' u_0 - u_k u_0')' = \underbrace{(u_k'' u_0 - u_k u_0'')}_{(1)}$$

[DERIVATION OF THE ERE] 6

⇒ We integrate:

$$-\left(u_k'' u_0 - u_k u_0''\right) = k^2 u_0(r) u_k(r)$$

exact derivative ⇒

$$-\left(u_k' u_0 - u_k u_0'\right)' = k^2 u_0(r) u_k(r)$$

integrate $\left(\int_{r_c}^{R} dr\right)$ ⇒

$$-\underbrace{\left(u_k' u_0 - u_k u_0'\right)}_{\text{Wronskian}}\Big|_{r_c}^{R} = k^2 \int u_0(r) u_k(r)\, dr$$

Reminder ⇒ The Wronskian $W(f,g)(x) = f(x)\,g'(x) - f'(x)\,g(x)$

[DERIVATION OF THE ERE] 7

$\Rightarrow$ Now, we repeat the same trick w/ free wave functions:

Eq.(1) & (2) with $v(r)=0$

$$\begin{cases} \text{Eq.(3)}: & -v_k'' = k^2 v_k(r), \quad v_k(r) = \dfrac{\sin(kr+\delta)}{\sin\delta} \\ \text{Eq.(4)}: & -v_0'' = 0, \quad v_0(r) = 1 - \dfrac{r}{a_0} \end{cases}$$

$\Rightarrow$ Then, we calculate [Eq.(3)] $\times v_0$ − [Eq.(4)] $\times v_k$ and use the "exact derivative" trick:

$$-\left(v_k' v_0 - v_k v_0'\right)\Big|_{r_c}^{R} = k^2 \int_{r_c}^{R} v_k(r) v_0(r)\, dr$$

[DERIVATION OF THE ERE] ③

⇒ Next, we calculate the difference:

$$- \left[-(u_k' u_0 - u_k u_0')\Big|_{r_c}^{R} = k^2 \int_{r_c}^{R} u_k(r) u_0(r)\, dr \right]$$

$$+ \left[-(v_k' v_0 - v_k v_0')\Big|_{r_c}^{R} = k^2 \int_{r_c}^{R} v_k(r) v_0(r)\, dr \right]$$

$$(u_k' u_0 - u_0 u_k')\Big|_{r_c}^{R} - (v_k' v_0 - v_k v_0')\Big|_{r_c}^{R}$$

$$= k^2 \int_{r_c}^{R} [v_k(r) v_0(r) - u_k(r) u_0(r)]\, dr$$

[DERIVATION OF THE ERE] (8)

⇒ Finally, we take the $r_c \to 0$ & $R \to \infty$ limits:

Notice that: $u_k(0) = 0$, $u_0(0) = 0$

$$\lim_{R\to\infty} u_k(R) = v_k(R), \quad \lim_{R\to\infty} u_0(R) = v_0(R)$$

Putting all the pieces together: (try yourself)

$$k\cot\delta_0(k) = -\frac{1}{a_0} + k^2\int_0^\infty [v_k v_0 - u_k u_0]\,dr$$

what does this imply?

[DERIVATION OF THE ERE] 9

⇒ We notice that the wave functions have a good expansion in powers of k^2:

$$u_k = u_0 + k^2 u_2 + k^4 u_4 + \dots$$
$$v_k = v_0 + k^2 v_2 + k^4 v_4 + \dots$$

if this is true, it is also true for $\int_0^\infty [v_k v_0 - u_k u_0]\,dr$

And we end up with:

$$k \cot \delta_0(k) = -\frac{1}{a_0} + \frac{1}{2} r_0 k^2 + \sum_{n=2}^{\infty} v_n k^{2n}$$

$$r_0 = 2\int_0^\infty [v_0^2 - u_0^2]\,dr$$

$$v_n = \int_0^\infty [v_{2n}(r) v_0(r) - u_{2n}(r) u_0(r)]\,dr$$

[INTERPRETATION OF THE ERE]

⇒ How do I interpret the ERE parameters?

$$k\cot\delta = -\frac{1}{\underline{\underline{a_0}}} + \frac{1}{2}\underline{\underline{r_0}}k^2 + \underline{\underline{v_2}}k^4 + \dots$$

a) Scattering length: $\sigma \xrightarrow[k\to 0]{} 4\pi|a_0|^2$

like the length below which the particles see each other at low energies

b) Effective range: $r_0 = 2\int_0^\infty [v_0^2(r) - u_0^2(r)]dr$

$$= 2\int_0^\infty \left[\left(1-\frac{r}{a_0}\right)^2 - u_0^2(r)\right]dr$$

Interpretation needed

(behaviour of this integral)

[THE EFFECTIVE RANGE] (1)

⇒ We have that: $r_0 = 2\int_0^\infty \left[\left(1-\frac{r}{a_0}\right)^2 - u_0^2(r)\right] dr$

$u_0(r) \simeq v_0(r) = (1 - r/a_0)$

∞ when $v(r) \simeq 0$

$\boxed{r = R}$

(1) → this contributes to r_0

range of the interaction

$$2\int_0^\infty \left[\left(1-\frac{r}{a}\right)^2 - u_0^2(r)\right] dr \simeq 2\int_0^R \left[\left(1-\frac{r}{a}\right)^2 - u_0^2(r)\right] dr \lesssim 2\int_0^R dr = 2R$$

$$\boxed{r_0 \lesssim 2R}$$

Related to range of interaction

[THE EFFECTIVE RANGE] 2

$\Rightarrow$ That is, the integral defining the effective range is only non-zero within the range of the potential

$$2\int_0^\infty [v_0^2(r) - u_0^2(r)]\,dr \simeq 2\int_0^R [v_0^2(r) - u_0^2(r)]\,dr \lesssim 2R$$

$\Rightarrow$ natural size of r_0 is about $2R$

$$r_0 \sim \mathcal{O}(1)\, 2R \quad \text{or} \quad r_0 \sim \mathcal{O}(1)\, R$$

For pion exchanges $R \sim \frac{1}{m_\pi} \sim 1.4\,\text{fm}$

[THE EFFECTIVE RANGE] 3

$\Rightarrow$ For pion exchanges $R_\pi \sim \frac{1}{m_\pi} \sim 1.4\,\text{fm}$

