NUCLEAR PHYSICS 12

a) THE GOLDSTONE THEOREM

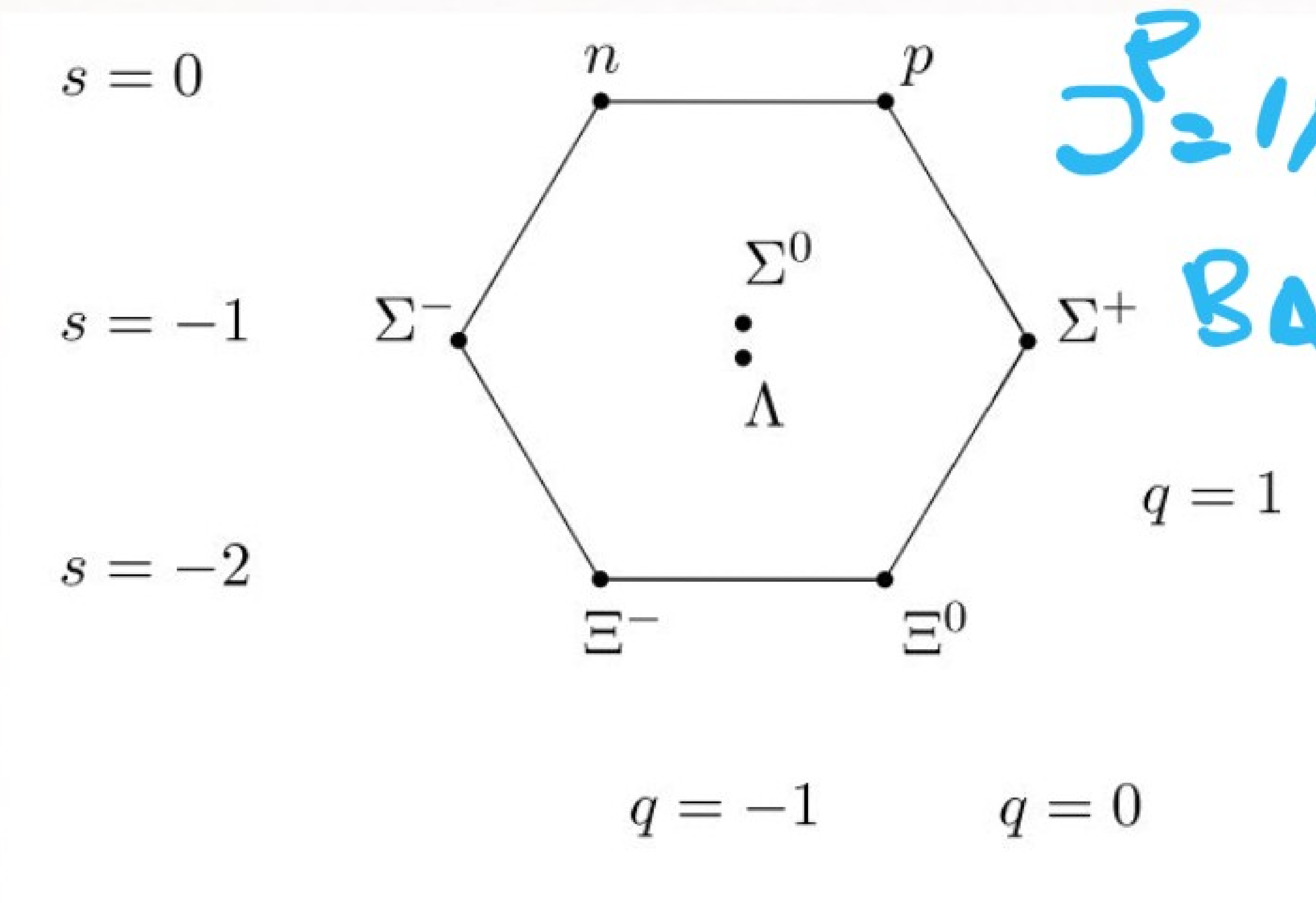
(the general explanation
of the mexican hat trick)

b) CHIRAL SYMMETRY

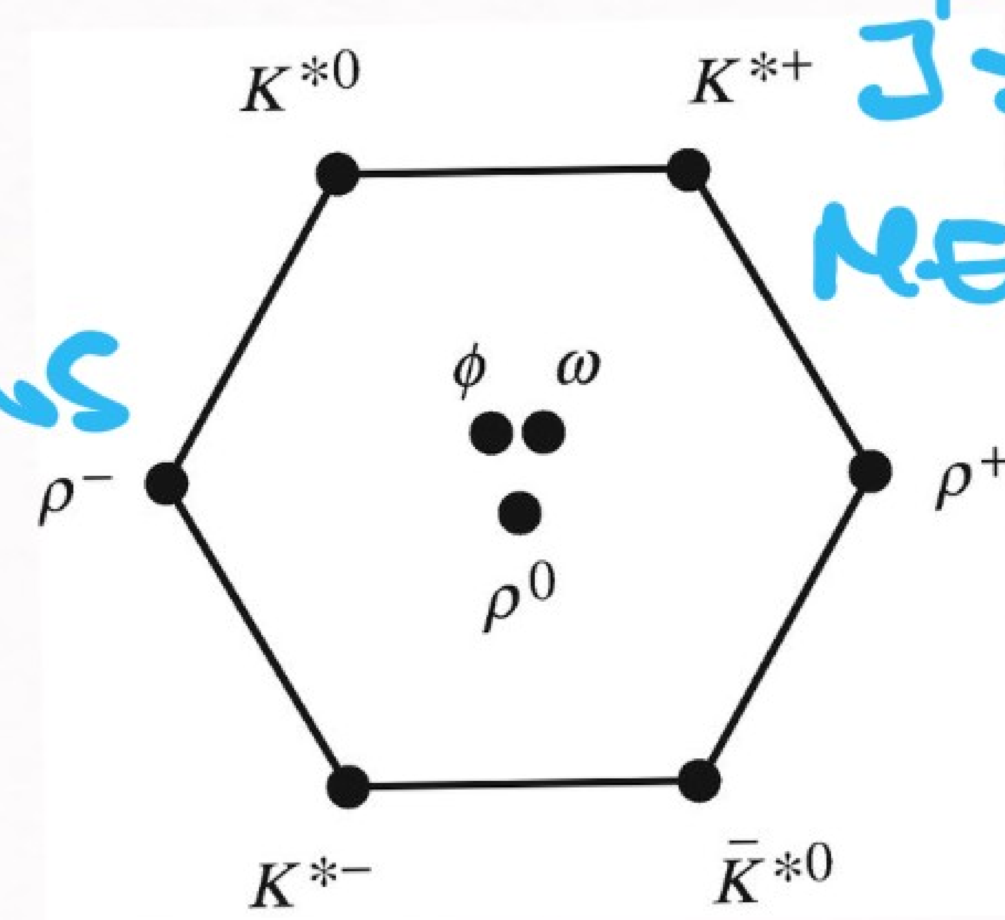
(the reason why the pion
is the lightest hadron)

RECAP

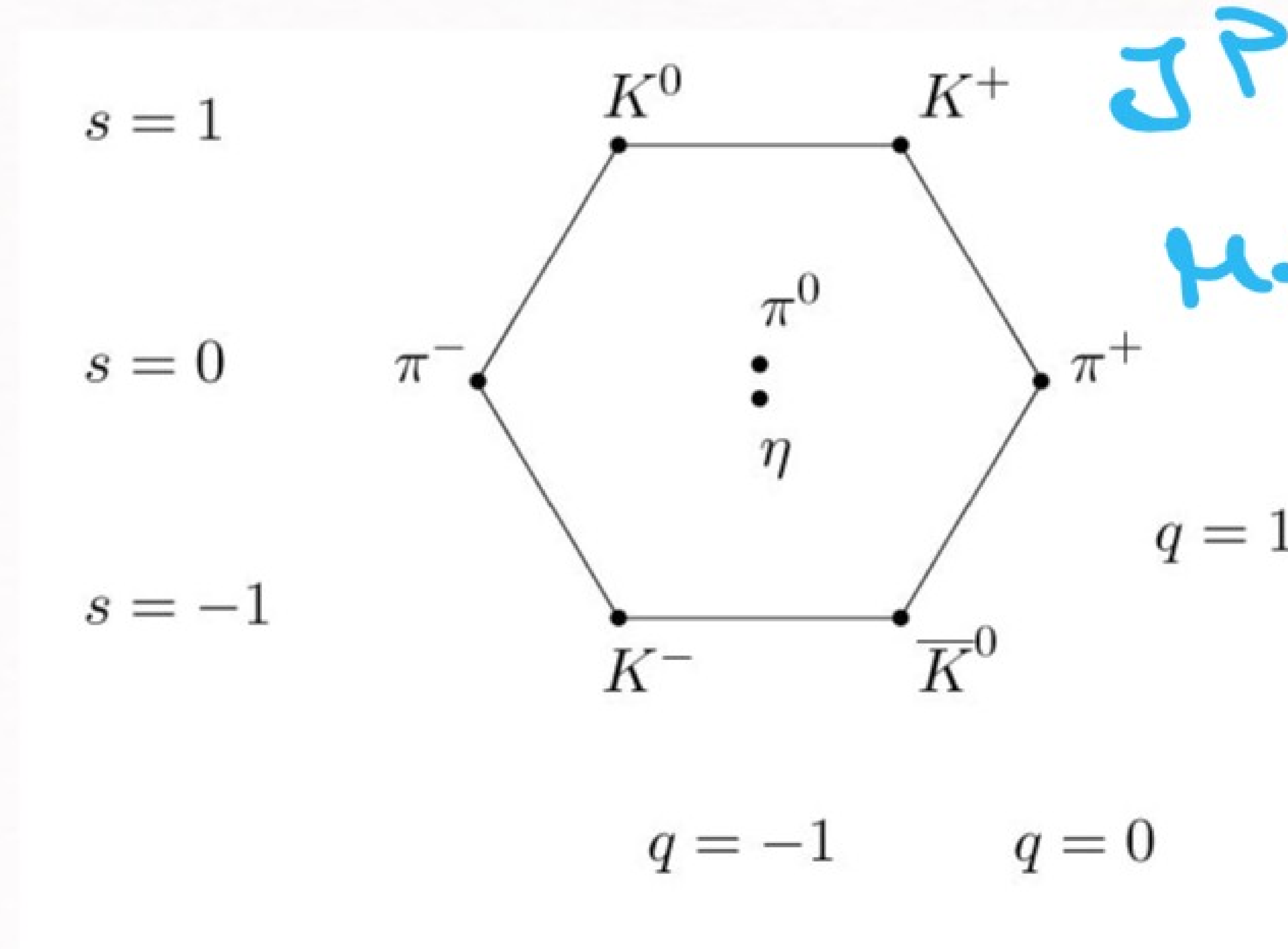
1) When studying the multiplets, we were confronted with a problem regarding their masses:



$J^P = 1/2^+$ BARYONS



$J^P = 1^-$ MESONS



$J^P = 0^-$ MESONS

$m_n \simeq m_p \simeq 940$ MeV

$m_\Lambda \simeq 1115$ MeV

$m_\Sigma \simeq 1193$ MeV

$m_\Xi \simeq 1318$ MeV

$m_\rho \simeq 770$ MeV

$m_\omega \simeq 780$ MeV

$m_{K^*} \simeq 890$ MeV

$m_\phi \simeq 1020$ MeV

$m_\pi \simeq 138$ MeV

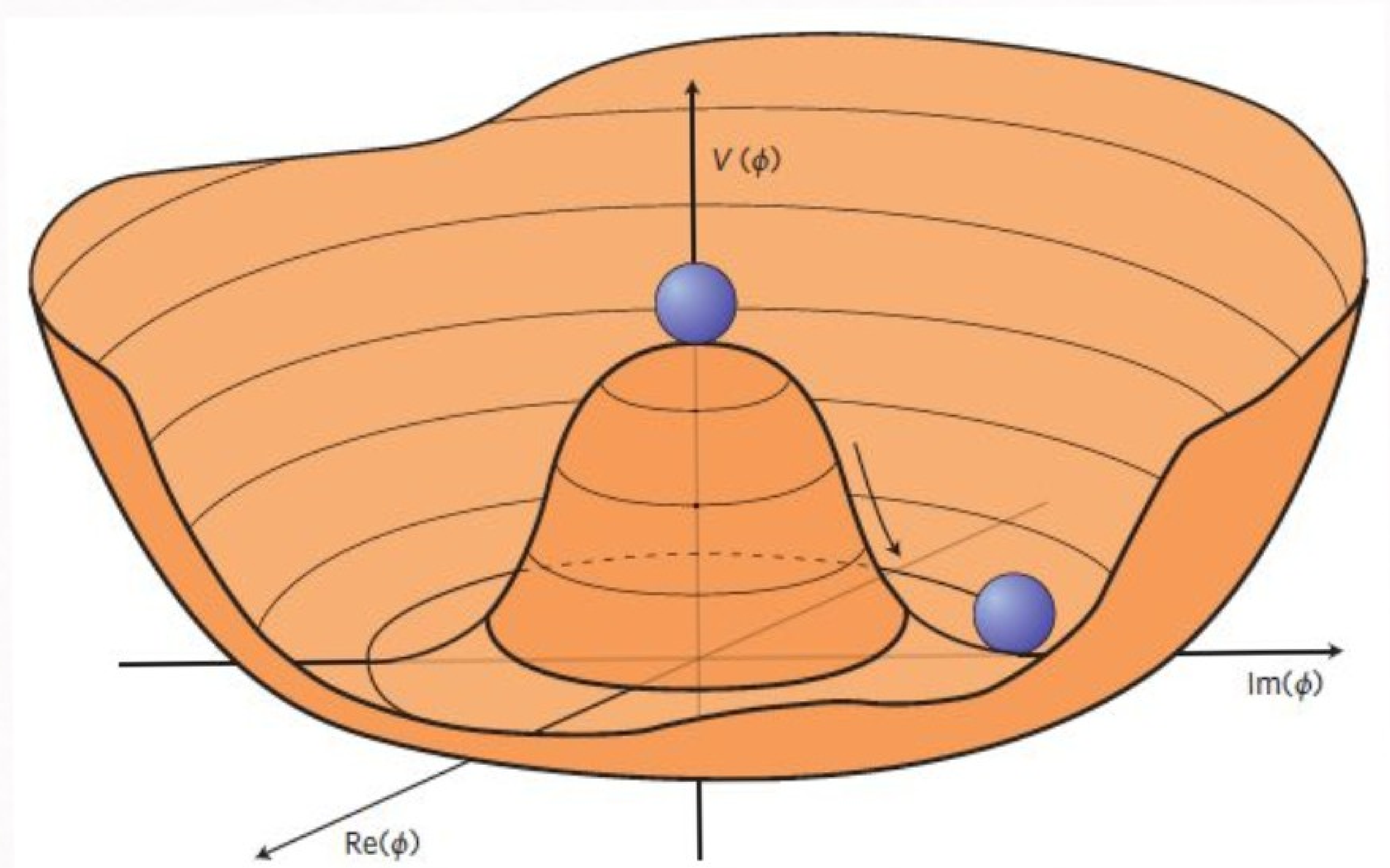
$m_K \simeq 495$ MeV

$m_\eta \simeq 548$ MeV

2) The pion is extremely light for a hadron

⇒ This requires an explanation

3) The linear sigma model is able to explain a light pion by using a mexican hat potential



⇒ One is able to create massless particles

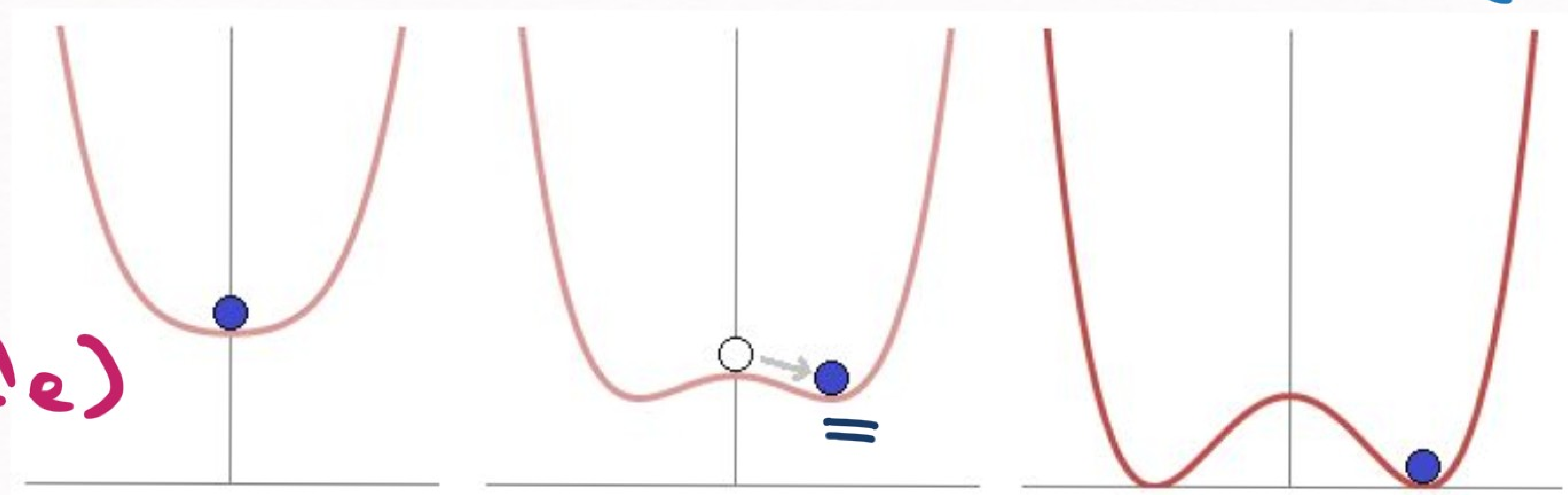
[GENERALIZING THE IDEAS BEHIND THE MEXICAN HAT POTENTIAL]

a) Let's review the mexican hat for one dimension:

$$V(\phi) = \frac{1}{2}\mu^2\phi^2 + \frac{\lambda}{4}\phi^4$$

$\lambda \geq 0$
(Choice)

Minimum energy state (ground state) changes w/ μ^2



→ [$\mu^2 > 0$ $\quad$ $\mu^2 \simeq 0$ $\quad$ $\mu^2 < 0$]

→ $\mu^2 \leq 0$

⇒ But you can understand this from the point of view of symmetry:

a.1) $V(\phi) = \frac{1}{2}\mu^2\phi^2 + \frac{\lambda}{4}\phi^4 \xrightarrow[\substack{\phi \to -\phi \\ \text{(parity)}}]{} V(-\phi) = \frac{1}{2}\mu^2\phi^2 + \frac{\lambda}{4}\phi^4$

a.2) Around the minimum: $\phi = \phi_{min} + \underline{\underline{\sigma}}$, $\phi_{min} = \pm\sqrt{-\frac{\mu^2}{\lambda}} = \underline{\underline{\pm v}}$
($\mu^2 < 0$)

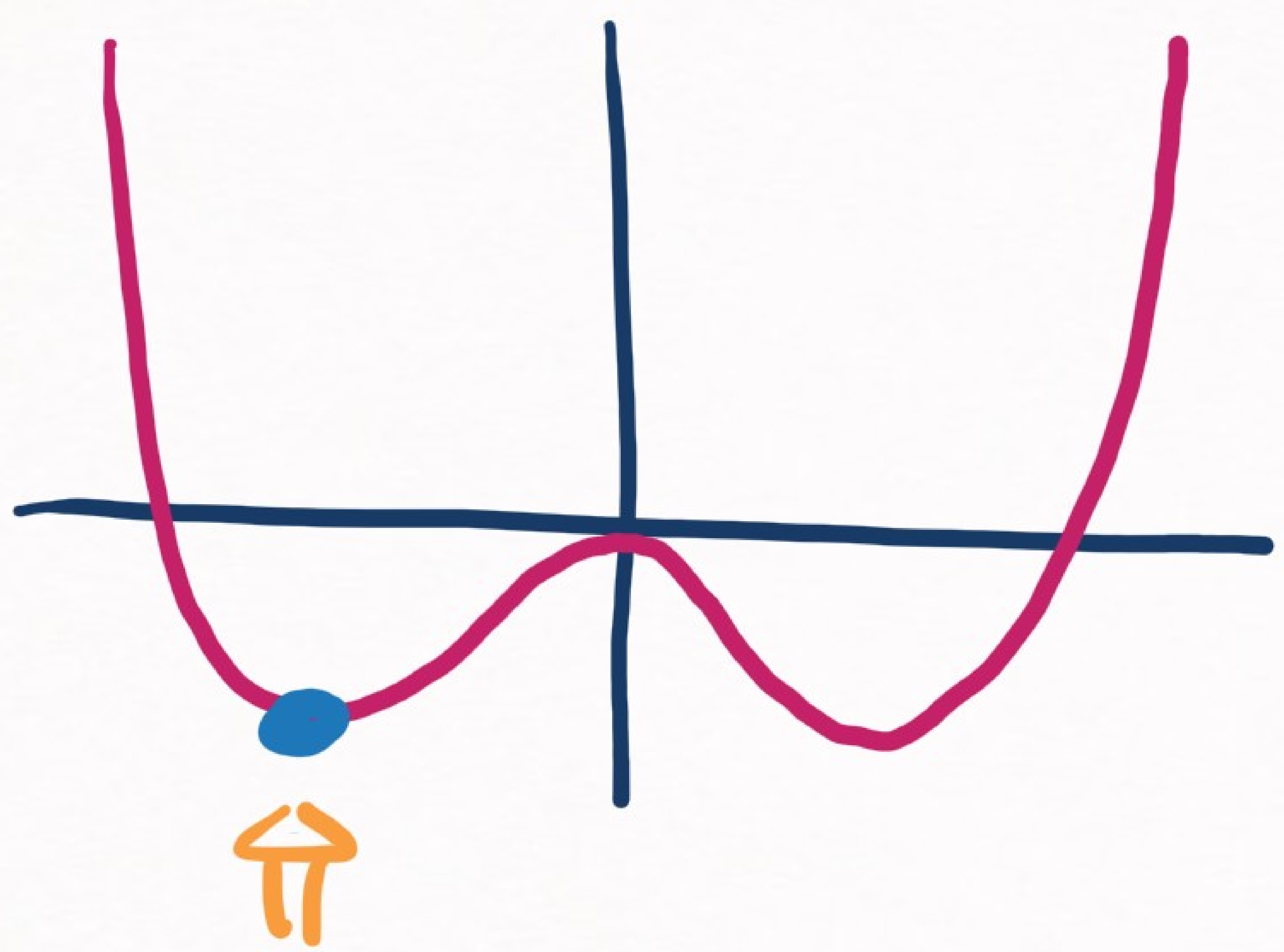
$\phi_{min} = +v \Rightarrow V_+(\sigma) = -\frac{v^2}{4} - \mu^2\sigma + \lambda v \sigma^3 + \frac{\lambda}{4}\sigma^4$

$\phi_{min} = -v \Rightarrow V_-(\sigma) = -\frac{v^2}{4} - \mu^2\sigma - \lambda v \sigma^3 + \frac{\lambda}{4}\sigma^4$

