NUCLEAR PHYSICS 14

a) SCATTERING THEORY (PART II)

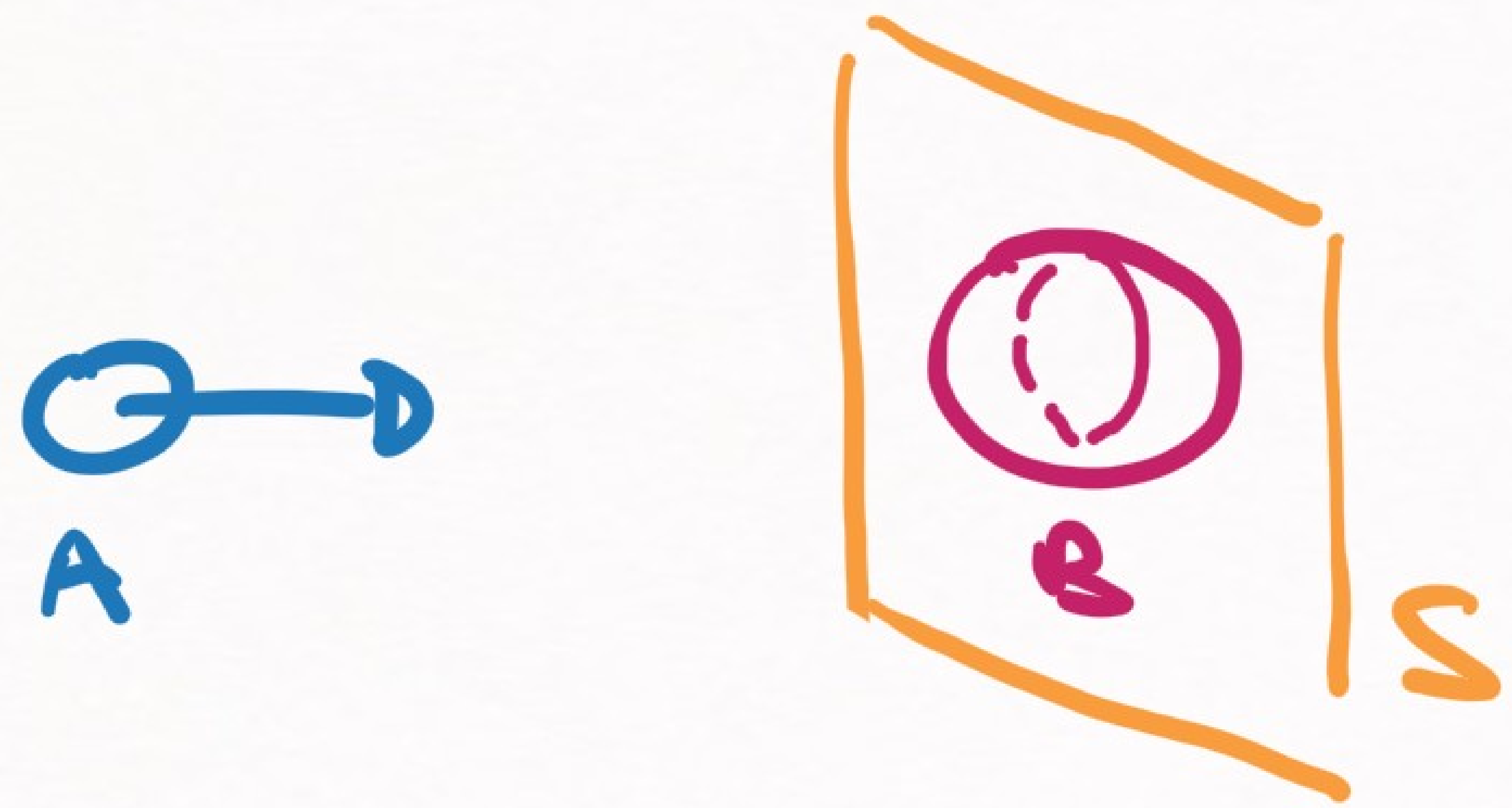
Spin & Scattering

Tensor forces & Scattering

NN Phase shifts

RECAP | [What is the cross section?]

a) Classical scattering:

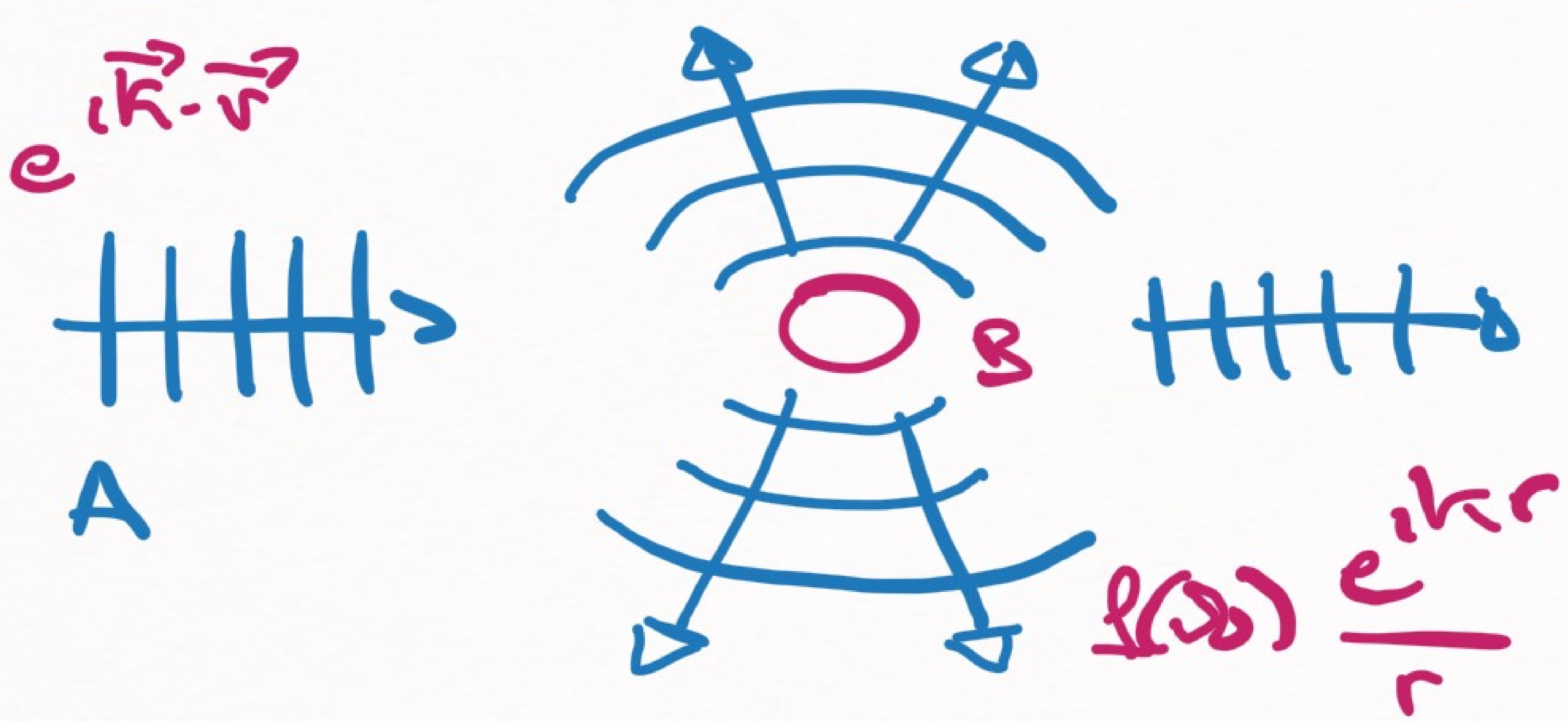


cross section

$$\Rightarrow \left[\sigma = \frac{N_S}{N_A N_B} S\right]$$

- N_A → number of projectiles
- N_B → number of targets
- N_S → number of scattered projectiles
- S → surface area of the experiment

b) Quantum scattering:



$$\psi_{\vec{k}}(\vec{r}) \xrightarrow[|\vec{r}|\to\infty]{} e^{i\vec{k}\cdot\vec{r}} + f(\Omega)\,\frac{e^{ikr}}{r}$$

$$\Rightarrow \left[\sigma = \int |f(\Omega)|^2 \, d\Omega\right]$$

RECAP | QUANTUM SCATTERING

1) Total & differential cross sections:

$$\sigma = \int |f(\Omega)|^2 d\Omega \,, \quad \frac{d\sigma}{d\Omega} = |f(\Omega)|^2$$

$\Omega \to$ solid angle $\underline{\underline{(\theta, \varphi)}}$

2) The scattering amplitude $f(\underline{\underline{\Omega}})$ is related to the asymptotic properties of the wave function:

$$\psi_{\vec{k}}(\vec{r}) \underset{\underline{\underline{|\vec{r}| \to \infty}}}{\longrightarrow} e^{i\vec{k}\cdot\vec{r}} + \underline{\underline{f(\Omega)}} \frac{e^{ikr}}{r}$$

$$u_\ell(r) \underset{\underline{\underline{r \to \infty}}}{\longrightarrow} \sin\left(kr - \ell\frac{\pi}{2} + \underline{\underline{\delta_\ell(k)}}\right)$$

$\Bigg\}$ $f(\Omega)$ and $\delta_\ell(k)$ must be related

$\Rightarrow$

RECAP | QUANTUM SCATTERING

3) How to uncover the relation?

$$e^{i\vec{k}\cdot\vec{r}} = \sum_{\ell}(2\ell+1)\,i^{\ell}\,j_{\ell}(kr)\,P_{\ell}(\hat{k}\cdot\hat{r})$$

$$\psi_{\vec{k}}(\vec{r}) = \sum_{\ell}(2\ell+1)\,i^{\ell}\,\frac{u_{\ell}(r)}{r}\,P_{\ell}(\hat{k}\cdot\hat{r})$$

with $\frac{u_{\ell}(r)}{r} \xrightarrow[r\to\infty]{} e^{i\delta_{\ell}}\left[\cos\delta_{\ell}\, j_{\ell}(kr) - \sin\delta_{\ell}\, y_{\ell}(kr)\right]$

$$\Rightarrow \left[\psi_{\vec{k}}(\vec{r}) - e^{i\vec{k}\cdot\vec{r}} \xrightarrow[|\vec{r}|\to\infty]{} f(\omega)\,\frac{e^{ikr}}{r}\right]$$

RECAP | QUANTUM SCATTERING

4) If we match coefficients in the previous expression:

$$\Rightarrow f(\Omega) = \sum_{\ell} (2\ell+1)\, f_\ell(k)\, P_\ell(\hat{k}\cdot\hat{r}) \quad \rightarrow \text{match}$$

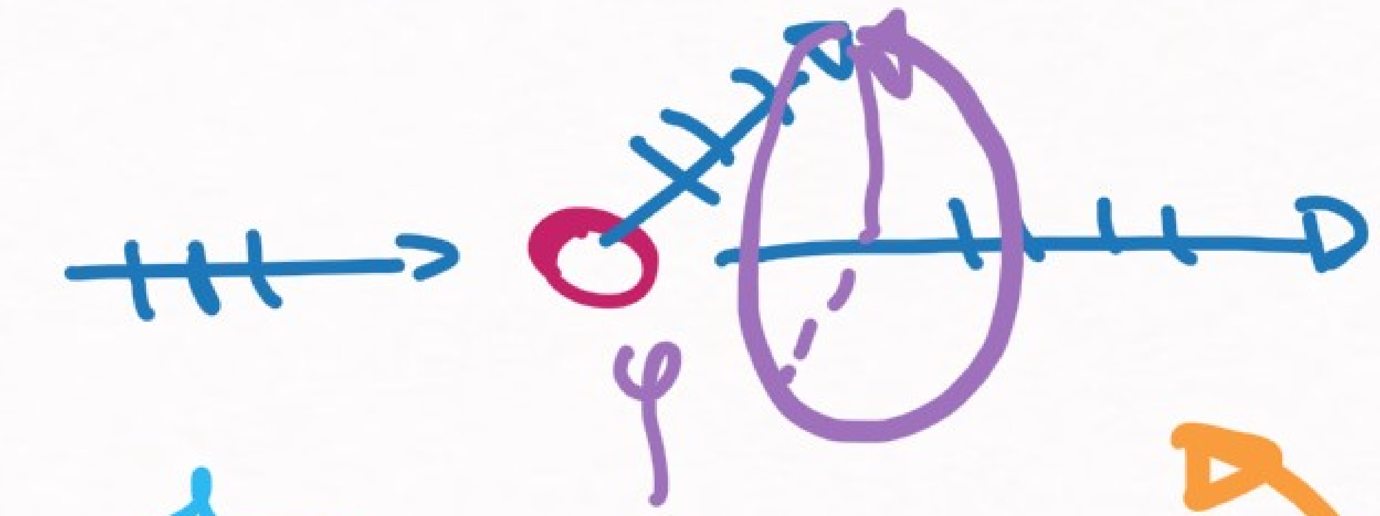
$$\text{with} \quad f_\ell(k) = \frac{e^{i\delta_\ell}\sin\delta_\ell}{k} = \frac{1}{k\cot\delta_\ell - ik}$$