$\Rightarrow \boxed{r_0 \sim (1.4 - 2.8)\,\text{fm}}$

$\boxed{{}^1S_0}$ singlet

$a_0 \simeq -23.7\,\text{fm}$

$r_0 \simeq 2.7\,\text{fm}$ ✓

$v_2 \simeq -0.5\,\text{fm}$

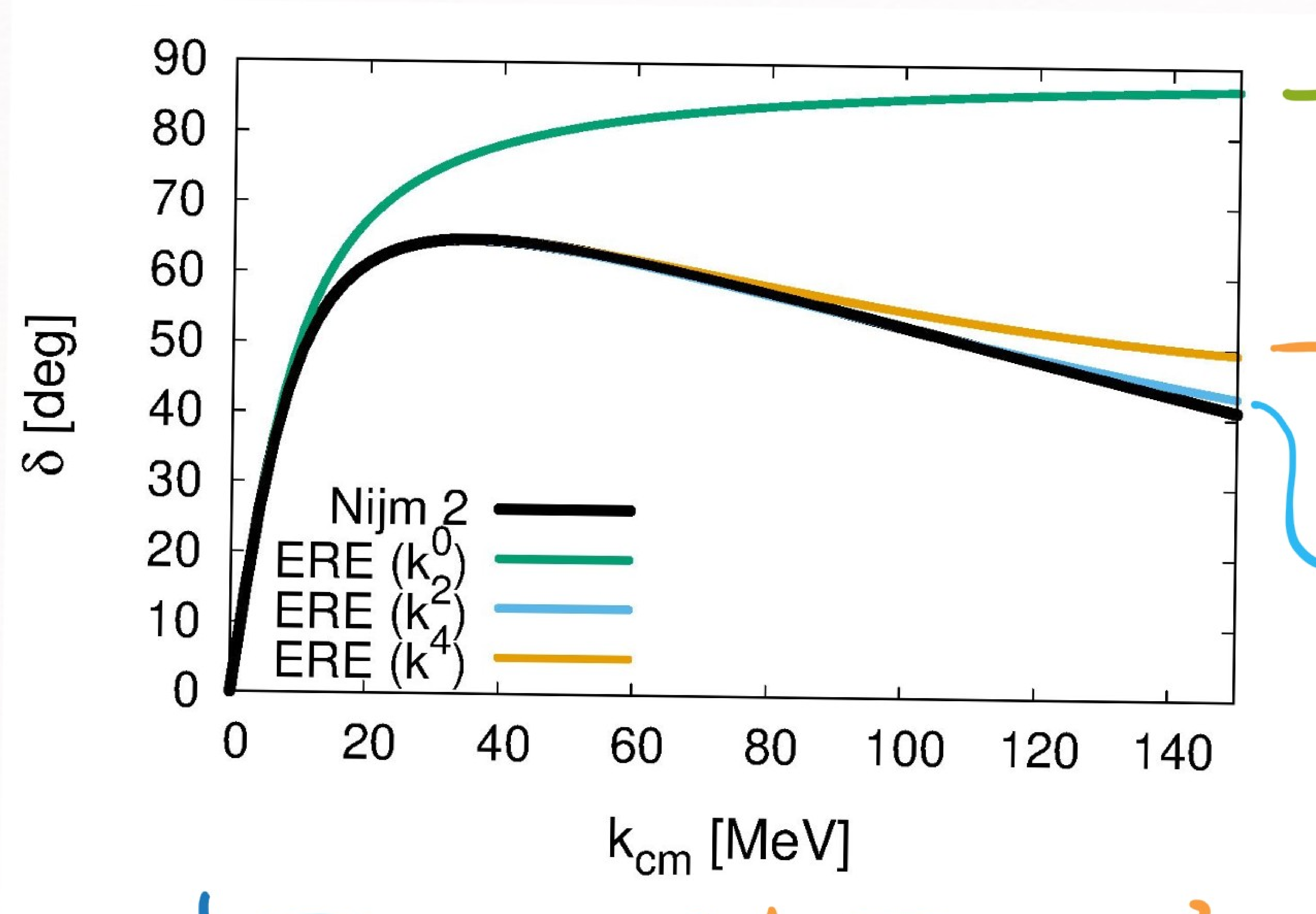
$\boxed{{}^3S_1}$ triplet

$a_0 \simeq 5.4\,\text{fm}$

$r_0 \simeq 1.8\,\text{fm}$ ✓

$v_2 \simeq 0.05\,\text{fm}$

[ERE & PHASE SHIFTS] ⇒ $^1S_0$ / singlet



→ $k\cot\delta = -\frac{1}{a_0}$

→ $k\cot\delta = -\frac{1}{a_0} + \frac{1}{2}r_0k^2 + v_2k^4$

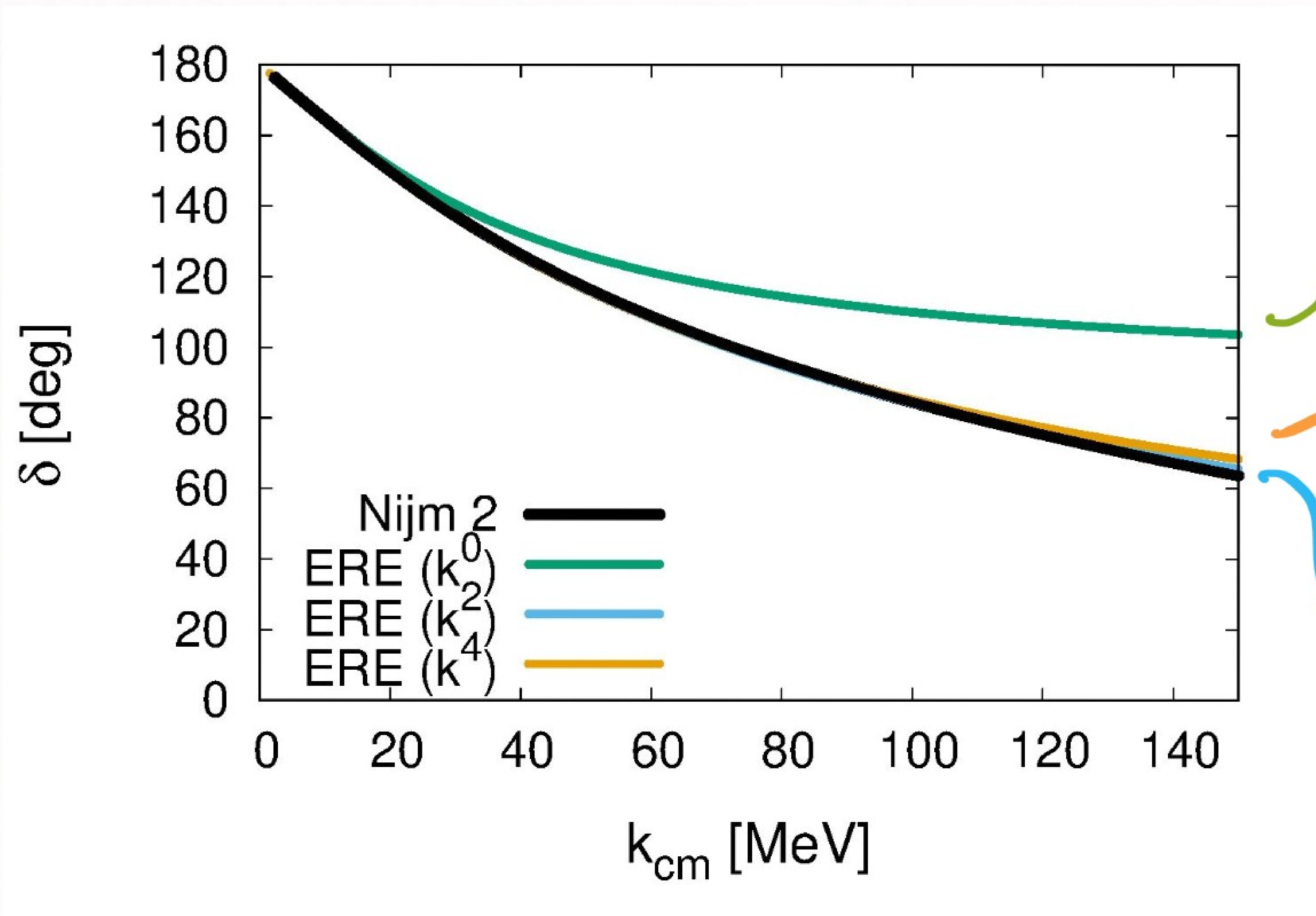
→ $k\cot\delta = -\frac{1}{a_0} + \frac{1}{2}r_0k^2$

here is the convergence region

this $k\cot\delta$ does worse for $k > 70$ MeV

(ERE only converges for $k < \frac{m_\pi}{2}$)

[ERE & PHASE SHIFTS] ⟹ $\boxed{{}^3S_1 \text{ / triplet}}$



⟹ $k \cot\delta = -\frac{1}{a_0}$

⟹ $k \cot\delta = -\frac{1}{a_0} + \frac{1}{2} r_0 k^2 + v_2 k^4$

⟹ $k \cot\delta = -\frac{1}{a_0} + \frac{1}{2} r_0 k^2$

⟹ Difficult to appreciate, but same comments as before apply for the convergence

RECAP!

a) The S-wave phase shifts admit the expansion:

$\rightarrow$ $k\cot\delta_0(k) = -\frac{1}{a_0} + \frac{1}{2}r_0k^2 + v_2k^4 + v_3k^6 + \dots$

which converges for $k < \frac{m}{2}$ $\left(V(r) \sim \frac{e^{-mr}}{r^\alpha} \right)$

b) a_0 can take any value ($a_0 \to \infty$ for a bound state w/ $B \to 0$)

c) $r_0 \sim \frac{\mathcal{O}(1)}{m}$ ($r_0 \sim (1.4-2.8)$ fm for nucleons)

$\Rightarrow$ Information about the range $\leftarrow$

$\rightarrow$ d) Good low energy description of phase shifts

[FORMAL SCATTERING THEORY]
(THE T-MATRIX)

⇒ Formal scattering view is about abstraction

Two views of quantum mechanics:

1) Wave functions obeying differential equations

$$\psi(\vec{r}),\quad \left[-\frac{\vec{\nabla}^2}{2\mu}+V(r)\right]\psi(\vec{r})=E\psi(\vec{r})$$

2) Linear operators acting on vectors on a Hilbert space

$$|\psi\rangle,\quad [H_0+V]|\psi\rangle=E|\psi\rangle$$ (eigenvalue equation)

$\Rightarrow$ What does this imply for scattering theory?