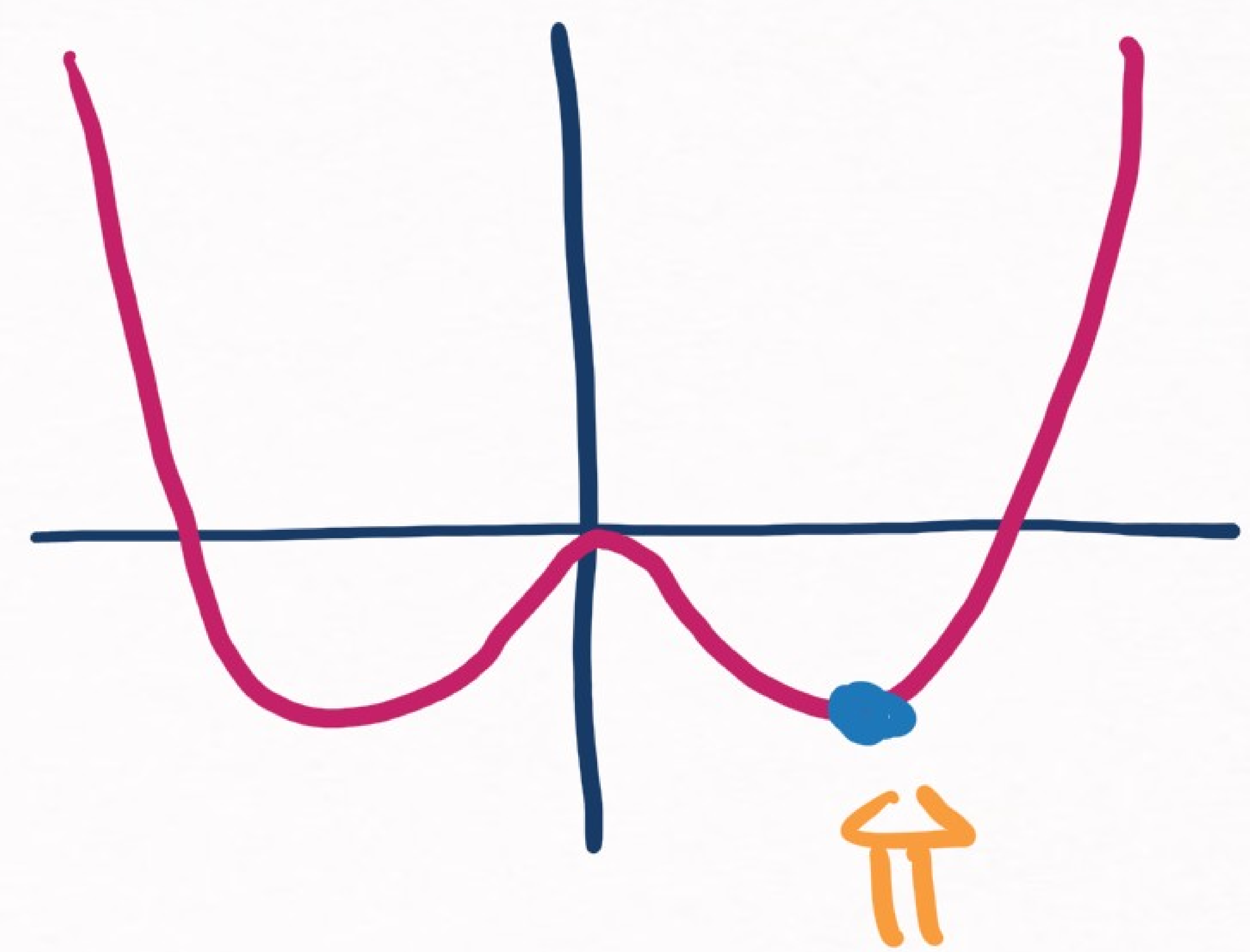
$\sigma \to -\sigma$: $V_+ \to V_-$ ($V_+ \not\to V_+$) ⇒ new theory lost this symmetry

[The original symmetry ($\phi \to -\phi$) is broken]

$\Rightarrow$ It needs to be like that, because there is only one vacuum state $\Rightarrow$ We have to choose our vacuum

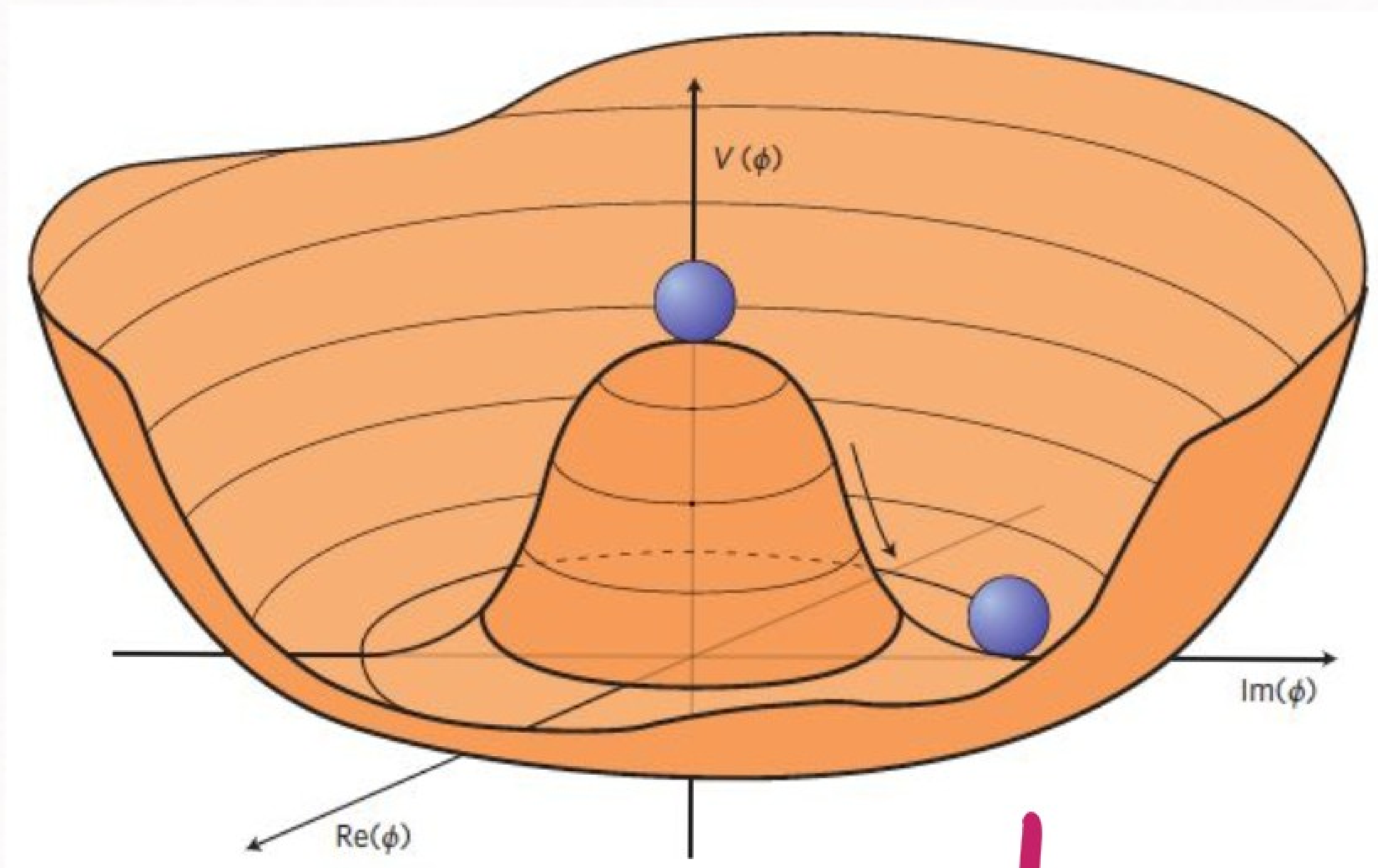


Choice #1



Choice #2

b) The same thing happens with ϕ complex:



$$V(|\phi|) = \frac{1}{2}\mu|\phi|^2 + \frac{\lambda}{4}|\phi|^4$$

$$\left[V(|\phi|) \xrightarrow[\phi \to e^{i\theta}\phi]{} V(|\phi|) \right]$$

Symmetry of the original potential

Choice of vaccum: $\phi_{min} = \underbrace{v e^{i\theta_0}}_{\text{Choice of a phase}}$

$$\phi_{min} \;\cancel{\xrightarrow[\phi \to e^{i\theta}\phi]{}}\; \phi_{min}$$

c) And the same happens with the LσM:

c.1) Original symmetry: $\phi = (\phi_0, \vec{\phi})$ $V = V(\sum_{i=0}^{3} \phi_i^2)$

$\phi_i \to R_{ij}\phi_i$ / $RR^T = 1$ + $|R\phi|^2 = |\phi|^2$ ⟹ a symmetry of V

$R \in O(4)$

c.2) New symmetry for $\underline{\mu^2 < 0}$: New vacuum

$$\begin{pmatrix}\phi_0\\ \vec{\phi}\end{pmatrix} \to \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & \check{R}\end{array}\right)\begin{pmatrix}\phi_0\\ \vec{\phi}\end{pmatrix}$$ still a symmetry

$\phi_0 = v + \sigma$

$\vec{\phi} = \vec{\pi}$

⇓

O(3) symmetry

$\vec{\phi} \to \underset{3\times 3}{\underline{\check{R}}}\vec{\phi}$ / $\check{R}\check{R}^T = 1$, $|\check{R}\vec{\phi}|^2 = |\vec{\phi}|^2$

⟹ $\check{R} \in O(3)$

RECAP (Spontaneous symmetry breaking)

	Original symmetry	Final symmetry	Massless fields
Real ϕ	$\phi \to -\phi$ [$\mathbb{Z}_2$]	$\sigma \to \sigma$ [$\mathbb{Z}_1$]	none
Complex ϕ	$\phi \to e^{i\theta}\phi$ [$U(1)$]	$\pi \to -\pi$ [$\mathbb{Z}_2$]	π (1)
$(\phi_0, \vec{\phi})$ [LσM]	$\phi \to R_{4\times4}\phi$ [$O(4)$]	$\vec{\pi} \to R_{3\times3}\vec{\pi}$ [$O(3)$]	$\vec{\pi}$ (3)

$$\left.\text{Original symmetry} > \text{Final symmetry}\right\}$$

Number of massless fields is related to this symmetry breaking pattern

[INTERESTING POINTS HERE]

a) Original → Final symmetry change has a name:

SPONTANEOUS SYMMETRY BREAKING

b) Number of massless particles related to the spontaneous symmetry breaking pattern

⇒ the question is how it is related?

