# NUCLEAR THEORY 3

a) REVIEW OF THE TWO-BODY PROBLEM

→ solutions for $E>0$, $E<0$, asymptotic properties, partial waves, etc.

b) SCATTERING IN QUANTUM MECHANICS

→ basic theory (definitions, partial wave expansion, etc.)

$\boxed{\text{Nuclear physics}} \rightarrow$ Nuclei

(Bound states of $A \geq 2$ nucleons)

a) $\underline{\underline{A = 2}} \rightarrow$ straight forward (realy easy to calculate)

deuteron

b) $A = 3, 4 \rightarrow$ doable w/ few-body techniques

$^3H$, $^3He$, $^4He$

c) $A > 4 \rightarrow$ increasingly difficult, eventually

we have to use nuclear models

$\boxed{A=2} \Rightarrow$ We will review the two-body problem

So where do we start?

$\boxed{\text{Schrödinger equation}}$ for two-body: (too many variables)

(simplify this) →

$$\Big[\underbrace{\frac{p_1^2}{2m_1}+\frac{p_2^2}{2m_2}}_{\text{momenta \& masses of particles 1 \& 2}}+\underbrace{V(\vec{r}_1-\vec{r}_2)}_{\text{potential}}\Big]\underbrace{\Psi(\vec{r}_1,\vec{r}_2)}_{\text{wave function}}=\underbrace{E_T}_{\text{energy of the system}}\Psi(\vec{r}_1,\vec{r}_2)$$

## [CENTER-OF-MASS COORDINATES] ← (our simplification)

a) p-space   a.1) total momentum: $\vec{P} = \vec{p}_1 + \vec{p}_2$

a.2) relative momentum: $\vec{p} = \dfrac{m_1\vec{p}_2 - m_2\vec{p}}{m_1 + m_2}$

b) r-space   b.1) c.m. radius: $\vec{R} = \dfrac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$

b.2) relative radius: $\vec{r} = \vec{r}_1 - \vec{r}_2$

c) total and reduce masses: $M = m_1 + m_2$

$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}$$

# [SCHRÖDINGER IN C.M. COORDINATES]

1) We begin with: $\left[\frac{p_1^2}{2m_1}+\frac{p_2^2}{2m_2}+V(\vec{r}_1-\vec{r}_2)\right]\Psi(\vec{r}_1,\vec{r}_2)$
$= E_T\Psi(\vec{r}_1,\vec{r}_2)$

Change of basis:

$$\vec{p}_1=\frac{m_1}{m_1+m_2}\vec{P}+\vec{p} \qquad \vec{r}_1=\vec{R}+\frac{m_2}{m_1+m_2}\vec{r}$$

$$\vec{p}_2=\frac{m_2}{m_1+m_2}\vec{P}-\vec{p} \qquad \vec{r}_2=\vec{R}-\frac{m_1}{m_1+m_2}\vec{r}$$

$$M=m_1+m_2$$

$$\frac{1}{\mu}=\frac{1}{m_1}+\frac{1}{m_2}$$

$\Rightarrow$ $\left[\frac{p^2}{2\mu}+\frac{P^2}{2M}+V(\vec{r})\right]\Psi(\vec{r},\vec{R})$
$= E_T\Psi(\vec{r},\vec{R})$

[NEXT : REMOVE THE C.M. MOVEMENT]

$$\psi(\vec{r},\vec{R}) = \psi(\vec{r})\,\Psi_{CM}(\vec{R})$$

$$\underline{\underline{\Psi_{CM}(\vec{R})}} = e^{i\vec{K}\cdot\vec{R}}, \quad \underline{\underline{E_T}} = E_{CM} + \frac{\vec{K}^2}{2M}$$

$$\left[\frac{p^2}{2\mu} + V(\vec{r})\right]\psi(\vec{r}) = E_{CM}\,\psi(\vec{r})$$

[NEXT TRICK : PARTIAL WAVE DECOMPOSITION]

$$\frac{\vec{p}^{\,2}}{2\mu} = -\frac{\vec{\nabla}^2}{2\mu} = -\frac{1}{2\mu}\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right) + \frac{\vec{L}^2}{2\mu} \longrightarrow (*)$$

$(*) \longrightarrow$ angular momentum is a good quantum number (if $V(\vec{r}) = V(r)$)

# PARTIAL WAVES

$\longrightarrow [\vec{L}^2, H] = 0 \longrightarrow$ good quantum number

$(V(\vec{r}) = V(r))$

a) Spherical harmonics:

$$Y_{\ell m}(\hat{r}) \;/\; \vec{L}^2 Y_{\ell m}(\hat{r}) = \ell(\ell+1) Y_{\ell m}(\hat{r})$$

$$L_z Y_{\ell m}(\hat{r}) = m\, Y_{\ell m}(\hat{r})$$

b) Reduced wave function:

$$\psi(\vec{r}) = \frac{u_\ell(r)}{r} Y_{\ell m}(\hat{r})$$

$u_\ell(r) \rightarrow$ reduced wave function

[SPHERICAL HARMONICS] (Review)

a) Orthogonality: $\int d^2\hat{r}\, Y^*_{\ell m}(\hat{r})\, Y_{\ell' m'}(\hat{r}) = \delta_{\ell\ell'}\delta_{mm'}$

b) Sum: $\displaystyle\sum_{m=-\ell}^{+\ell} Y^*_{\ell m}(\hat{x})\, Y_{\ell m}(\hat{y}) = \frac{2\ell+1}{4\pi}\, \underline{\underline{P_\ell(\hat{x}\cdot\hat{y})}}$ → Lagrange poly$\underline{\underline{\text{nomial}}}$

c) A few examples: $\underline{\underline{Y_{00}(\hat{r}) = \frac{1}{\sqrt{4\pi}}}}$ → probably the only one you need to memorize

$$Y_{11}(\hat{r}) = -\sqrt{\frac{3}{4\pi}}\,\frac{\sin\theta\, e^{i\varphi}}{\sqrt{2}}$$

$$Y_{10}(\hat{r}) = +\sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_{1-1}(\hat{r}) = +\sqrt{\frac{3}{4\pi}}\,\frac{\sin\theta\, e^{-i\varphi}}{\sqrt{2}}$$

[REDUCED SCHRÖDINGER EQUATION] → [Schrödinger eq. for $u_\ell(r)$]

$$-u_\ell''(r) + \left[2\mu V(r) + \frac{\ell(\ell+1)}{r^2}\right] u_\ell(r) = 2\mu E_{cm}\, u_\ell(r)$$

(ordinary diff. equation)

a) $E_{cm} > 0$ (scattering)

$$E_{cm} = \frac{k^2}{2\mu} \rightarrow -u_\ell''(r) + \left[2\mu V(r) + \frac{\ell(\ell+1)}{r^2}\right] u_\ell(r) = k^2 u_\ell(r)$$

b) $E_{cm} < 0$ (bound state)

$$E_{cm} = -\frac{\gamma^2}{2\mu} \rightarrow -u_\ell''(r) + \left[2\mu V(r) + \frac{\ell(\ell+1)}{r^2}\right] u_\ell(r) = -\gamma^2 u_\ell(r)$$

$\gamma$ → the wave number (of a bound state)

$\boxed{\text{SOLUTIONS}}$ $\rightsquigarrow$ ASYMPTOTIC SOLUTIONS $(r \to \infty)$

$\Rightarrow$ Condition: we will consider a finite-range potential

$$\lim_{r \to \infty} r^n V(r) \to 0 \quad \text{for } n > 0 \quad \left(\text{e.g. } V(r) \sim \frac{e^{-mr}}{r^\alpha}\right)$$

a) $E_{CM} < 0$ $\Rightarrow$ For $r \to \infty$ we have:

for $r \to \infty$, Schrödinger simplifies to: $-u_\ell'' + \frac{\ell(\ell+1)}{r^2} u_\ell(r) = -\gamma^2 u_\ell(r)$

$$\Rightarrow \left[ u_\ell(r) \xrightarrow[r \to \infty]{} A_\ell e^{-\gamma r} \left( 1 + \mathcal{O}\left(\frac{1}{\gamma r}\right)\right) \right]$$

[Later we'll see $r \to 0$]

# Asymptotic Solutions ②

$$u_\ell(r) \rightarrow \underbrace{A_\ell}_{} e^{-\gamma r}\left[1 + \mathcal{O}\left(\frac{1}{\gamma r}\right)\right]$$

→ Asymptotic normalization coefficient (ANC)

$\ell = 0$ (S-wave) $\longrightarrow$ $u_S(r) \rightarrow A_S e^{-\gamma r}$

→ CAVEAT: only true for finite-range potentials

Coulomb → infinite range → $u_S(r) \rightarrow \frac{2}{a_B^{3/2}}\, r\, e^{-(r/a_B)}$

$\left(\lim_{r\to\infty} r V_C(r) \neq 0\right)$ $\qquad \cancel{\rightarrow} \; A_S e^{-\gamma r}$

# ASYMPTOTIC SOLUTIONS (3)

b) $E_{CM} > 0 \Rightarrow$ For $r \to \infty$ we know have:

Schrödinger goes to $\to \left[ -u_\ell'' + \frac{\ell(\ell+1)}{r} u_\ell(r) = k^2 u_\ell(r) \right]$

Solutions $\to$ $u_\ell(r) \to a_\ell(k) \hat{j}_\ell(kr) + b_\ell(k) \hat{y}_\ell(kr)$

$$\left.\begin{aligned} \hat{j}_\ell(x) &= x\, j_\ell(x) \\ \hat{y}_\ell(x) &= x\, y_\ell(x) \end{aligned}\right\} \quad y_\ell(x), j_\ell(x) \text{ spherical Bessel functions}$$

# [SPHERICAL BESSEL FUNCTIONS] (review)

a) Rayleigh's formulas

$$j_\ell(x) = (-x)^\ell \left(\frac{1}{x}\frac{d}{dx}\right)^\ell \frac{\sin x}{x}$$

$$y_\ell(x) = -(-x)^\ell \left(\frac{1}{x}\frac{d}{dx}\right)^\ell \frac{\cos x}{x}$$