View 1) $\psi(\vec{r}) \xrightarrow[|\vec{r}| \to \infty]{} e^{i\vec{k}\cdot\vec{r}} + f(\Omega)\,\dfrac{e^{ikr}}{r}$

$$\Rightarrow \frac{d\sigma}{d\Omega} = |f(\Omega)|^2$$

View 2) $|\psi\rangle = |\vec{k}\rangle + T(E)\,G_0(E)\,|\vec{k}\rangle$

$\langle \vec{k}' | G_0 | \vec{k}\rangle = -\dfrac{\mu}{2\pi}\,\dfrac{e^{ikr}}{kr}$

$f(\Omega) = -\dfrac{\mu}{2\pi}\langle \vec{k}' | T(E) | \vec{k}\rangle$

(now a matrix element)

[$T(E)$, $G_0(E)$ are operators]

$\Rightarrow$ Why we would ever do something so complicated?

Just imagine that you have this kind of potential:

Non-local potential $\Rightarrow V^{NL} = \underline{\{\vec{\nabla}^2, \delta(r)\}} \Rightarrow \left[\begin{array}{l} \langle \vec{k}' | V^{NL} | \vec{k} \rangle \\ = \tilde{\delta}(\vec{k}' - \vec{k})(\vec{k}^2 + \vec{k}'^2) \end{array} \right]$

p-space

$$\langle \psi | V^{NL} | \psi \rangle = \int d^3\vec{r}\, \psi^*(\vec{r})\, \delta(r) \vec{\nabla}^2 \psi(\vec{r}) + \int d^3\vec{r}\, \psi^*(\vec{r}) \vec{\nabla}^2 (\delta \psi)$$

$$= \underline{\int d^3\vec{r}} \left[\psi^*(\vec{r})\, \delta(r) \vec{\nabla}^2 \psi(\vec{r}) - \vec{\nabla}^2 \psi^*(r)\, \delta(r)\, \psi(\vec{r}) \right]$$

Good luck solving them w/ Schrödinger

REDERIVING SCATTERING

⇒ We will try to derive scattering in a formal way

a) $H|\phi\rangle = E|\phi\rangle$

$|\phi\rangle$ → Wave function

H → Hamiltonian

} Schrödinger equation

+

b) $H = H_0 + V$

V → potential

H_0 → kinetic term $\quad H_0|\vec{p}\rangle = \frac{\vec{p}^{\,2}}{2\mu}|\vec{p}\rangle$

a+b) $(E - H_0)|\phi\rangle = V|\phi\rangle$

GREEN FUNCTIONS (3)

=D We want to solve $(E - H_0)|\phi\rangle = V|\phi\rangle$

[The Green function method]

=D first, we are going back to the language of wave functions & differential equations

a) $|\phi\rangle \rightarrow \phi(\vec{r}) = \langle \vec{r}|\phi\rangle$

b) Try the $\underbrace{\text{ansatz}}_{\text{proposal of a solution}}$: $\phi(\vec{r}) = e^{i\vec{k}\cdot\vec{r}} + \int d^3\vec{r}\,' \, \overbrace{G_0(\vec{r}-\vec{r}\,')}^{\text{[Green function]}} V(\vec{r}\,')\phi(\vec{r}\,')$

GREEN FUNCTIONS (2)

Q) When is $\phi(\vec{r}) = e^{i\vec{k}\cdot\vec{r}} + \int d^3\vec{r}\,' G_0(\vec{r}-\vec{r}\,')V(\vec{r}\,')\phi(\vec{r}\,')$

a solution of the Schrödinger equation?

A) You plug it into Schrödinger, and then you will find:

$$\left[(E-H_0)G_0(\vec{r}-\vec{r}\,') = \delta^{(3)}(\vec{r}-\vec{r}\,') \right]$$

Remember: $(E-H_0)|\phi\rangle = V|\phi\rangle$

GREEN FUNCTIONS ③

→ differential eq. w/ Dirac-deltas!

Solving $(E-H_0)\,G_0(\vec{r}-\vec{r}\,') = \delta^{(3)}(\vec{r}-\vec{r}\,')$ does not look trivial

$\Rightarrow$ Fourier-transform into p-space

$$G_0(\vec{q}\,) = \int \frac{d^3\vec{q}}{(2\pi)^3}\, G_0(\vec{r}\,)\, e^{-i\vec{q}\cdot\vec{r}}$$

$\Downarrow$

$$\left.\begin{aligned} &(E-H_0)\,G_0(\vec{q}\,) = 1 \\ &H_0 = \frac{\vec{q}^{\,2}}{2\mu} \end{aligned}\right\} \Rightarrow \quad G_0(q) = \frac{1}{E-\frac{\vec{q}^{\,2}}{2\mu}} \quad \text{or}$$

$$G_0(E) = \frac{1}{E-H_0}$$

(in operator language)

GREEN FUNCTIONS ④ Fourier-transform back into r-space

We have $G_0(\vec{q}) = \dfrac{1}{E - q^2/2\mu}$, but the orignal objective

still is: $\underline{\underline{G_0(\vec{r})}} = \displaystyle\int \frac{d^3\vec{q}}{(2\pi)^3} G_0(\vec{q}) e^{i\vec{q}\cdot\vec{r}}$

$$= \frac{1}{2\pi r} \int_0^\infty dq \, \frac{q \sin(qr)}{E - q^2/2\mu}$$

$$= \frac{1}{4\pi r} \mathrm{Im}\left[\int_{-\infty}^{+\infty} dq \, \frac{q e^{iqr}}{E - q^2/2\mu} \right] \Rightarrow$$ this can be solved by residues (complex analysis)

GREEN FUNCTIONS (5)

We have $G_0(\vec{r}) = \frac{1}{4\pi r} \,\mathrm{Im}\left[\int_{-\infty}^{\infty} dq \, \frac{q\, e^{iqr}}{E - q^2/2\mu} \right]$ and

we want to solve the $[\ldots]$ by residues:

a) Poles: $\int_{-\infty}^{\infty} dq \, \frac{q\, e^{iqr}}{E - q^2/2\mu} \Rightarrow q = \pm\sqrt{2\mu E} = \pm k$