⇒ this was solved by Nambu & Goldstone a long time ago

[NAMBU-GOLDSTONE THEOREM] (a schematic version) 1

a) Hamiltonian H / invariant under a group G

$[H, G] = 0$ → how we express a)

b) H has a $\underbrace{\text{vacuum state}}_{\text{fancy way of saying "ground state"}}$ $|0\rangle$

c) But $|0\rangle$ is not invariant under G,

but instead it will be invariant

under $\underline{\underline{F \subseteq G}}$ (F subgroup of G)

[NAMBU GOLDSTONE THEOREM] ②

If conditions a), b) and c) are met, then:

the quotient group $\frac{G}{F}$ can be used to generate states that are not the vacuum but have the same energy as the vacuum

(that is, these states have zero mass)

CAVEAT: G and F are continuous groups

EXTRA: the number of massless fields in $n_G - n_F$

$n_G \to$ # of generators of G $\quad n_F \to$ # of generators of F

REMINDER | What is a quotient group?

a) Group $(G, \cdot) \Rightarrow \exists e \in G \,/\, e \cdot a = a$ for $\forall a \in G$

$\forall a \in G, \exists b \in G \,/\, a \cdot b = e$

$\forall a, b, c \in G, \ (a \cdot b) \cdot c = a \cdot (b \cdot c)$

b) $F \subseteq G$ (F subgroup of G) if $\forall a, b \in F, \ a \cdot b \in F$

c) $G/F \Rightarrow p \equiv p'$ (p, p' are congruent)

if and only if $pF = p'F$ $(\exists a \in F \,/\, ap = p')$

$F \backslash G \Rightarrow$ same thing but with $Fp = Fp'$

What is a quotient group? 2

d) $G/F \Rightarrow$ We define $\overline{p} = pF$ (congruence class of p)

Now we define $\overline{p} \cdot \overline{q} = \overline{p \cdot q}$ ($\Rightarrow$ this is a good operation)

Then, if F is an invariant subgroup of G ($\forall a \in G, \forall f \in F, a \cdot f \cdot a^{-1} \in F$)

$\Rightarrow (G/F, \cdot)$ is a group called the quotient group of G & F

[QUOTIENT GROUPS: EXAMPLES]

a) $\mathbb{R}/\mathbb{Z} \cong U(1)$ $\quad a \in \mathbb{R},\ n \in \mathbb{Z} \Rightarrow \overline{(a+n)} = \overline{a}$

$\overline{a} \rightarrow e^{i\theta_a}$

$\overline{a} + \overline{b} \rightarrow e^{i(\theta_a + \theta_b)}$

b) $U(N)/U(1) \cong SU(N)$

c) $U(1)/\mathbb{Z}_n \cong U(1)$ $\quad \mathbb{Z}_n = \{ \mathbb{Z}$ but with equivalence based on mod $n \}$

b) $SO(4)/SO(3) \cong SO(3)$

$SO(n) \rightarrow$ proper rotations (rotations w/ reflections) in n dimensions

REMINDER | ② What are group generators?

→ basically, the number of independent parameters of a continuous transformation

(only applicable to continuous groups)

EXAMPLE:

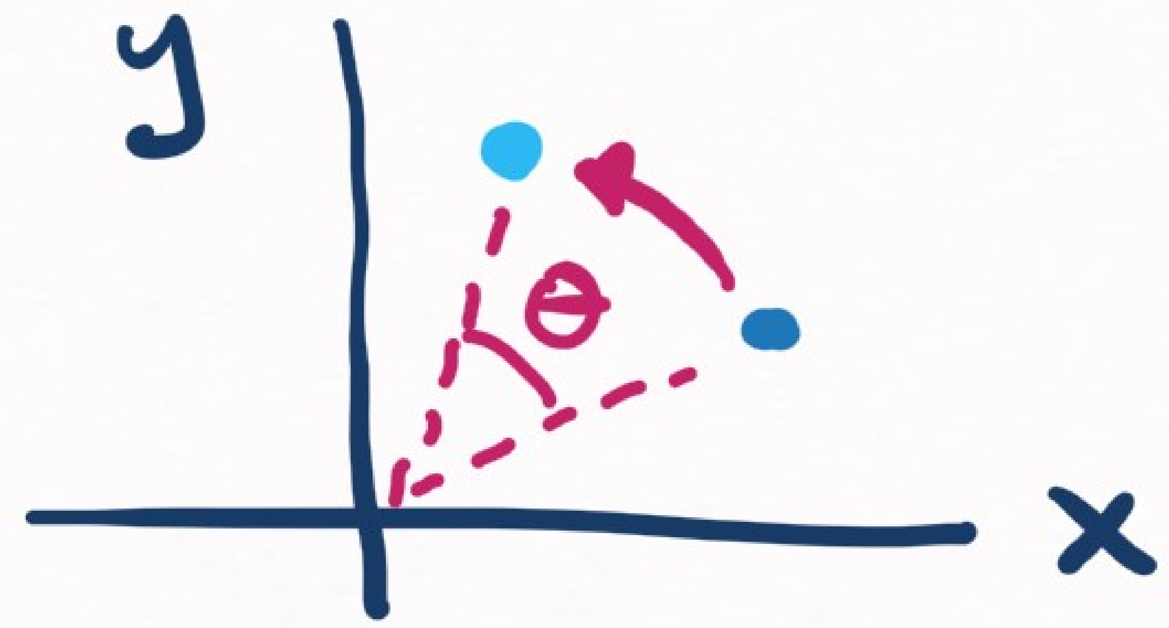
$U(1) = \{ U \in GL(1,\mathbb{C}) \,/\, U^\dagger U = 1 \}$

$\Downarrow$

$e^{i\theta}$ (element of $U(1)$) $\Rightarrow$ only 1 parameter (θ)

[GROUP GENERATORS]

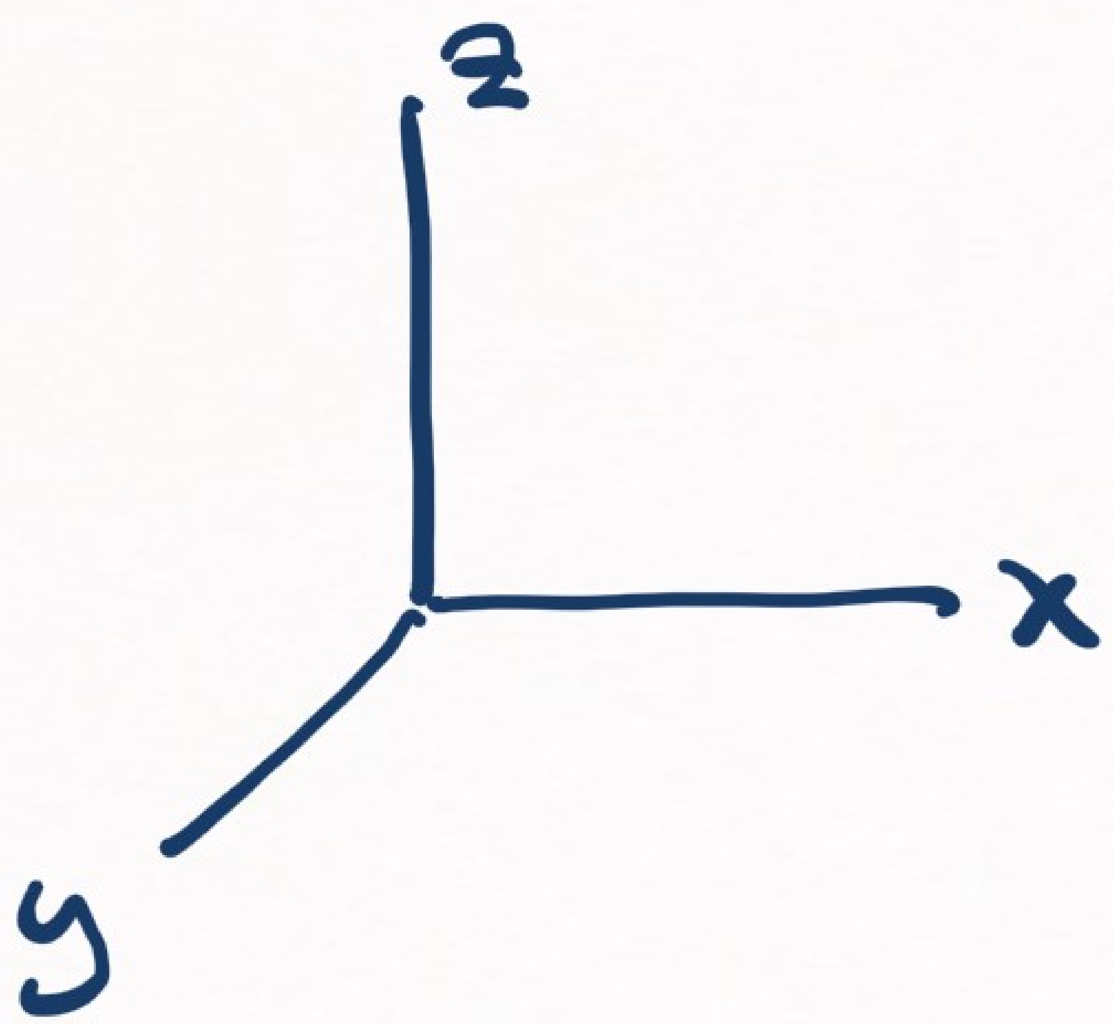
a) Rotations



a.1) two dimensions:

$$R(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$\Rightarrow$ only one parameter (generator)



a.2) three dimensions:

Euler angles $\rightarrow$ 3 angles

$\Rightarrow$ 3 generators

a.3) For n dimension $\Rightarrow$ consider an infinitesimal transformation

$RR^T = 1$ (rotation) $\quad R = 1 + \sum_a \delta\theta_a J_a$

$\Rightarrow \sum \delta\theta_a (J_a + J_a^T) = 0 \longrightarrow (J_a + J_a^T) = 0$

3-dim:

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

3 generators

[GROUP GENERATORS]

Solutions of $(J_a + J_a^\dagger) = 0$ in n-dim

a) Rotations $\to$ $SO(n)$, $n_{gen} = \frac{1}{2}n(n-1)$ $\Rightarrow$

b) Unitary transformations $\to$ $U(n)$, $SU(n)$

$$U(n) = \{ U \in GL(n,\mathbb{C}) \,/\, U^\dagger U = 1 \}$$

$$SU(n) = \{ U \in GL(n,\mathbb{C}) \,/\, U^\dagger U = 1 \,, \det U = 1 \}$$

$$U = 1 + \sum_a \delta\theta_a J_a \Rightarrow U^\dagger U = 1 \Rightarrow U^\dagger U = \mathbb{1} + \sum_a (\delta\theta_a J_a + \delta\theta_a^* J_a^\dagger)$$

Solutions of $J_a + J_a^\dagger = 0$ $\Big\} \to n(n-1) + n = n^2$

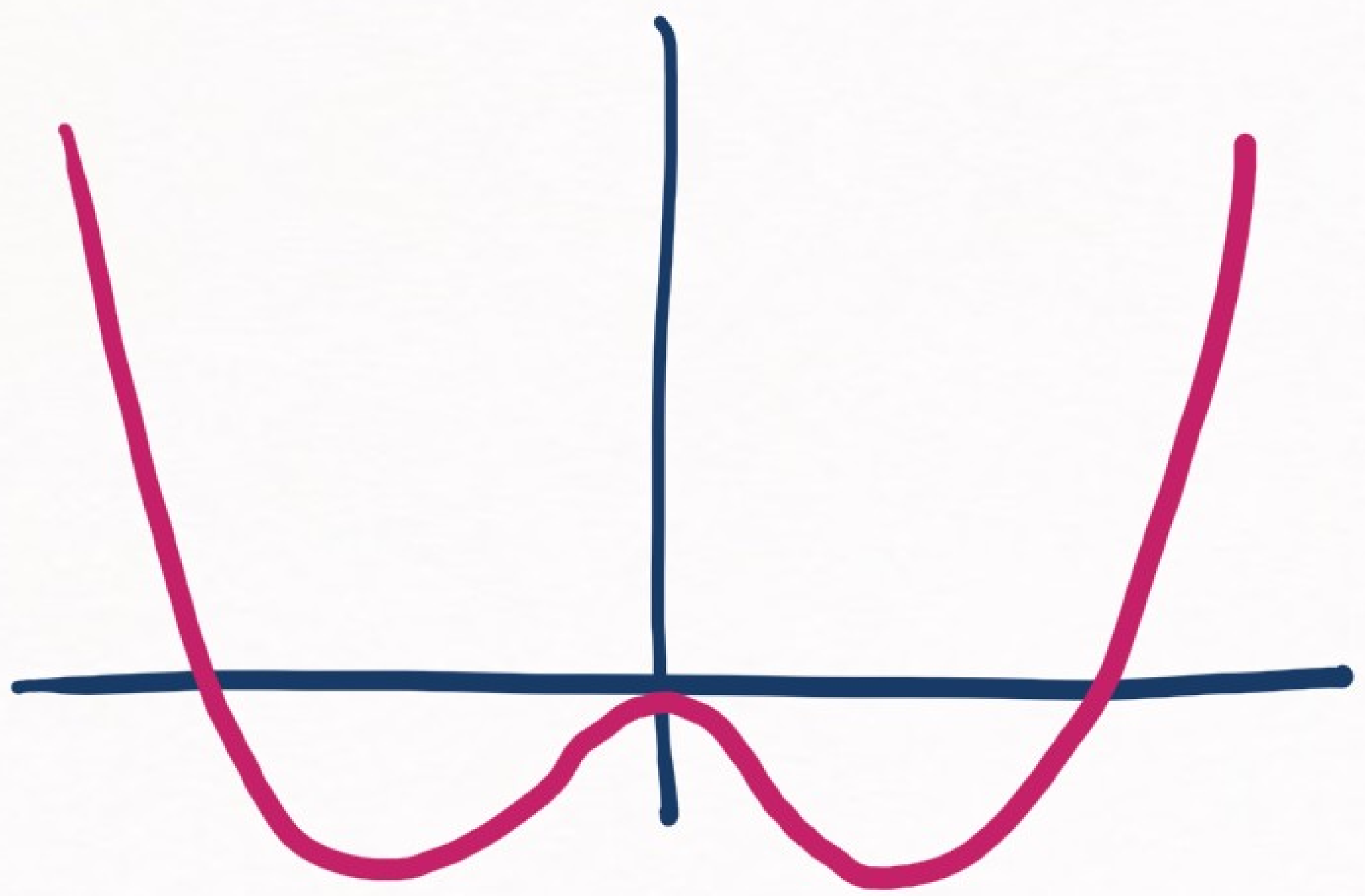
$$\left[\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ldots \right] \to 6 \qquad \begin{pmatrix} i & & \\ & 0 & \\ & & 0 \end{pmatrix}, \begin{pmatrix} 0 & & \\ & i & \\ & & 0 \end{pmatrix}, \begin{pmatrix} 0 & & \\ & 0 & \\ & & i \end{pmatrix} \to 3$$