[For $\ell=0$, these you might use very often]

b) A few examples: $\left[\, j_0(x) = \frac{\sin x}{x} \,,\; y_0(x) = -\frac{\cos x}{x} \,\right]$

$$\left[\begin{aligned} j_1(x) &= \frac{\sin x}{x^2} - \frac{\cos x}{x} \\ y_1(x) &= -\frac{\cos x}{x^2} - \frac{\sin x}{x} \end{aligned}\right] \quad \left[\begin{aligned} j_2(x) &= \left(\frac{3}{x^2}-1\right)\frac{\sin x}{x} - \frac{3}{x^2}\cos x \\ y_2(x) &= \left(-\frac{3}{x^2}+1\right)\frac{\cos x}{x} - \frac{3}{x^2}\sin x \end{aligned}\right]$$

# ASYMPTOTIC SOLUTIONS ④ ← $(r \to \infty)$

⇒ Asymptotic behavior: $j_\ell(x) \xrightarrow[x\to\infty]{} \frac{1}{x}\sin\left(x-\ell\frac{\pi}{2}\right)$

$$y_\ell(x) \xrightarrow[x\to\infty]{} -\frac{1}{x}\cos\left(x-\ell\frac{\pi}{2}\right)$$

[ This brings us to the concept of phase shifts: ]

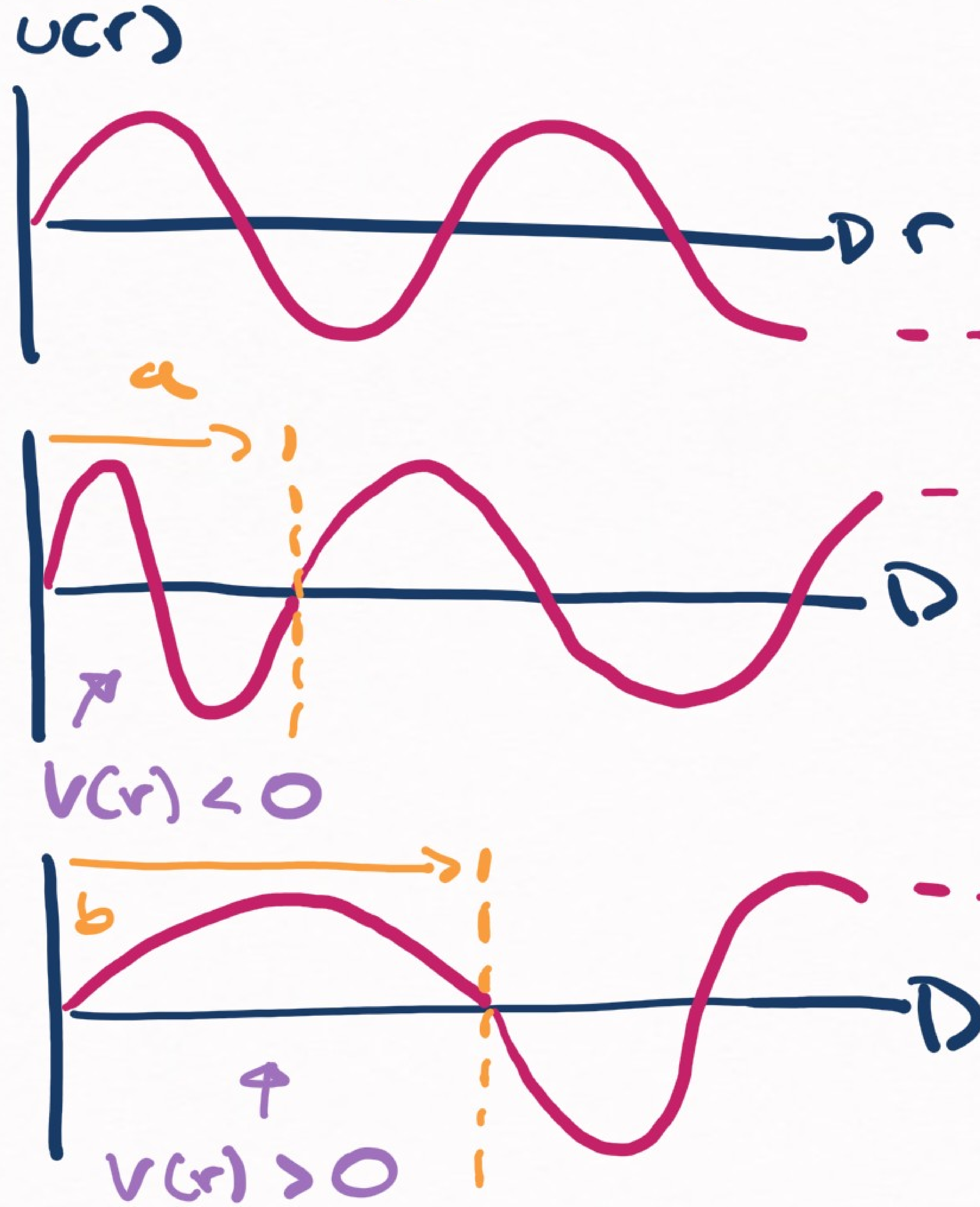
$$u_\ell(r) \longrightarrow \underbrace{N_\ell}_{\text{normalization factor}} \sin\Big(kr-\ell\frac{\pi}{2}+\underbrace{\delta_\ell(k)}_{\text{phase shift}}\Big)$$

→ Phase shift ←

(very important in scattering theory)

# PHASE SHIFTS

[Assume S-wave ($\ell = 0$)]




a) Free wave:

$V(r) = 0$

$\rightarrow \underline{\underline{\sin(kr)}}$

b) Attraction:

$V(r) < 0, \; r < a$

$(V(r) = 0, \; r > a)$

$\rightarrow \sin(kr + \underline{\underline{\delta}})$

$\underline{\underline{\delta > 0}}$

c) Repulsion:

$V(r) > 0, \; r < b$

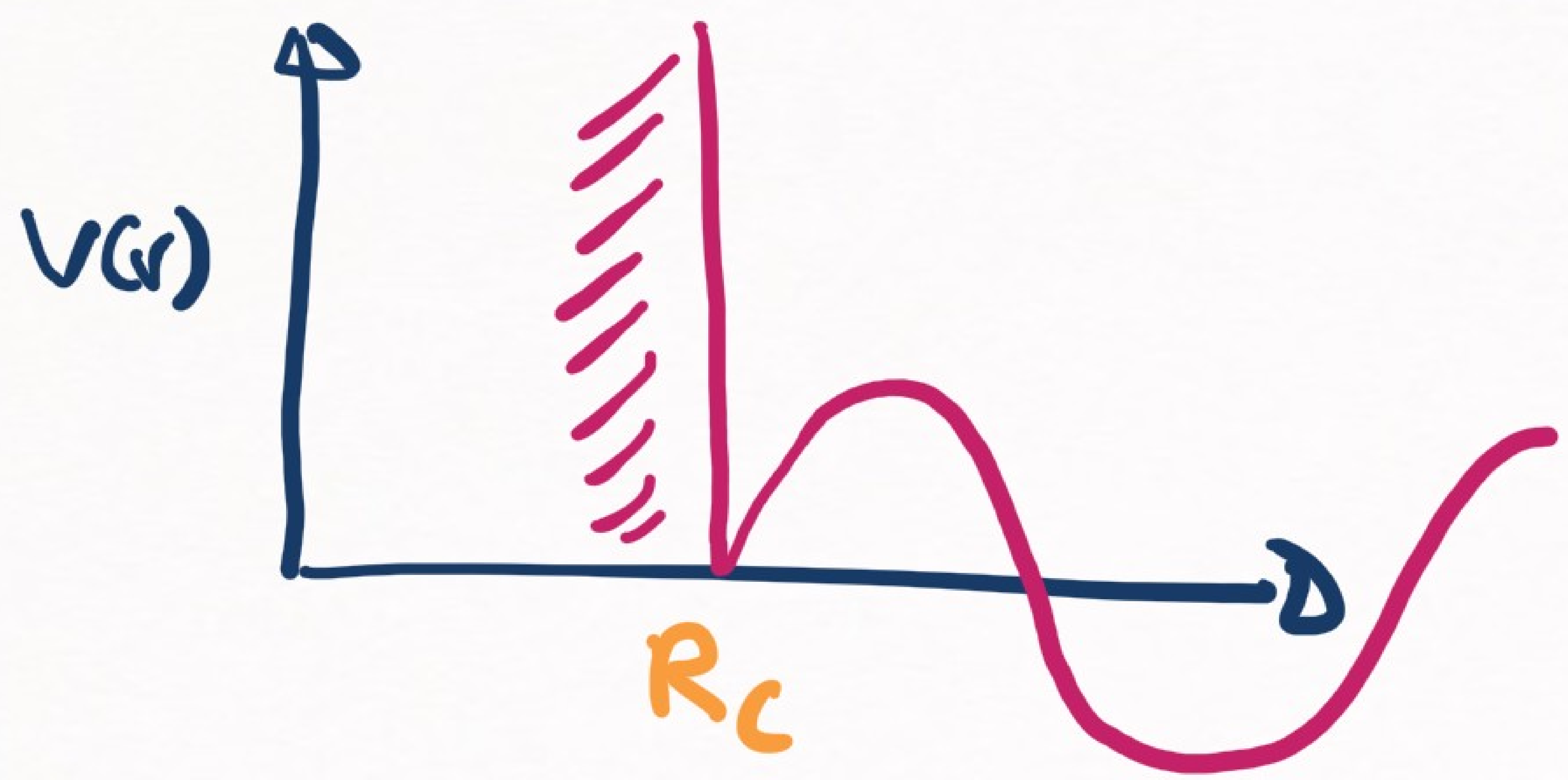
$(V(r) = 0, \; r > b)$

$\rightarrow \sin(kr + \underline{\underline{\delta}})$

$\underline{\underline{\delta < 0}}$

$\boxed{\text{PHASE SHIFT}}$ ⟹ Shift of the phase of the wave function with respect to the free case

