NUCLEAR PHYSICS 10

a) RENORMALIZATION &
EFFECTIVE FIELD THEORY (PART III)

b) ISOSPIN SYMMETRY

c) FLAVOR SYMMETRY (PART I)

RECAP

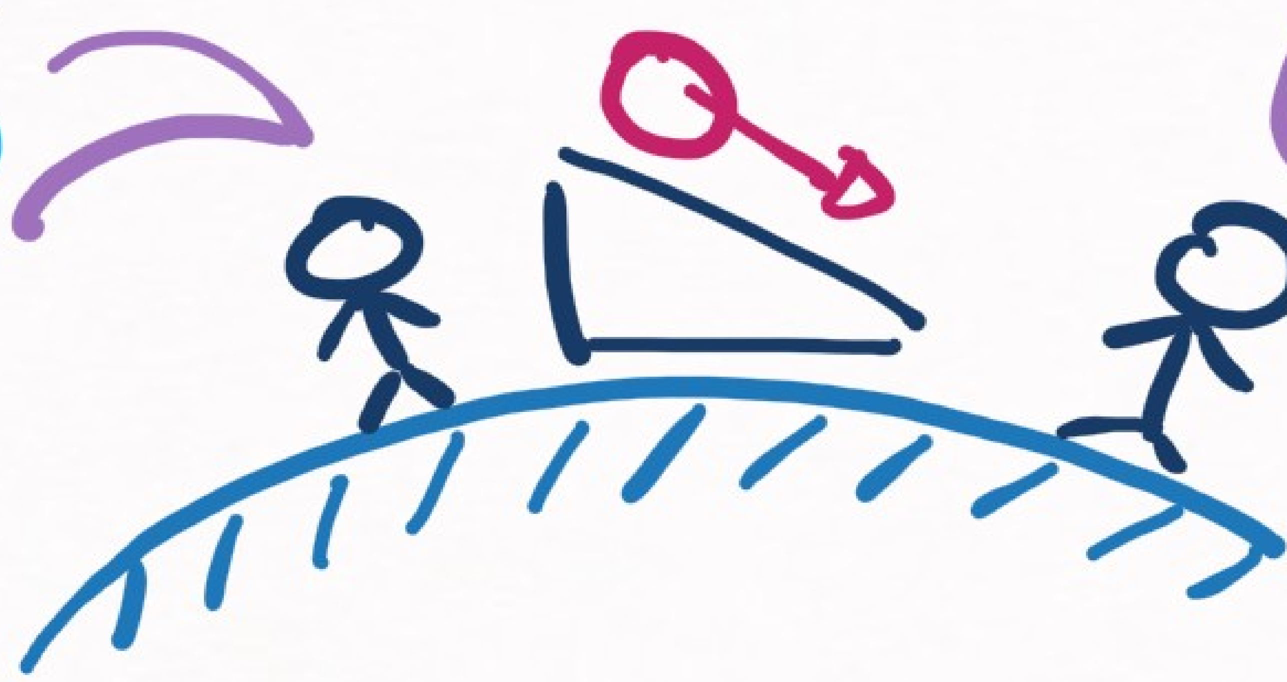
$$\left[\text{Physics at low energies does not depend on high energy details} \right]$$

FORMALIZATION

RENORMALIZATION

EFFECTIVE (FIELD) THEORIES

$\Delta V = mgh$



What about R_{earth}?

$$\boxed{V = -GMmf(r)} \longrightarrow \boxed{\Delta V = mgh + \text{corrections}}$$

NEW CONCEPTS

a) Underlying / fundamental theory

$$V = -G\frac{Mm}{r}$$

$c_1, c_2, c_3, \ldots$ low energy constants (LECs)

b) Effective theory ⟶ Power series in a small parameter

$$\Delta V = mgh\left[1 + \underline{\underline{c_1}}\left(\frac{h}{R}\right) + \underline{\underline{c_2}}\left(\frac{h}{R}\right)^2 + \underline{\underline{c_3}}\left(\frac{h}{R}\right)^3 + \ldots\right]$$

c) Power counting ⟶ Choice of the small parameter

in this example: $\frac{h}{R}$

Underlying theory #1

Underlying theory #2

Underlying theory #3

"The low energy physics (*) doesn't determine the high energy physics"

Effective theory A (**)

(*) → Taking the infrared limit (long-distance limit)

(**) → Infrared fixed point / universality class

There might be several infrared fix points

Underlying theory #1

Effective theory A

Effective theory B

RENORMALIZATION IN QM

a) Underlying theory → YUKAWA POTENTIAL

$$\left[V_Y(\vec{r}) = - \frac{g_Y^2}{4\pi} \frac{e^{-mr}}{r} \right] \leftarrow$$

b) If there is a shallow bound state ($\gamma \ll m$),
then the bound state will not <u>probe</u> $V_Y(r)$

c) Effective theory for this shallow bound state

$$\rightarrow \left[V_{Eff}(\vec{r}) = C_0 \, \delta^{(3)}(\vec{r}) + C_2 \vec{\nabla}^2 \delta^{(3)}(\vec{r}) + \dots \right]$$

REGULARIZATION (STEP 1)

a) Attractive Dirac-delta potential : not good (must be fixed)

$$V_{eff} = C_0 \delta^{(3)}(\vec{r}) \,, \quad \underline{C_0} < 0$$ $\Rightarrow$ predicts infinite binding

b) Regularization → making this potential finite

$$\rightarrow \quad \underline{\delta^{(3)}}(\vec{r}) \longrightarrow \underline{\delta^{(3)}_{\Lambda}}(\vec{r})$$ "smeared delta"

Example : $$\underline{\delta^{(3)}_{\Lambda}}(\vec{r}) = \frac{\Lambda^3}{\underline{8\pi^{3/2}}} e^{-\frac{1}{2}\Lambda^2 r^2}$$

c) There are infinite regularizations:

$[\delta^{(3)}(\vec{r}) = \infty$ for $\vec{r} = 0] \rightarrow$ not good

$\rightarrow \left[\delta^{(3)}_{\Lambda}(\vec{r}) \neq \infty \text{ for } \vec{r} = 0 + \int d^3\vec{r}\, \delta^{(3)}_{\Lambda}(\vec{r}) = 1 \right] \leftarrow$

$\rightarrow$ Any choice of $\delta^{(3)}_{\Lambda}(\vec{r})$ that fulfills these two conditions will be acceptable $\leftarrow$

$$\lim_{\Lambda \to \infty} \delta^{(3)}_{\Lambda}(\vec{r}) \longrightarrow \delta^{(3)}(\vec{r})$$

RENORMALIZATION (STEP 2)

a) make the coupling depend on Λ

→ (why? the potential is not an observable)

$$V_{eff}(\vec{r}) = C_0(\Lambda)\,\delta^{(3)}_{\Lambda}(\vec{r}) \qquad \underline{\underline{C_0 \to C_0(\Lambda)}}$$

b) for each $\underline{\underline{\Lambda}}$, determine the coupling from some observable

example → reproducing the binding energy

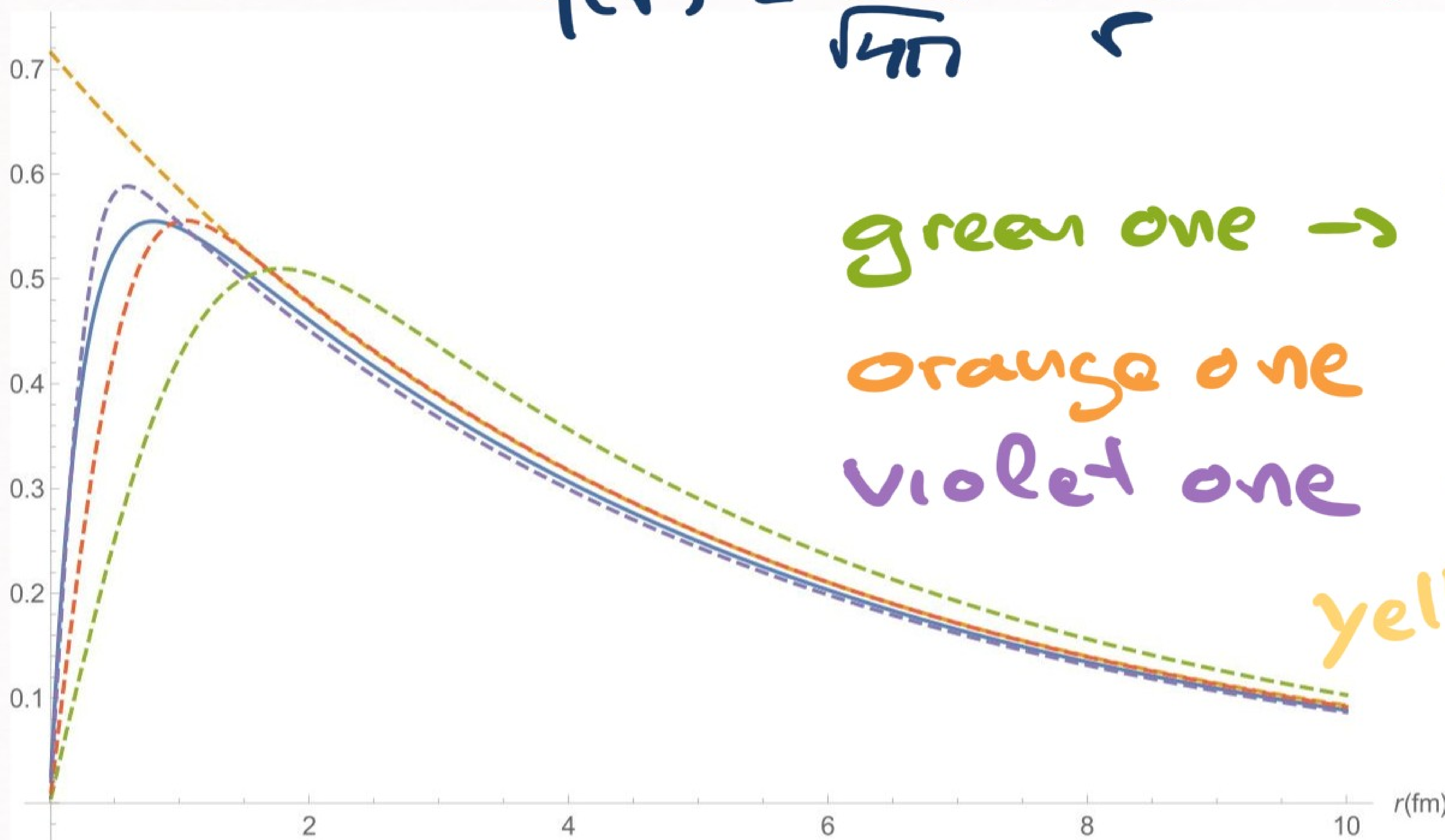
$$\left[E_B = -B = -\frac{\gamma^2}{2\mu}\right] \longrightarrow C_0(\Lambda) / \langle\psi|H|\psi\rangle = E_B$$

$$\left[\frac{d}{d\Lambda}\gamma = 0\right]$$

c) results are approximately independent of Λ

$$\psi(r) = \frac{1}{\sqrt{4\pi}} \frac{u(r)}{r} \rightarrow u(r) : \text{reduced wave function}$$

$u(r)$



r

green one $\rightarrow \delta_\Lambda(\vec{r}) + \Lambda = 250$ MeV

orange one $\rightarrow \Lambda = 500$ MeV

violet one $\rightarrow \Lambda = 1000$ MeV

yellow one $\rightarrow (\Lambda \rightarrow \infty)$

blue line $\rightarrow$ wf w/ Yukawa potential
(underlying theory)

$$\boxed{\text{FOR } \Lambda \gtrsim m, \text{ ALL THESE WFS LOOK SIMILAR}}$$

In general, $\exists$ many formulations of this idea

EFT with NDA: the algorithm

1. Identify the relevant degrees of freedom
2. Identify high- and low-energy scales → expansion parameters x
3. Identify symmetries of low-energy theory
4. Choose the accuracy required. This, together with the size of x, tells you the order, n, to which you must calculate.
5. Write down all possible local operators, that have naive dimensions up that order, and are consistent with symmetries
   "NDA"
6. Derive the behaviour of loops, and calculate them.