$(E = k^2/2\mu)$

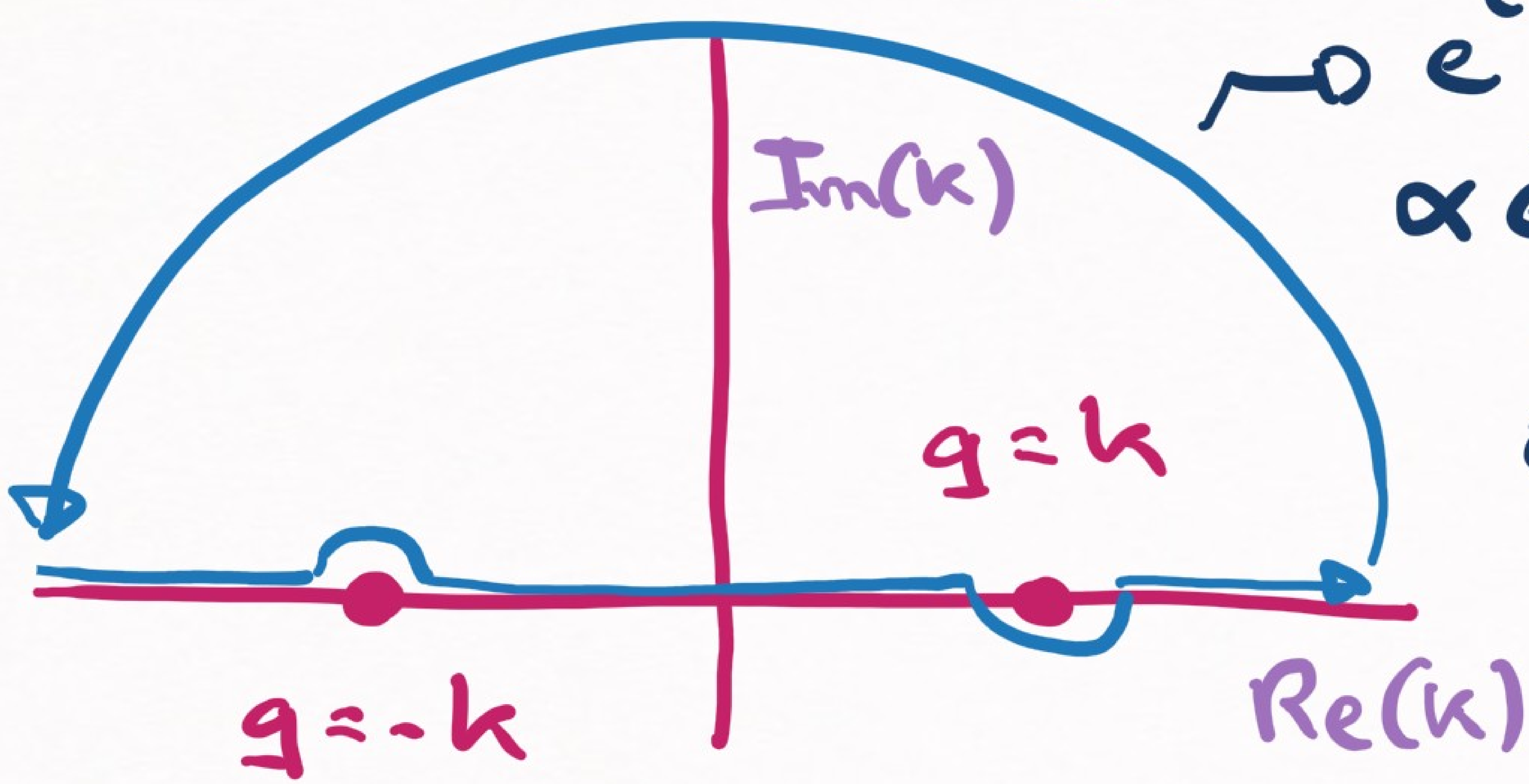
b) Contour integration: $\int_{-\infty}^{\infty} dq \, (\ldots) = \oint dq \, (\ldots)$

GREEN FUNCTIONS (6)

We have : $\oint dq \, \dfrac{q e^{iqr}}{E - q^2/2\mu}$ with poles: $q = \pm\sqrt{2\mu E} = \pm k$

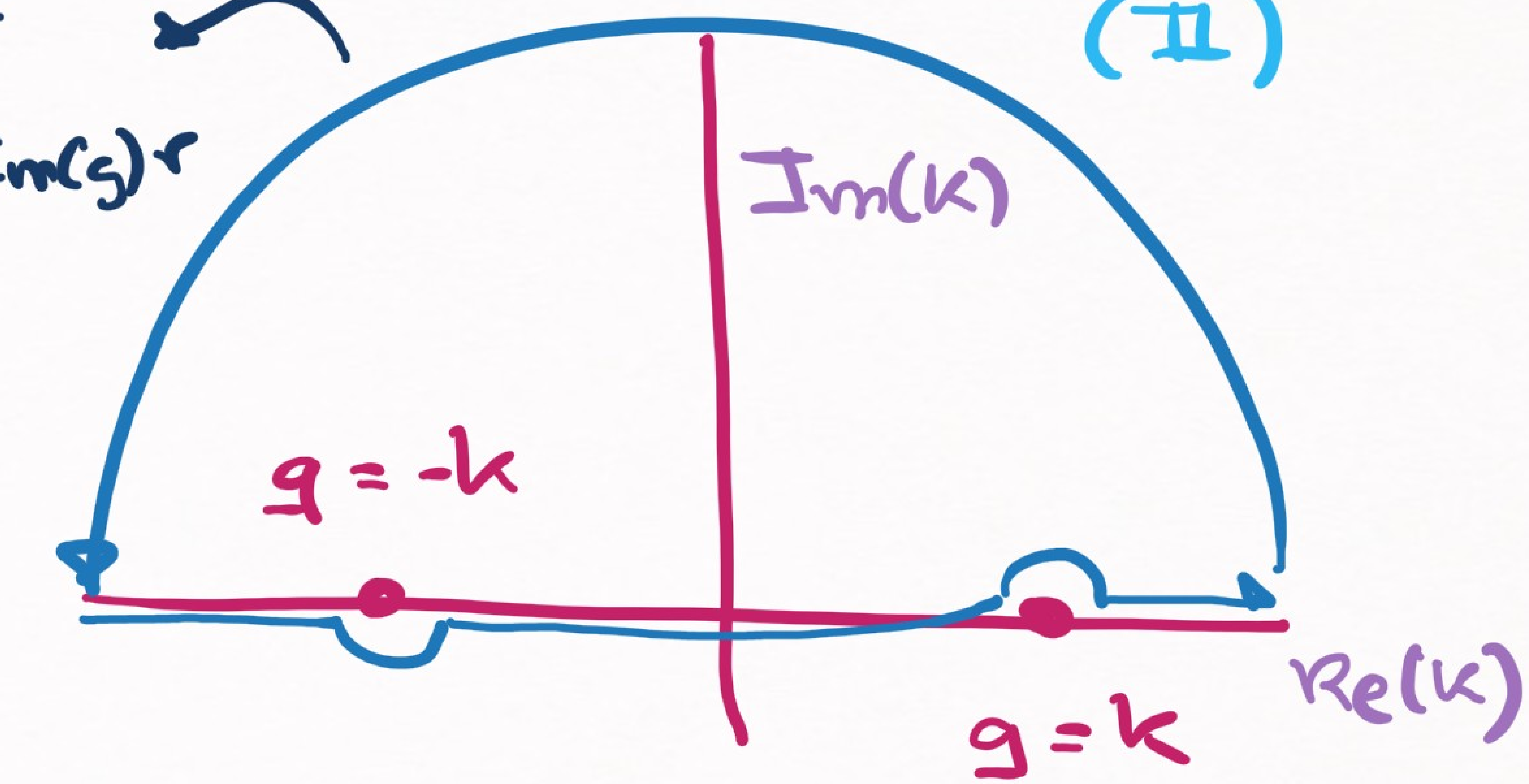
Two choices :

(I)



$\rightarrow e^{iqr} \propto e^{-\mathrm{Im}(q) r} \rightarrow 0$

(II)



GREEN FUNCTIONS ⑦

If we do the calculations, we find:

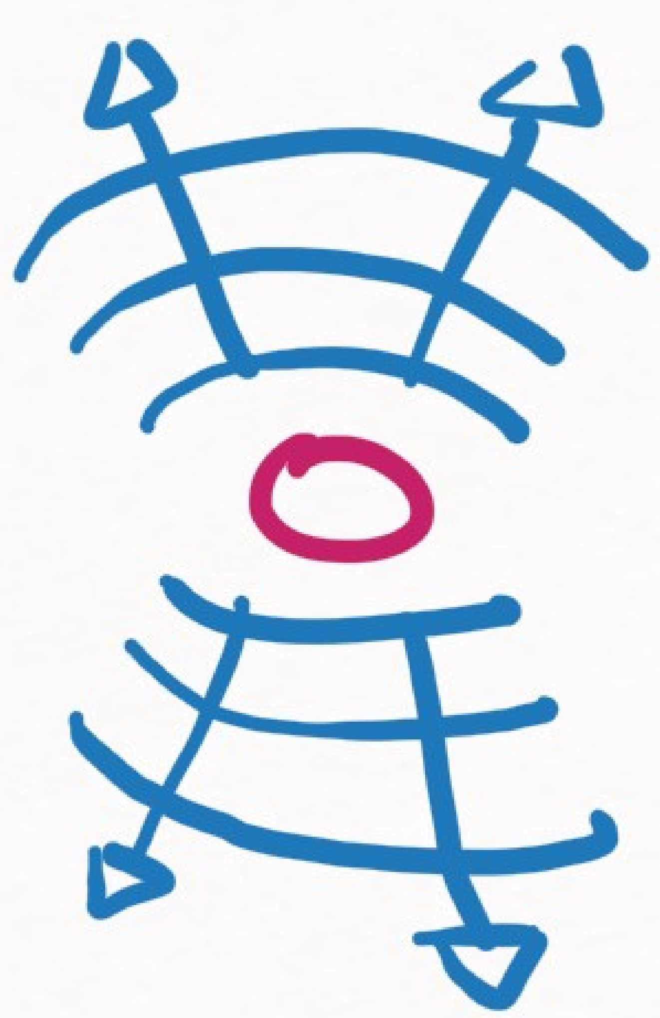
Contour I) $G_0^{(I)}(\vec{r}) = -\frac{\mu}{2\pi} \frac{e^{+ikr}}{r}$