Really easy example:




⟹ Hard-core potential: $\begin{bmatrix} V(r) = \infty \\ r < R_c \end{bmatrix}$

$u(R_c) = 0 \quad , \; u(r) = \sin(kr + \delta)$

$$\Rightarrow \boxed{\delta = -k R_c}$$

Reminder ⟹ Only for finite-range potentials:

Coulomb $\rightarrow u_\ell(r) \rightarrow \sin(kr - \eta \log(2kr) + \sigma_0 + \delta_c(k))$

# Asymptotic Solutions (S) $(r \to \infty)$

c) $E_{CM} = 0 \Rightarrow k = 0$, $\lim_{k\to 0} \delta_\ell(k) \to -\underline{\underline{a_\ell}}\, k^{2\ell+1} + \mathcal{O}(k^{2\ell+3})$

scattering hypervolume

$[a_\ell] = [L]^{2\ell+1}$

or in S-waves:

$$\lim_{k\to 0} \delta_0(k) \to -\underbrace{a_0}_{\text{scattering length}} k + \mathcal{O}(k^3)$$

$[a_0] = [L]$

$$u_S(r) \xrightarrow[k\to 0]{} 1 - \frac{r}{a_0}$$

// $a_0$ appears in low energy cross sections: $\sigma \to 4\pi |a_0|^2$

$E_{CM} \to 0$ $(k \to 0)$

NEXT ISSUE $\Rightarrow$ [WHAT HAPPENS NEAR THE ORIGIN?]

$\Rightarrow$ Regularity condition: $\boxed{u_\ell(0) = 0}$ $\rightarrow$ We usually impose this on the solutions

$\langle\psi|\psi\rangle = \int dr\, |u_\ell(r)|^2 < \infty$ for $E_{cm} < 0$ $\Rightarrow$ $u_\ell(0) = 0$ guarantees this condition

1) Regular potentials: $\left[\lim_{r\to 0} r^2 V(r) = 0\right]$

$\Rightarrow$ $\dfrac{\ell(\ell+1)}{r^2} \gg V(r)$ for $r \to 0$

$\Rightarrow$ For $r \to 0$, $u_\ell(r) = \overbrace{a_\ell r^{\ell+1}}^{\text{Regular solution}} + \overbrace{\dfrac{b_\ell}{r^\ell}}^{\text{Irregular solution}}$

$$\left[\begin{array}{l} u_\ell(0) = 0 \\ \Rightarrow b_\ell = 0 \end{array}\right]$$

1) Regular potential: $u_\ell(r) = a_\ell r^{\ell+1} + \frac{b_\ell}{r^\ell}$

+

Regularity condition: $u_\ell(0) = 0$

$\Downarrow$

$b_\ell = 0$ or $u_\ell(r) \xrightarrow[r\to 0]{} a_\ell r^{\ell+1}$

2) Singular potential: $\boxed{\lim_{r\to 0} r^2 V(r) \neq 0}$

$\Rightarrow$ If this happens, the $r\to 0$ $u_\ell(r)$ will be different

[BEHAVIOR NEAR THE ORIGIN]

2) Singular potentials: $\left[\lim_{r\to 0} r^2 V(r) \to \pm\infty\right]$ → more convenient than condition in previous slide

↳ easiest condition

2.a) Repulsive singular potentials:

$$2\mu V(r) \to + \frac{a^{n-2}}{r^n}$$ (with $n>2$)

$$2\mu V(r) \gg \frac{\vec{L}^2}{r^2} \text{ for } r\to 0$$ ⇒ Angular momentum become irrelevant for the $r\to 0$ solution

## [BEHAVIOR NEAR THE ORIGIN]

2.b) Singular $\Rightarrow$ no dependence on $\ell$ for $r \to 0$

Singular + repulsive, $+\frac{a^{n-2}}{r^n}$ type: (Semiclassical approximation)

$$u_\ell(r) \underset{r\to 0}{\longrightarrow} \underline{\underline{c_+}} \left(\frac{r}{a}\right)^{n/4} \exp\left[+\frac{2}{n-2}\left(\frac{a}{r}\right)^{\frac{n-2}{2}}\right] \longrightarrow (a)$$

$$+ \underline{\underline{c_-}} \left(\frac{r}{a}\right)^{n/4} \exp\left[-\frac{2}{n-2}\left(\frac{a}{r}\right)^{\frac{n-2}{2}}\right] \longrightarrow (b)$$

$$\left.\begin{array}{l} (a) \to \text{irregular solution} \\ (b) \to \text{regular solution} \end{array}\right\} \Rightarrow \boxed{u_\ell(0) = 0 \Rightarrow c_+ = 0}$$

## [BEHAVIOR NEAR THE ORIGIN]

2.b) Attractive singular potential: $2\mu V(r) \xrightarrow[r\to 0]{} -\dfrac{a^{n-2}}{r^{n-2}}$

$$u_\ell(r) \xrightarrow[r\to 0]{} c_1\left(\frac{r}{a}\right)^{n/4}\sin\left[\frac{2}{n-2}\left(\frac{a}{r}\right)^{\frac{n-2}{2}}\right] + c_2\left(\frac{r}{a}\right)^{n/4}\cos\left[\frac{2}{n-2}\left(\frac{a}{r}\right)^{\frac{n-2}{2}}\right]$$

(Semiclassical approximation)

$\Rightarrow$ there's something unusual with this solution

(all solutions are regular)

[ATTRACTIVE SINGULAR INTERACTIONS]

$\Rightarrow$ [All solutions are regular!] $\Rightarrow$ [$u_\ell(0) = 0$ for all values of $c_1, c_2$]

What is the meaning of this?

$\longrightarrow$ Singular potentials are incomplete:

a) Result of expanding the potential at $\underline{\underline{r \to \infty}}$

$$V(r) = \sum_{n=n_0}^{\infty} \frac{C_n}{r^n} \Rightarrow \text{not valid for } r \to 0$$

b) They have to be supplemented with short-range physics

## [ATTRACTIVE SINGULAR INTERACTIONS]

$$u(r) \sim \left(\frac{r}{a}\right)^{n/4} \exp\left[-\frac{2}{n-2}\left(\frac{a}{r}\right)^{\frac{n-2}{2}}\right]$$

a) Repulsive singular potential:

→ Particles repel, very low probability for these particles of being $\underbrace{\text{close to each other}}_{\underline{r \to 0}}$

→ It doesn't matter what is the true potential for $r \to 0$

b) Attractive singular potential:

→ Opposite situation: particles will tend to approach each other

→ It will be sensitive to short-range physics

[SINGULAR INTERACTIONS]

a) Repulsive → Insensitive to short distances

(short distances do not affect the solutions)

b) Attractive → Sensitive to short distances

(short distances determine the linear combination of the two solutions)

Notice that: sensitive to short distances, but not to its details! } renormalization

→ We only see: $c_2/c_1$ → not too many details in this ratio

## [BEHAVIOR NEAR THE ORIGIN]

(Singular, but it will behave as regular for certain values of g)

c) **Borderline case:** $2\mu V(r) = \frac{g}{r^2}$

c.1) $g > -\frac{1}{4} \Rightarrow u_0(r) = \underbrace{c_+ r^{\nu+\frac{1}{2}}}_{\text{regular}} + \underbrace{c_- r^{\nu-1/2}}_{\text{irregular}}$

$\nu = \sqrt{g + 1/4}$

c.2) $g < -\frac{1}{4} \Rightarrow u_0(r) = c\, r^{1/2} \sin(\alpha \log r + \varphi)$

$$= c\, r^{1/2} \sin\left(\alpha \log \frac{r}{R^*}\right)$$

$\alpha = \sqrt{-(g + \frac{1}{4})}$

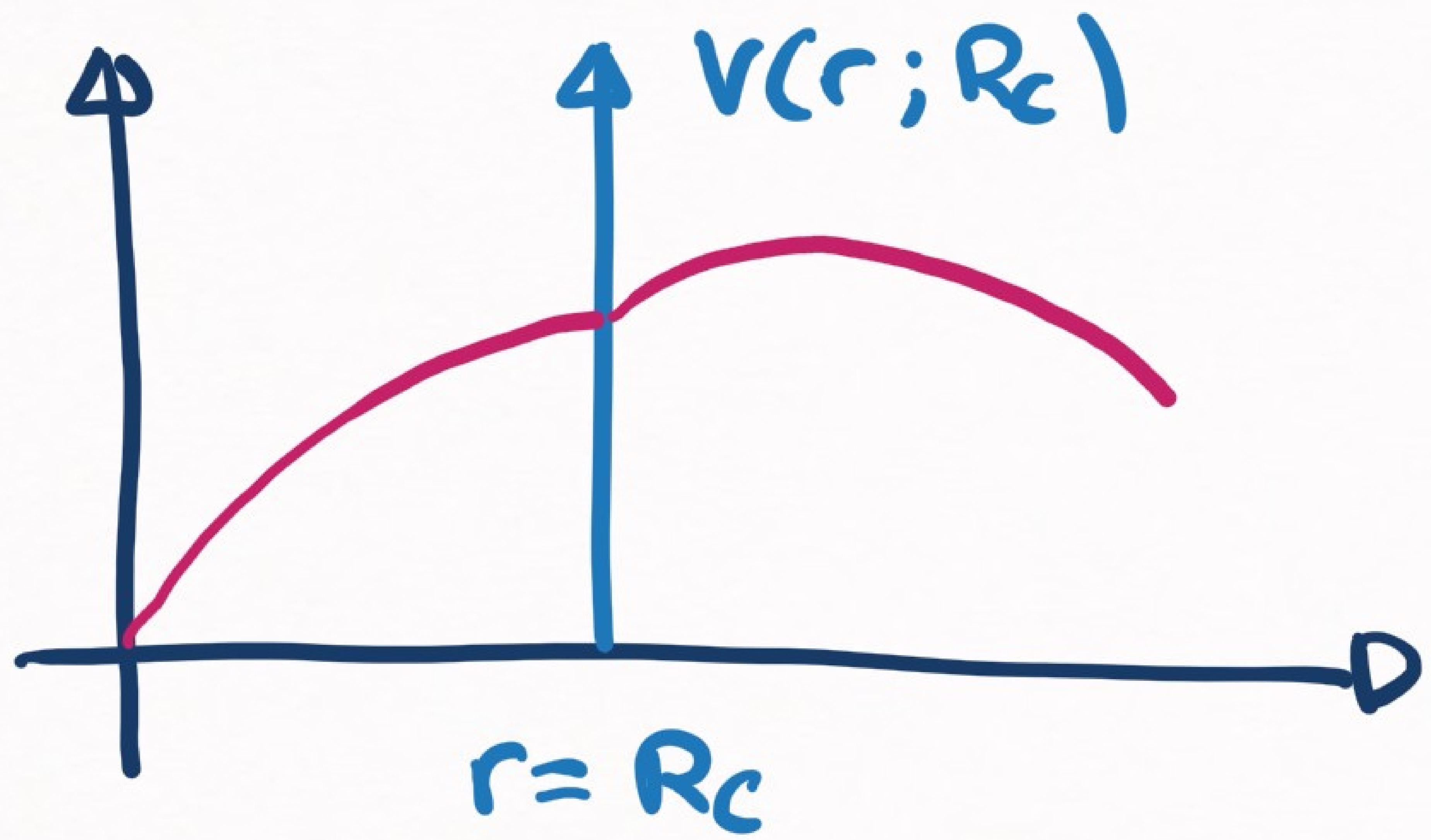
↳ like a typical attractive singular potential