All operators needed for renormalization at this order should be present →Model independence

→ nucleons + QM

→ γ/m

→ Central potential

→ $\mathcal{O}(\gamma/m)$

→ $V_C(\vec{r}) = C_0\,\delta^{(3)}(\vec{r})$

→ $C_0 = C_0(\Lambda)$

$\frac{d}{d\Lambda}\langle\psi|H|\psi\rangle = 0$

Also, many choices of regulator:

a) p-space (Λ) $\left[\lim_{\Lambda\to\infty}\right]$

$$V(\vec{q}) = C_0(\Lambda)\, e^{-\vec{q}^{\,2}/\Lambda^2} \quad \text{(Gaussian)}$$

$$V(\vec{q}) = C_0(\Lambda)\, \frac{\Lambda^2}{\Lambda^2+\vec{q}^{\,2}} \quad \text{(Lorentzian)}$$

b) r-space (R_c) $\left[\lim_{R_c\to 0}\right]$

$$V(r) = C_0(R_c)\, \frac{e^{-(r/R_c)^2}}{R_c^3\,\pi^{3/2}} \quad \text{(Gaussian again: } \underline{\underline{r = 2/\Lambda}}\text{)}$$

$$V(r) = C_0(R_c)\, \frac{\delta(r-R_c)}{4\pi R_c^2} \quad \text{(Delta-shell)}$$

$\to$ $\delta(r)$ in 3-dimension is not $\underbrace{\text{singular}}_{E_B\to-\infty}$ in QM $\leftarrow$

DELTA-SHELL REGULATOR

$$\delta^{(3)}(\vec{r}; R_c) = \frac{\delta(r - R_c)}{4\pi R_c^2} \longrightarrow$$ [Incredibly useful for analyzing things]

a) Description of bound states:

$$V_{LO}(r; R) = C_0(R)\,\frac{\delta(r - R_c)}{4\pi R_c^2} \Rightarrow \frac{1}{C_0(R_c)} = \frac{\mu}{2\pi}\left(\gamma - \gamma\coth(\gamma R_c)\right) \xrightarrow[R_c \to 0]{} \frac{\mu}{2\pi}\left(\gamma - \frac{1}{R_c}\right)$$

b) Description of scattering states:

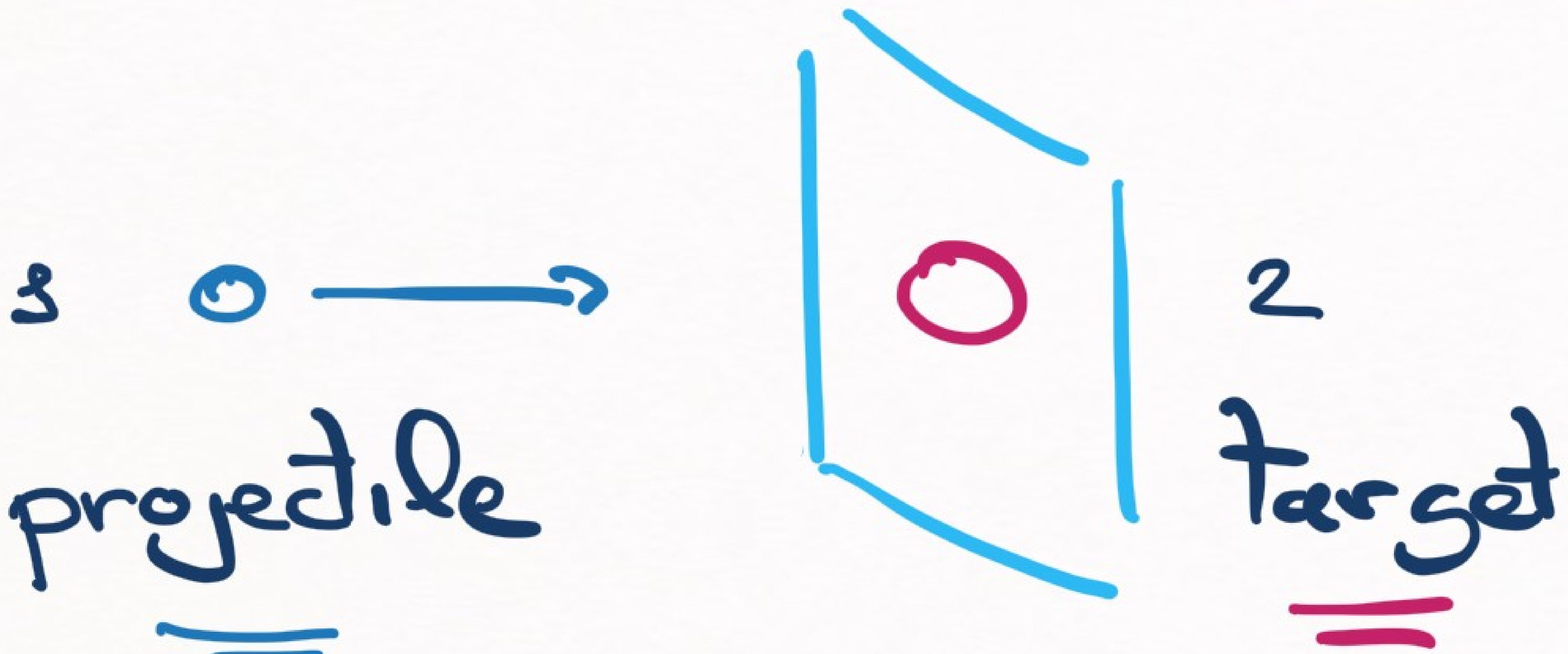
$$E = 0\,,\quad u_0(r) \xrightarrow[r \to \infty]{} (r - \underbrace{a_0}_{\text{constant}}) \Rightarrow \frac{1}{C_0(R_c)} = \frac{\mu}{2\pi}\left(\frac{1}{a_0} - \frac{1}{R_c}\right)$$

LOW ENERGY SCATTERING 1

Cross section:

[Effective area that the target offers to the projectile]

$\Downarrow$

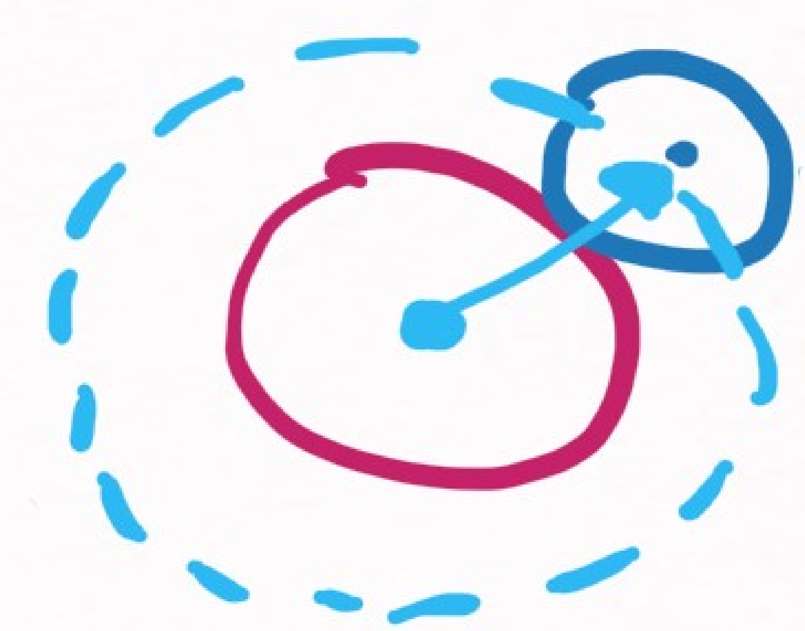


CLASSICALLY:

"If the center of the projectile fall within the light blue circle (→) then the projectile gets scattered"

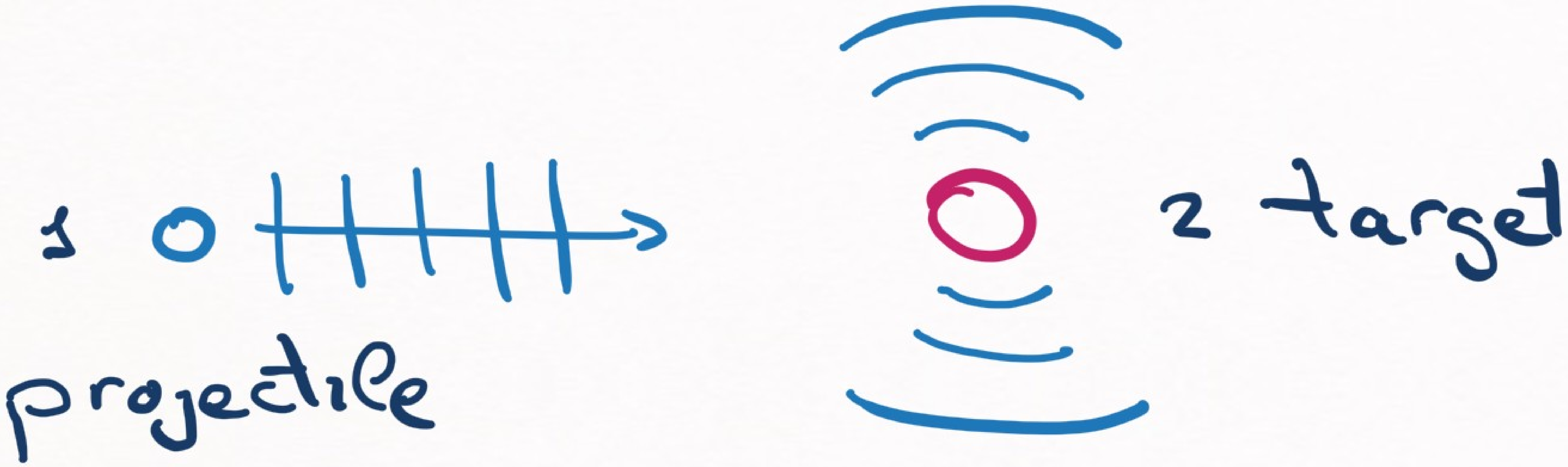
$R_1 \to$ radius of target

$R_2 \to$ radius of projectil



$$\sigma = \pi (R_1 + R_2)^2$$

LOW ENERGY SCATTERING ②



(we obtain waves)

$$\left[-\frac{\nabla^2}{M} + V(\vec{r})\right]\psi(\vec{r}) = 0$$

(zero energy)