Contour II) $G_0^{(II)}(\vec{r}) = -\frac{\mu}{2\pi} \frac{e^{-ikr}}{r}$

which of them corresponds to physical scattering

I)

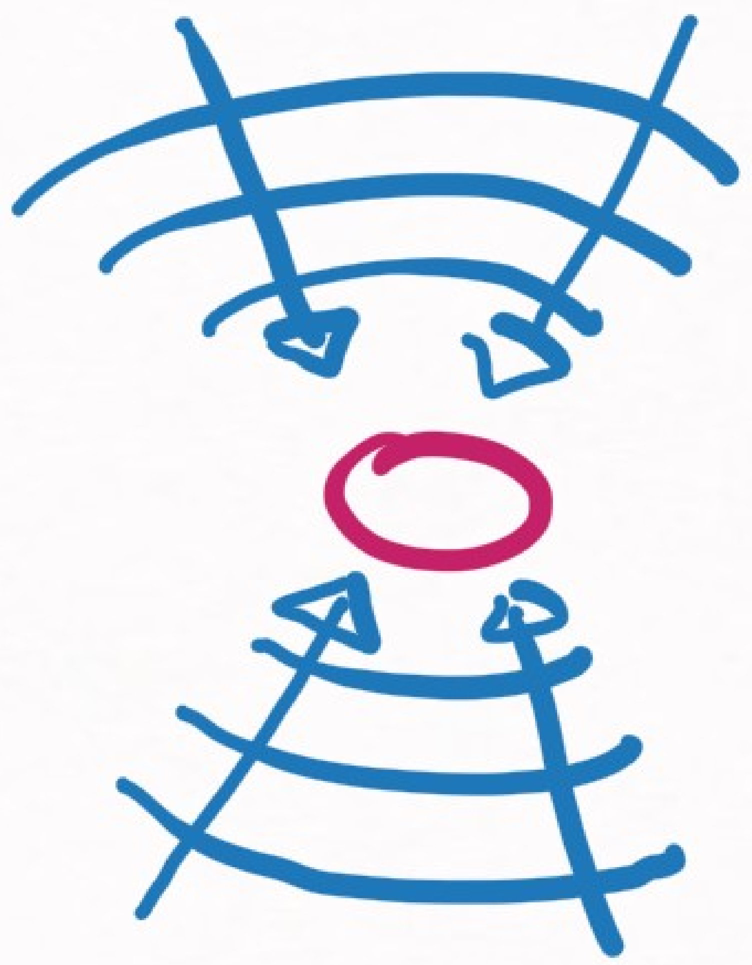
=



Sol:

outgoing spherical wave

II)



Sol:

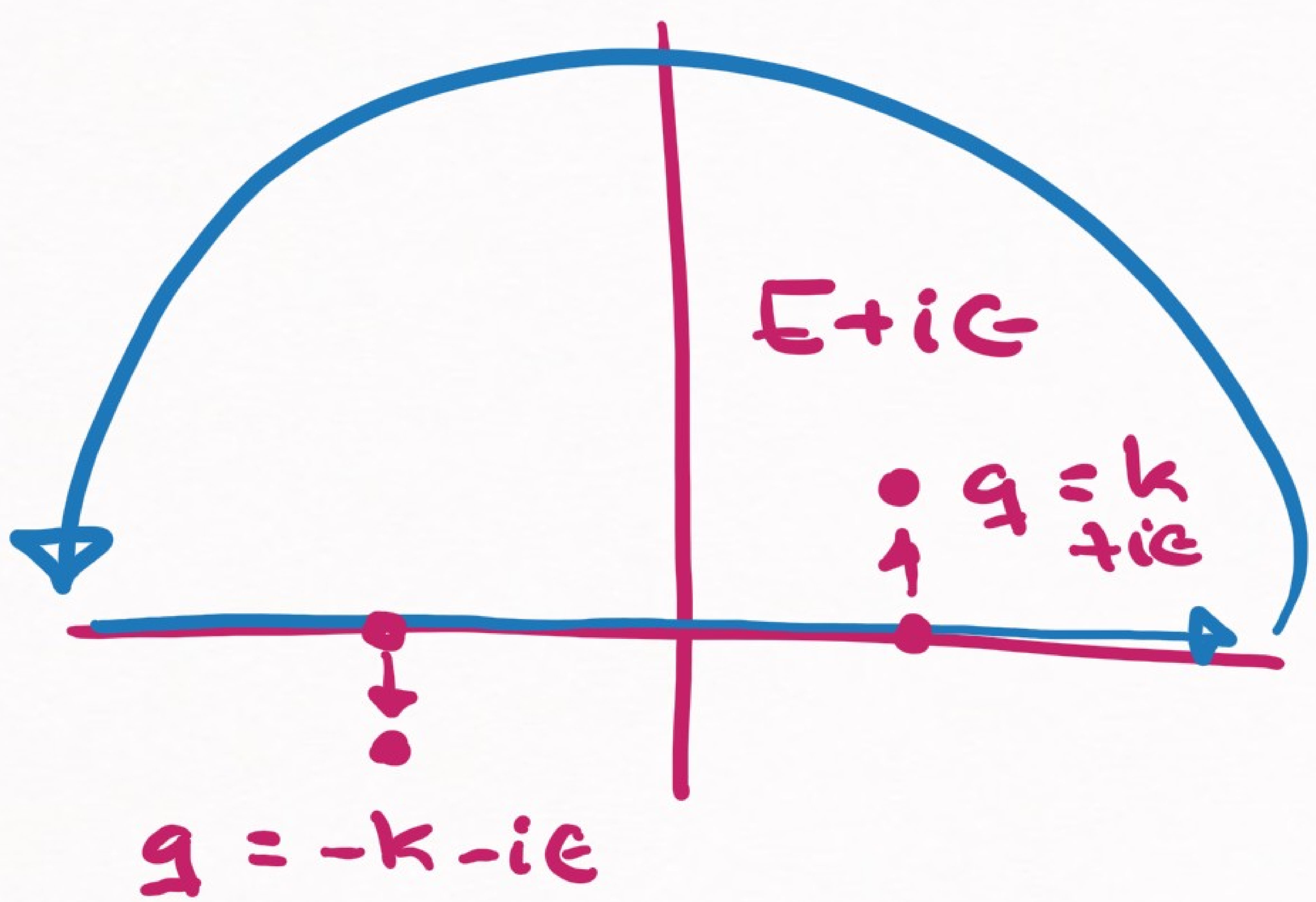
incoming spherical wave

GREEN FUNCTIONS (8)

Alternatively, we can modify the propagator $G_0(E)$:

$\Rightarrow \quad G_0(E) \longrightarrow G_0(E \pm i\epsilon) \qquad$ ($i\epsilon$ trick)

$\epsilon \to 0^+$



$q = \pm k \longrightarrow q = \pm(k \pm i\epsilon)$
(poles)

$$\Rightarrow \begin{cases} E+i\epsilon \longrightarrow \text{Contour (I)} \\ E-i\epsilon \longrightarrow \text{Contour (II)} \end{cases}$$

GREEN FUNCTIONS ⑨ (iε trick)