## [DELTA - SHELL POTENTIAL]

$$\Rightarrow \quad 2\mu V(r;R_c) = \frac{C_0(R_c)}{4\pi R_c^2}\,\delta(r-R_c)$$

$\Rightarrow$ Extremely useful

**How to solve it?**




$\Rightarrow$ At $r = R_c$ we have:

a) wave function is continuous at this point

b) derivative of the w.f. becomes discontinuous

[DELTA SHELL POTENTIAL] (s-wave solution)

a) $r < R_c \Rightarrow u(r) \propto \sin(kr)$ → usual free wave function

b) $r > R_c \Rightarrow u(r) \propto \sin(kr + \delta(k))$ → (but now with a phase)

c) $r \to R_c \Rightarrow \lim_{\epsilon \to 0^+} u(R_c - \epsilon) = \lim_{\epsilon \to 0^+} u(R_c + \epsilon)$

(continuity on $u(r)$)

and also that: (discontinuity on $u'(r)$)

$$\lim_{\epsilon \to 0^+} \int_{R_c - \epsilon}^{R_c + \epsilon} \left[ -u_k'' + \frac{2\mu}{4\pi R_c^2} C_o(R_c)\, \delta(r - R_c)\, u_k(r) \right] dr = \lim_{\epsilon \to 0^+} k^2 \int_{R_c - \epsilon}^{R_c + \epsilon} u_k(r)\, dr$$

## [DELTA SHELL POTENTIAL] (S-wave solution)

⇒ After some manipulations, we arrive at:

$$\lim_{\epsilon \to 0^+} u_k'(R_c+\epsilon) - u_k'(R_c-\epsilon) = 2\mu \frac{C_0(R_c)}{4\pi R_c^2} u_k(R_c)$$

⇒ Or, if we use a) and b) in previous slide:

$$k\cot(kR_c+\delta(k)) - k\cot(kR_c) = 2\mu \frac{C_0(R_c)}{4\pi R_c^2}$$

## [DELTA SHELL POTENTIAL] (scattering length)

$\Rightarrow$ Or, if we take the $k \to 0$ limit:

$$\cot x \xrightarrow[x \to 0]{} \frac{1}{x} \Rightarrow \frac{1}{R_c - a_0} - \frac{1}{R_c} = 2\mu \frac{C_0(R_c)}{4\pi R_c^2}$$

$$\Downarrow$$

$$\left[ \frac{1}{C_0(R_c)} = \frac{\mu}{2\pi}\left(\frac{1}{a_0} - \frac{1}{R_c}\right) \right] \to \text{"Renormalization group equation" for } C_0(R_c)$$

$\hookrightarrow$ In general, this potential is extremely useful

RECAP $\longrightarrow$ two-body problem in QM

1) Asymptotic solutions $(r \to \infty)$ $\Rightarrow$ Phase shifts & scattering lengths

$$u_k(r) \longrightarrow \sin(kr + \delta(k))$$
$$u_0(r) \longrightarrow r - a_0$$

Fundamental for scattering theory

2) Solutions near the origin

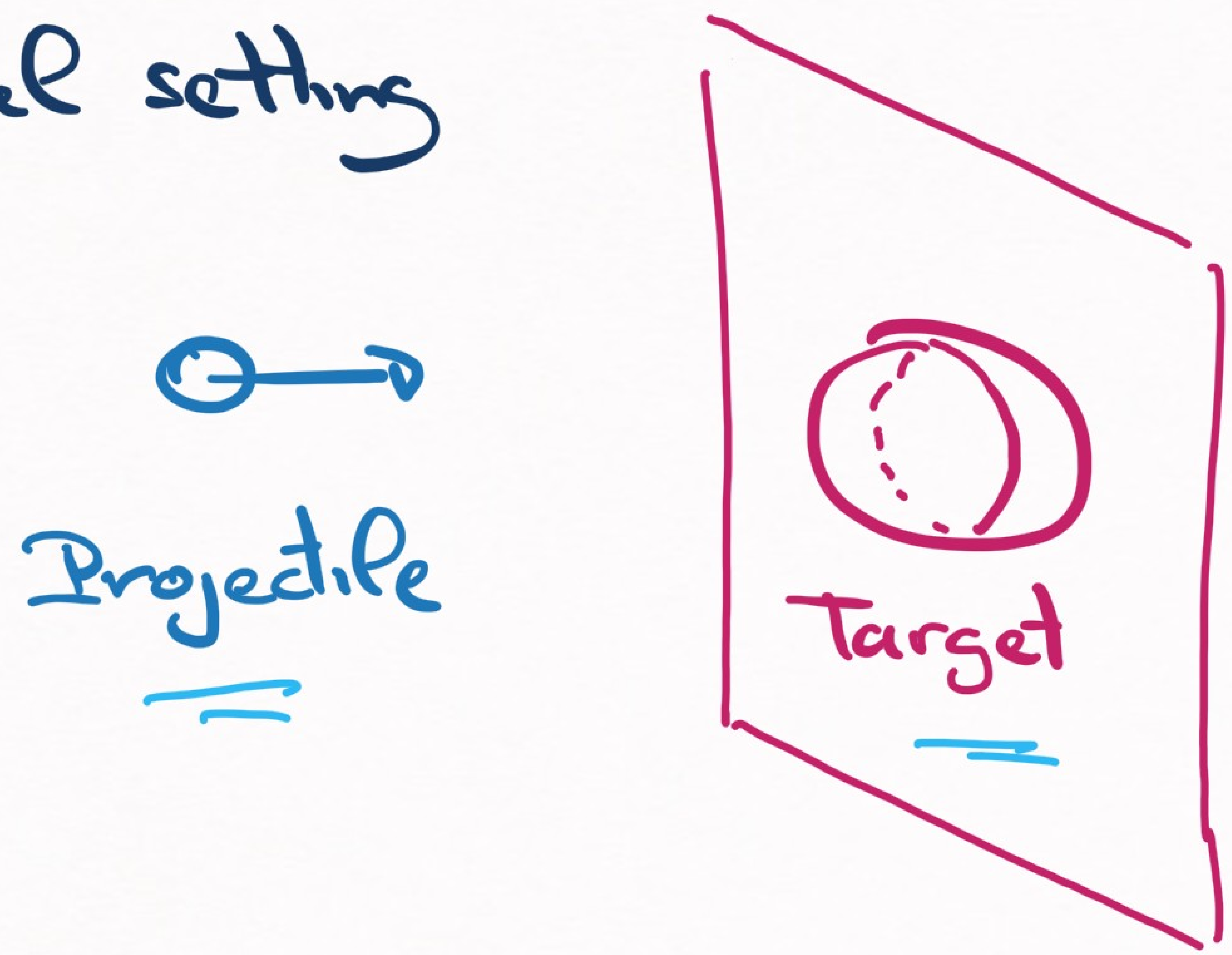
2.a) Regular potentials $\rightarrow$ $\exists$ a well-defined solution

2.b) Singular potentials $\rightarrow$ If attractive, solution not unique

# SCATTERING THEORY (1)

⇒ What is the cross section?