$\Downarrow$

$$\psi(\vec{r}) \xrightarrow[r\to\infty]{} N\,\frac{r-a_0}{r}$$

$\Rightarrow$ a_0 is the scattering length

Cross section in QM

$$\boxed{\sigma \xrightarrow[E_{cm}\to 0]{} 4\pi|a_0|^2}$$

$\Downarrow$

$\Leftarrow$ [a_0 is called the scattering length]

LOW ENERGY SCATTERING (3) → a_0 for a shallow bound state ($\gamma \to 0$)

a) Orthogonality of wave functions:

$$\langle \psi_{E=0} | \psi_{E=-B} \rangle = 0$$

$$\parallel$$

$$\int d^3\vec{r}\, \psi_0(\vec{r})\, \psi_B(\vec{r}) \propto \int dr\, \underbrace{(r-a_0)}_{\text{previous page}}\, \underbrace{e^{-\gamma r}}_{\psi(r) \propto \frac{e^{-\gamma r}}{r}} = \frac{1}{\gamma^2}(1-a_0\gamma)$$

Orthogonality $\Rightarrow (1-a_0\gamma)=0 \Rightarrow \boxed{a_0 \to \frac{1}{\gamma}}$

for $\gamma \to 0$

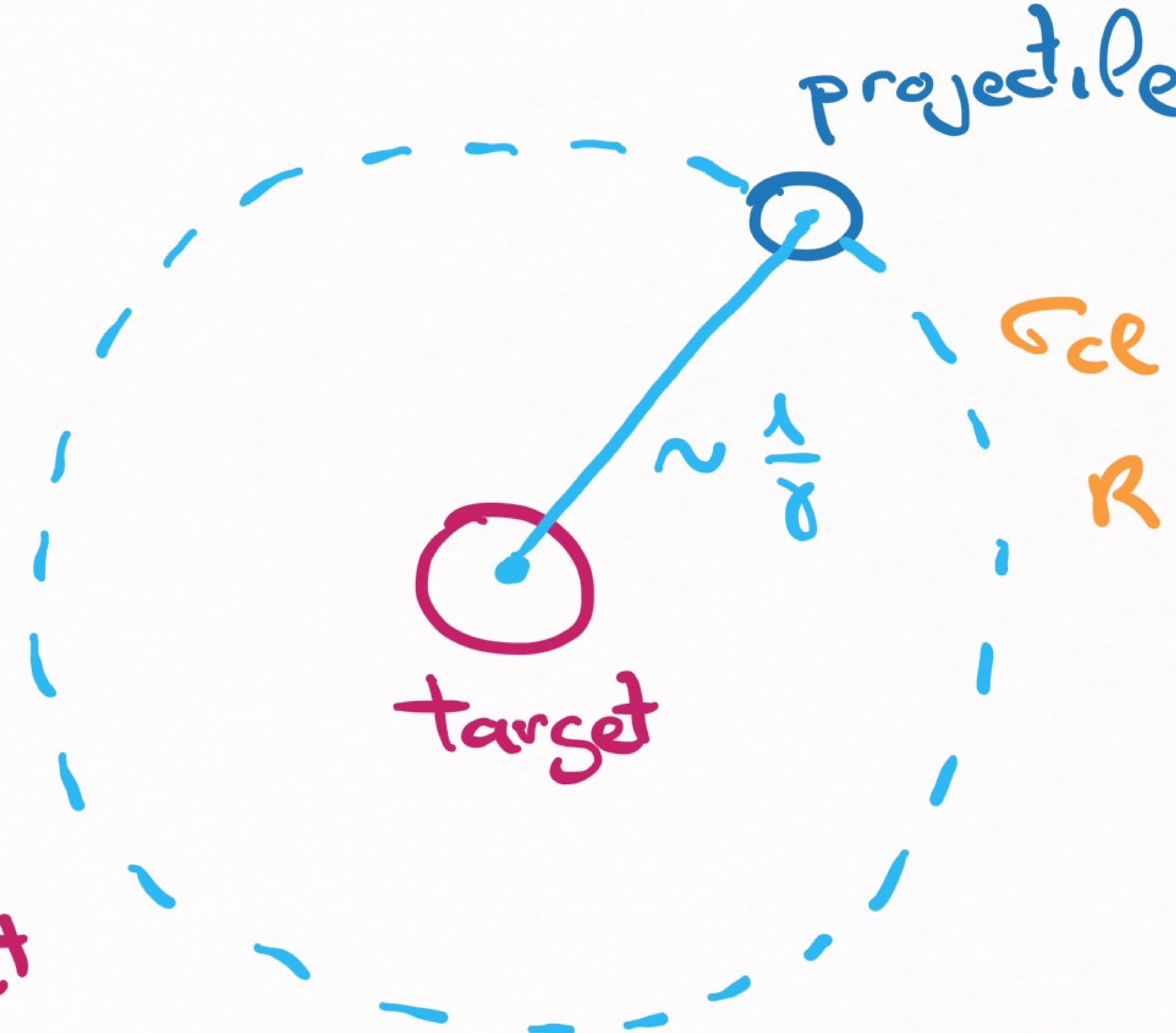
LOW ENERGY SCATTERING (4)

$\sigma = 4\pi |a_0|^2 = \frac{4\pi}{|\gamma|^2}$

b) For $\gamma \to 0$, $a_0 \to \frac{1}{\gamma}$

It makes sense intuitively:

if the projectile passes closer than $R < 1/\gamma$
$\Rightarrow$ it will be as if it was entangled with the target in a bound state



$\sigma_{el} = 4\pi |R|^2$

$R \sim \frac{1}{\gamma}$

$\underbrace{\langle r | 1\,0\,0 \rangle}_{\text{w.f.}} \sim \frac{1}{r} e^{-\gamma r}$ (size: $\frac{1}{\gamma}$)

$\boxed{\text{LOW ENERGY SCATTERING}}$ (5)

Two situations

a) $\gamma \to 0$ (strong, attractive potential)

$\Rightarrow a_0 \to \infty$

$$\frac{1}{C_0(R_c)} = \frac{\mu}{2\pi}\left(\frac{1}{a_0} - \frac{1}{R_c}\right) \longrightarrow$$

solution from renormalizing

$$V_{R_c}(r) = C_0(R_c)\,\frac{\delta(r-R_c)}{4\pi R_c^2}$$

b) no bound state + weak potential $\Rightarrow a_0 \to 0$

$$\frac{1}{a_0} \gg \frac{1}{R_c} \Rightarrow \frac{1}{C_0(R_c)} \to \frac{\mu}{2\pi}\frac{1}{a_0} \Rightarrow C_0(R_c) = \frac{2\pi}{\mu} a_0$$

LOW ENERGY SCATTERING ⑥

→ Actually, we can work out an approximation for weak potentials:

→ We want this

$$\left[\psi(r) \sim \frac{r-a_0}{r}\right] = \psi^{(0)} + \delta\psi \, , \quad \delta\psi \sim -\frac{a_0}{r}$$

$(E=0)$ $\qquad (\psi^{(0)} \sim 1)$

⇓

$$-\frac{\nabla^2}{2\mu}\left[\psi^{(0)} + \delta\psi\right] + V\psi^{(0)} = 0$$

$(V\delta\psi \ll V\psi^{(0)})$

⇓

$$\Rightarrow \; -\frac{\nabla^2}{2\mu}\delta\psi = V\psi^{(0)}$$

$$\left[\vec{\nabla}^2\left(\frac{1}{r}\right) = -4\pi\delta^{(3)}(\vec{r})\right] \Rightarrow \boxed{\left(\frac{2\pi}{\mu}a_0\right)\delta^{(3)}(\vec{r}) = V(\vec{r})}$$

→ we integrate it

LOW ENERGY SCATTERING ⑦

→ If we integrate:

$$\int d^3\vec{r}\left(\frac{2\pi}{\mu} a_0\, \delta^{(3)}(\vec{r})\right) = \int d^3\vec{r}\; V(\vec{r})$$

$$\Rightarrow \boxed{\frac{2\pi}{\mu} a_0 = \int d^3\vec{r}\; V(\vec{r})}$$

(Only works for a weak potential)

→ If $V(\vec{r}) = C_0\, \delta^{(3)}(\vec{r})$

$$\Rightarrow \quad \frac{2\pi}{\mu} a_0 = C_0 \underbrace{\int d^3\vec{r}\, \delta^{(3)}(\vec{r})}_{=1} = C_0$$

we recover previous formula in 18

$$\Rightarrow \boxed{\frac{2\pi}{\mu} a_0 = C_0}$$

LOW ENERGY SCATTERING ⑧

→ More generally: Fourier transform

$$\int d^3\vec{r}\, e^{i\vec{q}\cdot\vec{r}}\, V(\vec{r}) = \tilde{V}(\vec{q}) \Rightarrow \int d^3\vec{r}\, V(\vec{r}) = \tilde{V}(\vec{0})$$

$(|\vec{q}| \to 0)$

→ If we assume a Yukawa potential:

$$\tilde{V}_Y(\vec{q}) = -\frac{g_Y^2}{|\vec{q}|^2 + m^2} = -\frac{g_Y^2}{m^2}\left(1 - \frac{|\vec{q}|^2}{m^2} + \frac{|\vec{q}|^4}{m^4} - \cdots\right)$$

$$\tilde{V}_Y(\vec{0}) = -\frac{g_Y^2}{m^2} \Rightarrow \boxed{\frac{2\pi}{\mu} a_0 = -\frac{g_Y^2}{m^2} = C_0}$$