5) Cross section as a partial wave sum: $\hat{k}\cdot\hat{r} = \cos\theta$

$$\sigma = \int |f(\Omega)|^2 d\Omega = \int |f(\theta,\varphi)|^2 d(\cos\theta)\, d\varphi = \circledast$$

$$= \frac{4\pi}{k^2} \sum_{\ell} (2\ell+1) \sin^2\delta_\ell$$

RECAP | QUANTUM SCATTERING

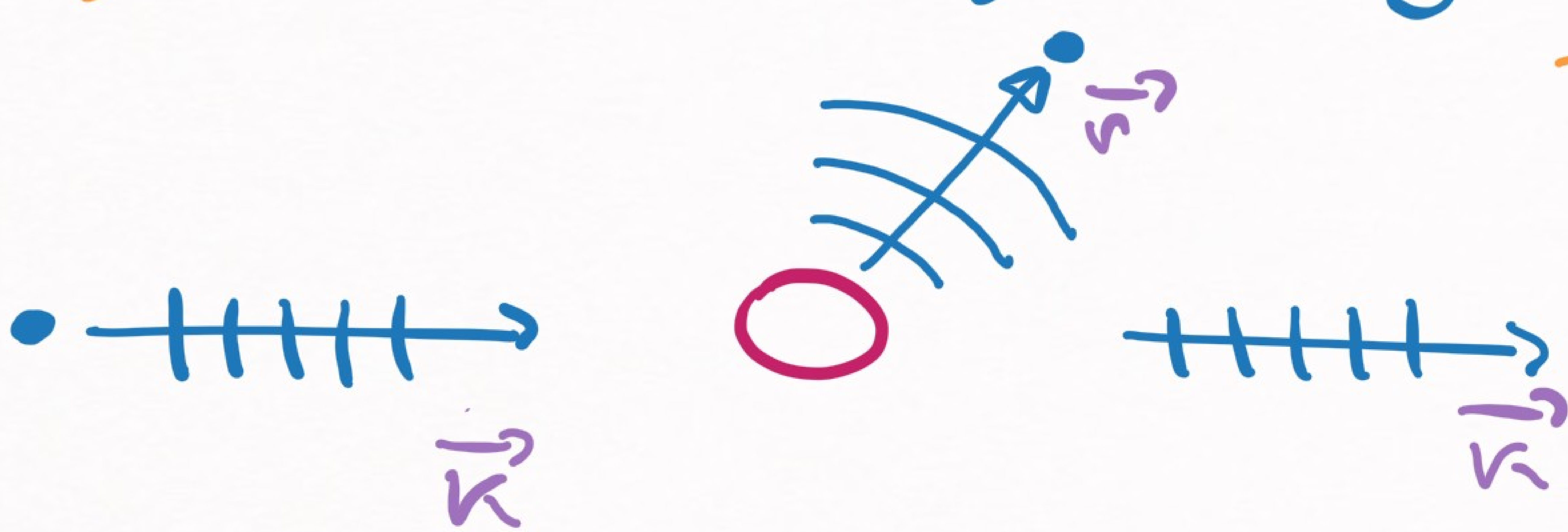


6) Low energy limit : $k \to 0$ $(E_{cm} \to 0)$

$\sigma \to 4\pi |a_0|^2 \Rightarrow a_0$ is the scattering length

$$\Rightarrow \left[\delta_\ell(k) \xrightarrow[k \to 0]{} a_\ell k^{2\ell+1} + O(k^{2\ell+3}) \right]$$

7) Visualization of the angles

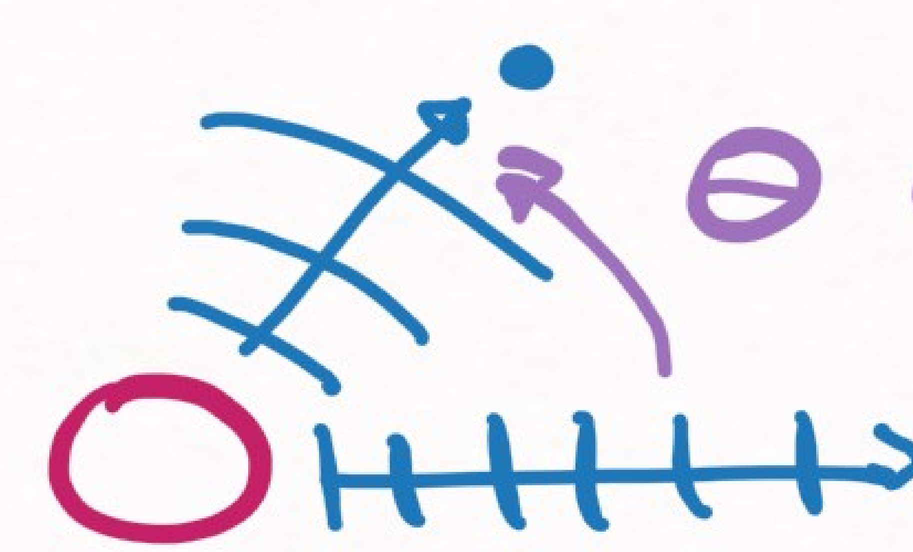


(θ, φ)

scattering usually symmetric on φ

$\vec{k} \cdot \vec{r} = kr\cos\theta$

$\hat{k} \cdot \hat{r} = \cos\theta$

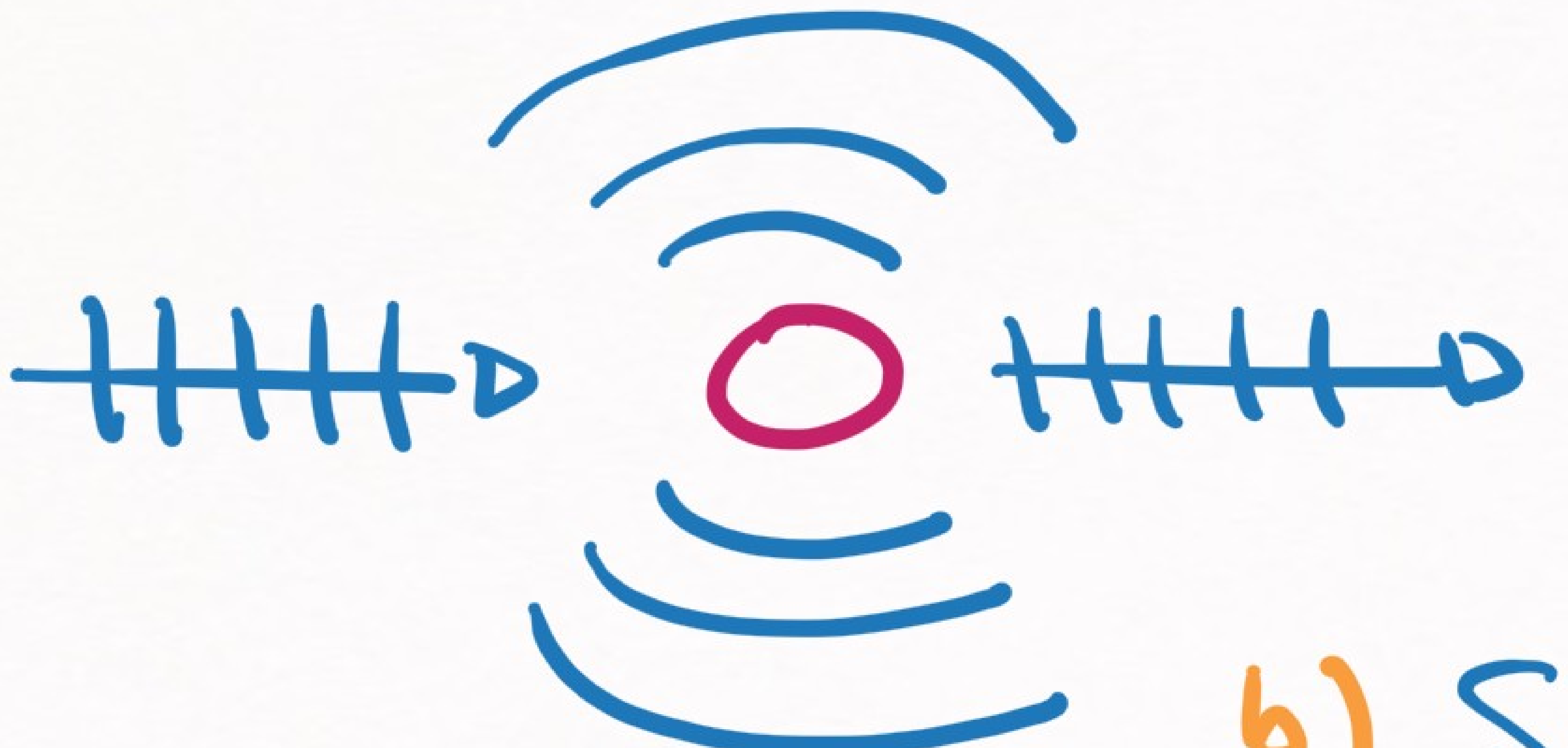


θ is just this angle

$\boxed{\text{WHAT ABOUT SPIN ?}}$ 3

⇒ Observation: nucleons have spin

a) No spin: ← base case



$$\psi_{\vec{k}}(\vec{r}) \longrightarrow e^{i\vec{k}\cdot\vec{r}} + f(\Omega)\,\frac{e^{ikr}}{r}$$

b) Spin: $|s\, m_s\rangle$ → total spin of the two particles

$$e^{i\vec{k}\cdot\vec{r}} \longrightarrow e^{i\vec{k}\cdot\vec{r}}\,|s\, m_s\rangle \qquad \longrightarrow \text{incoming wave}$$

$$f(\Omega)\,\frac{e^{ikr}}{r} \longrightarrow \sum_{m_s'} f_{m_s m_s'}(\Omega)\,\frac{e^{ikr}}{r} \qquad \longrightarrow \text{outgoing wave}$$

We add a spin index

WHAT ABOUT SPIN? (2)

b.1) Include spin in incoming wave:

$$e^{i\vec{k}\cdot\vec{r}} \longrightarrow e^{i\vec{k}\cdot\vec{r}}\,|s\,m_s\rangle$$

b.2) Include spin in outgoing wave:

$$f(\Omega)\frac{e^{ikr}}{r} \longrightarrow \sum_{m_s'} f^{s}_{m_s m_s'}(\Omega)\frac{e^{ikr}}{r}\,|s\,m_s'\rangle$$

purple indicates what happens if spin is not conserved

c) Putting all the pieces together:

$$\psi_{\vec{k}}(\vec{r}) \xrightarrow[|\vec{r}|\to\infty]{} e^{i\vec{k}\cdot\vec{r}}\,|s\,m_s\rangle + \sum_{s'm_s'} f^{s'}_{m_s m_s'}(\Omega)\frac{e^{ikr}}{r}\,|s'm_s'\rangle$$

WHAT ABOUT SPIN? 3

1) Differential + polarized cross section

$$\frac{d\sigma}{d\Omega}(m_s \to m_s') = |f^s_{m_s m_s'}(\Omega)|^2$$

2) Differential + unpolarized cross section

$$\frac{d\sigma}{d\Omega} = \frac{1}{2s+1} \sum_{m_s m_s'} |f^s_{m_s m_s'}(\Omega)|^2$$

3) Total cross section

$$\sigma = \frac{1}{2s+1} \sum_{m_s m_s'} \int |f^s_{m_s m_s'}(\Omega)|^2 d\Omega$$