Classical setting

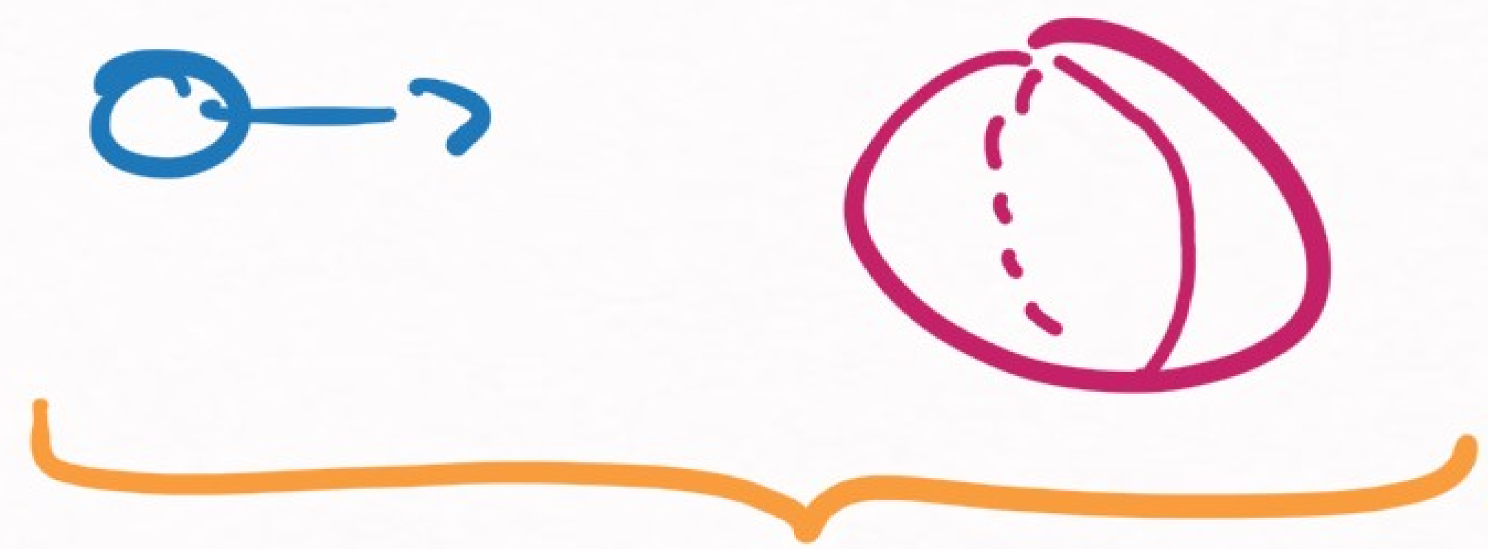



We want to know this:

the effective area that the target offers to the projectile (cross section $\sigma$)

# SCATTERING THEORY (2)

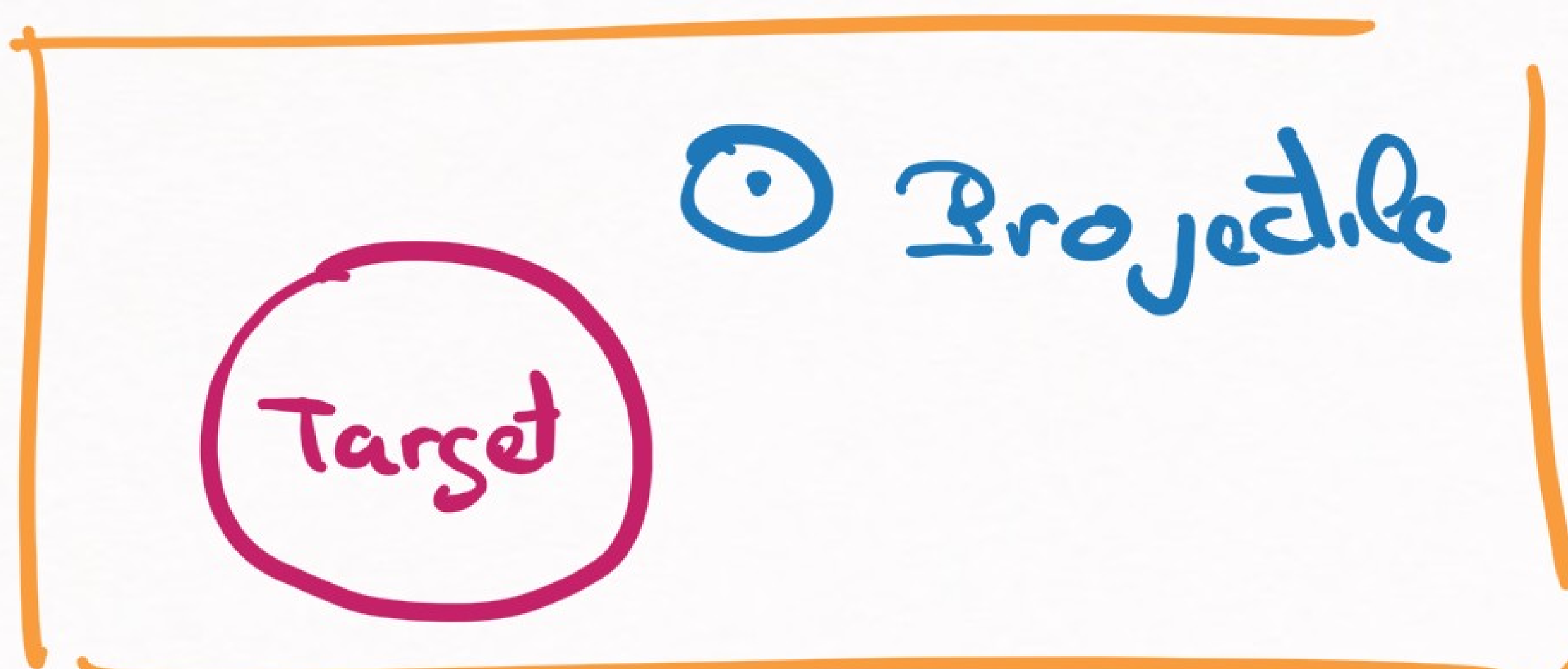
(scattering of rigid balls)



Experimental setup

Let's change the perspective (projectile's viewpoint)

(1)



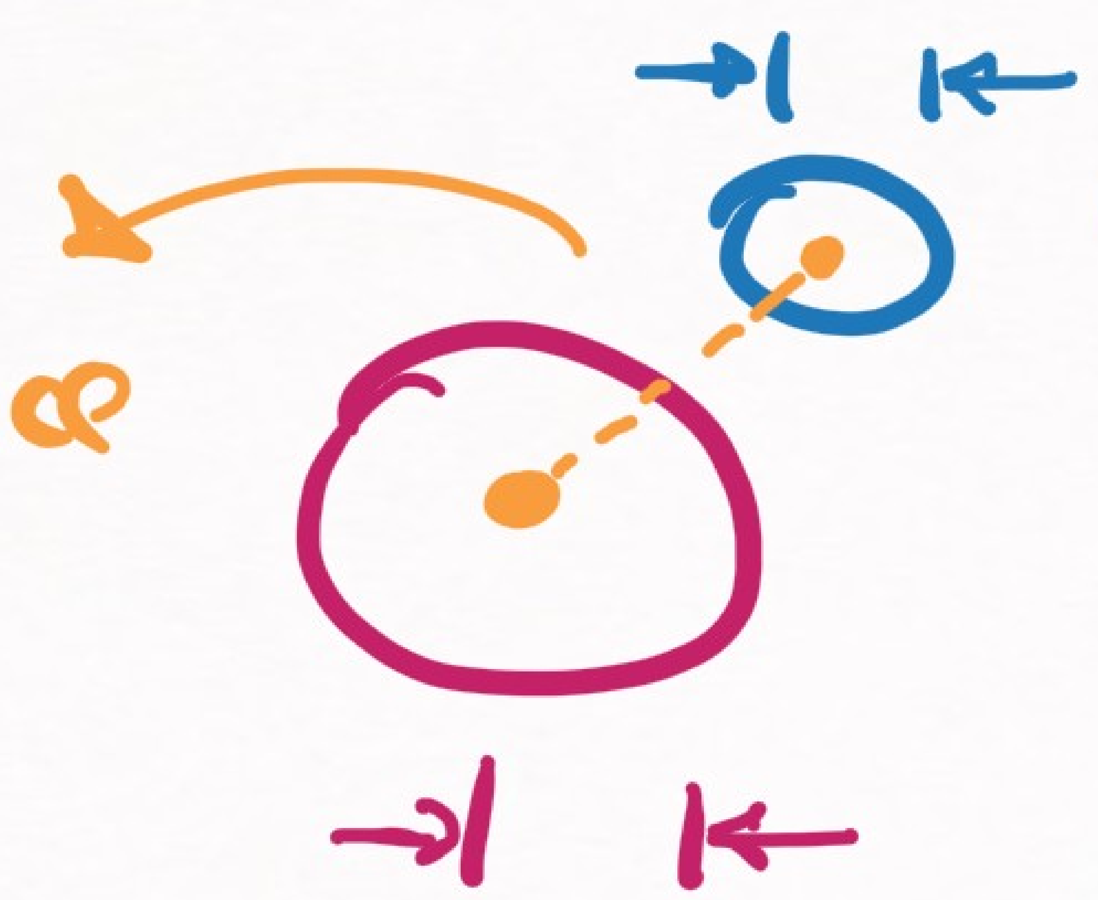

→ miss

OR


(2) → hit

# SCATTERING THEORY ③

For classical rigid balls, we will hit if:



distance $d$

$R_{projectile} = R_1$

$R_{target} = R_2$

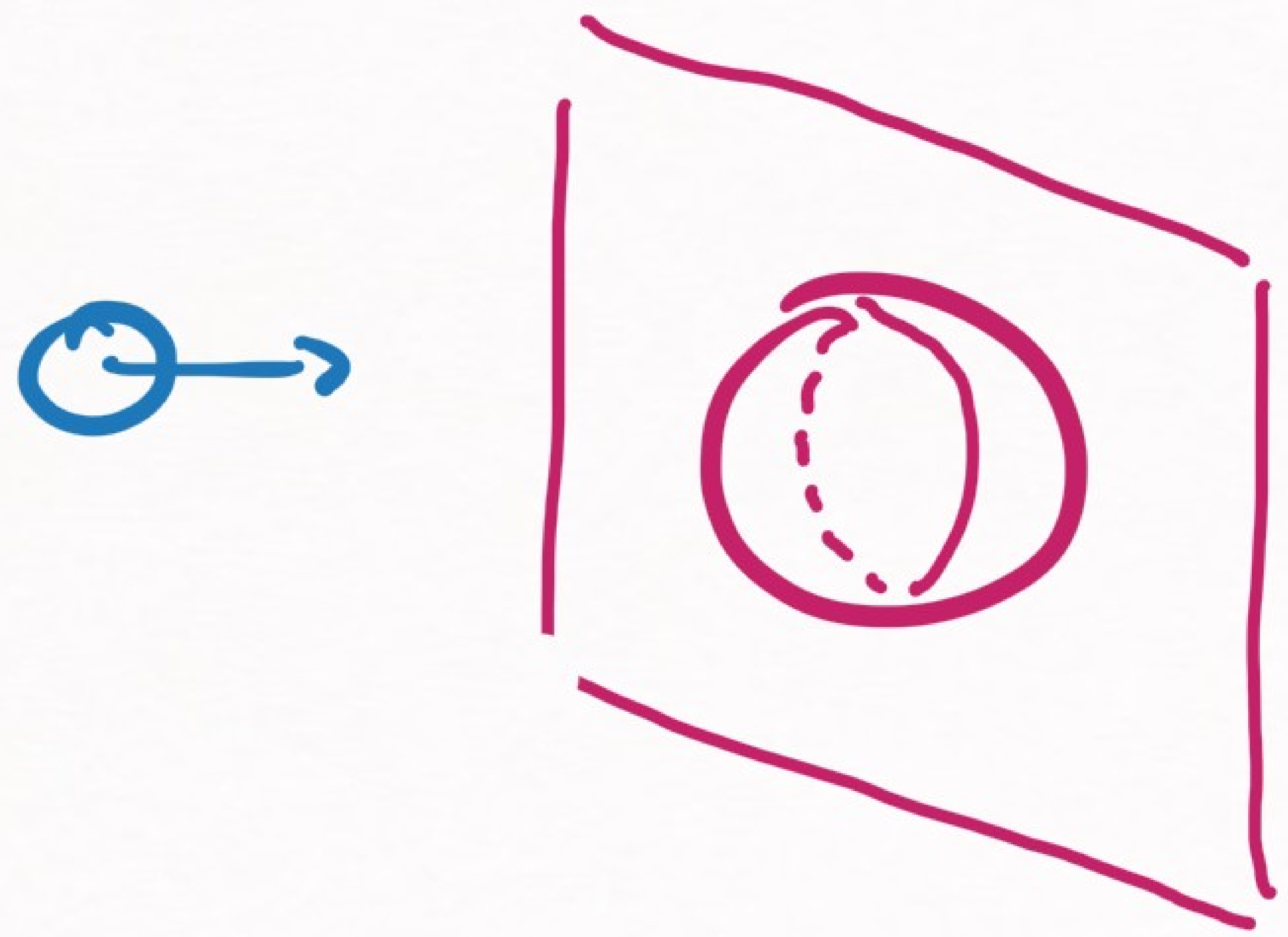
$\Rightarrow$ $d < (R_1 + R_2)$ → hit

Effective area comes from this

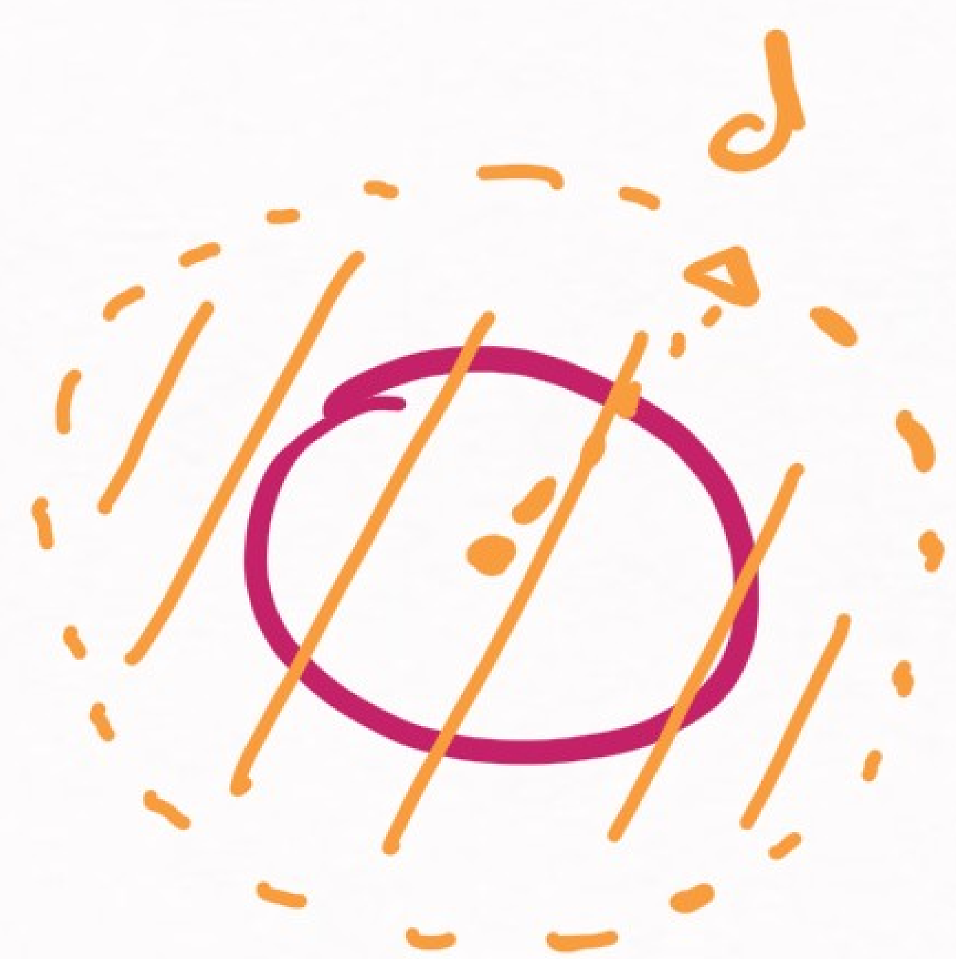
$d > (R_1 + R_2)$ → miss

# SCATTERING THEORY (4)

For rigid balls (classical):



$\Rightarrow$




Effective area of the target

$$\boxed{\begin{aligned}\sigma &= \pi d^2 \\ &= \pi (R_1 + R_2)^2\end{aligned}}$$

$\downarrow$

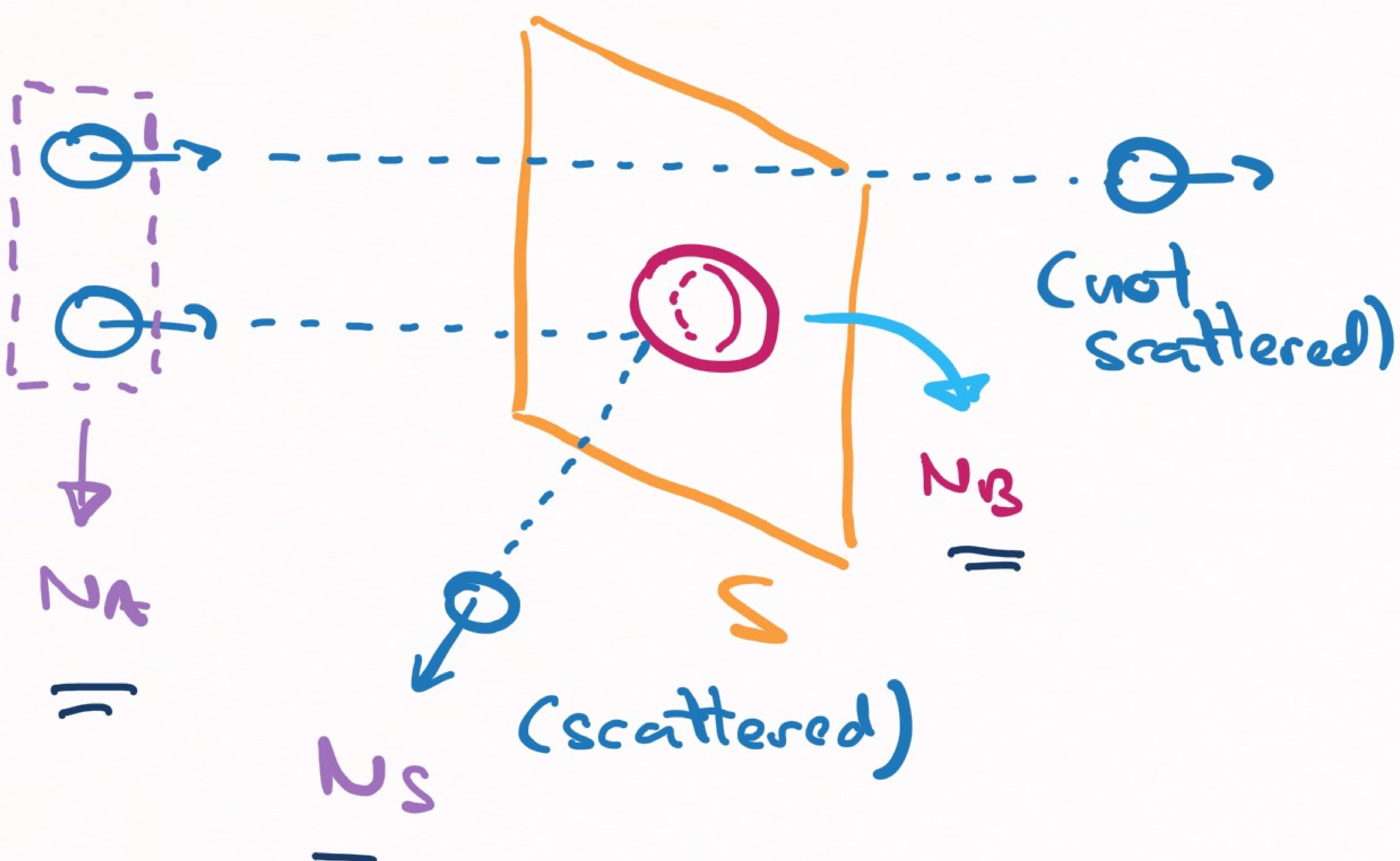
Cross section of these two rigid ball

We need to find a convenient way to extend this to other types of scatterings

# Scattering Theory (5)

How can we define $\sigma$ in terms of hits & misses?