[APPLYING THE NAMBU GOLDSTONE THEOREM]

a) Real ϕ field :

$$\left.\begin{array}{l} G = \mathbb{Z}_2 \\ F = \mathbb{Z}_1 \end{array}\right\} \Rightarrow$$ PROBLEM:

not continuous groups (but discrete)



$\Downarrow$

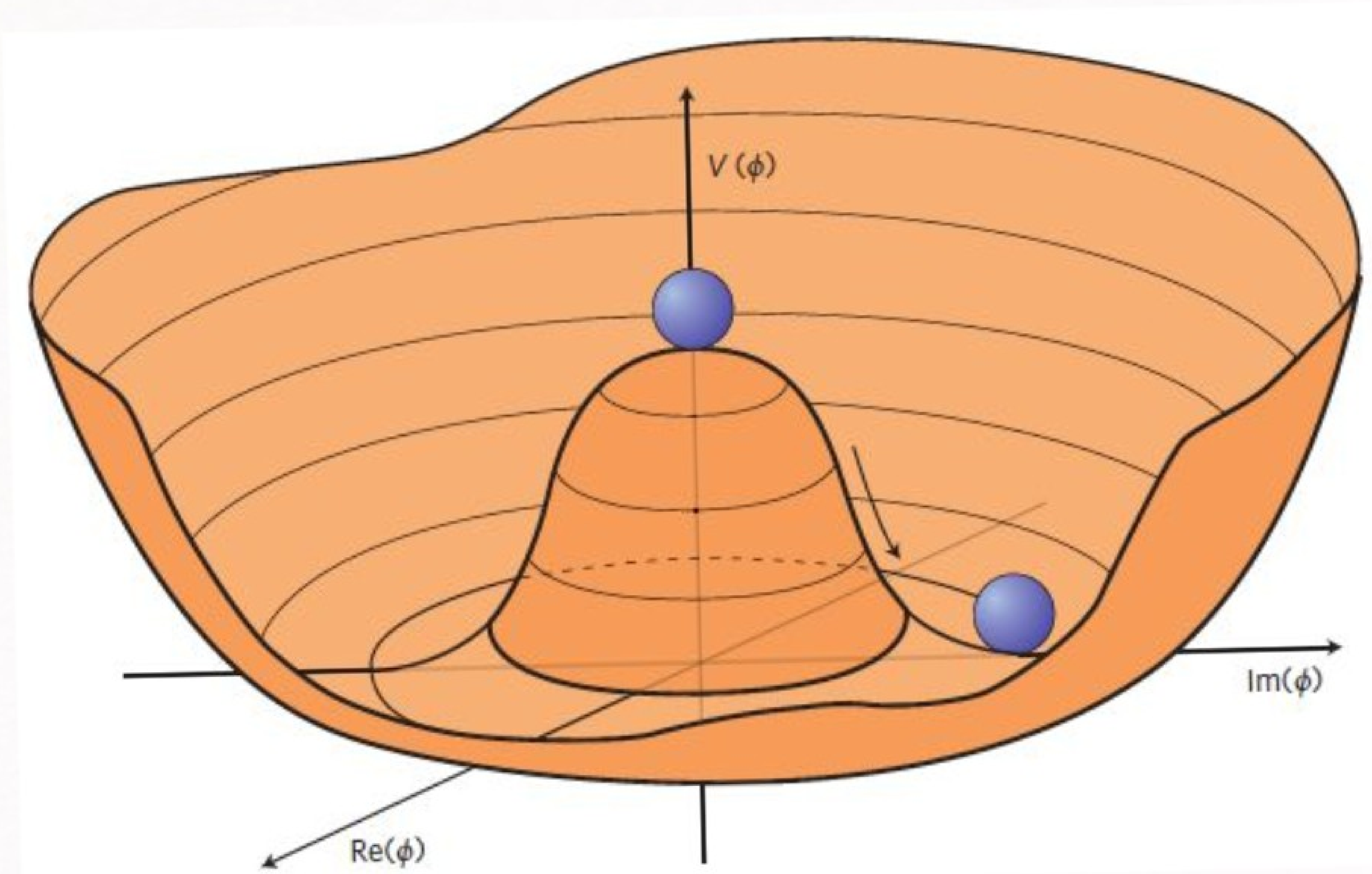
Nambu-Goldstone theorem does not apply

But we are physicists & like to ignore mathematical rigor

[$G/F \simeq \mathbb{Z}_2 \rightarrow$ discrete $(n_{G/F} = 0) \rightarrow$ 0 massless fields]

[APPLYING THE NAMBU-GOLDSTONE THEOREM]

b) Complex ϕ field:



$G = U(1)$ ✓ (continuous)

$F = \mathbb{Z}_1$ ✗ (discrete)

[Strictly speaking should not apply]

$\frac{G}{F} = U(1) \Rightarrow$ one generator

[Despite not being applicable still gives the correct number of massless fields]

[APPLYING THE NAMBU GOLDSTONE THEOREM]

c) ϕ_i, $i = 0,1,2,3$ in the linear sigma model

$G: \begin{pmatrix} \phi_0 \\ \vec{\phi} \end{pmatrix} \to \boxed{R_{4\times 4}} \begin{pmatrix} \phi_0 \\ \vec{\phi} \end{pmatrix} \Rightarrow O(4) = \{ R \in GL(4,\mathbb{R}) \,/\, R^T R = 1 \}$

$F: \begin{pmatrix} v \\ \vec{0} \end{pmatrix} \to \boxed{\begin{array}{c|c} 1 & 0 \\ \hline 0 & R_{3\times 3} \end{array}} \begin{pmatrix} v \\ \vec{0} \end{pmatrix} \Rightarrow O(3)$

$\Rightarrow \boxed{G/F \simeq O(3)} \Rightarrow O(n)$ has $\frac{1}{2}n(n-1)$ generator

$\Rightarrow G/F$ has 3 generator $\Rightarrow$ 3 massless field

$(\vec{\pi} = (\pi_1, \pi_2, \pi_3))$

[APPLYING THE NAMBU GOLDSTONE THEOREM]

$\Rightarrow$ You can apply it to any other combination of groups

3-component ϕ (real) $+$ $V(\vec{\phi}) = \frac{1}{2}\mu\vec{\phi}^2 + \frac{\lambda}{4}(\vec{\phi}^2)^2$

$$\left.\begin{array}{l} G = O(3) \\ F = O(2) \end{array}\right\} \; G/F \cong S^2$$

$\Rightarrow$ 2 massless fields

(JUST AN EXAMPLE)

$\cong$: this means isomorphic (in the geometric sense)

CHIRAL SYMMETRY

$\Rightarrow \left[\frac{m_u}{\Lambda_{QCD}}, \frac{m_d}{\Lambda_{QCD}} \ll 1 \text{ and } \frac{m_s}{\Lambda_{QCD}} < 1 \right]$

a) We already know that this gives rise to SU(2)-isospin and SU(3)-flavor symmetries

b) But it also triggers a more subtle type of symmetry

$\Rightarrow$ this new symmetry later breaks spontaneously & we will be able to apply Goldstone's theorem

[MASSLESS PARTICLES IN QM] $\Rightarrow$ There is a conserved quantity

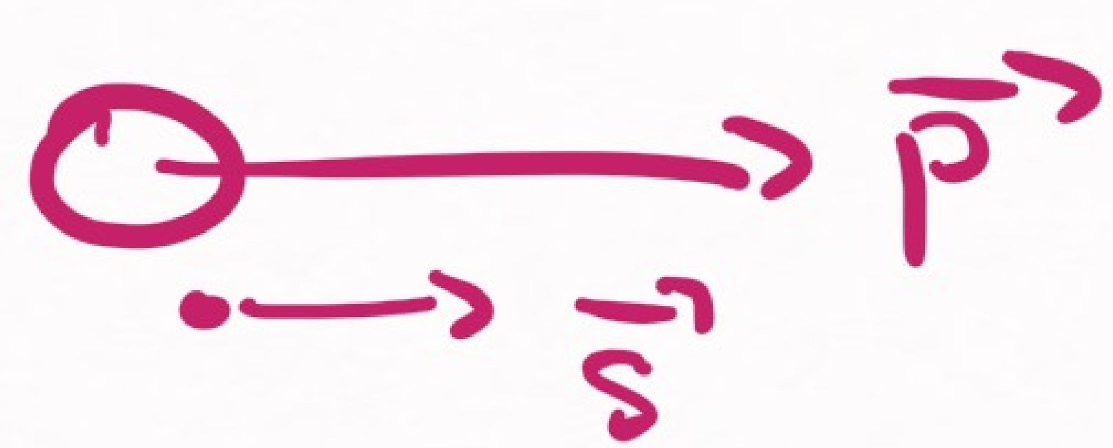
a) Particle with spin-$\frac{1}{2}$:

$\vec{\sigma} \rightarrow |+\rangle, |-\rangle$ (Spin) eigenvalues

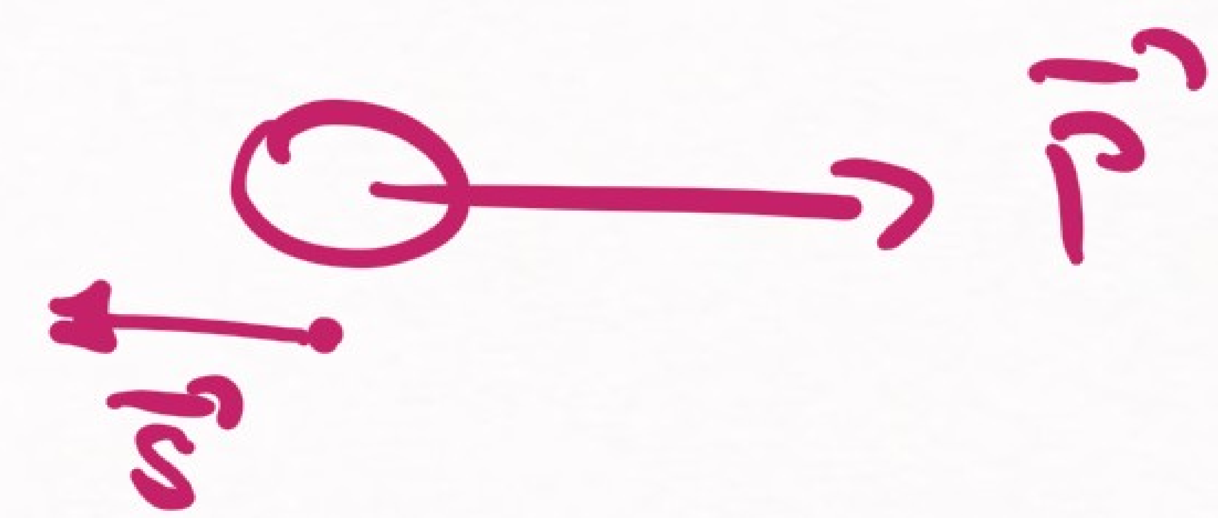
But we can also define a new quantity:

Helicity: $\lambda = \dfrac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}$ $\Rightarrow$ eigenvalues are $\lambda = \pm 1$

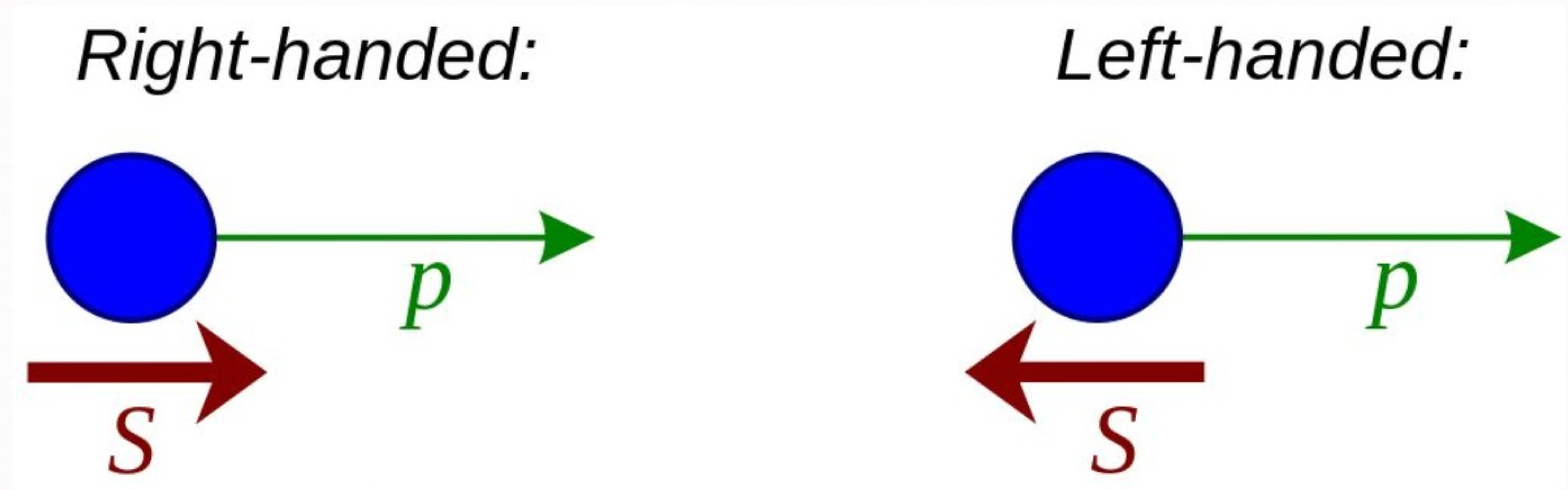
$\lambda = +1$



$\lambda = -1$



HELICITY



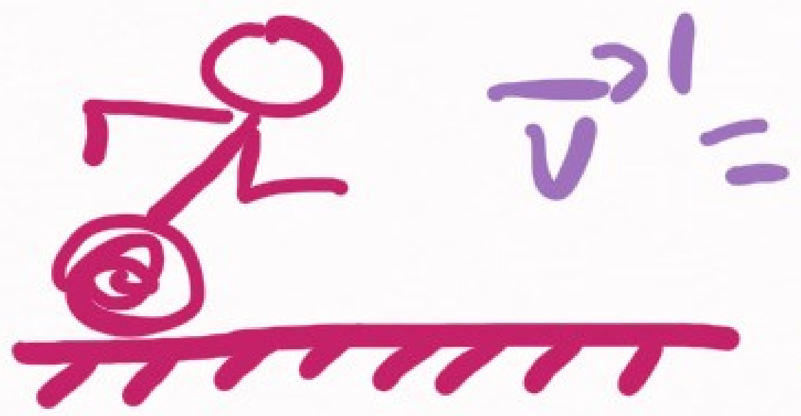
How does it behaves relative to boosts?

a) If the particle is massive ($m \neq 0$):

I can change the helicity of the particle by moving faster than the particle

$\vec{v} = 0$

=D



$\vec{v}' = \frac{\vec{p} - \vec{p}'}{m}$ + (relativistic corrections)

HELICITY

b) But if the particle is massless ($m=0$),

$\longrightarrow \vec{p}$ $(|\vec{p}| = E)$

$\longrightarrow \vec{s}$

$\vec{v} = 0$

$\lambda = +1$

$\Rightarrow$

$\longrightarrow \vec{p}$ $(|\vec{p}| = E)$

$\longrightarrow \vec{s}$

$\vec{v}' \neq 0$ $(\vec{v}' \parallel \vec{p})$

$\lambda = +1$

$\Rightarrow$ For massless particles helicity is conserved in boosts

[FROM HELICITY TO CHIRALITY] 1

⇒ What is the QFT version of helicity?

$$\mathcal{L}_{Dirac} = \bar{\psi}(i\not{\partial} - m)\psi \quad \text{(Dirac field)} \Leftarrow$$

Symmetries:

a) Global U(1) symmetry:

$$\psi(x) \to e^{i\alpha}\psi(x), \quad \psi^{\dagger}(x) \to e^{-i\alpha}\psi(x)$$

$$\mathcal{L} \to \mathcal{L}$$

[FROM HELICITY TO CHIRALITY] 2

⇒ Dirac field ψ contain four components

(two spin states for particles / for anti-particles)

⇒ Right- and left-handed components (definition)

$$\psi_R = P_R \psi = \frac{1}{2}(1+\gamma_5)\psi \qquad P_R^\dagger = P_R$$

$$\psi_L = P_L \psi = \frac{1}{2}(1-\gamma_5)\psi \qquad P_L^\dagger = P_L$$

[FROM HELICITY TO CHIRALITY] 3

$P_L + P_R = 1$

$$\Rightarrow \left[\begin{array}{lll} P_R^\dagger = P_R & P_R^2 = P_R & P_L P_R = 0 \\ P_L^\dagger = P_L & P_L^2 = P_L & P_L + P_R = 0 \end{array}\right] \Rightarrow \text{ they are } \underline{\text{projectors}}$$

a) $\underline{\underline{\psi}} = (P_L + P_R)\psi = \underline{\psi_L + \psi_R}$

b) We can try $U(1)$ symmetry but applied to $\underline{\underline{\psi_L}}$ & $\underline{\underline{\psi_R}}$:

$$\psi_L \to e^{i\alpha_L}\psi_L \qquad\qquad \psi_R \to e^{i\alpha_R}\psi_R$$

$$U(1)_L \qquad\qquad\qquad U(1)_R$$

[FROM HELICITY TO CHIRALITY] 4

$\Rightarrow$ Is this a symmetry of $\mathcal{L}_{Dirac} = \bar{\psi}(i\not{\partial} - m)\psi$?

a) kinetic term: $\mathcal{L}_1 = \bar{\psi} i \not{\partial} \psi$, $\not{\partial} = \gamma_\mu \partial^\mu$

$$\bar{\psi}\not{\partial}\psi = \bar{\psi}_L \not{\partial} \psi_L + \bar{\psi}_R \not{\partial} \psi_R \Rightarrow \bar{\psi}\not{\partial}\psi \xrightarrow[U(1)_L,\, U(1)_R]{} \bar{\psi}\not{\partial}\psi$$

b) Mass term: $\mathcal{L}_2 = \bar{\psi} m \psi = m\bar{\psi}\psi$

$\bar{\psi}\psi = \bar{\psi}_L \psi_R + \bar{\psi}_R \psi_L \longrightarrow$ mixes R & L components

$$\bar{\psi}\psi \longrightarrow e^{i(\alpha_R - \alpha_L)} \bar{\psi}_L \psi_R + e^{-i(\alpha_R - \alpha_L)} \bar{\psi}_R \psi_L$$

[FROM HELICITY TO CHIRALITY] 5

$\Rightarrow$ We see that: $\bar{\psi} m \psi \to e^{i(\alpha_R-\alpha_L)} \bar{\psi}_L m \psi_R + e^{-i(\alpha_R-\alpha_L)} \bar{\psi}_R m \psi_L$ } Does not conserve chirality

That is: $\bar{\psi} m \psi \not\to \bar{\psi} m \psi$ under $U(1)_R, U(1)_L$

However $\bar{\psi} m \psi \to \bar{\psi} m \psi$ $\underbrace{\text{for } \alpha_L = \alpha_R}$

We call this $U(1)_{L+R}$

[FROM HELICITY TO CHIRALITY] 6

a) If $m \neq 0 \Rightarrow U(1)_{L+R}$ symmetry ✓

b) If $m = 0 \Rightarrow \underbrace{U(1)_L \; \& \; U(1)_R \text{ symmetries}}_{\text{This theory has chiral symmetry}}$

$\Rightarrow$ But we want the application of this idea to QCD

[QCD & CHIRAL SYMMETRY] 5

a) How does this apply to QCD?

$$\mathcal{L}_{QCD} = \underbrace{-\frac{1}{4}G^a_{\mu\nu}G^{\mu\nu a}}_{1} + \overbrace{\bar{q}\,i\cancel{D}\,q}^{2} \overbrace{- \bar{q}\mathcal{M}q}^{3}$$

$$q = \begin{pmatrix} u \\ d \\ s \\ c \\ b \\ t \end{pmatrix} \longrightarrow \text{flavors} \qquad \mathcal{M} = \begin{pmatrix} m_u & & & & & 0 \\ & m_d & & & & \\ & & m_s & & & \\ & & & m_c & & \\ & & & & m_b & \\ 0 & & & & & m_t \end{pmatrix}$$

$\rightarrow$ the masses of each flavor

1 → Gluon field
2 → Kinetic term (quarks)
3 → Mass term (quarks)

[QCD & CHIRAL SYMMETRY] (2)

b) $\frac{m_u}{\Lambda_{QCD}}, \frac{m_d}{\Lambda_{QCD}} \ll 1$, $\frac{m_s}{\Lambda_{QCD}} < 1$ $\Rightarrow$ [We can take the approx.: $m_u = m_d = 0$ $(m_s = 0)$]

c) What happens if $m_q = 0$ for one flavor?

$$m_u = 0 \Rightarrow \begin{array}{l} u_L \to e^{i\alpha_L} u_L \\ u_R \to e^{i\alpha_L} u_L \end{array} \qquad \bar{u} i \not{\partial} u \to \bar{u} i \not{\partial} u$$

$\Rightarrow$ It will be a symmetry of QCD

[QCD & CHIRAL SYMMETRY] 3

d) What happens if $m_q = 0$ for two or more flavors?