This is just repeating what we did for the no spin case

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WHAT ABOUT SPIN? 4

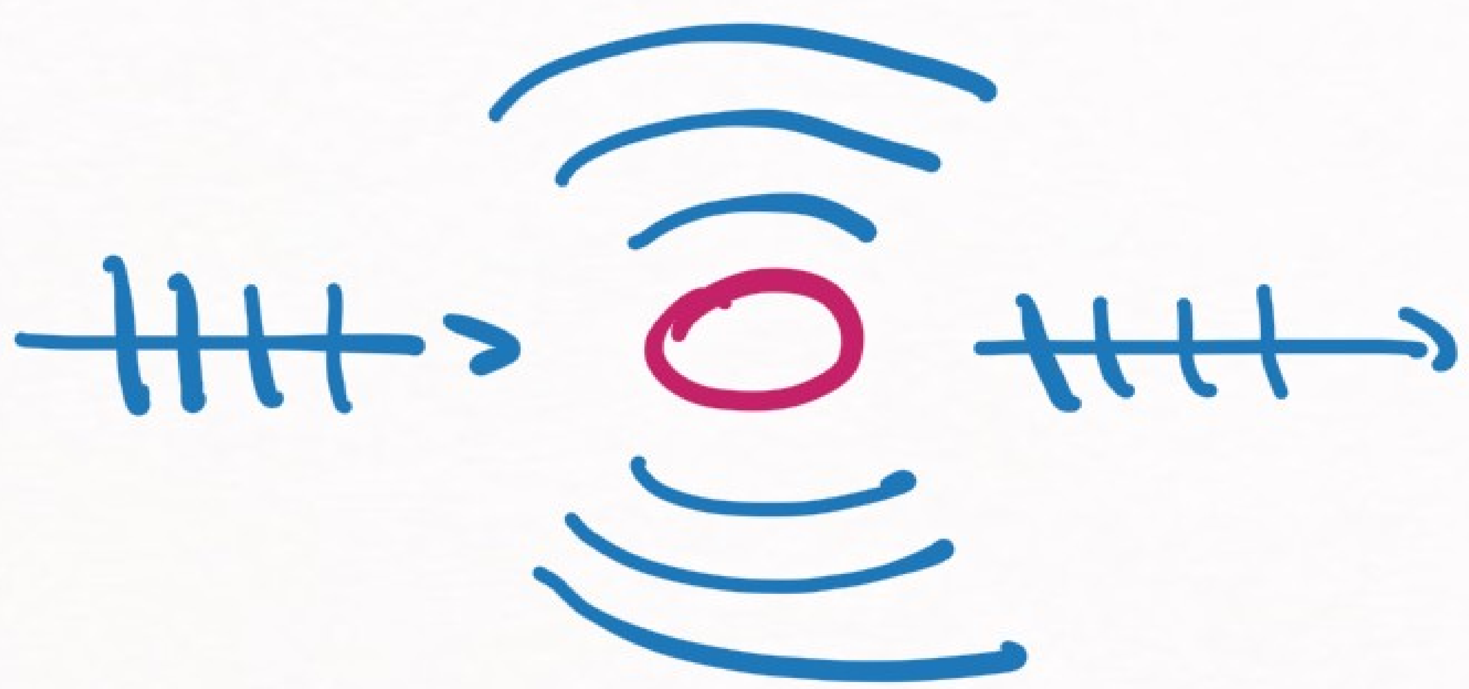
a) The previous only considers the two-body system already coupled to spin $|s\, m_s\rangle$

$\Rightarrow |s\, m_s\rangle$

b) It will more general to consider the spin of each particle separately

$\Rightarrow |s_A\, m_A\rangle |s_B\, m_B\rangle$ or $|s_1\, m_1\rangle |s_2\, m_2\rangle$

TWO PARTICLES W/ SPIN (1)



This is how it looks when considering each spin of each particle

$$\psi_{\vec{k}}(\vec{r}) \longrightarrow e^{i\vec{k}\cdot\vec{r}}\,|s_1 m_1\rangle|s_2 m_2\rangle + \sum_{m_1' m_2'} f_{m_1 m_2 m_1' m_2'}(\Omega)\,\frac{e^{ikr}}{r}\,|s_1 m_1'\rangle|s_2 m_2'\rangle$$

1) Polarized diff. cross section:

$$\frac{d\sigma}{d\Omega}(m_1 m_2 \rightarrow m_1' m_2') = |f_{m_1 m_2 m_1' m_2'}(\Omega)|^2$$

2) Unpolarized diff. cross section:

$$\frac{d\sigma}{d\Omega} = \frac{1}{(2S_1+1)(2S_2+1)} \sum_{\substack{m_1 m_2 \\ m_1' m_2'}} |f_{m_1 m_2 m_1' m_2'}(\Omega)|^2$$

TWO PARTICLES W/ SPIN (2)

$\longrightarrow \vec{S} = \vec{S}_1 + \vec{S}_2$

$\Rightarrow$ If total spin conserved ($[H, \vec{S}^2] = 0$), we simplify to:

$$\rho_{m_1 m_2 m_1' m_2'} = \sum_{S m_S m_S'} \langle s_1 m_1 s_2 m_2 | S m_S \rangle \langle s_1' m_1' s_2' m_2' | S m_S' \rangle \rho^S_{m_S m_S'}$$

$\Rightarrow$ If in addition we have $[H, S_z] = 0$,

we also have:

$$\rho^S_{m_S m_S'} = \rho^S \delta_{m_S m_S'}$$

TWO PARTICLES W/ SPIN 3

$\Rightarrow$ Example : $\left[V(\vec{r}) = V_c(\vec{r}) + \vec{S}_1 \cdot \vec{S}_2 \, V_s(\vec{r}) \right]$

$$[V, \vec{S}^2] = 0 \quad , \quad [V, S_z] = 0$$

$\Rightarrow$ Unpolarized cross section :

$$\frac{d\sigma}{d\Omega} = \frac{1}{(2S_1+1)} \frac{1}{(2S_2+1)} \sum_S (2S+1) |f_S|^2$$

(for concrete cases, we can simplify further)

TWO PARTICLES W/ SPIN ④

⇒ If we have two spin-$\frac{1}{2}$ particles (e.g. neutron-proton)

$$\left[V(\vec{r}) = V_C(\vec{r}) + \vec{\sigma}_1 \cdot \vec{\sigma}_2 \, V_S(\vec{r}) \right]$$

⇒ We write: $\dfrac{d\sigma}{d\Omega} = \dfrac{1}{4} |f_s(\Omega)|^2 + \dfrac{3}{4} |f_t(\Omega)|^2$

$f_s(\Omega) = f^{S=0}(\Omega) \rightarrow$ singlet ($S=0$) scattering amplitude

$f_t(\Omega) = f^{S=1}(\Omega) \rightarrow$ triplet ($S=1$) scattering amplitude

TWO PARTICLES w/ SPIN (S)

⇒ Provided we do not have pieces of the potential that mix orbital & intrinsic angular momentum

$\left(\text{e.g. } (3\vec{\sigma}_1\cdot\hat{r}\,\vec{\sigma}_2\cdot\hat{r} - \vec{\sigma}_1\cdot\vec{\sigma}_2)V_T(r) \right) \longrightarrow$ tensor force

all will be like before : → the case without spin

a) $\sigma = \frac{1}{4}\int |f_s(\Omega)|^2 d\Omega + \frac{3}{4}\int |f_t(\Omega)|^2 d\Omega$

$$\int |f_s|^2 d\Omega = \frac{4\pi}{k^2}\sum_\ell (2\ell+1)\sin^2\delta_\ell^{\text{singlet}}(k)\,,\quad \int |f_t|^2 d\Omega = \frac{4\pi}{k^2}\sum_\ell (2\ell+1) \times \sin^2\delta_\ell^{\text{triplet}}(k)$$

b) $$\sigma \xrightarrow[k\to 0]{} 4\pi\left\{ \frac{1}{4}|a_s|^2 + \frac{3}{4}|a_t|^2 \right\}$$

TWO PARTICLES W/ SPIN ⑥

1) Spin does not mix w/ orbital angular momentum:

$$\sigma = \frac{1}{4}\underbrace{\int |f_s(\Omega)|^2 d\Omega}_{= \frac{4\pi}{k^2}\sum_{\ell}(2\ell+1)\sin^2\delta^s_\ell(k)} + \frac{3}{4}\underbrace{\int |f_t(\Omega)|^2 d\Omega}_{= \frac{4\pi}{k^2}\sum_{\ell}(2\ell+1)\sin^2\delta^t_\ell(k)}$$

$$\sigma \xrightarrow[k\to 0]{} 4\pi \left\{ \frac{1}{4}|a_s|^2 + \frac{3}{4}|a_t|^2 \right\}$$

For np we have:

$a_s \simeq -23.7$ fm

$a_t \simeq 5.4$ fm

TWO PARTICLES W/ SPIN (7)

2) But if we have mixing:

[TENSOR FORCE]

$$V(\vec{r}) = V_c(\vec{r}) + V_S(\vec{r})\,\vec{\sigma}_1\cdot\vec{\sigma}_2$$

$$+ V_T(\vec{r})\left(3\,\vec{\sigma}_1\cdot\hat{r}\;\vec{\sigma}_2\cdot\hat{r} - \vec{\sigma}_1\cdot\vec{\sigma}_2\right)$$

$\Rightarrow$ everything will become much more complicated

(we will see how ... don't expect to follow the details)

TENSOR FORCES (1)

Reminder $\Rightarrow$ Nuclear forces are not central (quadrupole moment of deuteron)

$\Rightarrow$ There is a tensor term

$\Downarrow$

mixing of S & L

$$S_{12}(\hat{r}) = (3\,\vec{\sigma}_1\cdot\hat{r}\;\sigma_2\cdot\hat{r} - \vec{\sigma}_1\cdot\vec{\sigma}_2)$$

Commutators $\Rightarrow$

$$[S_{12}, \vec{L}^2] \neq 0$$

$$[S_{12}, \vec{S}^2] \neq 0$$

$$[S_{12}, \vec{J}^2] = 0$$

$\Rightarrow$ total angular momentum $\vec{J} = \vec{S} + \vec{L}$ is conserved

TENSOR FORCES ②

a) We have that $[S_{12}(\hat{r}), \vec{L}^2] \neq 0$, $[S_{12}(\hat{r}), \vec{S}^2] \neq 0$

but $[S_{12}(\hat{r}), \vec{J}^2] = 0$ $\qquad \vec{J} = \vec{L} + \vec{S}$

$\Rightarrow$ the total angular momentum is still a good quantum number

b) $S_{12}(\hat{r}) = 3\, \vec{S}_1 \cdot \hat{r}\, \vec{S}_2 \cdot \hat{r} - \vec{S}_1 \cdot \vec{S}_2 = 3 \underbrace{\left(\hat{r}_i \hat{r}_j - \frac{1}{3}\delta_{ij}\right)}_{b.1} \underbrace{\left(S_{1i} S_{2j} - \frac{1}{3} \vec{S}_1 \cdot \vec{S}_2\, \delta_{ij}\right)}_{b.2}$

why? $\rightarrow$ b.1) this term has $\ell = 2$ $\Rightarrow$ mix ℓ, ℓ' with $\ell' = \ell \pm 2$

b.2) this term has $s = 2$ $\Rightarrow$ mix s, s' with $s' = s \pm 2$

(np $s, s' = 0, 1$
$\Rightarrow$ no mixing of spin)

$\boxed{\text{R-SPACE TENSORS}}$ $\Rightarrow$ What's the relation between $\hat{r}_{i_1}\hat{r}_{i_2}\cdots\hat{r}_{i_\ell}$ and $Y_{\ell m}(\vec{r})$?

a) Spherical coordinates:

$$x = r\sin\theta\cos\varphi$$
$$y = r\sin\theta\sin\varphi$$
$$z = r\cos\theta$$

b) Spherical harmonics

$$Y_{11}(\vec{r}) = -\sqrt{\frac{3}{4\pi}}\,\frac{\sin\theta\, e^{i\varphi}}{\sqrt{2}}$$
$$Y_{10}(\vec{r}) = +\sqrt{\frac{3}{4\pi}}\cos\theta$$
$$Y_{1-1}(\vec{r}) = +\sqrt{\frac{3}{4\pi}}\,\frac{\sin\theta\, e^{-i\varphi}}{\sqrt{2}}$$

c) The $\hat{r}_i$ vector:

$\boxed{\ell = 1}$

$$\hat{r}_{+1} = -\frac{1}{\sqrt{2}}(\hat{r}_x + i\hat{r}_y)$$
$$\hat{r}_0 = \hat{r}_z$$
$$\hat{r}_{-1} = +\frac{1}{\sqrt{2}}(\hat{r}_x - i\hat{r}_y)$$

$\hat{r}_x, \hat{r}_y, \hat{r}_z$ cartesian

$\hat{r}_0, \hat{r}_{\pm 1}$ spherical basis

d) Correspondence:

$$\hat{r}_\mu = \sqrt{\frac{4\pi}{3}}\, Y_{1\mu}(\vec{r})$$

or

$$Y_{1\mu}(\vec{r}) = \sqrt{\frac{3}{4\pi}}\,\hat{r}_\mu$$

R-SPACE TENSORS ②

$\Rightarrow$ The same thing happens w/ $Q_{ij} = \hat{r}_i \hat{r}_j - \frac{1}{3}\delta_{ij} \underbrace{\hat{r}\cdot\hat{r}}_{=1}$

a) Spherical harmonic: $Y_{20}(\hat{r}) = +\sqrt{\frac{5}{4\pi}}\,\frac{3}{2}\left(\cos^2\theta - \frac{1}{3}\right)$

b) Spin-2 tensor: $Q_{zz} = \cos^2\theta - \frac{1}{3}$

$\Rightarrow \left[\, Y_{20}(\hat{r}) = +\sqrt{\frac{5}{4\pi}}\,\frac{3}{2}\,Q_{zz} \,\right]$ Q_{zz} has $\ell = 2, m_\ell = 0$

And it is possible to find the equivalence for all other cases (though tedious) $\Rightarrow$ $\boxed{Q_{ij} \text{ has } L = 2}$

the details not important

TENSOR FORCES ③

a) Mixing pattern :

$\rightarrow$ $S_{12}(\vec{r}) = 3\,\vec{\sigma}_1\cdot\hat{r}\,\vec{\sigma}_2\cdot\hat{r} - \vec{\sigma}_1\cdot\vec{\sigma}_2 = 3\underbrace{\left(\hat{r}_i\hat{r}_j - \frac{1}{3}\delta_{ij}\right)}_{a.1)}\underbrace{\left(\sigma_{1i}\sigma_{2j} - \frac{1}{3}\vec{\sigma}_1\cdot\vec{\sigma}_2\,\delta_{ij}\right)}_{a.2}$

a.1) $L, L' \,/\, L' = L \pm 2$

a.2) $S, S' \,/\, S' = S \pm 2$ $\Rightarrow$ But $S = 0, 1$ only $\Rightarrow$ $S' = S$

(n, p have $s_1 = s_2 = 1/2$)

b) What happens with $S = 0$?

$S_{12}(\vec{r}) = 3\,\vec{\sigma}_1\cdot\hat{r}\,\vec{\sigma}_2\cdot\hat{r} - \vec{\sigma}_1\cdot\vec{\sigma}_2 = 2\left[3(\vec{S}\cdot\hat{r})^2 - \vec{S}^2\right]$

$\Rightarrow$ If $S = 0$, then $S_{12}(\vec{r}) = 0$

$\vec{S} = \frac{1}{2}(\vec{\sigma}_1 + \vec{\sigma}_2)$

TENSOR FORCES ④

c) What happens with $\underline{\underline{S=1}}$?

$$S \otimes L = 1 \otimes L = (L-1) \oplus L \oplus (L+1) \Rightarrow J = L, L \pm 1$$

c.1) If $L=J$, this partial wave will not mix with any other $\Rightarrow$ uncoupled wave

c.2) If $L = J \pm 1$, then these two partial waves will mix w/ each other $\Rightarrow$ coupled waves

TENSOR FORCES (5)

$\Rightarrow$ Spectroscopic notation

$[\,{}^{2S+1}L_J\,]$

$\hookrightarrow$ this is how we name partial waves

$L = S, P, D, F, \ldots$

$L = 0, 1, 2, 3 \ldots$

c) A concrete example:

c.0) $S=0, L=0 \Rightarrow J=0$

$S_{12}=0 \leftarrow {}^1S_0$ (singlet) $\Rightarrow$ does not mix

c.1) $S=1, L=1, J=1 \Rightarrow {}^3P_1$

${}^3P_1 \Rightarrow$ does not mix $(S_{12} \neq 0)$

c.2) $S=1, L=0, J=1$ $\underline{\underline{({}^3S_1)}}$ $\Rightarrow$ it has to mix w/ $S=1, L=2, J=1$

$\underline{\underline{({}^3D_1)}}$

Mixing: ${}^3S_1 - {}^3D_1$ (deuteron)

TENSOR FORCES ⑥

$\Rightarrow$ Evaluation of $S_{12}(\hat{r})$:

a) Singlets $({}^1S_0, {}^1P_1, {}^1D_2, \dots)$: $\langle {}^1L_L | S_{12}(\hat{r}) | {}^1L_L \rangle = \underline{\underline{0}}$

b) Uncoupled triplets $({}^3P_0, {}^3P_1, {}^3D_2, \dots)$:

$\langle {}^3P_0 | S_{12} | {}^3P_0 \rangle = -4 \qquad \langle {}^3L_{\underline{L}} | S_{12} | {}^3\underline{\underline{L}}_L \rangle = +1$

c) Coupled triplets $({}^3S_1 - {}^3D_1, {}^3P_2 - {}^3F_2, {}^3D_3 - {}^3G_3, \dots)$

$$S_J = \begin{pmatrix} -2\,\frac{J-1}{2J+1} & 6\,\frac{\sqrt{J(J+1)}}{2J+1} \\ 6\,\frac{\sqrt{J(J+1)}}{2J+1} & -2\,\frac{J+2}{2J+1} \end{pmatrix}$$

[TENSOR FORCES & THE DEUTERON] $\Rightarrow$ IDEAL EXAMPLE

$\Rightarrow$ The deuteron is a good example to understand the effects of the tensor force

a) Spin : $S = 1$, $L = 0$ (and $L = 2$)

b) Without tensor forces: $|1 m_d\rangle \to$ spin wavefunction

$$\left[\ \psi_d(\vec{r}) = \frac{u(r)}{r} Y_{00}(\hat{r})\, \underline{|1 m_d\rangle}\ \right]$$

[TENSOR FORCES & THE DEUTERON] 2

c) With tensor forces:

$|1\, m_s\rangle$

c.1) $|1\, m_d\rangle$ can't coincide w/ the np spin w.f. because we are coupling w/ orbital angular momentum

$L=0 \Rightarrow m_d = m_s$, but for $L \neq 0$ this is not true in general

c.2)

$$\psi_d(\vec{r}) = \frac{u(r)}{r} Y_{00}(\hat{r})\,|1\, m_d\rangle$$

$$+ \frac{w(r)}{r} \sum_{m_s m_\ell} Y_{2 m_\ell}(\hat{r})\,|1\, m_s\rangle \langle 2 m_\ell\, 1 m_s | 1\, m_d\rangle$$

[TENSOR FORCES & THE DEUTERON] 3

d) If we define $|^3S_1(1m_d)\rangle = Y_{00}(\hat{r})\,|1\,m_d\rangle$

$$|^3D_1(1m_d)\rangle = \sum_{m_s m_\ell} Y_{2m_\ell}(\hat{r})\,|1\,m_s\rangle \times \langle 2m_\ell\,1m_s|1m_d\rangle$$

d.1) $$\psi_d(\vec{r}) = \frac{u(r)}{r}|^3S_1\rangle + \frac{w(r)}{r}|^3D_1\rangle$$

d.2) calculate tensor force in the $\mathcal{B} = \{|^3S_1\rangle, |^3D_1\rangle\}$

$$\langle S_{12}(\hat{r})\rangle = \begin{pmatrix} 0 & 2\sqrt{2} \\ 2\sqrt{2} & -2 \end{pmatrix}$$

[TENSOR FORCES & THE DEUTERON] 4

e) How do we write the Schrödinger equation?

e.1) Imagine we have the potential:

$$V(r) = V_C(r) + V_T(r)\, S_{12}(\hat{r})$$

Rewrite this as a matrix in $|{}^3S_1\rangle, |{}^3D_1\rangle$ basis

e.2) Then, we will have:

$$-\begin{pmatrix} u \\ w \end{pmatrix}'' + 2\mu V_C(r) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ w \end{pmatrix} + 2\mu V_T(r) \begin{pmatrix} 0 & 2\sqrt{2} \\ 2\sqrt{2} & -2 \end{pmatrix} \begin{pmatrix} u \\ w \end{pmatrix}$$

$\langle S_{12}(\hat{r}) \rangle$

$$+ \begin{pmatrix} 0 & 0 \\ 0 & \frac{6}{r^2} \end{pmatrix} \begin{pmatrix} u \\ w \end{pmatrix} = -\gamma^2 \begin{pmatrix} u \\ w \end{pmatrix}$$

$\frac{\ell(\ell+1)}{r^2}$

[TENSOR FORCES & THE DEUTERON] 5

f) And we can compare without/with tensor force:

f.1) Without: $-u'' + 2\mu V_c(r) u(r) = -\gamma^2 u(r)$

Uncoupled differential equation

f.2) With: $-\begin{pmatrix} u \\ w \end{pmatrix}'' + 2\mu V_c(r) \begin{pmatrix} u \\ w \end{pmatrix} + 2\mu V_T(r) \begin{pmatrix} 0 & 2\sqrt{2} \\ 2\sqrt{2} & -2 \end{pmatrix} \begin{pmatrix} u \\ w \end{pmatrix}$

$+ \begin{pmatrix} 0 & 0 \\ 0 & 6/r^2 \end{pmatrix} \begin{pmatrix} u \\ w \end{pmatrix} = -\gamma^2 \begin{pmatrix} u(r) \\ w(r) \end{pmatrix}$

Coupled differential equation

(provided $V_T(r) \neq 0$)

[TENSOR FORCES & THE DEUTERON] 6

g) Normalization :

$$\langle\psi|\psi\rangle = \int_0^\infty dr\, u^2(r) \longrightarrow \langle\psi|\psi\rangle = \int_0^\infty dr\,(u^2 + w^2)$$

→ w/ tensor forces

h) Asymptotic behavior of the wave functions :

$$\lim_{r\to\infty} r^n V(r) = 0 \implies$$ For $r \to \infty$, we have:

(finite-range condition)

$$-\begin{pmatrix} u \\ w \end{pmatrix}'' + \begin{pmatrix} 0 & 0 \\ 0 & \frac{6}{r^2} \end{pmatrix}\begin{pmatrix} u \\ w \end{pmatrix} = -\gamma^2 \begin{pmatrix} u \\ w \end{pmatrix}$$

two uncoupled differential equations

[TENSOR FORCES & THE DEUTERON] 7

h) Asymptotic behavior

For $r \to \infty$, we have $-\begin{pmatrix} u \\ w \end{pmatrix}'' + \begin{pmatrix} 0 & 0 \\ 0 & 6/r^2 \end{pmatrix}\begin{pmatrix} u \\ w \end{pmatrix} = -\gamma^2 \begin{pmatrix} u \\ w \end{pmatrix}$

which implies:

$$u(r) \xrightarrow[r \to \infty]{} A_S e^{-\gamma r}$$

$$w(r) \xrightarrow[r \to \infty]{} A_D e^{-\gamma r}\left(1 + \frac{3}{\gamma r} + \frac{3}{(\gamma r)^2}\right)$$

the asymptotic behavior of an $\ell=0$ w.f. and an $\ell=2$ w.f.