⇒ Our result : $G_0(\vec{r}; \underbrace{E \pm i\epsilon}) = -\frac{\mu}{2\pi}\frac{e^{\pm ikr}}{r}$

this is the usual procedure

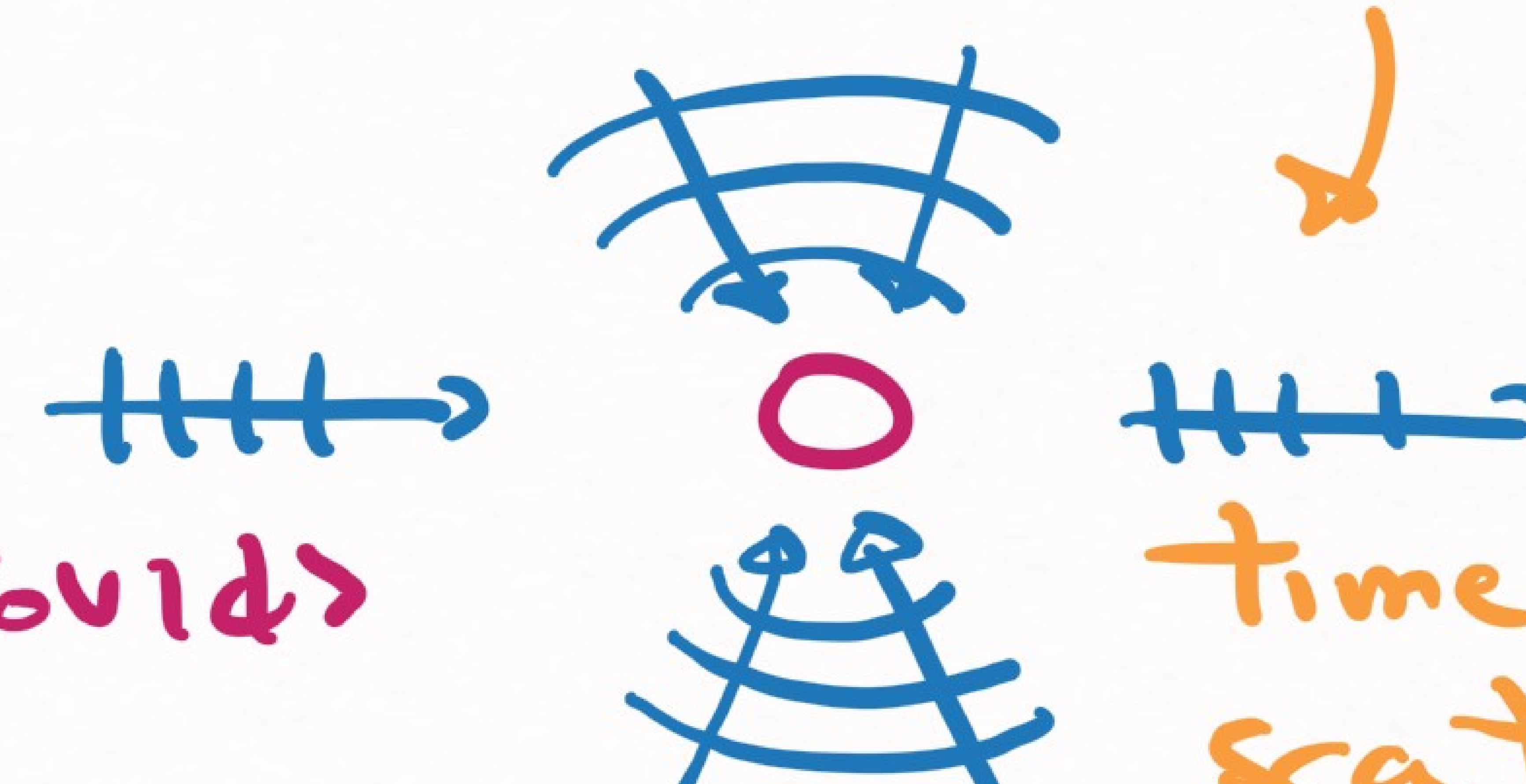
we will chose $+i\epsilon$

⇒ And now we go back to $\underline{\underline{|\phi\rangle}} \longrightarrow \underline{\underline{|\phi^{\pm}\rangle}}$

a) $|\phi^+\rangle$ corresponds to: b) $|\phi^-\rangle$ corresponds to:

usual scattering

$|\phi\rangle = |\vec{k}\rangle + G_0 V|\phi\rangle$



time-reversed scattering

[THE SCATTERING AMPLITUDE]

RECAP

1) Schrödinger's equation: $(E-H_0)|\phi\rangle = V|\phi\rangle$

2) Ansatz for $|\phi\rangle$: $|\phi\rangle = |\vec{k}\rangle + G_0 V|\phi\rangle$

$$\Rightarrow \phi(\vec{r}) = e^{i\vec{k}\cdot\vec{r}} + \int d^3\vec{r}\,' G_0(\vec{r}-\vec{r}\,')V(\vec{r}\,')\phi(\vec{r}\,')$$

3) Solve $G_0(\vec{r}; E \pm i\epsilon) = -\frac{\mu}{2\pi}\frac{e^{\pm ikr}}{r}$

or $G_0(E) = \frac{1}{E-H_0}$

$+i\epsilon$ is the physical solution

NEXT → 4) Check if $\underline{\phi^+(\vec{r})} \to e^{i\vec{k}\cdot\vec{r}} + f(\Omega)\frac{e^{ikr}}{r}$

(scattering solution)

[THE SCATTERING AMPLITUDE] (2)

⇒ Let's see what happens:

$$\left[\ \phi^+(\vec{r}) = e^{i\vec{k}\cdot\vec{r}} - \frac{\mu}{2\pi}\int d^3\vec{r}\,'\ \frac{e^{ik|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|}\,V(\vec{r}\,')\,\phi^+(\vec{r}\,')\ \right]$$

a) $\lim_{r\to\infty}$: $|\vec{r}-\vec{r}\,'| \to r(1-\hat{r}\cdot\hat{r}\,')$

$k|\vec{r}-\vec{r}\,'|$
$\to kr - \vec{k}\cdot\vec{r}\,'$
(play w/ angles)

$$\phi^+(\vec{r}) \to e^{i\vec{k}\cdot\vec{r}} - \frac{\mu}{2\pi}\frac{e^{ikr}}{r}\int d^3\vec{r}\,'\, e^{-i\vec{k}\cdot\vec{r}\,'}\,V(\vec{r}\,')\,\phi^+(\vec{r}\,')$$

$$\to e^{i\vec{k}\cdot\vec{r}} + f(\Omega)\frac{e^{ikr}}{r}$$

→ you have $f(\Omega)$!

[THE SCATTERING AMPLITUDE] 3

b) If we now match the $r \to \infty$ limits:

$$f(\Omega) = -\frac{\mu}{2\pi}\int d^3\vec{r}\,' e^{-i\vec{k}\cdot\vec{r}\,'} V(\vec{r}\,')\, \phi^+(\vec{r}\,')$$

$$= -\frac{\mu}{2\pi}\langle \vec{k}' | V | \phi^+ \rangle$$ (now a matrix element)

$\Rightarrow$ Problem: this still depends on $\phi^+(\vec{r})$ (the full wave function)

usally, we don't know $\phi^+(\vec{r})$

[THE SCATTERING AMPLITUDE] 4

c) We have arrived at $f(\Omega) = -\frac{\mu}{2\pi}\langle \vec{k}'|V|\psi^+\rangle$,

but we would like an expression w/o ψ^+

Let's see: $|\psi^+\rangle = |\vec{k}\rangle + G_0(E+i\epsilon)V|\psi^+\rangle$

$= |\vec{k}\rangle + G_0(E+i\epsilon)V|\vec{k}\rangle + G_0VG_0V|\psi^+\rangle$

$= (1 + G_0V + G_0VG_0V + \ldots\ldots)|\vec{k}\rangle \Rightarrow$ iterative equation

I can remove ψ^+

THE T-MATRIX (1)

a) Iterative definition of $|\phi^+\rangle$, we arrive at

$$|\phi^+\rangle = (1 + \underbrace{G_0 V + G_0 V G_0 V + \dots}_{G_0 \underbrace{(V + V G_0 V + V G_0 V G_0 V + \dots)}_{[T] \text{ (the T-matrix)}}})|\vec{k}\rangle$$

$$|\phi^+\rangle = |\vec{k}\rangle + G_0 T |\vec{k}\rangle \leftarrow [\,T\,] \text{ (the T-matrix)}$$

$\Rightarrow$ Alternatively: $T = V + V G_0 V + V G_0 V G_0 V + \dots$

$$= V + V G_0 (V + V G_0 V + \dots)$$

$$= V + V G_0 T$$

$$\boxed{T = V + V G_0 T}$$

THE T-MATRIX ②

b) We could have also defined it as:

$$\langle \vec{k}' | V | \phi^{+}_{\vec{k}} \rangle = \langle \vec{k}' | T(E+i\epsilon) | \vec{k} \rangle$$