$$\begin{pmatrix} u \\ d \end{pmatrix} = \begin{pmatrix} u \\ d \end{pmatrix}_L + \begin{pmatrix} u \\ d \end{pmatrix}_R$$

$$\begin{pmatrix} u \\ d \end{pmatrix}_L \to V_L \begin{pmatrix} u \\ d \end{pmatrix}_L \quad , \begin{pmatrix} u \\ d \end{pmatrix}_R \to V_R \begin{pmatrix} u \\ d \end{pmatrix}_R$$

$$(\bar{u}\ \bar{d}) i \not{\partial} \begin{pmatrix} u \\ d \end{pmatrix} = (L+R) \longrightarrow (\bar{u}\ \bar{d}) i \not{\partial} \begin{pmatrix} u \\ d \end{pmatrix}$$

if $V_L^\dagger V_L = 1$

$V_R^\dagger V_R = 1$

$$\Rightarrow \boxed{V_L, V_R \in U(2)}$$

[QCD & CHIRAL SYMMETRY] 4

⇒ Now with 3 light flavors:

$$\begin{pmatrix} u \\ d \\ s \end{pmatrix} = \begin{pmatrix} u \\ d \\ s \end{pmatrix}_R + \begin{pmatrix} u \\ d \\ s \end{pmatrix}_L \quad , \quad (\bar{u}\,\bar{d}\,\bar{s})\, i\not{\partial} \begin{pmatrix} u \\ d \\ s \end{pmatrix} = (\bar{u}\,\bar{d}\,\bar{s})_R\, i\not{\partial} \begin{pmatrix} u \\ d \\ s \end{pmatrix}_R + (\bar{u}\,\bar{d}\,\bar{s})_L\, i\not{\partial} \begin{pmatrix} u \\ d \\ s \end{pmatrix}_L$$

$$\begin{pmatrix} u \\ d \\ s \end{pmatrix}_{L(R)} \longrightarrow V_{L(R)} \begin{pmatrix} u \\ d \\ s \end{pmatrix}_{L(R)}$$

$$\left.\begin{aligned} V_L^\dagger V_L &= 1 \\ V_R^\dagger V_R &= 1 \end{aligned}\right\} \Rightarrow \quad \underline{\underline{V_L, V_R \in U(3)}}$$

[QCD & CHIRAL SYMMETRY] (5)

e) For n light quark flavors

naive answer

$\Rightarrow$ $\mathcal{L}_{QCD}$ invariant under $\underline{\underline{G = U_L(n) \otimes U_R(n)}}$

$\Rightarrow$ But as often happens, there is going to be a complication

[CHIRAL ANOMALY] $\Rightarrow$ Reduces the symmetry

[CHIRAL ANOMALY]

a) If $G = U(n)_L \otimes U(n)_R \cong \underline{\underline{U(n)}}_{\underline{\underline{L+R}}} \otimes \underline{\underline{U(n)}}_{\underline{\underline{L-R}}}$

$\Rightarrow$ then we have (classically)

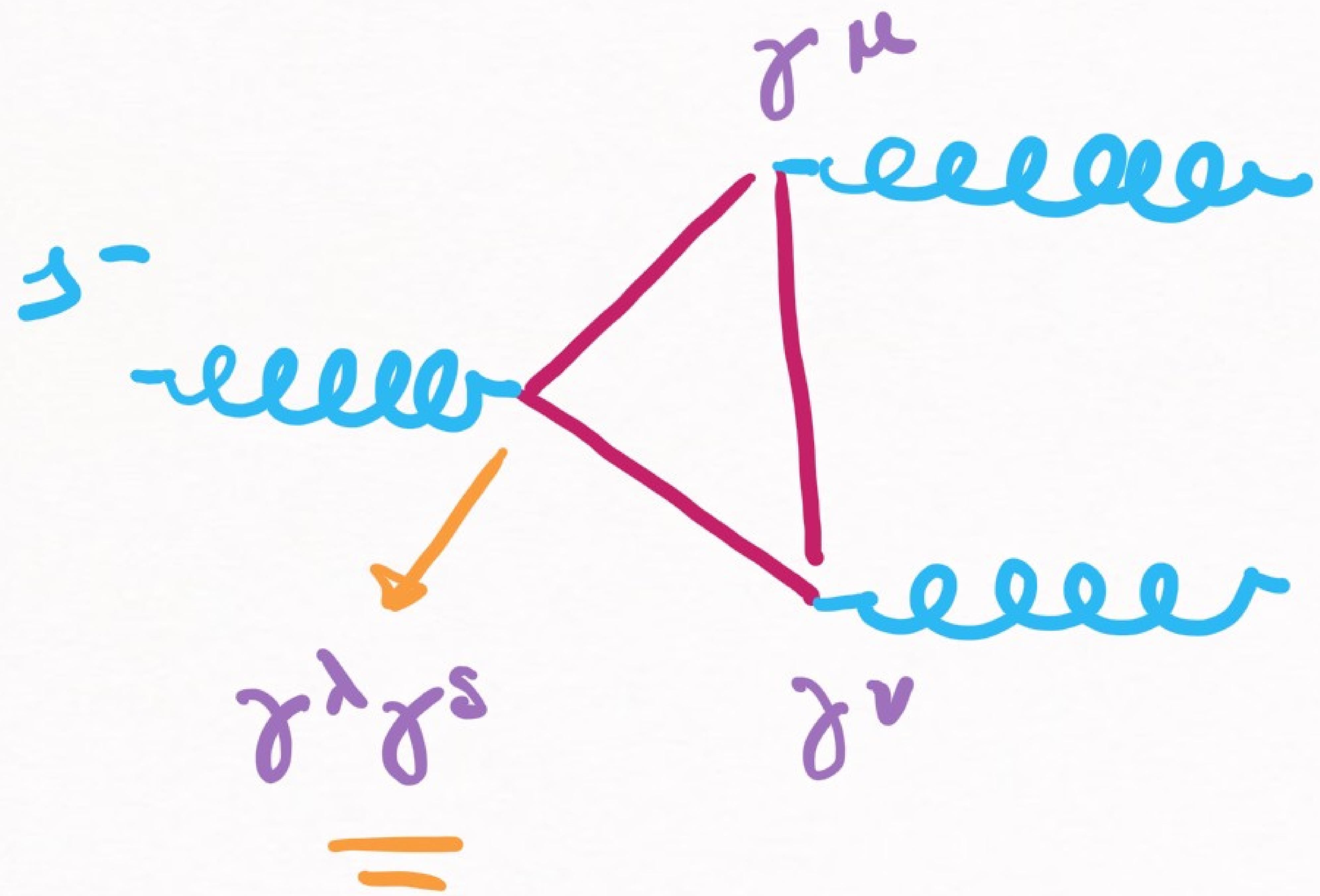
$$\left[\begin{array}{lll} J_\mu = \bar{q}\gamma^\mu q & \longrightarrow & \partial^\mu J_\mu = 0 \\ J_\mu^5 = \bar{q}\gamma^\mu\gamma^5 q & \longrightarrow & \partial^\mu J_\mu^5 = 0 \end{array}\right]$$

$$\bar{q}\gamma^\mu q = \bar{q}_R\gamma^\mu q_R + \bar{q}_L\gamma^\mu q_L \quad (L+R)$$

$$\bar{q}\gamma^\mu\gamma^5 q = \bar{q}_R\gamma^\mu q_R - \bar{q}_L\gamma^\mu q_L \quad (L-R)$$

[CHIRAL ANOMALY] 2

b) However, there is an issue with this QFT diagram:



b.1) Classically:

$$\partial_\mu(\text{diag}) = 0 \quad \partial_\nu(\text{diag}) = 0$$

$$\partial_\lambda(\text{diag}) = 0$$

b.2) QFT:

$$\partial_\mu(\text{diag}) = 0 \quad \partial_\nu(\text{diag}) = 0$$

$$\underline{\partial_\lambda}(\text{diag}) \neq 0$$

$\Rightarrow \exists$ a problem with J^5_μ (L-R)

[CHIRAL ANOMALY] 3

⇒ In QFT we have :

$$\partial^\mu J_\mu = 0 \quad \& \quad \underline{\underline{\partial^\mu J^5_\mu \neq 0}}$$

this is the operator that allows
the $\pi \to \gamma\gamma$ (in case that you have studied this)

⇒ [In QFT, $U(1)_{L-R}$
is not conserved]

[QCD & CHIRAL SYMMETRY]

a) Classically, if we have n massless flavors:

$$\mathcal{L}_{QCD} \xrightarrow[G]{} \mathcal{L}_{QCD}, \quad G = U(n)_{L+R} \otimes U(n)_{L-R}$$

b) Quantum-mechanically, $U(1)_{L-R}$ has to be broken:

$$\mathcal{L}_{QCD} \xrightarrow[G]{} \mathcal{L}_{QCD}, \quad G = SU(n)_{L+R} \otimes SU(n)_{L-R} \otimes U(1)_{L+R}$$

(we removed the $U(1)_{L-R}$)

[QCD : GROUND STATE SPECTRUM] 1

$V^\mu \to$ vector field

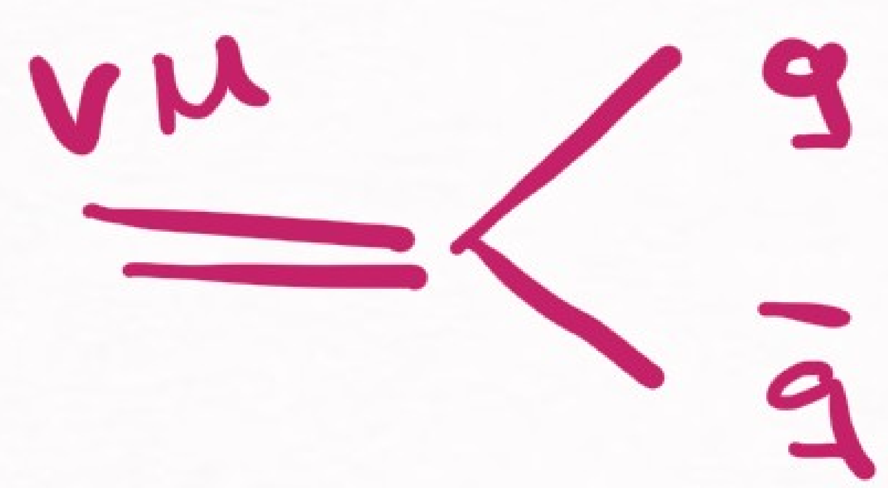
$A^\mu \to$ axial field

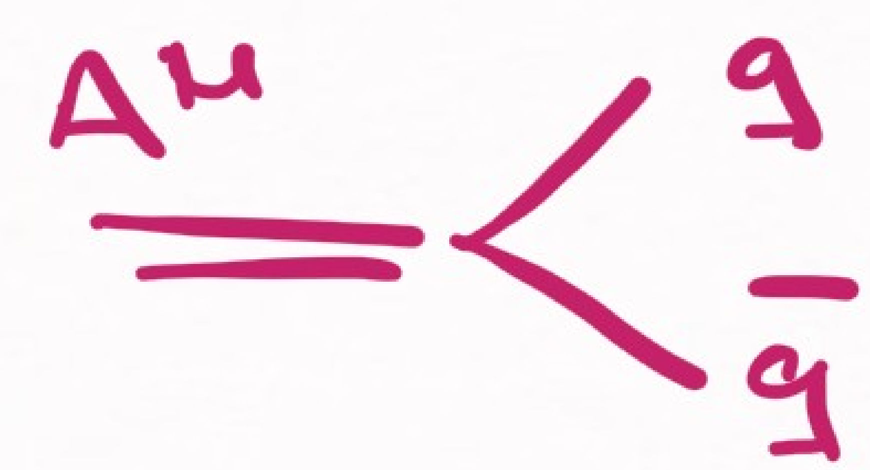
a) $G = SU(n)_L \otimes SU(n)_R \otimes U(1)_{L+R}$

$\simeq (SU(n)_{L+R} \otimes U(1)_{L+R}) \otimes SU(n)_{L-R}$

a.1) $\underbrace{\bar{q}\gamma^\mu q}_{U(1)_{L+R},\, SU(n)_{L+R}}, \quad \underbrace{\bar{q}\gamma^\mu\gamma^5 q}_{SU(n)_{L-R}}$ $\Rightarrow$ Should have consequences

$\Downarrow$

a.2) 