[TENSOR FORCES & THE DEUTERON] 8

i) Observables:

$$r_m^2 = \frac{\langle r^2 \rangle}{4} = \frac{1}{4}\int_0^\infty r^2 (u^2 + w^2)\, dr \longrightarrow \text{matter radius}$$

$$Q_d = \frac{1}{20}\int_0^\infty r^2 \underline{\underline{w}} \,(\underline{\underline{2\sqrt{2}\, u}} - w)\, dr \longrightarrow \text{quadrupole moment}$$

$$P_D = \int_0^\infty dr\, w^2(r) \longrightarrow \text{D-wave probability}$$

(Caveat: not an observable once we consider QFT or relativistic corrections)

[TENSOR FORCES & THE DEUTERON] 9

i) Observables (continued):

$$\left.\begin{aligned} r_m &= 1.9754(9)\ \text{fm} \\ Q_d &= 0.2859(3)\ \text{fm}^2 \end{aligned}\right\}$$ from electromagnetic properties

$P_D \sim (3-5)\%$ } from magnetic moment of deuteron

$\eta = \dfrac{\Delta_D}{\Delta_S} = 0.0256(4)$ } from low-energy scattering

$\Rightarrow$ Test case to try to understand the complications of the tensor force

[TENSOR FORCES & THE DEUTERON] 10

SUMMARY

1) Tensor force $\Rightarrow$ only affects $S=1$ partial waves

$\Rightarrow$ only mixes $S=1$, $L=J\pm1$ PW

$^3S_1 - {^3D_1}$, $^3P_2 - {^3F_2}$, $^3D_3 - {^3G_3}$, etc.

2) Deuteron + tensor force $\Rightarrow$ adds a D-wave

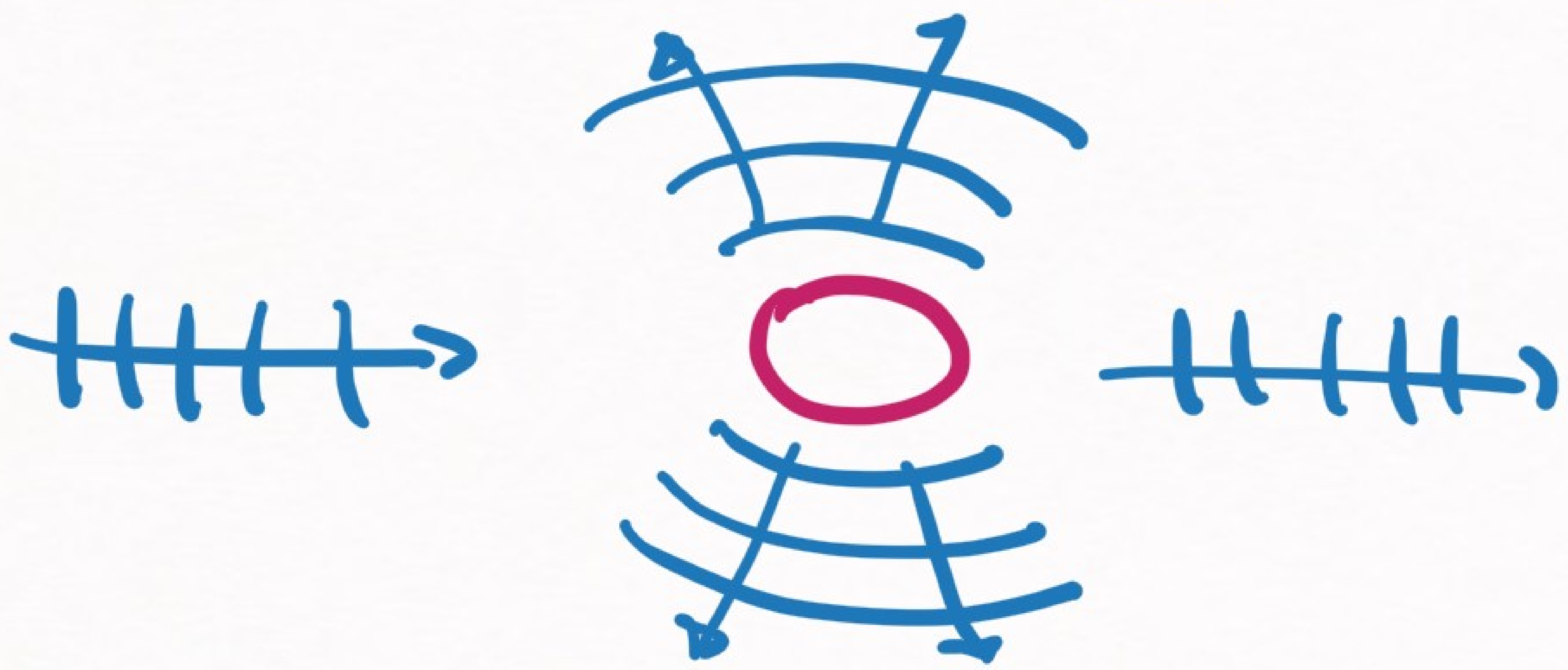
$$\psi_d(\vec{r}) = \frac{u(r)}{r}|^3S_1\rangle \Rightarrow \psi_d(\vec{r}) = \frac{u(r)}{r}|^3S_1\rangle + \frac{w(r)}{r}|^3D_1\rangle$$

(no tensor force) (w/ tensor force)

[SCATTERING & TENSOR FORCES]

WARNING: REALLY COMPLICATED

$\Rightarrow$ How does $S_{12}(\hat{r})$ affects scattering?



Still scattering w/spin

$$\psi_{\vec{k}}(\vec{r}) \xrightarrow[|\vec{r}|\to\infty]{} e^{i\vec{k}\cdot\vec{r}}\,|s_1 m_1\rangle|s_2 m_2\rangle + \sum_{m_1' m_2'} f_{m_1 m_2 m_1' m_2'}(\Omega)\,\frac{e^{ikr}}{r}\,|s_1 m_1'\rangle|s_2 m_2'\rangle$$

$\Rightarrow$ $\boxed{\text{Still the same}}$

[SCATTERING & TENSOR FORCES]

a) If we have

$$\psi_{\vec{k}}(\vec{r}) \xrightarrow[|\vec{r}| \to \infty]{} e^{i\vec{k}\cdot\vec{r}} |s_1 m_1\rangle |s_2 m_2\rangle + \sum_{m_1' m_2'} \underbrace{f_{m_1 m_2 m_1' m_2'}(\Omega)}_{\text{this is not affected by the tensor force}} \frac{e^{ikr}}{r} |s_1 m_1'\rangle |s_2 m_2'\rangle$$

b) $[S_{12}(\vec{r}), \vec{S}^2] \neq 0$

in general, but for $1/2 \otimes 1/2$ (np), S is conserved

$$f_{m_1 m_2 m_1' m_2'}(\Omega) \longrightarrow \underline{\underline{f^S_{m_S m_S'}(\Omega)}}$$

[SCATTERING & TENSOR FORCES] $\Rightarrow$ Two cases

a) Spin singlets $(S=0)$ $\longrightarrow$ tensor force vanishes

$$\psi_{\vec{k}}(\vec{r}) \longrightarrow \left[e^{i\vec{k}\cdot\vec{r}} + f^{s}(\Omega)\frac{e^{ikr}}{r}\right] \times |00\rangle$$

$f^{s}(\Omega)$ admits the usual partial wave expansion:

$$f^{s}(\Omega) = \sum_{\ell=0}^{\infty}(2\ell+1)\, f^{s}_{\ell}(k)\, P_{\ell}(\hat{k}\cdot\hat{r})$$

$$f^{s}_{\ell}(k) = \frac{1}{k\cot\delta^{s}_{\ell} - ik}$$

Everything is just as in spinless scattering

(this case is easy)

[SCATTERING & TENSOR FORCES] $\Rightarrow$ Second case

b) Spin triplets $(S=1)$: $\rightarrow$ tensor force does not vanish

$$\psi_k(\vec{r}) \rightarrow e^{i\vec{k}\cdot\vec{r}}\,|1 m_s\rangle + \sum_{m_s'} f^{t}_{m_s m_s'}(\Omega)\,\frac{e^{ikr}}{r}\,|1 m_s'\rangle$$

most important diff. : $\left[f^{t}_{m_s m_s'}(\Omega) \neq \sum_{\ell} (2\ell+1)\, f^{s}_{\ell\, m_s m_s'}(k) \times P_\ell(\hat{k}\cdot\hat{r}) \right]$

$\rightarrow$ We can't use this expansion because the ℓ's are mixed by the tensor force

$\rightarrow$ We can't expand in $\underline{\underline{\ell}}$ alone (we will have to expand in $\underline{\underline{J}}$)

[SCATTERING & TENSOR FORCES] $\mathcal{Y}^{l}_{jm}(\hat{r}) \rightarrow$ generalized spherical harmonic

$\Rightarrow$ l is not conserved, but j is conserved:

$\Rightarrow$ we have to expand in j, but how?

$\Rightarrow$ Example: plane wave x spinor $\leftarrow$ easy example

$$e^{i\vec{k}\cdot\vec{r}} |s m_s\rangle = 4\pi \sum_{l m_l} i^l j_l(kr) Y^*_{l m_l}(\hat{k}) Y_{l m_l}(\hat{r}) |s m_s\rangle$$ (usual exp. in l)

We define: $$\left[\mathcal{Y}^{l}_{jm}(\hat{r}) = \sum_{m_l m_s} \langle l m_l\, s m_s | j m\rangle Y_{l m}(\hat{r}) |s m_s\rangle \right]$$

$$\Rightarrow \left[Y_{l m_l}(\hat{r}) |s m_s\rangle = \sum_{j=l-1}^{l+1} \mathcal{Y}^{l}_{jm}(\hat{r}) \langle j m | l m_l\, s m_s\rangle \right]$$

[SCATTERING & TENSOR FORCES]

(if you expand using $\mathcal{Y}^{\ell}_{jm}$)

$\Rightarrow$ We include this in the expansion:

$$\left[\; e^{i\vec{k}\cdot\vec{r}}|s m_s\rangle = 4\pi \underline{\underline{\sum_{jm}}} \sum_{\ell=j-1}^{j+1} i^{\ell} j_\ell(kr) \langle jm|\ell m_\ell\, s m_s\rangle Y^{*}_{\ell m_\ell}(\hat{k})\, \mathcal{Y}^{\ell}_{jm}(\hat{r}) \;\right]$$

and then define : $\left[\; Z^{\ell s m_s}_{jm}(\hat{k}) = \langle \ell m_\ell\, \underline{\underline{s m_s}}|jm\rangle Y_{\ell m_\ell}(\hat{k}) \;\right]$

$$\Rightarrow \left[\; e^{i\vec{k}\cdot\vec{r}}|s m_s\rangle = 4\pi \underline{\underline{\sum_{jm}}} \sum_{\ell=j-1}^{j+1} i^{\ell} j_\ell(kr)\, Z^{\ell s m_s *}_{jm}(\hat{k})\, \mathcal{Y}^{\ell}_{jm}(\hat{r}) \;\right]$$

[expansion in j]

[SCATTERING & TENSOR FORCES]

$\Rightarrow$ Next, it will be convenient to redefine $j_\ell(kr)$ as:

(will be useful later)

$$\underline{j_\ell(kr)} = \frac{h_\ell^{(+)}(kr) - h_\ell^{(-)}(kr)}{2i}$$

$$\underline{h_\ell^{(+)}(kr)} \xrightarrow[kr\to\infty]{} \frac{e^{i(kr - \ell\frac{\pi}{2})}}{kr} \qquad \underline{h_\ell^{(-)}} \xrightarrow[kr\to\infty]{} \frac{e^{-i(kr - \ell\frac{\pi}{2})}}{kr}$$

$h_\ell^{(+)}, h_\ell^{(-)}$ are very similar to the spherical Hankel functions (only difference is a phase)

[SCATTERING & TENSOR FORCES] $\forall$ equivalent to the expansion with $j_\ell(kr)$

$\Rightarrow$ Why this change? Consider first the spinless case:

$$e^{i\vec{k}\cdot\vec{r}} = 4\pi \sum_{\ell m} i^\ell \, \frac{h_\ell^{(+)}(kr) - h_\ell^{(-)}(kr)}{2i} \, Y^*_{\ell m}(\hat{k}) Y_{\ell m}(\hat{r})$$

$$\Rightarrow \psi_{\vec{k}}(\vec{r}) \xrightarrow[|\vec{r}|\to\infty]{} 4\pi \sum_{\ell m} i^\ell \, \frac{S_\ell(k)\, h_\ell^{(+)}(kr) - h_\ell^{(-)}(kr)}{2i} \, Y^*_{\ell m}(\hat{k}) Y_{\ell m}(\hat{r})$$