TWO INFRARED FIXED POINTS

Underlying theory: scattering by a potential $\tilde{V}(\mathbf{q})$

Effective theory A

Effective theory B

$$V_{eff} = C_0(R_c)\,\delta^{(3)}_{R_c}(\vec{r})$$

$$\frac{1}{C_0(R_c)} = \frac{\mu}{2\pi}\left(\frac{1}{a_0} - \frac{1}{R_c}\right)$$

strong scattering / weakly bound state

$$C_0 = \frac{2\pi}{\mu} a_0$$

→ weak scattering

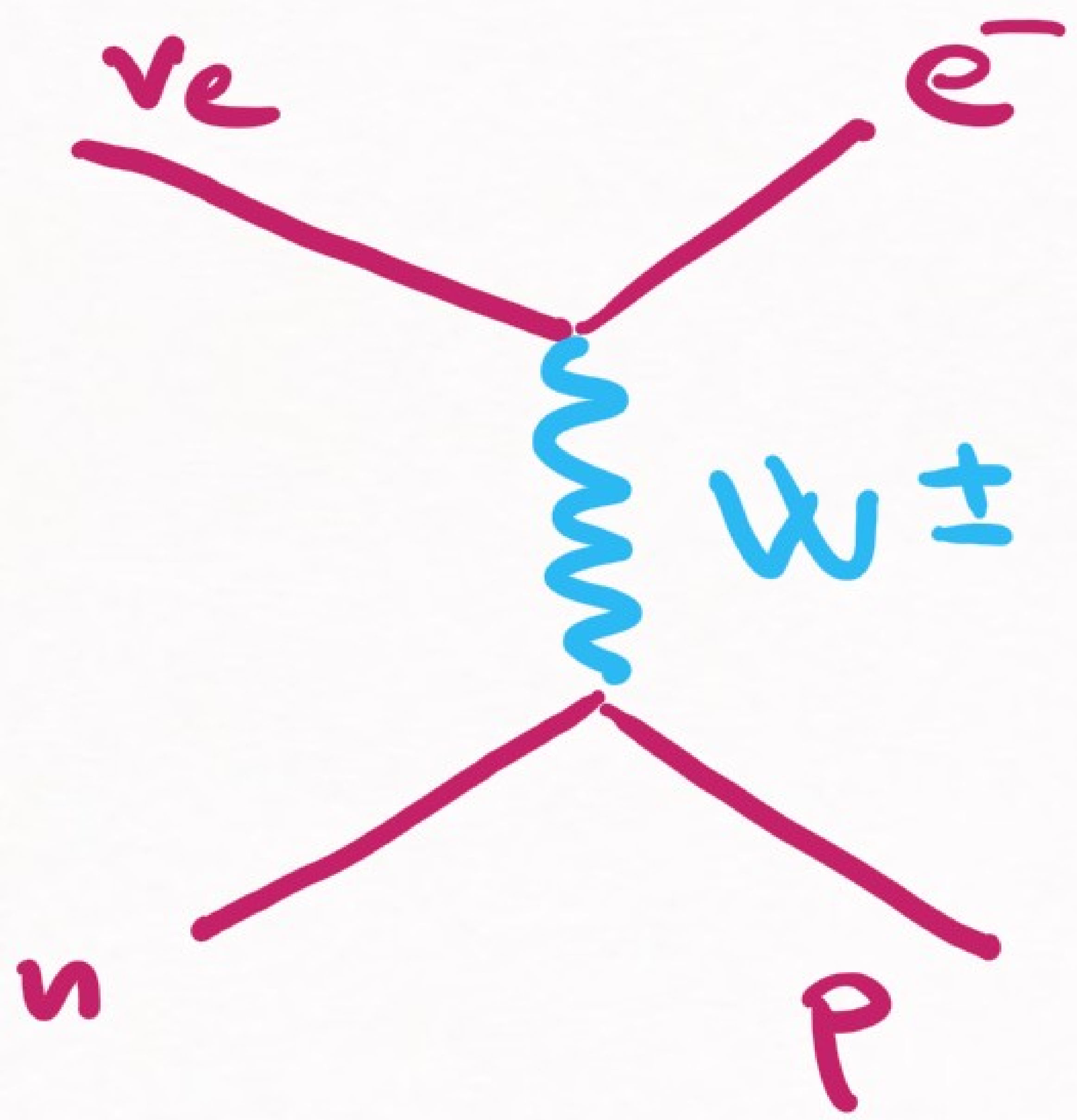
SUMMARY | For two-body systems,

$\exists$ two low energy effective theories

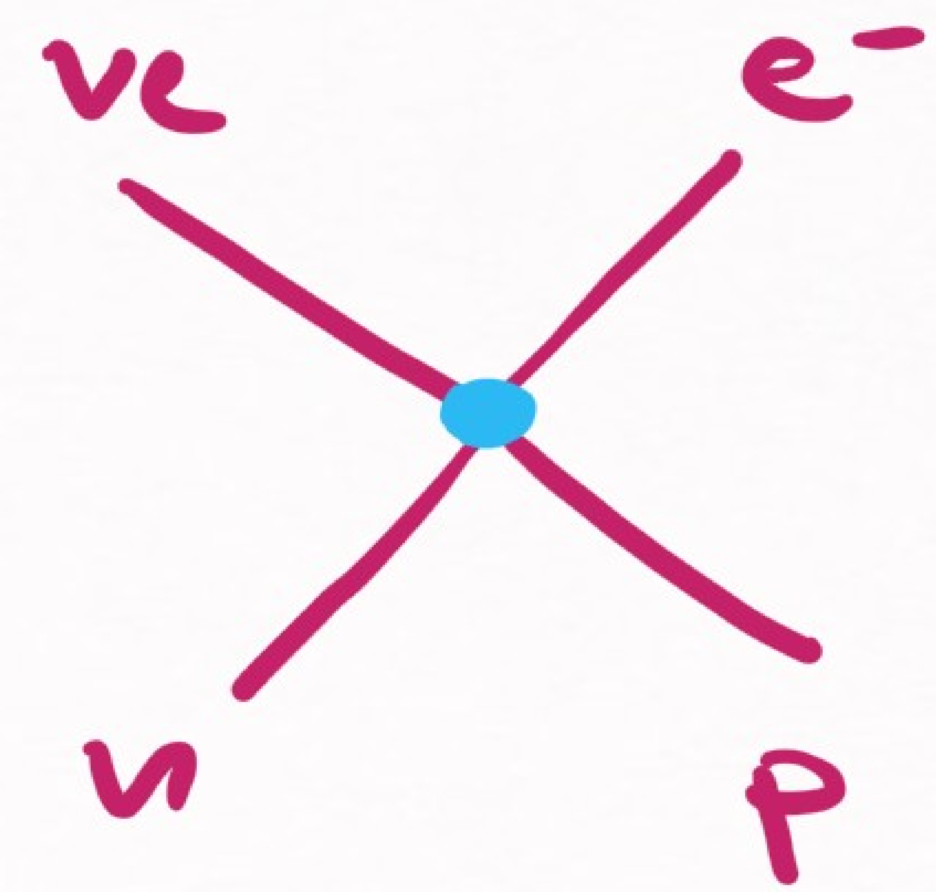
a) Effective theory for systems w/ shallow bound states

b) Effective theory for systems w/ a very weak potential

CURIOSITY $\rightarrow$ THE EFFECTIVE THEORY FOR WEAK POTENTIALS IS OF INTEREST FOR WEAK INTERACTIONS



$\Longrightarrow$



(Fermi theory of weak interactions)

Effective theories & Renormalization

$\boxed{\text{THE END}}$

Now we will be back to standard nuclear physics

$\boxed{\text{ISOSPIN SYMMETRY}}$

Reminder: the nuclear force does not distinguish neutrons & protons

How to take this into account? $\Rightarrow$ Symmetry

$$\begin{pmatrix} p \\ n \end{pmatrix} \to U \begin{pmatrix} p \\ n \end{pmatrix}$$

$\to$ this should not change the properties of the system (provided this doesn't change the normalization of the states)

$$|\det U| = 1$$

ISOSPIN TRANSFORMATIONS ①

$$\begin{pmatrix} p \\ n \end{pmatrix} \to U \begin{pmatrix} p \\ n \end{pmatrix}$$

/ (a) U doesn't change the norm of the state:

$$|\det U| = 1$$

(b) Global phase changes are trivial:

$$\begin{pmatrix} p \\ n \end{pmatrix} \to e^{i\phi} \begin{pmatrix} p \\ n \end{pmatrix} \Rightarrow \det U = 1$$

Not an isospin transformation, but a trivial change of phase ⇐

(not part of isospin symmetry)

ISOSPIN TRANSFORMATIONS (2)

$$(p\ n)^\dagger \begin{pmatrix} p \\ n \end{pmatrix} \to (p\ n)^\dagger \begin{pmatrix} p \\ n \end{pmatrix} \Rightarrow \boxed{U^\dagger U = 1}$$

$$\Rightarrow U \in GL(2, \mathbb{C}) \,/\, U^\dagger U = 1 \,,\ \det U = 1$$

In group theory, this group is called $\underline{\underline{SO(2)}}$

ISOSPIN STATES

[SU(2)-isospin isomorphic to $\underbrace{\text{SU(2)-spin}}_{\text{spin rotations belonged to SU(2)}}$]

$$|p\rangle = |\tfrac{1}{2}\ \tfrac{1}{2}\rangle_I = |+\rangle_I$$

$$|n\rangle = |\tfrac{1}{2}\ -\tfrac{1}{2}\rangle_I = |-\rangle_I$$

$$|0\,0\rangle_I = \frac{1}{\sqrt{2}}\left(|+-\rangle_I - |-+\rangle_I\right) = \frac{1}{\sqrt{2}}\left(|pn\rangle - |np\rangle\right)$$

$$|1\,1\rangle_I = |++\rangle = |pp\rangle$$

$$|1\,0\rangle_I = \frac{1}{\sqrt{2}}\left(|+-\rangle_I + |-+\rangle_I\right) = \frac{1}{\sqrt{2}}\left(|pn\rangle + |np\rangle\right)$$

$$|1\,-1\rangle_I = |--\rangle = |nn\rangle$$

ISOSPIN STATES

a) $|p\rangle, |n\rangle \longrightarrow |p\rangle = |\tfrac{1}{2}\,\tfrac{1}{2}\rangle_I,\ |n\rangle = |\tfrac{1}{2}\,-\tfrac{1}{2}\rangle_I$

b) $|\Delta^{++}\rangle, |\Delta^{+}\rangle, |\Delta^{0}\rangle, |\Delta^{-}\rangle \longrightarrow |\Delta^{++}\rangle = |\tfrac{3}{2}\,\tfrac{3}{2}\rangle_I,$

$|\Delta^{+}\rangle = |\tfrac{3}{2}\,\tfrac{1}{2}\rangle_I, \ldots$

c) $|\pi^{+}\rangle, |\pi^{0}\rangle, |\pi^{-}\rangle$

$|\pi^{+}\rangle = |1\,1\rangle_I$

$|\pi^{0}\rangle = |1\,0\rangle_I$

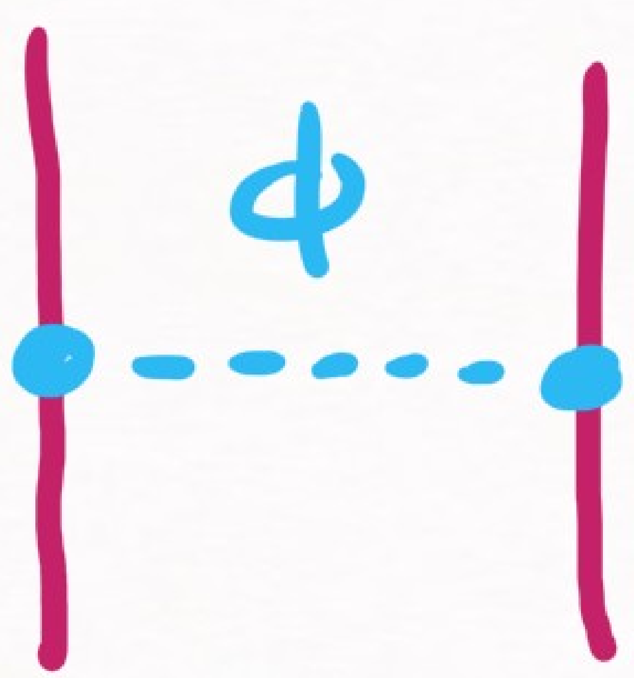
You can change this sign depending on conventions $\longleftarrow$ $|\pi^{-}\rangle = \pm\,|1\,-1\rangle_I$

(not important for us right now)

NUCLEAR FORCES & ISOSPIN (1)

Spherical basis

1) How do we take isospin into account?