, 

We coupled a field to these currents

$V^\mu J_\mu = V^\mu(\bar{q}\gamma_\mu q)$ $\quad$ $A^\mu J^5_\mu = A^\mu(\bar{q}\gamma_\mu\gamma^5 q)$

[QCD : GROUND STATE SPECTRUM] ②

b) $V^\mu = \rho^\mu, \partial_\mu \sigma, \ldots$

$\Delta^\mu = a_1^\mu, \partial_\mu \pi, \ldots$

$G = SU(n)_L \otimes SU(n)_R \otimes U(1)_{L+R}$

$\Rightarrow \Big[m(\text{meson}, P = +1)$

$= m(\text{meson}, P = -1) \Big]$

$\Downarrow$

consequence for the spectrum

If V^μ, Δ^μ represent meson,

$V^\mu \to J^P = 1^-$, $\Delta^\mu \to J^P = 1^+$

c) General rule : $m(H, J^+) = m(H, J^-)$

Experimentally $\Rightarrow$

$m(\rho) = 770\,\text{MeV}\ (1^-)$

$m(a_1) = 1230\,\text{MeV}\ (1^+)$

$\left.\begin{aligned} \rho &\to J^P = 1^- \\ a_1 &\to J^P = 1^+ \end{aligned}\right\}$ (PDG)

[QCD: GROUND STATE SPECTRUM] 3

d) Experimentally:

$$\underbrace{m(\rho)}_{\substack{\text{couples to} \\ \bar{q}\gamma^\mu q \ (L+R)}} < \underbrace{m(a_1)}_{\substack{\text{couples to} \\ \bar{q}\gamma^\mu\gamma^5 q \ (L-R)}}$$

More masses:

$$m(N, \tfrac{1}{2}^+) \simeq 940\,\text{MeV} \qquad m(\Delta, \tfrac{3}{2}^+) \simeq 1230\,\text{MeV}$$

$$m(N, \tfrac{1}{2}^-) \simeq 1535\,\text{MeV} \qquad m(\Delta, \tfrac{3}{2}^-) \simeq 1700\,\text{MeV}$$

[QCD : GROUND STATE SPECTRUM] (4)

e) Conclusion : $\underbrace{\bar{q}\gamma^\mu q \text{ operators}}_{\substack{\text{lower masses} \\ (L+R) \\ SU(N)_{L+R} \otimes U(1)_{L+R}}} < \underbrace{\bar{q}\gamma^\mu\gamma^5 q \text{ operators}}_{\substack{\text{higher masses} \\ (L-R) \\ SU(N)_{L-R}}}$

f) What does it remind you?

$$\left.\begin{array}{l} \text{LσM} \rightarrow G = SO(4) \\ \text{vacuum} \Rightarrow F = SO(3) \end{array}\right\} \Rightarrow \text{Goldstone theorem}$$

[QCD : GROUND STATE SPECTRUM] (5)

$\Rightarrow$ a) Symmetry from $m_u = m_d = m_s = 0$:

$$G = SU(3)_{L+R} \otimes SU(3)_{L-R} \otimes U(1)_{L+R}$$

b) Symmetry of the lowest mass hadrons :

$$F = SU(3)_{L+R} \otimes U(1)_{L+R}$$

Then, from Goldstone theorem :

$G/F \simeq SU(3)_{L-R}$ that generates massless particles

$SU(3)$ has 8 generators $\Rightarrow$ 8 massless particles

[CHIRAL SYMMETRY BREAKING
& NAMBU - GOLDSTONE THEOREM]

a) $\underline{\underline{m_u = m_d = 0}} \Rightarrow \frac{G}{F} \simeq SU(2)_{L-R} \Rightarrow$ 3 massless bosons (π^+, π^0, π^-)

b) $m_u = m_d = m_s = 0 \Rightarrow \frac{G}{F} \simeq SU(3)_{L-R} \Rightarrow$ 8 massless bosons (π, K, η)

[NOTICE THIS] G/F bosons couple to $\bar{q}\gamma_\mu\gamma_5 q$ $\quad (L-R) \simeq G/F$

$(\underline{\underline{\bar{q}\gamma_\mu\gamma_5 q}} \to 1^+) \times \underline{\underline{\partial_\mu \phi}}$ $(\phi \to J=0)$

if ϕ is $0^+ \Rightarrow \partial_\mu \phi \to 1^-$ $(\partial_\mu \to 1^-) \Rightarrow$ won't work

$\Rightarrow \boxed{\phi \text{ is } 0^-}$

[COUPLING OF QUARKS
TO THE PION]

a) Pions couple to the $SU(2)_{L-R}$ current:

$$\bar{q}\gamma^\mu\gamma^5 q \Rightarrow \bar{q}\gamma^\mu\gamma^5 q \boxed{\partial_\mu\pi} \quad \text{or} \quad \bar{q}\gamma^\mu\gamma^5\vec{\tau} q \cdot \partial_\mu\vec{\pi}$$

$$\pi = \vec{\tau}\cdot\vec{\pi}$$

That is, we have derivative couplings

b) What about nucleons? $\Rightarrow$ Exactly the same as with quarks

$$\bar{N}\gamma^\mu\gamma^5 N\,\partial_\mu\pi \rightarrow \text{how pions couple to nucleon} \qquad (N \sim q)$$

[NON-CHIRAL VS CHIRAL PIONS] (1)

most simple assumption

1) Before chiral symmetry we had:

$$\mathcal{L}_{\pi NN} = g_{\pi NN} \bar{N} i \gamma_5 \vec{\tau} \cdot \vec{\pi} N$$

← non-derivative interaction

2) After chiral symmetry:

$$\mathcal{L}_{\pi NN} = \frac{g_A}{2 f_\pi} \bar{N} \gamma^5 \gamma_\mu \vec{\tau} \cdot \partial_\mu \vec{\pi} N$$

$g_A \simeq 1.26$

$f_\pi \simeq 92.4$ MeV

1+2) Comparison:

Goldberger-Treimann Relation

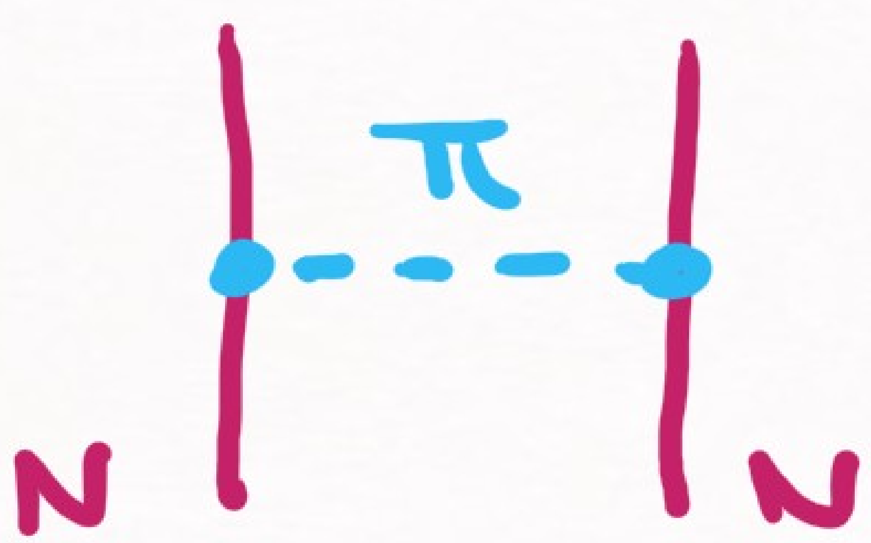
$$\boxed{\frac{g_A}{f_\pi} = \frac{g_{\pi NN}}{M_N}}$$

$g_{\pi NN} \simeq 13.4$

connects old pion theories with the new ones

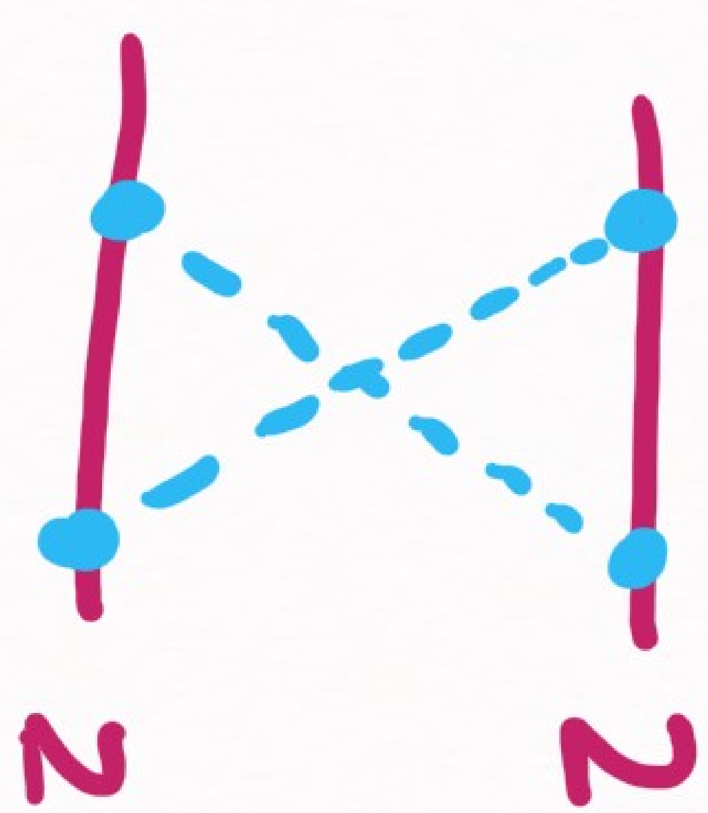
[NON-CHIRAL VS CHIRAL PIONS] 2

1) For OPE (one-pion exchange) there is no difference:



$$= -\frac{g_{\pi NN}^2}{4M_N^2}\,\frac{\vec{\sigma}_1\cdot\vec{q}\;\vec{\sigma}_2\cdot\vec{q}}{\vec{q}^{\,2}+m_\pi^2} = -\frac{g_A^2}{4f_\pi^2}\,\frac{\vec{\sigma}_1\cdot\vec{q}\;\vec{\sigma}_2\cdot\vec{q}}{\vec{q}^{\,2}+m_\pi^2}$$

2) For TPE (two-pion exchange) diagrams, it makes a big difference



$\rightarrow$ (too long & complicated to show)

derivative pion $\Rightarrow$ we will be able to obtain good results with TPE

CHIRAL PIONS

⇒ Current NN calculations use the pion dynamics derived from chiral symmetry breaking (derivative interaction)

⇒ In addition, the exploit the separation of scales between m_π & M_N ($m_\pi \ll M_N$) to formulate an expansion in power of $\frac{m_\pi}{M_N}$

CHIRAL PERTURBATION THEORY

RECAP

CHIRAL SYMMETRY BREAKING

a) $\frac{m_u}{\Lambda_{QCD}}, \frac{m_d}{\Lambda_{QCD}}, \frac{m_s}{\Lambda_{QCD}} < 1 \Rightarrow m_u = m_d = m_s = 0$ possible approximation

b) n massless flavors imply chiral symmetry in QCD

$$G = SU(n)_L \otimes SU(n)_R \otimes U(1)_{L+R}$$

c) ground states of hadrons have a different symmetry

$$F = SU(n)_{L+R} \otimes U(1)_{L+R}$$

RECAP

d) From Nambu-Goldstone theorem:

$$\frac{G}{F} \simeq SU(2)_{L-R}$$ generates massless states

$\Rightarrow$ 8 massless bosons $\Rightarrow$ π, $\underline{K}$, η

e) This explains why they are so light

f) It also provides the correct pion-nucleon dynamics $\Rightarrow$ derivative interactions ✓

- General introduction
- The two-nucleon system
- Nuclear Structure

Additional materials: SU(3) Clebsch-Gordan coefficients, ChPT Reviews.

If you want to know more