(so we can apply it to other situations)




$$\boxed{\sigma = \frac{N_S}{N_A N_B} S} \Rightarrow$$

$N_S \rightarrow$ scattered projectiles

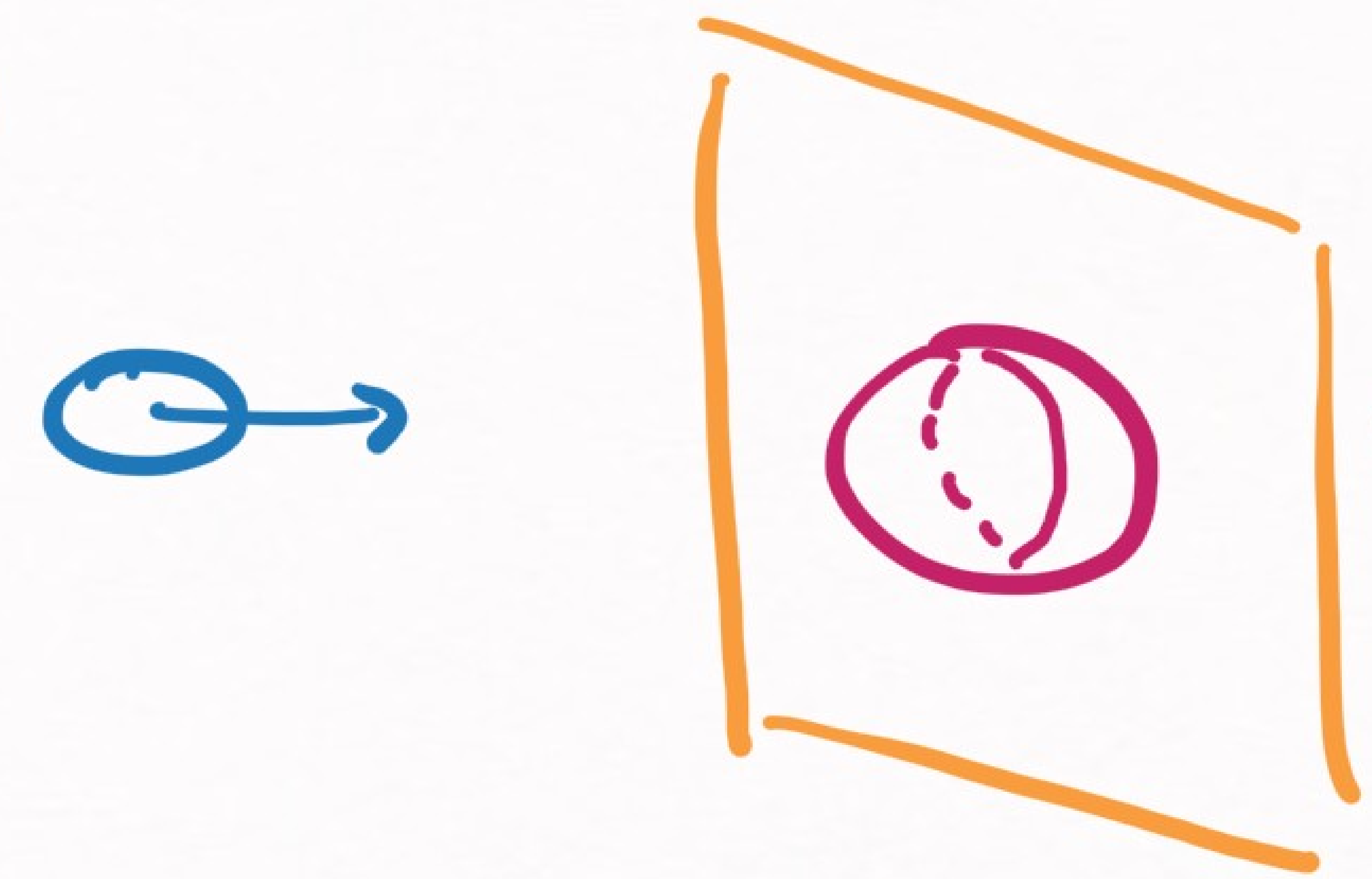
$N_A \rightarrow$ # of projectiles

$N_B \rightarrow$ # of targets

$S \rightarrow$ area on which we do the experiment

# SCATTERING THEORY (6)

Example: rigid balls



S

$S_B^{eff} = \pi(R_A+R_B)^2$

$S_B^{eff} < S$

$N_A \to$ input (our choice)

$N_B \to$ input (our choice)

$S_B^{eff} = \pi(R_A+R_B)^2 \to$ from our previous arguments

$N_S \to$ we can calculate this

$\frac{S_B^{eff}}{S} \to$ probability of a single projectile hitting a single target

$$N_S = N_A N_B \left(\frac{S_B^{eff}}{S}\right)$$

## SCATTERING THEORY (7)

$\Rightarrow$ Putting all the pieces together:

$$\left.\begin{aligned} \sigma &= \frac{N_S}{N_A N_B}\, S \\ N_S &= N_A N_B \frac{S_B^{eff}}{S} \end{aligned}\right\} \Rightarrow \sigma = S_B^{eff} = \pi (R_A + R_B)^2$$

Our previous result (except for the change in the names of $R_A$ & $R_B$)

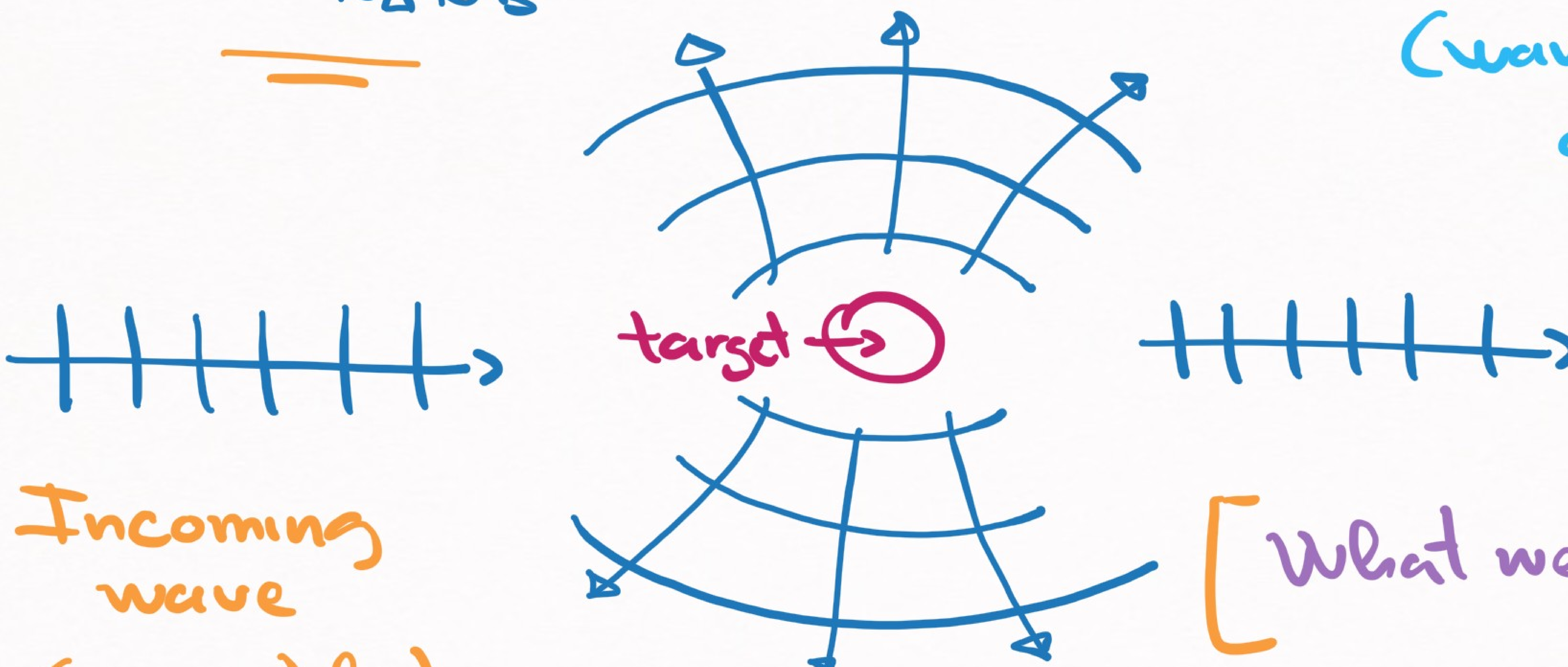
NEXT THING:

WHAT IS THE QM VERSION OF THIS?

# QUANTUM SCATTERING (1)

$\Rightarrow \sigma = \dfrac{N_S}{N_A N_B} S$ ) $\Rightarrow$ What is its quantum equivalent?