$\left[S_\ell(k) = e^{2i\delta_\ell(k)} \right] \Rightarrow$ S-matrix (or its partial wave projection)

(this will probe to be easier later)

[SCATTERING & TENSOR FORCES]

⇒ Now we will build $\psi_{\vec{k}}(\vec{r})$ for the triplet in analogy to the expansion of $e^{i\vec{k}\cdot\vec{r}}|1 m_s\rangle$ (as in previous slide)

$$e^{i\vec{k}\cdot\vec{r}}|1 m_s\rangle = 4\pi\sum_{Jm} i^J j_J(kr)\, \mathcal{Z}^{J s m_s *}_{Jm}(\hat{k})\, \mathcal{Y}^{J}_{Jm}(\hat{r}) \Big\} \Rightarrow \ell = j$$

(slide 43) (s = 1) (no mixing)

$$+\,4\pi\sum_{Jm}\Big\{ i^{J-1} j_{J-1}(kr)\, \mathcal{Z}^{J-1, s m_s *}_{Jm}(\hat{k})\, \mathcal{Y}^{J-1}_{Jm}(\hat{r}) + i^{J+1} j_{J+1}(kr)\, \mathcal{Z}^{J+1, s m_s *}_{Jm}(\hat{k})\, \mathcal{Y}^{J+1}_{Jm}(\hat{r}) \Big\}$$

$\ell = j-1$ $\qquad$ $\ell = j+1$

they mix

[SCATTERING & TENSOR FORCES] $j_\ell(x) \to h_\ell^{(+)}(x), h_\ell^{(-)}(x)$

⇒ Now, we make the following changes: $(e^{i\vec{k}\cdot\vec{r}}|sm_s\rangle)$

$$\ell = j-1$$

$$\mathcal{Z}_{jm}^{j-1,sm_s *}(\hat{k})\, \mathcal{Y}_{jm}^{j-1}(\hat{r})\, j_{j-1}(kr) = \mathcal{Z}_{jm}^{j-1,sm_s *}(\hat{k})\, \mathcal{Y}_{jm}^{j-1}(\hat{r})\, \frac{h_{j-1}^{(+)}(kr) - h_{j-1}^{(-)}(kr)}{2i}$$

$+$ (the same expression for $\ell = j+1$)

⇒ And for the $e^{i\vec{k}\cdot\vec{r}}|sm_s\rangle \to \psi_{\vec{k}}(\vec{r})$ transformation:

$$\mathcal{Z}_{jm}^{j-1,sm_s *}(\hat{k})\, \mathcal{Y}_{jm}^{j-1}(\hat{r})\, \frac{h_{j-1}^{(+)} - h_{j-1}^{(-)}}{2i} \longrightarrow \mathcal{Z}_{jm}^{j-1,sm_s *}(\hat{k}) \left\{ \mathcal{Y}_{jm}^{j-1}(\hat{r})\, \frac{S_{j-1,j-1} h_{j-1}^{(+)} - h_{j-1}^{(-)}}{2i} + \mathcal{Y}_{jm}^{j+1}(\hat{r})\, \frac{S_{j+1,j-1}\, h_{j+1}^{(+)}}{2i} \right\}$$

like slide 43 but for coupled

[SCATTERING & TENSOR FORCES]

$\Rightarrow$ We do the following substitutions

a) $\ell = J$: $\mathcal{Z}_{Jm}^{JSm_s *}(\hat{k})\, \mathcal{Y}_{Jm}^{J}(\hat{r}) \dfrac{h_J^{(+)} - h_J^{(-)}}{2i} \rightarrow \mathcal{Z}_{Jm}^{JSm_s^*}\, \mathcal{Y}_{Jm}^{J}(\hat{r}) \dfrac{S_{J,J}\, h_J^{(+)} - h_J^{(-)}}{2i}$

b) $\ell = J-1$: $(\ldots) \rightarrow \mathcal{Z}_{Jm}^{J-1\,Sm_s^*}(\hat{k}) \Big\{ \mathcal{Y}_{Jm}^{J-1}(\hat{r}) \dfrac{S_{J-1,J-1}\, h_{J-1}^{(+)} - h_{J-1}^{(-)}}{2i}$

$+ \mathcal{Y}_{Jm}^{J+1}(\hat{r}) \dfrac{S_{J+1,J-1}}{2i} h_{J+1}^{(+)} \Big\}$

c) $\ell = J+1$: $(\ldots)$

$\hookrightarrow \mathcal{Z}_{Jm}^{J+1\,Sm_s^*}(\hat{k}) \Big\{ \mathcal{Y}_{Jm}^{J+1}(\hat{r}) \dfrac{S_{J+1,J+1}\, h_{J+1}^{(+)} - h_{J+1}^{(-)}}{2i}$

$+ \mathcal{Y}_{Jm}^{J-1}(\hat{r}) \dfrac{S_{J-1,J+1}}{2i} h_{J-1}^{(+)} \Big\}$

Complicated, but mechanical

[SCATTERING & TENSOR FORCES]

⇒ And now we build $\psi_{\vec{k}}(\vec{r})$ from analogy to this:

$$e^{i\vec{k}\cdot\vec{r}}|s\,m_s\rangle = 4\pi\sum_{jm} i^j j_j(kr)\, \mathcal{Z}^{js\,m_s *}_{jm}(\hat{k})\, \mathcal{Y}^{j}_{jm}(\hat{r})$$

$$+4\pi\sum_{jm}\left\{ i^{j-1} j_{j-1}(kr)\, \mathcal{Z}^{j-1\,s\,m_s}_{jm}\, \mathcal{Y}^{j-1}_{jm} + i^{j+1} j_{j+1}(kr)\, \mathcal{Z}^{j+1\,s\,m_s}_{jm}\, \mathcal{Y}^{j+1}_{jm} \right\}$$

Leading to:

$$\psi_{\vec{k}}(\vec{r}) = 4\pi\sum_{jm} i^j \times \Big(\text{(a) in previous slide}\Big) \quad \text{with } \ell = j \rightarrow \text{does not mix}$$

$$+4\pi\sum_{jm}\Big\{ i^{j-1}\times\big(\text{(b) in prev.}\big) + i^{j+1}\times\big(\text{(c) in prev.}\big)\Big\} \rightarrow \text{mix}$$

$\ell = j-1$ (for (b)), $\ell = j+1$ (for (c))

[SCATTERING & TENSOR FORCES]

$\Rightarrow$ The expression for $\psi_{\vec{k}}(\vec{r})$ is not particularly simple, but gives us a pattern:

a) $$\mathcal{Y}^{J}_{Jm}\,\frac{h_J^{(+)}-h_J^{(-)}}{2i} \longrightarrow \mathcal{Y}^{J}_{Jm}\,\frac{S_{JJ}\,h_J^{(+)}-h_J^{(-)}}{2i}\,,\quad S_{JJ}=e^{2i\delta_J(k)}$$

b) $$\mathcal{Y}^{J-1}_{Jm}\,\frac{h_{J-1}^{(+)}-h_{J-1}^{(-)}}{2i} \longrightarrow \mathcal{Y}^{J-1}_{Jm}\,\frac{S_{J-1,J-1}h_{J-1}^{(+)}-h_{J-1}^{(-)}}{2i}+\mathcal{Y}^{J+1}_{Jm}\,\frac{S_{J+1,J-1}h_{J+1}^{(+)}}{2i}$$

c) $$\mathcal{Y}^{J+1}_{Jm}\,\frac{h_{J+1}^{(+)}-h_{J+1}^{(-)}}{2i} \longrightarrow \underbrace{\mathcal{Y}^{J+1}_{Jm}\,\frac{S_{J+1,J+1}h_{J+1}^{(+)}-h_{J+1}^{(-)}}{2i}}_{\ell'=\ell}+\underbrace{\mathcal{Y}^{J-1}_{Jm}\,\frac{S_{J-1,J+1}h_{J-1}^{(+)}}{2i}}_{\ell'\neq\ell}$$

$$\mathcal{Y}^{\ell}_{Jm} \longrightarrow$$

[SCATTERING & TENSOR FORCES]

⇒ But $\underline{\underline{\psi_{\vec{k}}}}(\vec{r})$ contains a simple pattern inside:

$$\underline{\underline{\mathcal{Y}^{j-1}_{jm}}}(\hat{r})\,\frac{b_{j-1}h^{(+)}_{j-1}-a_{j-1}h^{(-)}_{j-1}}{2i}+\mathcal{Y}^{j+1}_{jm}(\hat{r})\,\frac{b_{j+1}h^{(+)}_{j+1}-a_{j+1}h^{(-)}_{j+1}}{2i}$$

where the relation among the coefficients is:

$$\left\{\underbrace{\begin{pmatrix} b_{j-1} \\ b_{j+1} \end{pmatrix}}_{\substack{h^{(+)}\text{ coeff.}\\ \text{(outgoing wave)}}}=\underline{\underline{\underbrace{\begin{pmatrix} S_{j-1,j-1} & S_{j-1,j+1} \\ S_{j+1,j-1} & S_{j+1,j+1} \end{pmatrix}}_{\text{S-matrix}}}}\underbrace{\begin{pmatrix} a_{j-1} \\ a_{j+1} \end{pmatrix}}_{\substack{h^{(-)}\text{ coefficients}\\ \text{(incoming wave)}}}\right\}$$

[SCATTERING & TENSOR FORCES]

$\Rightarrow$ This is exactly the same as in the spinless case:

$$\mathcal{Y}_{\ell m}(\hat{r})\,\frac{b_\ell h_\ell^{(+)}(kr) - a_\ell h_\ell^{(-)}(kr)}{2i} \quad \Rightarrow \quad b_\ell = \underline{\underline{e^{2i\delta_\ell(k)}}}\,a_\ell$$

$\Rightarrow$ It just means an asymptotic wave function:

$$\psi_{\ell m}(r) = \frac{u_\ell(r)}{r}\,\mathcal{Y}_{\ell m}(\hat{r})$$

$$\frac{u_\ell(r)}{r} \longrightarrow b_\ell h_\ell^{(+)}(kr) - a_\ell h_\ell^{(-)}(kr)$$

$$\longrightarrow a_\ell\left[\underline{\underline{e^{2i\delta_\ell}}}\,h_\ell^{(+)}(kr) - h_\ell^{(-)}(kr)\right]$$

[SCATTERING & TENSOR FORCES]

$\Rightarrow$ This form allows us to generalize $e^{2i\delta_\ell}$ to couple channels:

$$\frac{1}{r}\begin{pmatrix} u \\ w \end{pmatrix} \longrightarrow \begin{pmatrix} h^{(+)}_{J-1} & 0 \\ 0 & h^{(+)}_{J+1} \end{pmatrix}\begin{pmatrix} b_{J-1} \\ b_{J+1} \end{pmatrix} - \begin{pmatrix} h^{(-)}_{J-1} & 0 \\ 0 & h^{(+)}_{J+1} \end{pmatrix}\begin{pmatrix} a_{J-1} \\ a_{J+1} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} b_{J-1} \\ b_{J+1} \end{pmatrix} = S_J \begin{pmatrix} a_{J-1} \\ a_{J+1} \end{pmatrix}$$

$$\left[\psi_{Jm}(\vec{r}) = \frac{u(r)}{r} \mathcal{Y}^{J-1}_{Jm}(\hat{r}) + \frac{w(r)}{r} \mathcal{Y}^{J+1}_{Jm}(\hat{r}) \right]$$

(same as in deuteron, but for all Jm)