$\Longrightarrow$

$|\phi^+\rangle = |1\,1\rangle_I$, $|\phi^0\rangle = |1\,0\rangle_I$, ...

or in the Cartesian basis:

$|\phi^+\rangle = -\frac{1}{\sqrt{2}}\left(|\phi_1\rangle + i|\phi_2\rangle\right)$, $|\phi^0\rangle = |\phi_3\rangle$

....

$\searrow$ $|\phi^a\rangle$, $a = 1, 2, 3$ (or $a = x, y, z$)

$[\langle P\phi|H|P\rangle \propto g \times$ (other factors)$]$

$\Longrightarrow \langle P\phi^a|H|P\rangle \propto [\langle I'm'|T^a|I\,m\rangle]$

$\langle I'm'|$ $\qquad$ $|I\,m\rangle$

$\times\, g \times$ (other factors)

NUCLEAR FORCES & ISOSPIN (2)

Reminder

- → Spherical basis: $|\phi^+\rangle, |\phi^0\rangle, |\phi^-\rangle$
- → Cartesian basis: $|\phi_a\rangle \to |\phi_x\rangle, |\phi_y\rangle, |\phi_z\rangle$ (or $|\phi_1\rangle, |\phi_2\rangle, |\phi_3\rangle$)

$$|\phi^+\rangle = -\frac{1}{\sqrt{2}}\left(|\phi_1\rangle + i|\phi_2\rangle\right)$$

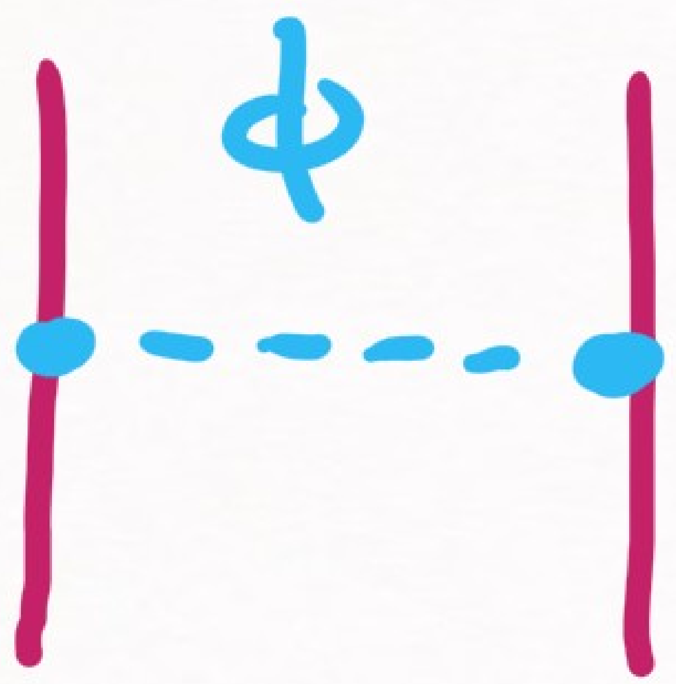
$$|\phi^0\rangle = |\phi_3\rangle$$

$$|\phi^-\rangle = +\frac{1}{\sqrt{2}}\left(|\phi_1\rangle - i|\phi_2\rangle\right)$$

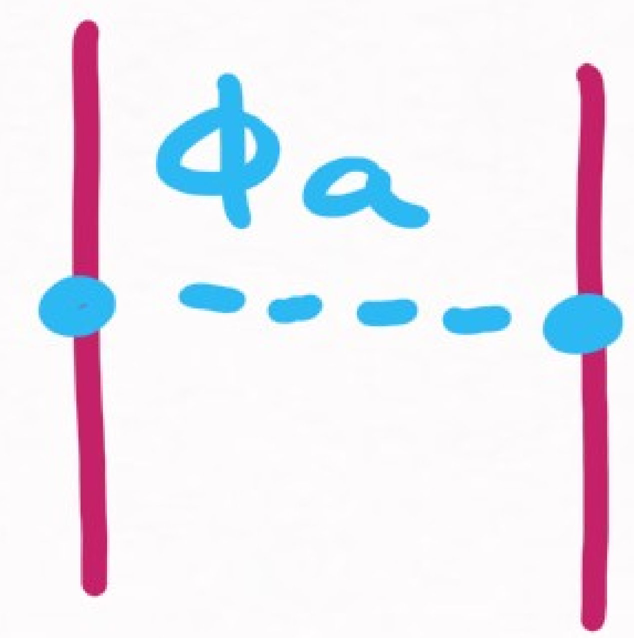
Cartesian basis is more convenient for calculating the potential

NUCLEAR FORCES & ISOSPIN (3)

2) What is the practical effect of isospin?



$= V(\vec{q})$ $\Rightarrow$



$= \vec{\tau}_1 \cdot \vec{\tau}_2 \, V(\vec{q})$

(no isospin)

$|I_1 m_1\rangle$ $|I_2 m_2\rangle$

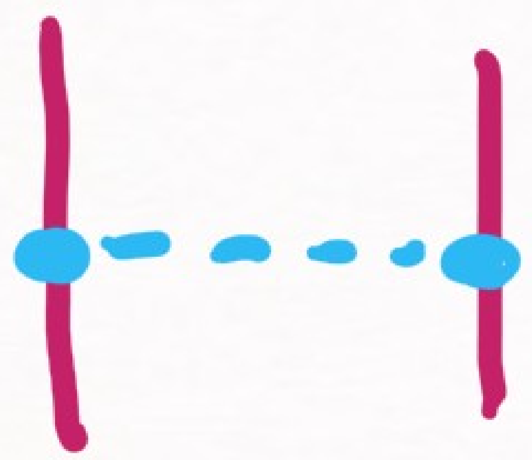
($\exists$ isospin)

$\vec{\tau}_1, \vec{\tau}_2$ -> isospin matrices for particles 1 & 2

$\vec{\tau}_2 \rightarrow (2I_2+1)\times(2I_2+1)$

$\vec{\tau}_1 \rightarrow (2I_1+1)\times(2I_1+1)$

2.a) Nucleons & pions:

 $= \vec{\tau}_1 \cdot \vec{\tau}_2 \, V(\vec{q})$

nucleons

$\rightarrow$ isospin-$1/2$

$\Downarrow$

$\vec{\tau} = \underbrace{\{\tau_x, \tau_y, \tau_z\}}_{\text{Pauli matrices}}$

$\{\sigma_x, \sigma_y, \sigma_z\}$

Spin | $\vec{\sigma}_1 \cdot \vec{\sigma}_2 = \begin{cases} 1 & S=1 \\ -3 & S=0 \end{cases}$

$\Downarrow$

Isospin | $\vec{\tau}_1 \cdot \vec{\tau}_2 = \begin{cases} 1 & I=1 \\ -3 & I=0 \end{cases}$

Example: Deuteron, $I=0 \Rightarrow \vec{\tau}_1 \cdot \vec{\tau}_2 = -3$
(check past lessons)

2.a) Nucleons & pions (continuation)

Two-nucleon isospin states:

$$|11\rangle = |pp\rangle$$
$$|10\rangle = \frac{1}{\sqrt{2}}(|pn\rangle + |np\rangle)$$
$$|1-1\rangle = |nn\rangle$$

$$|00\rangle = \frac{1}{\sqrt{2}}(|pn\rangle - |np\rangle)$$

protons → fermions
neutrons → fermions
⇓
nucleons
→ fermions

Extended Fermi-Dirac statistics:

(isospin wf) × (spin wf) × (spatial wf) → antisymmetric

2.a) Nucleons & pions (continuation)

Antisymmetry $\rightarrow$ $\underbrace{(\text{isospin w.f.})}_{(-1)^{I+1}} \times \underbrace{(\text{spin w.f.})}_{(-1)^{S+1}} \times \underbrace{(\text{spatial w.f.})}_{(-1)^{L}}$

$L \rightarrow$ orbital angular momentum

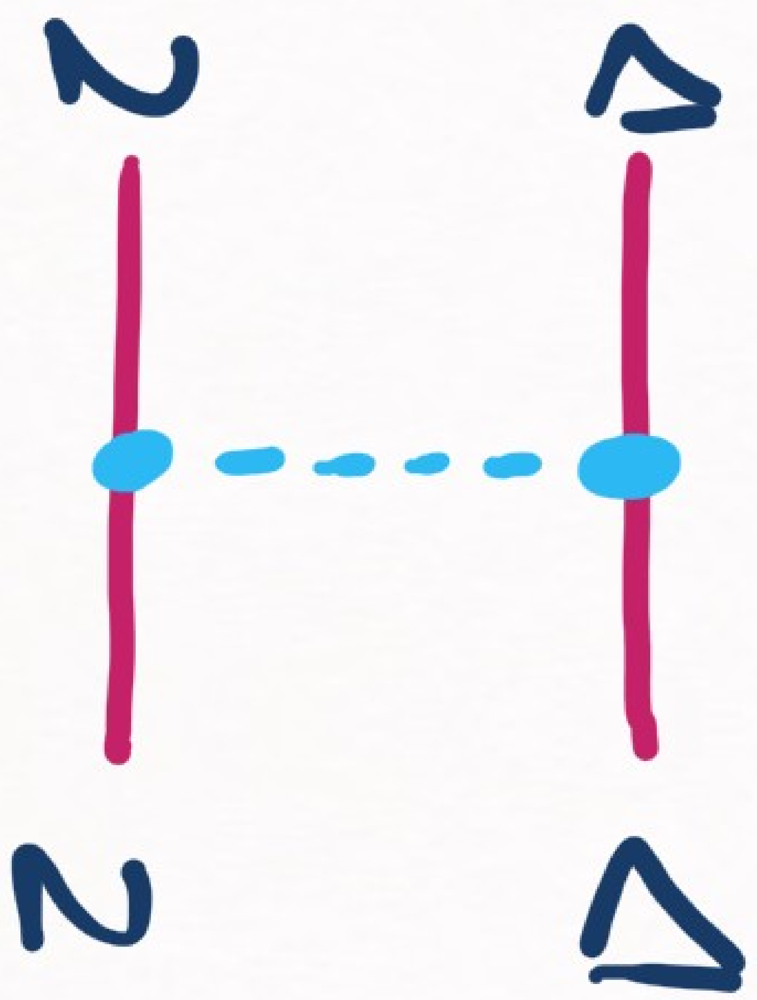
$\Downarrow$

Total symmetry: $\boxed{I+S+L \text{ odd}}$

Deuteron $\rightarrow$ $S=1$, $L=0$ $\Rightarrow$ $I=0$

Singlet state $\rightarrow$ $S=0$, $L=0$ $\Rightarrow$ $I=1$

2.b) Nucleons, deltas & pions:

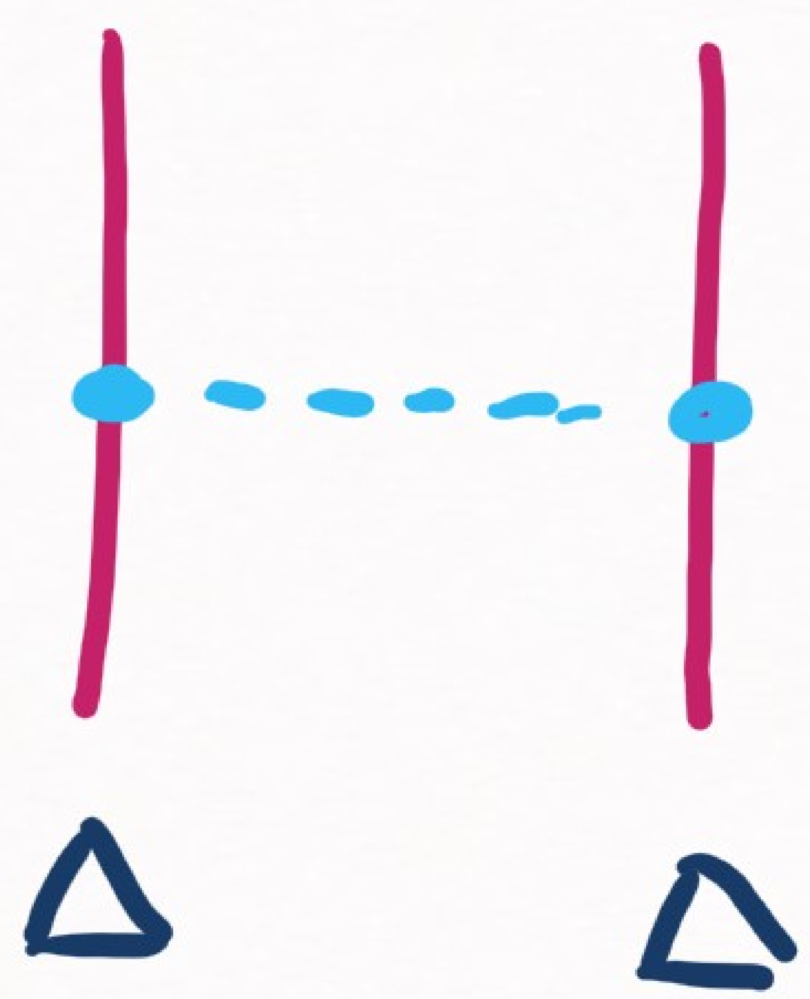


$$= \vec{\tau}_1 \cdot \vec{T}_2 \, V(\vec{q})$$

$(|\tfrac{1}{2} m\rangle)$ $(|\tfrac{3}{2} m'\rangle)$

$\vec{T} = \{T_x, T_y, T_z\}$

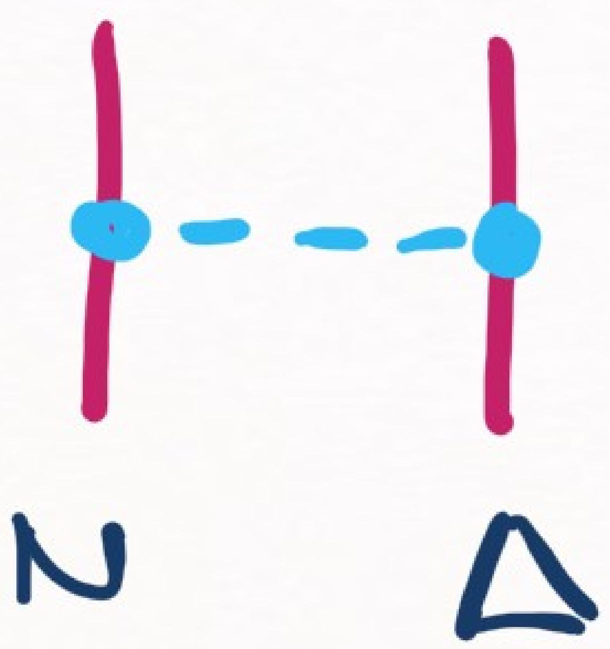
the same as the spin-3/2 matrices



$$= \vec{T}_1 \cdot \vec{T}_2 \, V(\vec{q})$$

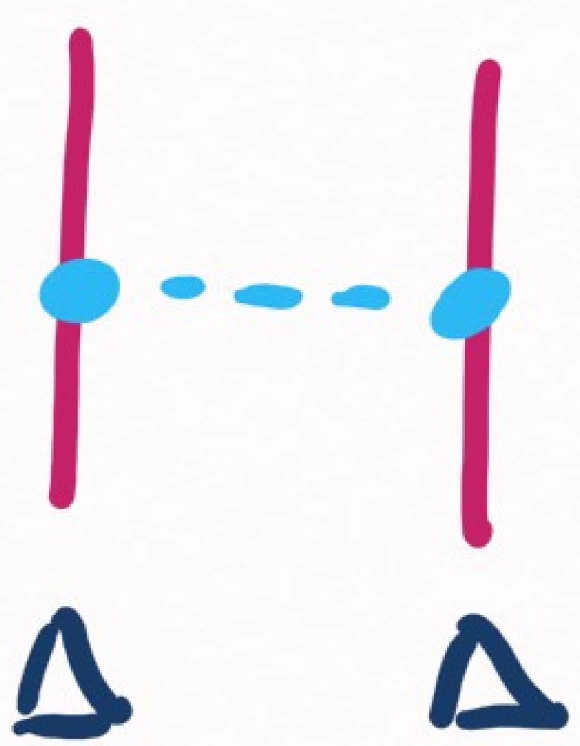
2.b) Nucleons, deltas & pions

$\vec{\tau} = 2\vec{T}$ (spin-1/2)



$$\longrightarrow \underline{\underline{\vec{\tau}_1\cdot\vec{T}_2}} = \begin{cases} 3/2 & \text{for } T=2 \\ -5/2 & \text{for } T=1 \end{cases}$$

$$\vec{T}_1\cdot\vec{T}_2 = \frac{1}{2}\left\{\vec{T}^2 - \vec{T}_1^2 - \vec{T}_2^2\right\} = \frac{1}{2}\left\{T(T+1) - T_1(T_1+1) - T_2(T_2+1)\right\}$$



$$\longrightarrow \underline{\underline{\vec{T}_1\cdot\vec{T}_2}} = \begin{cases} 9/4 & \text{for } T=3 \\ -3/4 & \text{for } T=2 \\ -11/4 & \text{for } T=1 \\ -15/4 & \text{for } T=0 \end{cases} \qquad \frac{1}{2}\left(T(T+1) - \frac{15}{2}\right)$$

ORIGIN OF ISOSPIN (1)

a) QCD $\longrightarrow$ $\Lambda_{QCD} \simeq (200-350)\,\mathrm{MeV}$ as natural scale

b) $\left.\begin{array}{l} m_u, m_d \ll \Lambda_{QCD} \\ |m_u - m_d| \ll \Lambda_{QCD} \end{array}\right\} \Rightarrow$ u, d quarks are essentially massless

$\Rightarrow$ no practical distinction between them

c) $m_u, m_d, m_s < \Lambda_{QCD} \Rightarrow$ FLAVOR SYMMETRY
(works a bit worse than isospin)

ORIGIN OF ISOSPIN ②

1) $\underbrace{\frac{m_u}{\Lambda_{QCD}}, \frac{m_d}{\Lambda_{QCD}}}_{\text{QCD doesn't distinguish } u,d \text{ quarks}} \ll 1 \Rightarrow \boxed{\begin{pmatrix} u \\ d \end{pmatrix} \to U \begin{pmatrix} u \\ d \end{pmatrix}}$ SYMMETRY

$\Rightarrow$ (Repeating same arguments) $\Rightarrow$ $U \in SU(2)$

$$\left.\begin{aligned} |u\rangle &= |\tfrac{1}{2}\ \tfrac{1}{2}\rangle_I = |+\rangle_I \\ |d\rangle &= |\tfrac{1}{2}\ -\tfrac{1}{2}\rangle_I = |-\rangle_I \end{aligned}\right\} \text{ just as in the } p, n \text{ cases}$$

ORIGIN OF ISOSPIN (3) → a few complications

2) What happen with antiquarks?

2.a) Quarks → Antiquarks: $\hat{C}|u\rangle = |\bar{u}\rangle$

$\hat{C}|d\rangle = |\bar{d}\rangle$

$\hat{C}$ → particle-antiparticle conjugation operator

2.b) Consider the isospin transformation:

$$\begin{pmatrix} u \\ d \end{pmatrix} \longrightarrow \underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{U} \begin{pmatrix} u \\ d \end{pmatrix}$$

→ $U^{+}U = 1$

$\det U = 1$

2.c) How would this transformation look for antiquarks: (more + charge → + m_I)

Naively → $|\bar{u}\rangle = |\tfrac{1}{2}\, -\tfrac{1}{2}\rangle_I$ $\quad |\bar{d}\rangle = |\tfrac{1}{2}\, \tfrac{1}{2}\rangle_I$

But if $\begin{pmatrix} u \\ d \end{pmatrix} \to \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} u \\ d \end{pmatrix} \underset{(\bar{e})}{\Rightarrow} \begin{pmatrix} \bar{u} \\ \bar{d} \end{pmatrix} \to \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \bar{u} \\ \bar{d} \end{pmatrix}$

In isospin basis:

→ Particles:

$$\begin{pmatrix} |+\rangle \\ |-\rangle \end{pmatrix} \to \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} |+\rangle \\ |-\rangle \end{pmatrix}$$

→ Antiparticles:

$$\begin{pmatrix} |+\rangle \\ |-\rangle \end{pmatrix} \to \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{\text{different!}}\begin{pmatrix} |+\rangle \\ |-\rangle \end{pmatrix}$$

a) Particles:

$|+\rangle \to -|-\rangle$

$|-\rangle \to +|+\rangle$

b) Antiparticles:

$|+\rangle \to +|-\rangle$ $\quad (|\bar{u}\rangle \to -|\bar{d}\rangle)$

$|-\rangle \to -|+\rangle$ $\quad (|\bar{d}\rangle \to +|\bar{u}\rangle)$

(isospin basis) particle basis

→ [But we would like a) & b) to be the same in the isospin basis]

Solution ⇒ $|\bar{d}\rangle = -|\tfrac{1}{2}\,\tfrac{1}{2}\rangle_I$, $|\bar{u}\rangle = |\tfrac{1}{2}\,-\tfrac{1}{2}\rangle_I$ (new definition)

a, b) Particles & Antiparticles $\quad |+\rangle \to -|-\rangle$, $\quad |-\rangle \to +|+\rangle$ or $\begin{pmatrix} |+\rangle \\ |-\rangle \end{pmatrix} \to \underline{\underline{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}} \begin{pmatrix} |+\rangle \\ |-\rangle \end{pmatrix}$

ISOSPIN & ANTIPARTICLES (1)

$$|u\rangle = |+\rangle\,,\quad |d\rangle = |-\rangle \;\longrightarrow\; \begin{pmatrix}|+\rangle\\ |-\rangle\end{pmatrix} \rightarrow U\begin{pmatrix}|+\rangle\\ |-\rangle\end{pmatrix}$$

$$\Downarrow$$

$$\underbrace{|\bar{u}\rangle = |-\rangle\,,\quad |\bar{d}\rangle = -|+\rangle}_{} \;\longrightarrow\; \begin{pmatrix}|+\rangle\\ |-\rangle\end{pmatrix} \rightarrow U\begin{pmatrix}|+\rangle\\ |-\rangle\end{pmatrix}$$