That is, $V | \phi^{+}_{\vec{k}} \rangle = T(E+i\epsilon) | \vec{k} \rangle$

c) And now the scattering amplitude reads:

$$f(\Omega) = -\frac{\mu}{2\pi} \langle \vec{k}' | T(E+i\epsilon) | \vec{k} \rangle$$

with $\quad T(E) = V + V G_0(E) T(E)$

[THE LIPPMANN-SCHWINGER EQUATION]

a) This is the equation that defines the T-matrix

$$T = V + V G_0 T \longrightarrow \text{Equivalent to Schrödinger}$$

(though not obvious)

b) We can sandwhich it in between $\langle \vec{k}'|$ and $|\vec{k}\rangle$:

$$\langle \vec{k}'|T|\vec{k}\rangle = \langle \vec{k}'|V|\vec{k}\rangle + \langle \vec{k}'|V G_0 T|\vec{k}\rangle = \bullet$$

$$\bullet = \left\{ \underbrace{\mathbb{1}}_{\text{identity operator}} = \int \frac{d^3\vec{q}}{(2\pi)^3} |\vec{q}\rangle\langle\vec{q}| \right\} = \langle \vec{k}'|V|\vec{k}\rangle + \int \frac{d^3\vec{q}}{(2\pi)^3} \frac{\langle \vec{k}'|V|\vec{q}\rangle\langle\vec{q}|T|\vec{k}\rangle}{E - \frac{\vec{q}^{\,2}}{2\mu}}$$

[THE LIPPMANN-SCHWINGER EQUATION] ②

Technical notes: a) $G_0(E)|\vec{q}\rangle = \dfrac{1}{E - q^2/2\mu}|\vec{q}\rangle$

b) $\langle \vec{k}' | \vec{k} \rangle = (2\pi)^3 \delta^{(3)}(\vec{k}' - \vec{k})$

Also implies the identity:

$$\mathbb{1} = \int \frac{d^3\vec{q}}{(2\pi)^3} |\vec{q}\rangle\langle\vec{q}|$$

[THE LIPPMANN-SCHWINGER EQUATION] (3)

⇒ But, how do we solve it?

a) General method is discretization
(we make it into a linear system
over some mesh points)

b) Luckily, $\exists$ a few potentials for which
it is possible to solve it exactly
→ Separable potentials

[THE LIPPMANN-SCHWINGER EQUATION] 4

⇒ Separable potentials:

[$\langle \vec{k}' | V | \vec{k} \rangle = g\, f(\vec{k}')\, f(\vec{k})$] (definition)

a) Ansatz: $\langle \vec{k}' | T(E) | \vec{k} \rangle = \tau(E)\, f(\vec{k}')\, f(\vec{k})$

b) Solution:

$$\tau(E) = \frac{1}{\frac{1}{g} - \int \frac{d^3\vec{q}}{(2\pi)^3} \frac{f^2(\vec{q})}{E - \vec{q}^2/2\mu}}$$

(check slide 47)

$\boxed{\text{T-MATRIX}}$ (cont'ed)

⇒ Two interesting connections:

a) Perturbative series

$$T = V + VG_0T = V + VG_0V + VG_0VG_0V + \cdots$$

b) Feynman diagrams

T = (sum of bubble diagrams)

[T-MATRIX AND PERTURBATIONS] 1

$\Rightarrow$ If we undo the iteration:

$$T = V + V G_0 T$$

$$= V + V G_0 V + V G_0 V G_0 V + \dots$$

$\Rightarrow$ This is just the perturbative series

We can recover the Born approx. $(T = V + \mathcal{O}(V^2))$

$$f(\Omega) = -\frac{\mu}{2\pi} \langle \vec{k}' | V | \vec{k} \rangle + \mathcal{O}(V^2)$$

$$= -\frac{\mu}{2\pi} \int d^3\vec{r}\, e^{i(\vec{k}-\vec{k}')\cdot\vec{r}}\, V(\vec{r}) + \mathcal{O}(V^2)$$

[T-MATRIX AND PERTURBATIONS] 2

$\Rightarrow$ We can recover Rutherford scattering for example:

$V_c \to$ Coulomb (two charged particles)

$$f(\Omega) = -\frac{\mu}{2\pi}\langle \vec{k}' | V_c | \vec{k} \rangle + \dots$$

$$\langle \vec{k}' | V_c | \vec{k} \rangle = V_c(\vec{k}' - \vec{k}) \,, \quad V_c(\vec{q}) = 4\pi \frac{\alpha}{q^2} \underbrace{Z_1 Z_2}_{\text{charges}}$$

$$\Rightarrow f(\Omega) = -2\mu \frac{Z_1 Z_2 \alpha}{|\vec{k} - \vec{k}'|^2} \Rightarrow \boxed{\frac{d\sigma}{d\Omega} = \frac{4\mu^2\alpha^2 (Z_1 Z_2)^2}{|\vec{k}' - \vec{k}|^4}}$$

$\to$ Rutherford scattering

[T-MATRIX AND PERTURBATIONS] 3

⇒ Alternatively, we can calculate Yukawa scattering in the Born approximation

$$f(\Omega) = -\frac{\mu}{2\pi}\langle \vec{k}' | V_Y | \vec{k} \rangle + \cdots$$

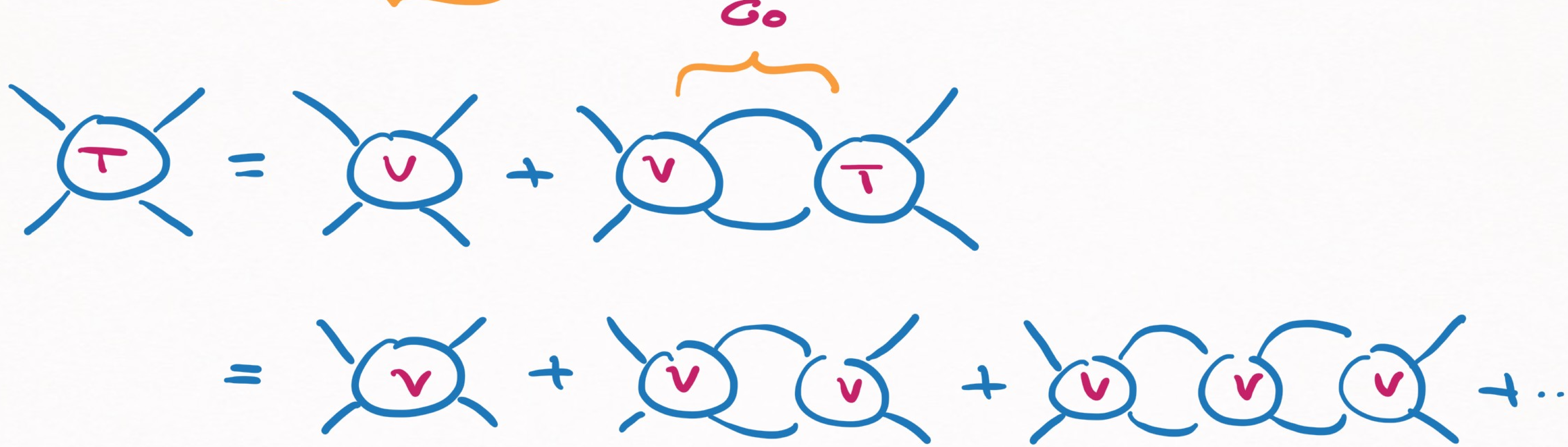
$$= -\frac{\mu}{2\pi} \frac{g_Y^2}{m_Y^2 + (\vec{k}-\vec{k}')^2}$$

$$V_Y(\vec{q}) = \frac{g_Y^2}{m_Y^2 + \vec{q}^{\,2}}$$

$$\frac{d\sigma}{d\Omega} = \frac{\mu^2}{4\pi^2} \frac{g_Y^4}{[m_Y^2 + (\vec{k}-\vec{k}')^2]^2}$$

[DIAGRAMMATIC REPRESENTATION OF THE T-MATRIX]

$\Rightarrow$ $T = \underbrace{V + V G_0 T} = V + V G_0 V + V G_0 V G_0 V + \dots$