[SCATTERING & TENSOR FORCES] $\Rightarrow$ Finally we build $e^{i\vec{k}\cdot\vec{r}}|sm_s\rangle$ and $\psi_{\vec{k}}(\vec{r})$

$$e^{i\vec{k}\cdot\vec{r}}|sm_s\rangle = 4\pi \sum_{jm} \sum_{\ell=j-1}^{j+1} i^\ell j_\ell(kr)\, Z^{\ell s m_s *}_{jm}(\hat{k})\, \mathcal{Y}^\ell_{jm}(\hat{r})$$

$$\psi_{\vec{k}}(\vec{r}) = 4\pi \sum_{jm} \sum_{\ell=j-1} i^\ell Z^{\ell s m_s *}_{jm}(\hat{k})\, \psi^\ell_{jm}(\vec{r})$$

with:
$$\psi^{\ell=j}_{jm}(\vec{r}) \xrightarrow[|\vec{r}|\to\infty]{} \mathcal{Y}^j_{jm}(\hat{r})\, \frac{e^{2i\delta_j} h^{(+)}_j - h^{(-)}_j}{2i}$$

$$\psi^{\ell=j\pm1}_{jm}(\vec{r}) \xrightarrow[|\vec{r}|\to\infty]{} \left(\mathcal{Y}^{j-1}_{jm}(\hat{r}) \;\; \mathcal{Y}^{j+1}_{jm}(\hat{r})\right) \left[\begin{pmatrix} h^{(+)}_{j-1} & \\ & h^{(+)}_{j+1} \end{pmatrix} S^j - \begin{pmatrix} h^{(-)}_{j-1} & \\ & h^{(-)}_{j+1} \end{pmatrix} \right] \begin{pmatrix} a_{j-1} \\ a_{j+1} \end{pmatrix}$$

$$+ \begin{pmatrix} a_{j-1} \\ a_{j+1} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\ell=j-1} \text{ or } \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\ell=j+1}$$

[SCATTERING & TENSOR FORCES]

⇒ From the previous expressions, it should be possible to find $f_{m_s m_s'}(\Omega)$ by taking the limit:

$$\lim_{|\vec{r}|\to\infty}\left[\psi_{\vec{k}}(\vec{r}) - e^{i\vec{k}\cdot\vec{r}}|s\,m_s\rangle\right] = f_{m_s m_s'}(\Omega)\frac{e^{ikr}}{r}$$

⇒ It's really complicated, but most of you will never need the full expression

(we will not expand this here)

[SCATTERING & TENSOR FORCES] → How to define phase shifts

a) Single channel → Phase shift uniquely defined

$$S_\ell = e^{2i\delta_\ell}$$

b) Coupled channels → many definitions possible

b.1) Eigen phase shifts: $S = \begin{pmatrix} \cos\epsilon_J & -\sin\epsilon_J \\ \sin\epsilon_J & \cos\epsilon_J \end{pmatrix} \begin{pmatrix} e^{2i\delta_{J-1}} & 0 \\ 0 & e^{2i\delta_{J+1}} \end{pmatrix} \times \begin{pmatrix} \cos\epsilon_J & \sin\epsilon_J \\ -\sin\epsilon_J & \cos\epsilon_J \end{pmatrix}$

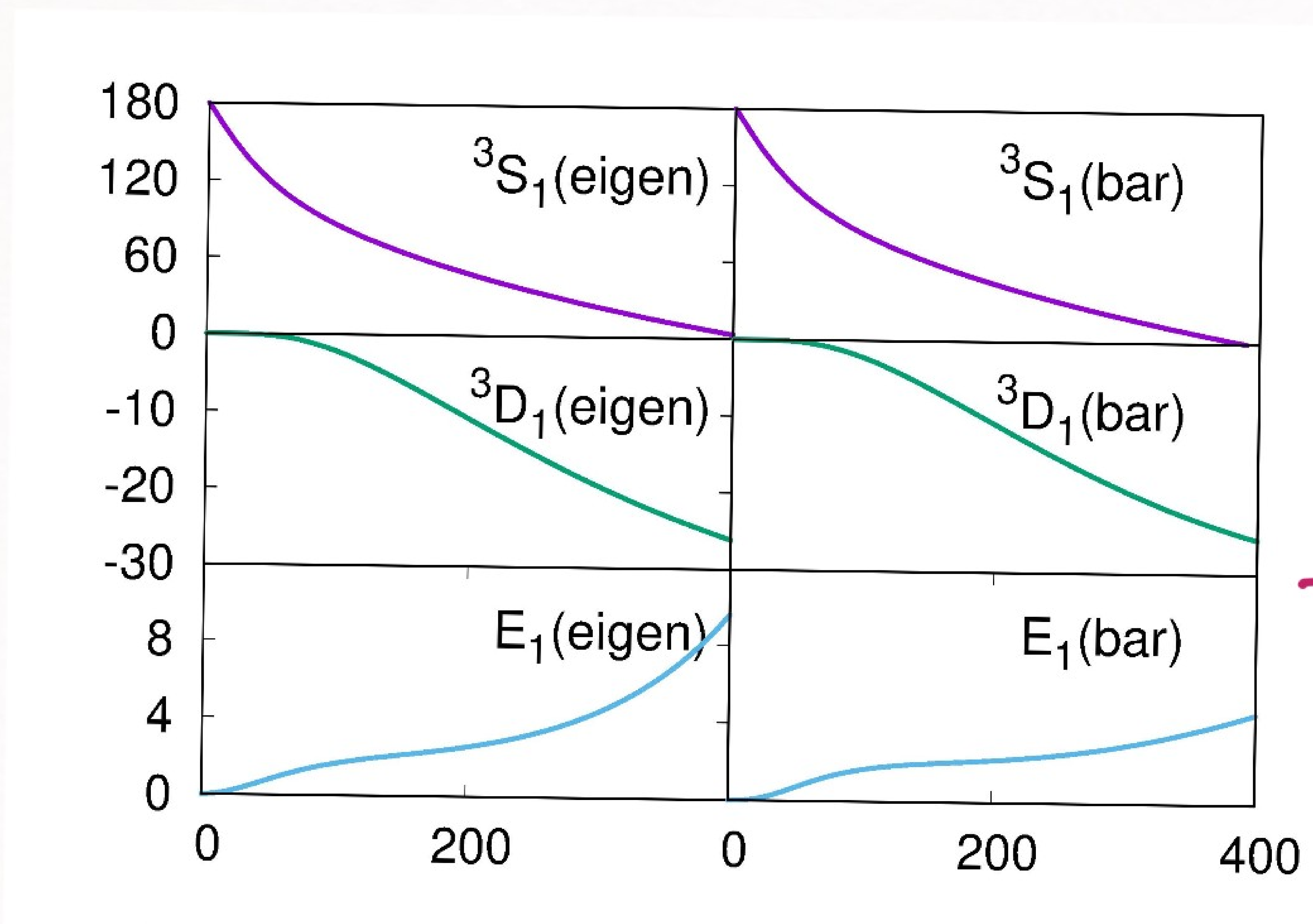
b.2) Nuclear bar phase shifts:

$$S = \begin{pmatrix} e^{i\bar{\delta}_{J-1}} & 0 \\ 0 & e^{i\bar{\delta}_{J+1}} \end{pmatrix} \begin{pmatrix} \cos(2\bar{\epsilon}_J) & i\sin(2\bar{\epsilon}_J) \\ i\sin(2\bar{\epsilon}_J) & \cos(2\bar{\epsilon}_J) \end{pmatrix} \begin{pmatrix} e^{i\bar{\delta}_{J-1}} & 0 \\ 0 & e^{i\bar{\delta}_{J+1}} \end{pmatrix}$$

[SCATTERING & TENSOR FORCES]

=> Eigen & Nuclear bar are the most common parametrizations

=> They look similar, but not identical



$^3S_1 - {}^3D_1$ partial wave (Nijmegen II potential)

this changes a bit more w/ parametrization choice

NN PHASE SHIFTS

$\Rightarrow$ How do they look like?

$\Downarrow$

$\sim$ 8000 np scattering data

$$\underbrace{\frac{d\sigma}{d\Omega}(m_1 m_2 \to m_1' m_2')}$$

fit to a model

$\Downarrow$

You obtain phase shifts

$\Rightarrow$

CAVEAT They are not directly observable, but extracted from the differential cross section

$\Downarrow$

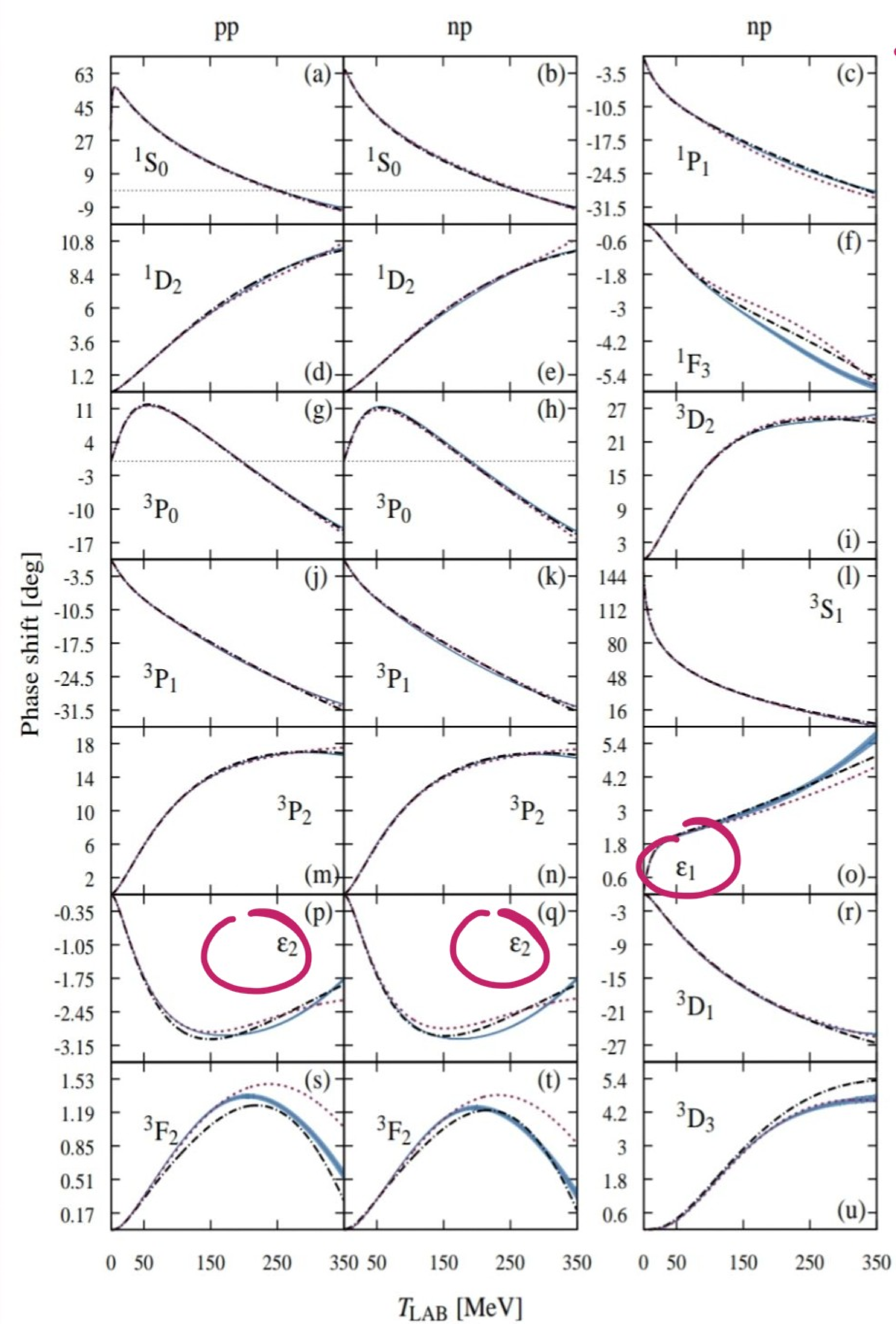
They have a certain degree of model dependence

$\Rightarrow$

NN PHASE SHIFTS

⇒ Partial wave analysis (the Granada PWA)