This change ensures the same type of isospin transformation for particles & antiparticles

(really convenient)

[$|\bar{u}\rangle = -|-\rangle$, $|\bar{d}\rangle = |+\rangle$ also works]

ISOSPIN & ANTIPARTICLES (2)

a) Particle-particle

$$|11\rangle = |uu\rangle$$

$$|10\rangle = \frac{1}{\sqrt{2}}(|ud\rangle + |du\rangle)$$

$$|1-1\rangle = |dd\rangle$$

$$|00\rangle = \frac{1}{\sqrt{2}}(|ud\rangle - |du\rangle)$$

b) Particle-antiparticle

$$|11\rangle = -|u\bar{d}\rangle$$

$$|10\rangle = \frac{1}{\sqrt{2}}(|u\bar{u}\rangle - |d\bar{d}\rangle)$$

$$|1-1\rangle = |d\bar{u}\rangle$$

$$|00\rangle = \frac{1}{\sqrt{2}}(|u\bar{u}\rangle + |d\bar{d}\rangle)$$

$\boxed{\text{FLAVOR SYMMETRY}} \to \boxed{\text{SU(3)-FLAVOR}}$

a) $\dfrac{m_u}{\Lambda_{QCD}}, \dfrac{m_d}{\Lambda_{QCD}} \ll 1 \Rightarrow \begin{pmatrix} u \\ d \end{pmatrix} \to U \begin{pmatrix} u \\ d \end{pmatrix}$ is a (pretty) good symmetry

b) $\dfrac{m_s}{\Lambda_{QCD}} < 1 \Rightarrow \begin{pmatrix} u \\ d \\ s \end{pmatrix} \to U \begin{pmatrix} u \\ d \\ s \end{pmatrix}$ is (sort of a) good symmetry

$m_s \sim 95\,\text{MeV}$

What about U? $\Rightarrow U \in GL(3,\mathbb{C}),\ U^\dagger U = 1,\ \det U = 1$

$\Rightarrow \boxed{U \in SU(3)}$

$\boxed{\text{SU(3) FLAVOR}}$ → How to classify baryons?

a) SU(2) - spin & SU(2) - isospin

[2 spin-½ particles ⇒ total spin is 0 or 1]

our notation usually is: $\frac{1}{2} \otimes \frac{1}{2} = \underset{J=0}{\underline{\underline{0}}} \oplus \underset{J=1}{\underline{\underline{1}}}$

$J=0 \rightarrow |00\rangle = \frac{1}{\sqrt{2}}\left(|+-\rangle - |-+\rangle\right)$ → antisymmetric (multiplicity of 1)

$J=1 \rightarrow |10\rangle = \frac{1}{\sqrt{2}}\left(|+-\rangle + |-+\rangle\right)$ → symmetric (multiplicity of 3)

$|11\rangle, |10\rangle, |1-1\rangle$

a) SU(2)-spin & SU(2)-isospin

$\frac{1}{2} \otimes \frac{1}{2} = 0_A \oplus 1_S$ is based on the total spin/isospin of the system

We could have used multiplicity instead: $\underline{\underline{2 \otimes 2 = 1 \oplus 3}}$

$\Rightarrow$ For SU(3)-flavor we will classify the states using multiplicity

b) How do we extend this to SU(3)?

$$\begin{pmatrix} u \\ d \\ s \end{pmatrix} \rightarrow \underline{\underline{3}}$$ (because its multiplicity is 3)

b.1) Now we have two quarks: $\underline{\underline{|q_1 q_2\rangle}}$, $q_i = u, d, s$

$$|q_1 q_2 (S)\rangle = \frac{1}{\sqrt{2}}\left(|q_1 q_2\rangle + |q_2 q_1\rangle\right) \Rightarrow \underline{\underline{6 \text{ states}}}$$

$$|q_1 q_2 (A)\rangle = \frac{1}{\sqrt{2}}\left(|q_1 q_2\rangle - |q_2 q_1\rangle\right) \Rightarrow \underline{\underline{3 \text{ states}}}$$

SU(3) REPRESENTATIONS

a) Notice that SU(3) respects symmetry:

$$|q_1 q_2\rangle \longrightarrow (U|q_1\rangle)(U|q_2\rangle)$$

$$|q_1 q_2(S)\rangle \longrightarrow |q_1 q_2(S)\rangle \quad \text{(still symmetric)}$$

$$|q_1 q_2(A)\rangle \longrightarrow |q_1 q_2(A)\rangle \quad \text{(still antisymmetric)}$$

$3 \otimes 3 \sim 3 \oplus 6$ except for a detail

SU(3) REPRESENTATIONS

b) Basically, $\exists$ two representations of $|q_1 q_2\rangle$

b.1) Antisymmetric $\rightarrow$ 3 states

$$|q_1 q_2 (A)\rangle \rightarrow |(Uq_1)(Uq_2)(A)\rangle$$

b.2) Symmetric $\rightarrow$ 6 states

$$|q_1 q_2 (S)\rangle \rightarrow |(Uq_1)(Uq_2)(S)\rangle$$

$\boxed{\text{TRIPLETS \& ANTITRIPLETS}}$

This can only be done with SU(2)

$(n_F = 2)$

1) Remember SU(2)-isospin?

$$\begin{pmatrix} u \\ d \end{pmatrix} \rightarrow U_I \begin{pmatrix} u \\ d \end{pmatrix} \Rightarrow \overbrace{\begin{pmatrix} -\bar{d} \\ \bar{u} \end{pmatrix} \rightarrow U_I \begin{pmatrix} -\bar{d} \\ \bar{u} \end{pmatrix}}$$

2) What happens with SU(3)-flavor?

$$\boxed{\hat{C}\,\lambda\,|P\rangle = \lambda^{*}\,|\bar{P}\rangle}$$

$$\begin{pmatrix} u \\ d \\ s \end{pmatrix} \rightarrow U_F \begin{pmatrix} u \\ d \\ s \end{pmatrix} \xRightarrow{\hat{C}} \begin{pmatrix} \bar{u} \\ \bar{d} \\ \bar{s} \end{pmatrix} \rightarrow U_F^{*} \begin{pmatrix} \bar{u} \\ \bar{d} \\ \bar{s} \end{pmatrix}$$

$\boxed{\text{TRIPLET \& ANTITRIPLET}}$

Example

$$\begin{pmatrix} u \\ d \\ s \end{pmatrix} \to \underbrace{\begin{pmatrix} 0 & 0 & i \\ 0 & -1 & 0 \\ -i & 0 & 0 \end{pmatrix}}_{U} \begin{pmatrix} u \\ d \\ s \end{pmatrix}$$

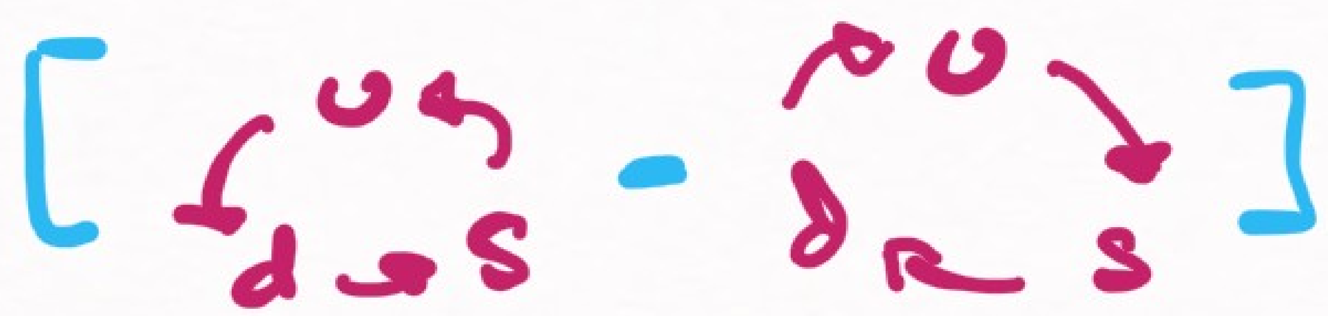
$$U^{\dagger}U = \mathbb{1}, \quad \det U = 1$$

$$U \in SU(3)$$

$$\begin{aligned} u &\to is \\ d &\to -d \\ s &\to -iu \end{aligned} \quad \to \boxed{\text{C.C.}}$$

$$\boxed{\text{C.C.}} \to \begin{aligned} \bar{u} &\to -i\bar{s} \\ \bar{d} &\to -\bar{d} \\ \bar{s} &\to +i\bar{u} \end{aligned} \to \begin{pmatrix} \bar{u} \\ \bar{d} \\ \bar{s} \end{pmatrix} \to \begin{pmatrix} 0 & 0 & -i \\ 0 & -1 & 0 \\ i & 0 & 0 \end{pmatrix} \begin{pmatrix} \bar{u} \\ \bar{d} \\ \bar{s} \end{pmatrix}$$

[TRIPLET OR ANTITRIPLET?] (1)



→ signs

$$|1\rangle = \frac{1}{\sqrt{2}}\left(|ud\rangle - |du\rangle\right) \longrightarrow (+i)\,\frac{1}{\sqrt{2}}\left(|ds\rangle - |sd\rangle\right) = i|2\rangle$$

$$|2\rangle = \frac{1}{\sqrt{2}}\left(|ds\rangle - |sd\rangle\right) \longrightarrow (-i)\,\frac{1}{\sqrt{2}}\left(|ud\rangle - |du\rangle\right) = (-i)|1\rangle$$

$$|3\rangle = \frac{1}{\sqrt{2}}\left(|su\rangle - |us\rangle\right) \longrightarrow -\frac{1}{\sqrt{2}}\left(|su\rangle - |us\rangle\right) = -|3\rangle$$

$$\begin{pmatrix} u \\ d \\ s \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 0 & i \\ 0 & -1 & 0 \\ -i & 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ d \\ s \end{pmatrix}$$

(for particles)

[TRIPLET OR ANTITRIPLET ?] (2)

$|1\rangle \to +i|2\rangle$

$|2\rangle \to -i|1\rangle$

$|3\rangle \to -|3\rangle$

+

$|1\rangle = |s\rangle$ or $|\bar{s}\rangle$

$|2\rangle = |u\rangle$ or $|\bar{u}\rangle$

$|3\rangle = |d\rangle$ or $|\bar{d}\rangle$

$|\bar{u}\rangle \to -i|\bar{s}\rangle$

$|\bar{d}\rangle \to -|\bar{d}\rangle$

$|\bar{s}\rangle \to +i|\bar{u}\rangle$

$\Rightarrow$

Only possible match:

$|1\rangle = |\bar{s}\rangle$

$|2\rangle = |\bar{u}\rangle$

$|3\rangle = |\bar{d}\rangle$

$|\bar{q}\bar{q}\rangle$ + antisymmetric $\Rightarrow$ $\boxed{\text{ANTITRIPLET}}$

[∃ ALSO A VISUAL PROOF]