(waves instead of particles)




Incoming wave (projectile)

$e^{i\vec{k}\cdot\vec{r}}$

Outgoing wave $\propto \dfrac{e^{ikr}}{r}$

[What we do is to write a wave function:]

$$\psi_{\vec{k}}(\vec{r}) \to e^{i\vec{k}\cdot\vec{r}} + f(\Omega)\frac{e^{ikr}}{r}$$

# QUANTUM SCATTERING ②

$$\Rightarrow \quad \psi_{\vec{k}}(\vec{r}) \xrightarrow[|\vec{r}| \to \infty]{} \underbrace{e^{i\vec{k}\cdot\vec{r}}}_{\text{incoming}} + \underbrace{f(\Omega)\frac{e^{ikr}}{r}}_{\text{outgoing}}$$

$$\longrightarrow \quad \psi_{in}(\vec{r}) + \psi_{out}(\vec{r})$$

→ a way of rewriting the scattering wave function

Conjecture: there should be some relation among these quantities

$$\left.\begin{aligned} N_A N_B &\sim \psi_{in}(\vec{r}) \\ S N_S &\sim \psi_{out}(\vec{r}) \end{aligned}\right\}$$

We still need to figure it out

# QUANTUM SCATTERING ③

$N_A N_B \sim \psi_{in}(\vec{r})$
$N_B S \sim \psi_{out}(\vec{r})$

⇒ Let's figure out the relation: ⟶ probably related to the flux

a) $N_B = 1$ (for simplicity)

influx of projectiles: $\frac{N_A}{T} \propto \Phi_{in}$ (incoming flux)

b) outflux of scattered particles

$$S \frac{N_S}{T} \propto \int d\vec{S} \cdot \vec{\Phi}_{out}$$

# QUANTUM SCATTERING ④ $[N_B = 1,\ \frac{N_A}{T} \propto \Phi_{in}]$

⇒ So far we have: $N_B = 1,\ \frac{N_A}{T} \propto \Phi_{in},\ \frac{N_S}{T} \propto \int d\vec{S} \cdot \vec{\Phi}_{out}$

a) Incoming flux: $\Phi_{in} = |\vec{\Phi}_{in}|$ ←

$$\vec{\Phi}_{in} = -\frac{i}{2m}\left[\psi_{in}^* \vec{\nabla} \psi_{in} - \psi_{in} \vec{\nabla} \psi_{in}^*\right] \quad (\psi_{in} = e^{i\vec{k}\cdot\vec{r}})$$

$$\Rightarrow \quad \vec{\Phi}_{in} = \frac{\vec{k}}{m} \quad \text{or} \quad |\vec{\Phi}_{in}| = \frac{k}{m}$$

b) Outgoing flux: $\hat{r} \cdot \vec{\Phi}_{out} = -\frac{i}{2m}\left[\psi_{out}^* \hat{r} \cdot \vec{\nabla} \psi_{out} - \psi_{out} \hat{r} \cdot \vec{\nabla} \psi_{out}^*\right]$

$$\left(\psi_{out} = f(\Omega)\frac{e^{ikr}}{r}\right) \quad \Rightarrow \quad \vec{\Phi}_{out} = \frac{k}{m}|f(\Omega)|^2 \frac{\hat{r}}{R^2}$$

# Quantum Scattering (5)

$$\left[ S \frac{N_S}{T} \propto \int d\vec{S} \cdot \vec{\Phi}_{out} \right]$$

b) (continuation) $\lim_{R \to \infty} \int d\vec{S} \cdot \vec{\Phi}_{out} = \left\{ d\vec{S} = R^2 \, d\Omega \, \hat{n} \right\} =$

$$= \frac{K}{m} \int |f(\Omega)|^2 \, d\Omega$$

c) Putting all the pieces together:

$$\sigma = \frac{N_S}{N_A N_B} S = \frac{1}{N_B} \left( \frac{T}{N_A} \right) \left( S \frac{N_S}{T} \right) = \int |f(\Omega)|^2 \, d\Omega$$

# QUANTUM SCATTERING ⑥

$f(\Omega) \to$ scattering amplitude

$\Rightarrow$ We have found that: $$\boxed{\sigma = \int |f(\Omega)|^2 \, d\Omega}$$

Total cross section

$\Rightarrow$ Or, if we want the angular dependence:

$$\boxed{\frac{d\sigma}{d\Omega} = |f(\Omega)|^2} \to \text{Differential cross section}$$

But we still need more detail $\to$ relation of $f(\Omega)$ w/ $\delta_\ell(k)$ or $a_\ell$, etc.

# QUANTUM SCATTERING ⑦

⇒ How do we connect this $f(\Omega)$ with things we can calculate? $(\delta_\ell(k))$

## [PARTIAL WAVE EXPANSION]

a) We expand in partial waves $\underbrace{\psi(\vec{r}) = \sum_{\ell m} \psi_\ell(r) Y_{\ell m}(\hat{r})}$ (this still lacks something)

b) But ... we also need to include $\vec{k}$:

$$e^{i\vec{k}\cdot\vec{r}} = 4\pi \sum_{\ell m} i^\ell j_\ell(kr) Y^*_{\ell m}(\hat{k}) Y_{\ell m}(\hat{r})$$

$$= \sum_\ell (2\ell+1) i^\ell j_\ell(kr) P_\ell(\hat{k}\cdot\hat{r})$$

PW expansion of the plane wave

Legrange polynomials

# [PARTIAL WAVE EXPANSION] 1

Lagrange polynomials

⇒ In analogy to: $e^{i\vec{k}\cdot\vec{r}} = \sum_{\ell} (2\ell+1)\, i^{\ell} j_{\ell}(kr) \underbrace{P_{\ell}(\hat{k}\cdot\hat{r})}$

we expand: $f(\omega) = \sum_{\ell} (2\ell+1) f_{\ell}(k) P_{\ell}(\hat{k}\cdot\hat{r})$

⇒ Or even better, we can expand the full wave function:

$$\psi_k(\vec{r}) = 4\pi \sum_{\ell m} i^{\ell} \frac{u_{\ell}(r)}{r} Y^*_{\ell m}(\hat{k}) Y_{\ell m}(\hat{r})$$

$$= \sum_{\ell} (2\ell+1)\, i^{\ell} \frac{u_{\ell}(r)}{r} P_{\ell}(\hat{k}\cdot\hat{r})$$

≅ expression for $e^{i\vec{k}\cdot\vec{r}}$ but:

$j_{\ell}(kr) \rightarrow \frac{u_{\ell}(r)}{r}$

# PARTIAL WAVE EXPANSION ②

⇒ For the moment we have: $\psi_k(\vec{r}) = \sum_{\ell} (2\ell+1) i^{\ell} \frac{u_\ell(r)}{r} P_\ell(\hat{k}\cdot\hat{r})$

and we also know that: $u_\ell(r) \underset{r\to\infty}{\longrightarrow} \sin(kr - \ell\frac{\pi}{2} + \delta_\ell(k))$

⇒ It will be better to use the normalization below:

$$\frac{u_\ell(r)}{r} \rightarrow e^{i\delta_\ell}[\cos\delta_\ell(k) j_\ell(kr) - \sin\delta_\ell(k) y_\ell(kr)]$$

⇒ With this is easier to show that: $\psi_k(\vec{r}) \underset{r\to\infty}{\longrightarrow} e^{i\vec{k}\cdot\vec{r}} + f(\Omega)\frac{e^{ikr}}{r}$

# PARTIAL WAVE EXPANSION ③

⇒ We have to match these:

$$\left[ \psi_{\vec{k}}(\vec{r}) \longrightarrow e^{i\vec{k}\cdot\vec{r}} + f(\Omega)\frac{e^{ikr}}{r} \right]$$

a) $\psi_k(\vec{r}) = \sum_{\ell} (2\ell+1)\, i^{\ell}\, \frac{u_\ell(r)}{r}\, P_\ell(\cos\theta)$

b) $e^{i\vec{k}\cdot\vec{r}} = \sum_{\ell} (2\ell+1)\, i^{\ell}\, \frac{j_\ell(kr)}{r}\, P_\ell(\cos\theta)$

c) $f(\Omega) = \sum_{\ell} (2\ell+1)\, f_\ell(k)\, P_\ell(\cos\theta)$

+ asymptotic form of $u_\ell(r)$ (check it in previous slide)

(requires a few detailed calculations, but it's easy)

# PARTIAL WAVE EXPANSION (4)

⇒ Putting all the pieces together

$$\left[ f_\ell(k) = \frac{e^{i\delta_\ell}\sin\delta_\ell}{k} = \frac{1}{k\cot\delta_\ell - ik} \right] \rightarrow$$

Requires a few calculations

We can do even more calculation & find the expression for $\sigma = \int |f(\Omega)|^2 d\Omega$

$$f(\Omega) = \sum_\ell (2\ell+1) f_\ell(k) P_\ell(\cos\theta)$$

# QUANTUM SCATTERING

⇒ If we go back to our definition of $\sigma$:

$$\sigma = \int |f(\Omega)|^2 \, d\Omega = \frac{4\pi}{k^2} \sum_{\ell=0}^{\infty} (2\ell+1) \sin^2 \delta_\ell(k)$$

⇒ In the $k \to 0$ limit: $\delta_\ell(k) \to -a_\ell k^{2\ell+1} + \mathcal{O}(k^{2\ell+3})$

$$\boxed{\sigma \xrightarrow[k \to 0]{} 4\pi |a_0|^2 + \mathcal{O}(k^2)}$$

→ Scattering length is a bit like an effective radius at low energies

## RECAP

1) For classical scattering: $\left[\sigma = S \, \frac{N_S}{N_A N_B}\right]$ ←

2) The quantum mechanical version requires:

2.a) A scattering wave function

$$\left[\ \psi_k(\vec{r}) \xrightarrow[|\vec{r}| \to \infty]{} e^{i\vec{k}\cdot\vec{r}} + f(\Omega)\,\frac{e^{ikr}}{r}\ \right]$$

2.b) A classical-to-quantum "dictionary":

$$N_B = 1, \quad \frac{N_A}{T} \sim \Phi_{in} = \frac{k}{m}, \quad S\,\frac{N_S}{T} \sim \int d\vec{S}\cdot\vec{\Phi}_{out} = \frac{k}{m}\int |f(\Omega)|^2 \, d\Omega$$

# RECAP!

3) The QM cross section is thus:

$$\sigma = \int |f(\Omega)|^2 \, d\Omega \qquad \frac{d\sigma}{d\Omega} = |f(\Omega)|^2$$

4) The connection to the phase shifts is done by using the partial wave expansion:

4.a) $f(\Omega) = \sum_{\ell} (2\ell+1) f_\ell(k) P_\ell(\cos\theta)$, $f_\ell(k) = \dfrac{1}{k\cot\delta_\ell - ik}$

4.b) $\sigma = \dfrac{4\pi}{k^2} \sum_{\ell} (2\ell+1) \sin^2\delta_\ell$, $\sigma \xrightarrow[k\to 0]{} 4\pi |a_0|^2$

4.c) Reminder: $u_\ell(r) \xrightarrow[r\to\infty]{} \sin\left(kr - \frac{\ell\pi}{2} + \delta_\ell(k)\right)$

See you on Friday

15:50