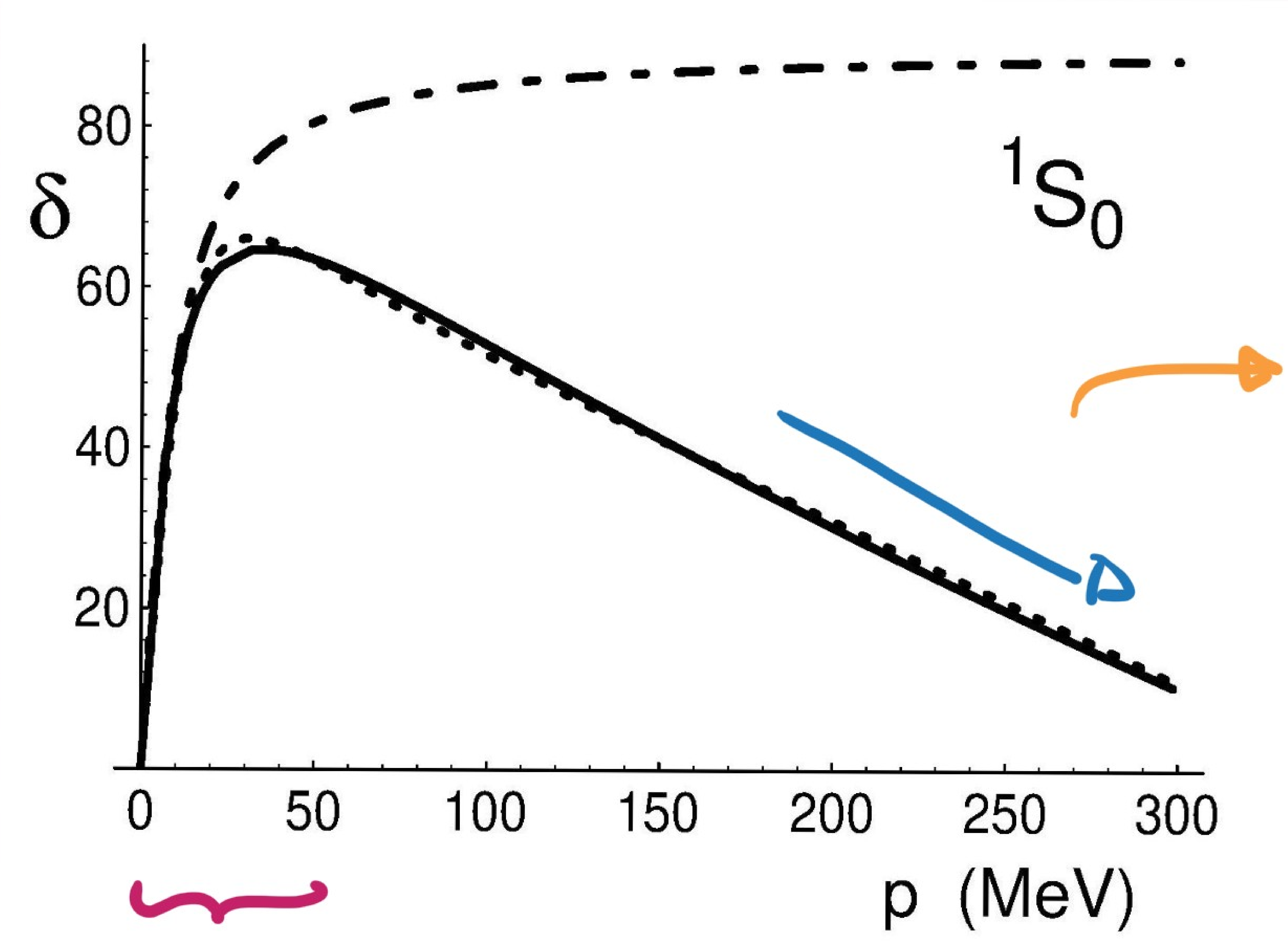
Other PWAs look really similar



singlets

triplets

[SINGLET PHASE SHIFTS] → worth looking in more detail



$L=0$ is usually important

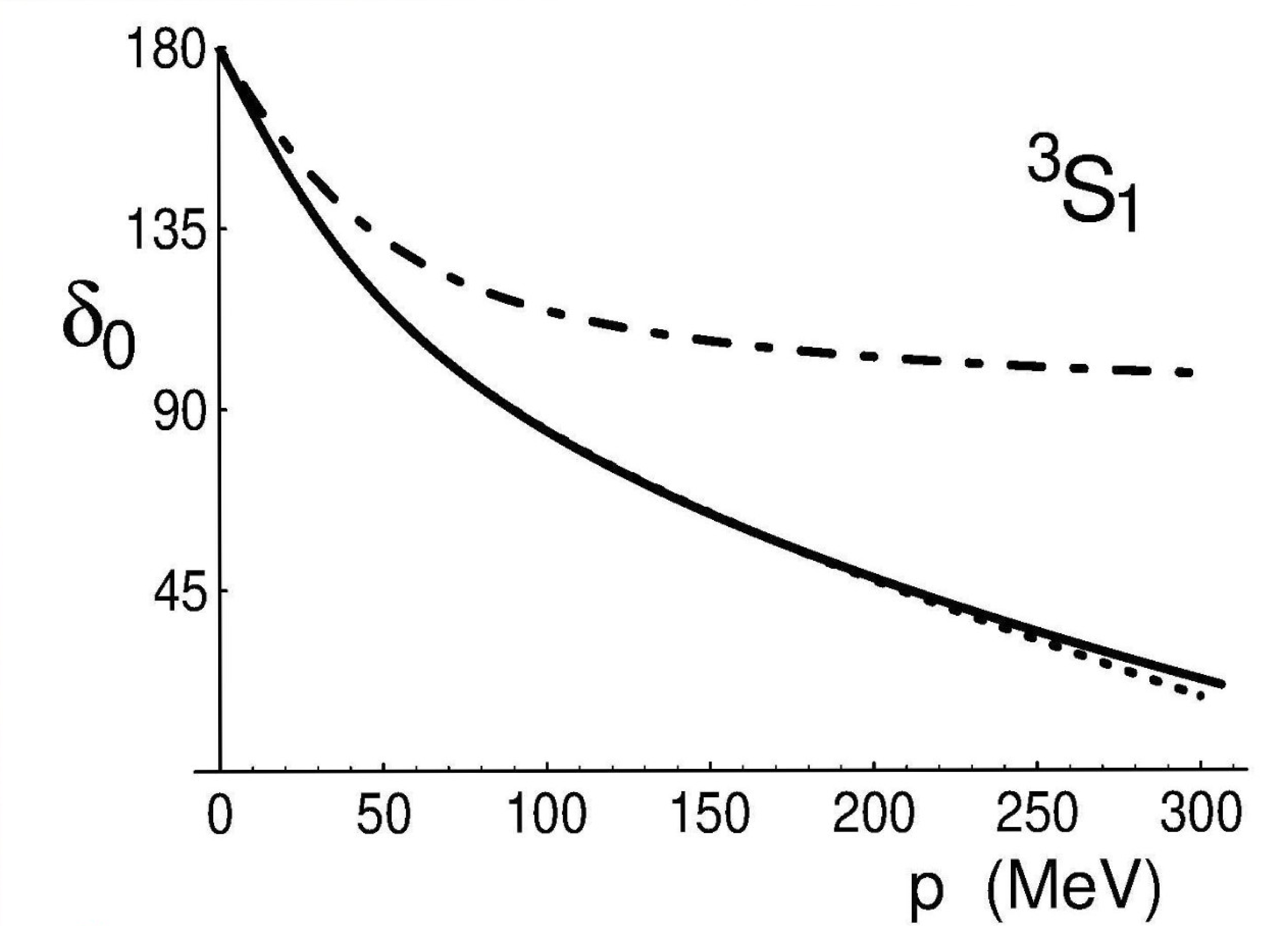
$$\boxed{\delta(p) \sim \delta_0 - p R_c}$$

⇓

Like for a hard-core potential ⇒ indicates short-range repulsion

$\delta \to -a_0 p + O(p^3) \Rightarrow a_0 < 0$ $\quad (a_0 \simeq -23.7 \text{ fm})$

[TRIPLET PHASE SHIFTS]



$\delta(p) \to \pi - a_0 p + \mathcal{O}(p^3)$

$\Rightarrow a_0 > 0$ ($a_0 \simeq 5.4$ fm)

$$\delta(p) \xrightarrow[p \to 0]{} \pi - a_0 p + \mathcal{O}(p^3)$$

For bound states,

$\delta(p) \to \pi$ for $p \to 0$

(Levinson theorem:

$$\delta(0) - \delta(\infty) = n\pi$$

w/ n number of bound states)

Recall

1) Spin generates a few changes for scatterings

$$\psi_k(\vec{r}) \to e^{i\vec{k}\cdot\vec{r}} + f(\Omega)\frac{e^{ikr}}{r}$$

$$\Rightarrow \psi_k(\vec{r}) \to e^{i\vec{k}\cdot\vec{r}}|s m_s\rangle + \sum_{m_s'} f_{m_s m_s'}(\Omega)\frac{e^{ikr}}{r}|s m_s'\rangle$$

2) This changes are rather trivial in most situation

3) Exception $\Rightarrow$ tensor force

Why? $\Rightarrow$ tensor force mixes orbital angular momentum

RECAP!

4) However, tensor forces still preserve total angular momentum $\Rightarrow \vec{J} = \vec{L} + \vec{S}$

5) Trick: organize the expansion of the scattering wave function in terms of jm (instead of ℓm_ℓ)

$$e^{i\vec{k}\cdot\vec{r}} |s m_s\rangle = 4\pi \sum_{jm} \sum_{\ell} i^{\ell} j_\ell(kr) \mathcal{Z}_{jm}^{\ell s m_s *}(\hat{k}) \mathcal{Y}_{jm}^{\ell}(\hat{r})$$

$s = 1$

$$\sum_{\ell} = \sum_{\ell = j-1}^{j+1}$$

$$\psi_{\vec{k}}(\vec{r}) = 4\pi \sum_{jm} \sum_{\ell} \mathcal{Z}_{jm}^{\ell s m_s *}(\hat{k}) \psi_{jm}^{\ell}(\vec{r})$$

$$\Rightarrow \lim_{r \to \infty} \psi_{\vec{k}}(\vec{r}) - e^{i\vec{k}\cdot\vec{r}} |s m_s\rangle = \sum_{m_s'} f_{m_s m_s'}(\Omega) \frac{e^{ikr}}{r} |s m_s'\rangle$$

RECAP!

6) How to define phase shifts in this case?

w/o tensor force: $\psi_{\ell m}(\vec{r}) = \frac{u_\ell(r)}{r} Y_{\ell m}(\hat{r})$

$$\frac{u_\ell(r)}{r} \to \frac{S_\ell(k)\, h_\ell^{(+)}(kr) - h_\ell^{(-)}(kr)}{2i}$$

Equivalent to:

$$\frac{u_\ell(r)}{r} \to e^{i\delta_\ell}\left(\cos\delta_\ell\, j_\ell(kr) - \sin\delta_\ell\, y_\ell(kr)\right)$$

w/ tensor force: $\psi_{jm}^{\ell=j\pm1}(\vec{r}) = \frac{u(r)}{r} \mathcal{Y}_{jm}^{\ell=j-1}(\hat{r}) + \frac{w(r)}{r} \mathcal{Y}_{jm}^{\ell=j+1}(\hat{r})$

$$\frac{1}{r}\begin{pmatrix} u \\ w \end{pmatrix} \to \left[\begin{pmatrix} h_{j-1}^{(+)} & 0 \\ 0 & h_{j+1}^{(+)} \end{pmatrix} \underbrace{S}_{} - \begin{pmatrix} h_{j-1}^{(-)} & 0 \\ 0 & h_{j+1}^{(-)} \end{pmatrix}\right] \begin{pmatrix} a_{j-1} \\ a_{j+1} \end{pmatrix}$$

[S-matrix]

RECAP!

7) S-matrix is always the same, but phase shifts admit several definitions:

one channel: $S_\ell(k) = e^{2i\delta_\ell(k)}$ ✓ easy

two channels:

7.a) Eigen: $S^J(k) = \begin{pmatrix} \cos\epsilon & -\sin\epsilon \\ \sin\epsilon & \cos\epsilon \end{pmatrix} \begin{pmatrix} e^{2i\delta_1} & 0 \\ 0 & e^{2i\delta_2} \end{pmatrix} \begin{pmatrix} \cos\epsilon & \sin\epsilon \\ -\sin\epsilon & \cos\epsilon \end{pmatrix}$

7.b) Bar: $S^J(k) = \begin{pmatrix} e^{i\bar\delta_1} & 0 \\ 0 & e^{i\bar\delta_2} \end{pmatrix} \begin{pmatrix} \cos 2\bar\epsilon & i\sin 2\bar\epsilon \\ i\sin 2\bar\epsilon & \cos 2\bar\epsilon \end{pmatrix} \begin{pmatrix} e^{i\bar\delta_1} & 0 \\ 0 & e^{i\bar\delta_1} \end{pmatrix}$

=D And you can invent many more!

SEE YOU ON
TUESDAY
15:50