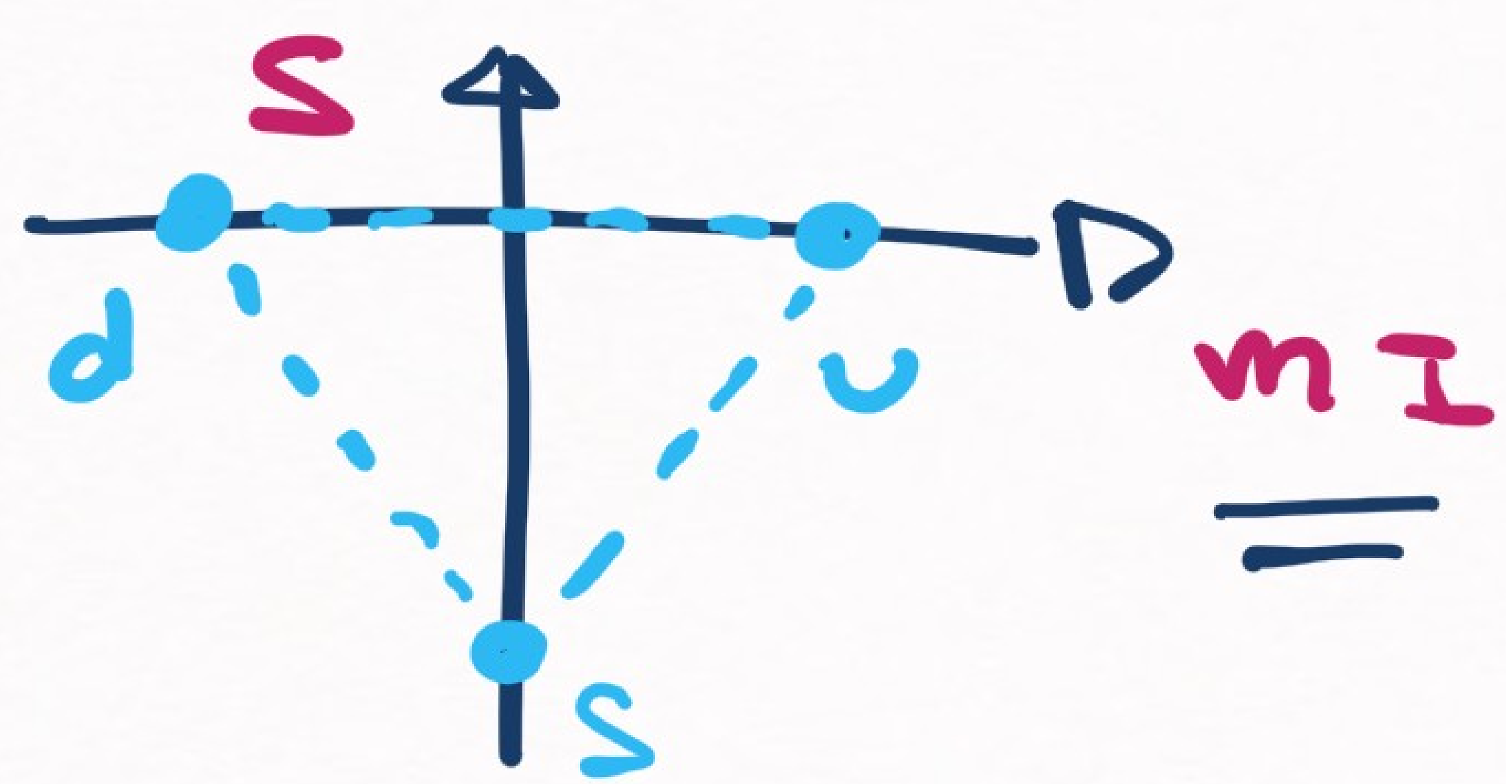
a) Notice that $SU(2)_I \subseteq SU(3)_F$ (subgroup)

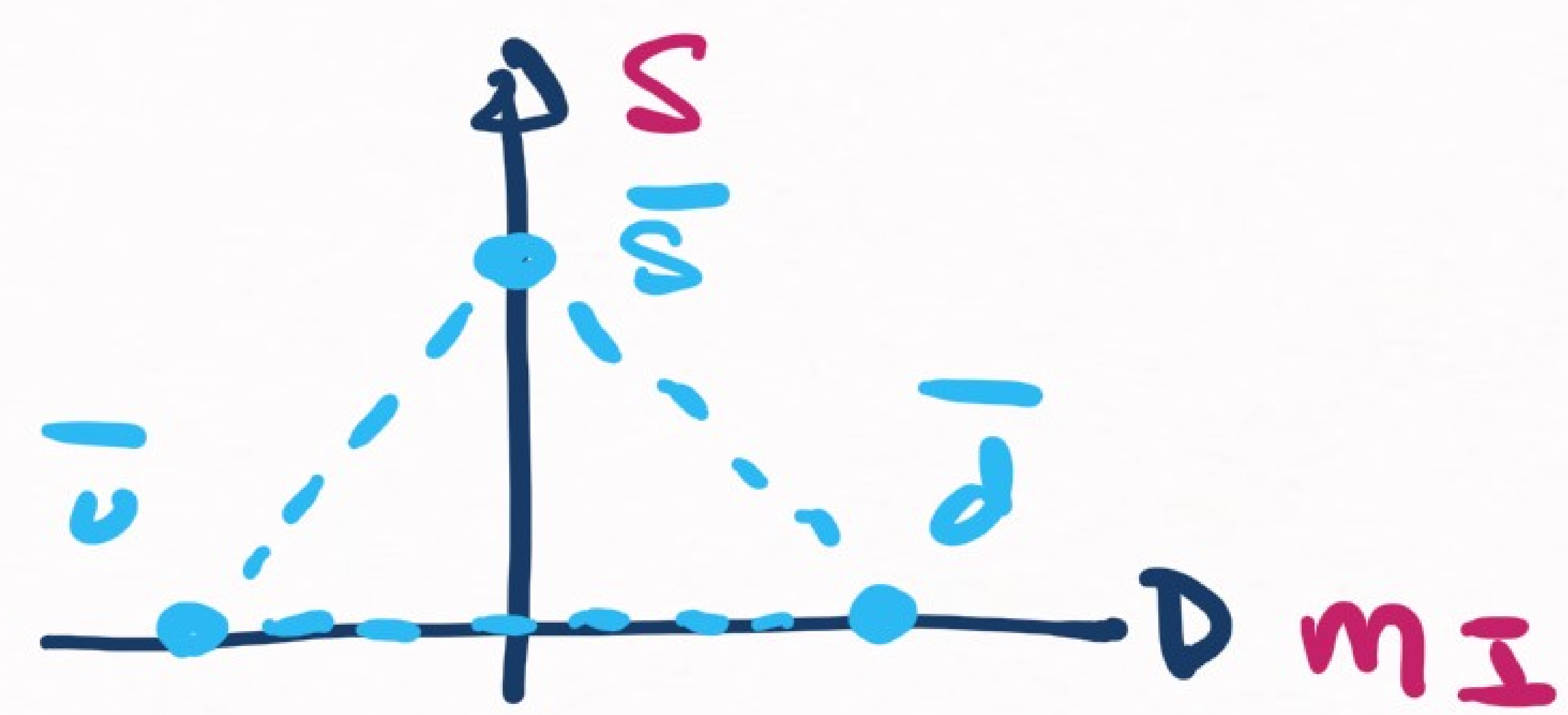
b) Representations of $SU(3)_F$ → I, m_I + (a second quantum number)

our choice → strangness s-quark has $S=-1$ (convention)

(other choice → hypercharge)

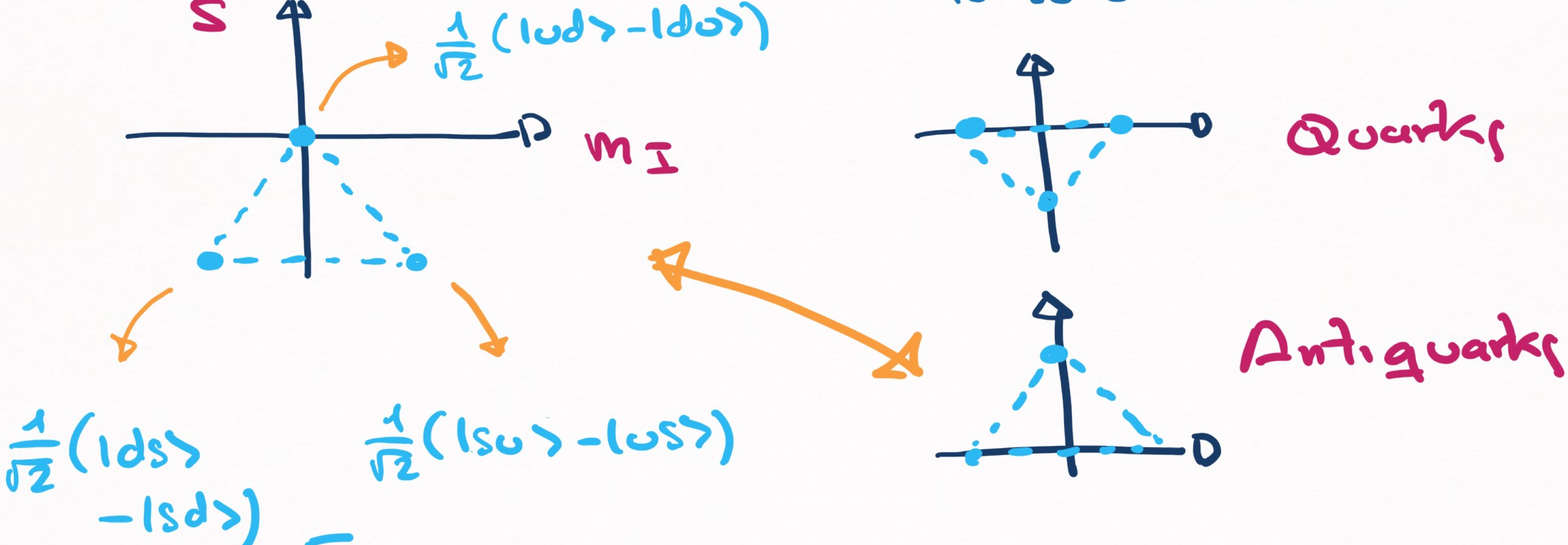
c) Visual representation of $3(u,d,s)$ & $\bar{3}(\bar{u},\bar{d},\bar{s})$





HOW DOES $\frac{1}{\sqrt{2}}(|q_1 q_2\rangle - |q_2 q_1\rangle)$ LOOK LIKE?

TO BE COMPARED WITH:



[Antisymmetric 2-quark looks a lot like the antiquark]

[TO SUMMARIZE ...]

$\Rightarrow$ SU(3) representations of two (u, d, s) quarks:

a) Antisymmetric $\frac{1}{\sqrt{2}}(|q_1 q_2\rangle - |q_2 q_1\rangle) \rightarrow$ antitriplet

b) Symmetric $\frac{1}{\sqrt{2}}(|q_1 q_2\rangle + |q_2 q_1\rangle) \rightarrow$ sextet

$\Rightarrow$ $\boxed{3 \otimes 3 = \bar{3} \oplus 6}$

FOR THE NEXT LESSON:

a) How to include one extra quark

$3 \otimes 3 \otimes 3$

b) How to build a baryon from this

SEE YOU

ON TUESDAY

15:50