$\Rightarrow$ Just Feynman diagrams as applied to QM

[T-MATRIX AND BOUND STATES] 1

a) We can rewrite $T = V + VG_0T$ as follows:

$$T = V + VG_0V + VG_0VG_0V + \ldots\ldots \qquad \text{(you already know)}$$

$$= V + V(G_0 + \underline{G_0VG_0} + \cdots)V$$

$$= V + VGV \quad \text{with} \quad G = G_0 + G_0VG$$

b) Notice that: $\quad G(E) = \dfrac{1}{E-H} = \dfrac{1}{E-H_0} + \dfrac{1}{E-H_0}V\dfrac{1}{E-H_0}$

$+ \ldots\ldots$ (infinite sum)

$H|\psi_B\rangle = E_B|\psi_B\rangle$ → bound state energy

$$\langle\psi_B|G(E)|\psi_B\rangle = \frac{1}{E-E_B} \xrightarrow{E\to E_B} \infty \;!!$$

[T-MATRIX AND BOUND STATES] 2

c) If $\langle \psi_B | G(E) | \psi_B \rangle = \frac{1}{E - E_B}$, then the T-matrix will behave funny for $\underline{\underline{E \to E_B}}$

Let's see: $\to$ we write the identity for the eigenstates of $H = H_0 + V$

$$\mathbb{1} = \sum_{i=1}^{n_B} |\psi_{B_i}\rangle\langle\psi_{B_i}| + \int \frac{d^3\vec{q}}{(2\pi)^3} |\psi_{\vec{q}}^{+}\rangle\langle\psi_{\vec{q}}^{+}|$$

(for $V \to 0$, $\mathbb{1} \to \int \frac{d^3\vec{q}}{(2\pi)^3} |\vec{k}\rangle\langle\vec{k}|$)

[T-MATRIX AND BOUND STATES] 3

d) Let's insert the identity in $T = V + VGV$:

$$\Rightarrow G(E) = \sum_{i=1}^{n_B} \frac{|\psi_{B_i}\rangle\langle\psi_{B_i}|}{E - E_{B_i}} + (\text{contributions from the continuum})$$

$E > 0$

$$\Rightarrow T(E \to E_{B_i}) \longrightarrow V \frac{|\psi_{B_i}\rangle\langle\psi_{B_i}|}{E - E_{B_i}} V$$

[T-MATRIX AND BOUND STATES] 4

$\Rightarrow$ The T-matrix has a pole for $E \to E_B$

$$\left[\; T(E) \xrightarrow[E \to E_B]{} V \frac{|\psi_B\rangle\langle\psi_B|}{E - E_B} V \; \right]$$

take the residue and find that:

$$\operatorname*{Res}_{E = E_B} T(E) = V \underbrace{|\psi_B\rangle\langle\psi_B|}_{} V$$

I'm simplifying to just one bound state

[T-MATRIX AND BOUND STATES] 3

⇒ We have that: $\underset{E=E_B}{\mathrm{Res}}\, T(E) = V|\psi_B\rangle\langle\psi_B|V$

which we can compare with: $T(E) = V + VG_0(E)T(E)$

⇒ I can find the equation for a bound state:

$$\underset{E=E_B}{\mathrm{Res}}\, T(E) = \lim_{E\to E_B} (E-E_B)V + VG_0(E_B)\underset{E=E_B}{\mathrm{Res}}\, T(E)$$

⇒ $\underset{E=E_B}{\mathrm{Res}}\, T(E) = VG_0(E_B)\underset{E=E_B}{\mathrm{Res}}\, T(E)$ ⇒ $V|\psi_B\rangle\cancel{\langle\psi_B|V} =$

$VG_0(E_B)V|\psi_B\rangle\cancel{\langle\psi_B|V}$

[T-MATRIX AND BOUND STATES] 6

$\Rightarrow$ We find that: $\left[|\psi_B\rangle = G_0(E_B) V |\psi_B\rangle \right]$

$\Rightarrow$ Or, if I write the matrix elements:

$$\langle \vec{p} | \psi_B \rangle = - \frac{2\mu}{p^2 + \gamma^2} \int \frac{d^3\vec{q}}{(2\pi)^3} \langle \vec{p} | V | \vec{q} \rangle \langle \vec{q} | \psi_B \rangle$$

$\gamma = \sqrt{-2\mu E_B}$, $E_B = -\dfrac{\gamma^2}{2\mu}$

[T-MATRIX AND BOUND STATES] 7

$\Rightarrow$ Particular case: $\left[\langle \vec{p}\,' | V | \vec{p} \rangle = g\, f(\vec{p}\,')\, f(\vec{p}) \right]$ separable potential

$$\Rightarrow \langle \vec{p} | \psi_B \rangle = -\frac{2\mu}{\gamma^2 + p^2} f(\vec{p}) \underbrace{\int \frac{d^3\vec{q}}{(2\pi)^3}\, g\, f(\vec{q}) \langle \vec{q} | \psi_B \rangle}_{\text{a constant (no } \vec{p} \text{ dependence)}}$$

$$\Rightarrow \psi_B(\vec{p}) = \underbrace{\mathcal{N}}_{\text{normalization}} \frac{f(\vec{p})}{\gamma^2 + p^2}$$

normalization $\rightarrow \int \frac{d^3\vec{p}}{(2\pi)^3} |\psi_B(\vec{p})|^2 = 1$

[T-MATRIX AND BOUND STATES] ⑧

⇒ Even more particular case: $\langle \vec{p}\,'|V|\vec{p}\rangle = g$

$(f(\vec{k}) = 1)$

(pure contact-range potential)

$$\Rightarrow \psi_B(\vec{p}) = \frac{\mathcal{N}}{p^2+\gamma^2} = \frac{\sqrt{8\pi\gamma}}{p^2+\gamma^2}$$

$$\int \frac{d^3\vec{p}}{(2\pi)^3} |\psi_B(\vec{p})|^2 = 1$$

[T-MATRIX AND BOUND STATES] 9

$\Rightarrow$ However, it is more convenient to write $|\psi_B\rangle$ in terms of the vertex function $|\phi_B\rangle$

$$|\psi_B\rangle = G_0(E)|\phi_B\rangle \Rightarrow |\phi_B\rangle = VG_0(E)|\phi_B\rangle$$

or $$\phi_B(\vec{p}) = \int \frac{d^3q}{(2\pi)^3} \frac{\langle\vec{p}|V|\vec{q}\rangle \phi_B(\vec{q})}{E_B - \vec{q}^2/2\mu}$$

For separable potentials $\Rightarrow$ $\left[\phi_B(\vec{p}) = N f(\vec{p})\right]$

SUMMARY

1) Effective range expansion $\Rightarrow$ efficient description of scatterings at low energies

$$k\cot\delta_0(k) = -\frac{1}{a_0} + \frac{1}{2} r_0 k^2 + \sum_{n=2}^{\infty} v_n k^{2n}$$

$r_0 \Rightarrow$ contains info about the range of $V(r)$

2) Formal scattering theory $\Rightarrow$ Quantum mechanics in operator language

$$\left[-\frac{\nabla^2}{2\mu} + V(\vec{r})\right]\psi(\vec{r}) = E\psi(\vec{r}) \Rightarrow (H_0 + V)|\psi\rangle = E|\psi\rangle$$

SUMMARY

3) Reformulation of scattering:

$$\psi_k(\vec{r}) \to e^{i\vec{k}\cdot\vec{r}} + f(\Omega)\frac{e^{ikr}}{r} \Rightarrow |\psi_{\vec{k}}\rangle = |\vec{k}\rangle + G_0 V |\psi_{\vec{k}}\rangle$$

4) T-matrix and the Lippmann-Schwinger equation:

$$\underline{\underline{f(\Omega)}} = -\frac{\mu}{2\pi}\langle \vec{k}' | T(E) | \vec{k} \rangle$$

with $T(E) = V + V G_0(E) T(E)$

SUMMARY

5) Bound states in formal scattering theory

⇒ Poles in the T-matrix

$$T(E) \underset{E \to E_B}{\longrightarrow} \frac{V|\psi_B\rangle\langle\psi_B|V}{E-E_B} \quad , \quad \underset{E=E_B}{\mathrm{Res}}\, T(E) = V|\psi_B\rangle\langle\psi_B|V$$

Bound state equation: $|\psi_B\rangle = G_0(E)V|\psi_B\rangle$

⇒ Equivalent to the usual Schrödinger equation

See you Friday

at 15:50