NUCLEAR PHYSICS (21)

RESIDUAL INTERACTIONS IN THE SHELL MODEL

RECAP !

NUCLEAR MODELS

1) SHELL MODEL ⟶ (RECAP IN NEXT PAGE)

2) ROTATIONAL MODEL

3) VIBRATIONAL MODEL

} COLLECTIVE MODEL

2, 3) ⟶ SPECTRUM OF HEAVY NUCLEI

SHELL MODEL ⇒

- ⊛ explains magic numbers (2, 8, 20, 28, 50, 82, ...)
- ⊛ separation energies
- ⊛ ↑ of nuclei close to (N, Z) magic
- ...

⇓

Relies on a series of assumptions

⇓

1) protons & neutrons are fermions (each have to have diff. quantum numbers)

2) ⊃ mean field potential

$$H = \sum_i T_i + \underbrace{\sum_{ij} V^{2B}_{ij} + \sum_{jk} V^{3B}_{ijk} + \dots}$$

kinetic potential (2B, 3B, 4B ...)

$$\longrightarrow H = \underbrace{\sum_i (T_i + V_i^{MF})}_{\text{kinetic + mean field potential}} + \underbrace{\Delta V}_{\text{perturbation (small)}}$$

3) What we do is to fill the energy levels of the mean field potential

EASY + GOOD FEATURES :

- JT near closed shells ($^{16}O \rightarrow {}^{15/17}O$)
- Excited spectra (near closed shells) (^{41}Ca, ^{38}Ar)
- Pairing interaction (trivial example of ΔV)

→ [TWO BIG OPEN PROBLEMS W/ SHELL MODEL] ←

1) How do we find a $V^{MF}(\vec{r})$?

So far, we have simply assumed a V^{MF}

$$V^{MF} = \frac{1}{2} m\omega^2 r^2 - S\vec{e}\cdot\vec{r}$$ (it works)

→ We didn't give a derivation for the

→ $U^{MF} = \frac{1}{2} m(\omega r)^2 - \sum \vec{e}_i \cdot \vec{r}_i$

this is phenomenological (ad-hoc)

but it works well.

what we want → some methods to obtain the mean field pot.

METHODS → Hartree Fock, Skyrme, Gogny

[FOR NEXT LESSON]

2) How do we deal w/ ΔV ?

(TODAY'S LESSON)

Pairing interaction → a very simple example

In general ΔV is going to be much more complex to deal with

[How to calculate the effects of ΔV?]

Review the basics:

1) Mean field potential $H = \sum_i h_i + \Delta V$

2) Monoparticular wave function

$[h_i \phi^{(n)}(\vec{r}) = \varepsilon_n \phi^{(n)}(\vec{r})]$

5) Construct the full wave function

$$\Psi_\alpha = \mathcal{A}\left[\prod_{i=1}^{N} \phi_i^{(n_i)}\right]$$

- $\mathcal{A}$: antisymmetrize (bc we have fermions)
- $\prod_{i=1}^{N} \phi_i^{(n_i)}$: wave function factorizes (product of ϕ_i's)
- $N = \sum_i n_i$

→ WHEN $\Delta V = 0$, WHAT WE DID IS:

$\underline{\underline{\phi(n)}}$ → n is energy level

⋮ ⋮

2 (D_2 states)

1 (D_1 states)

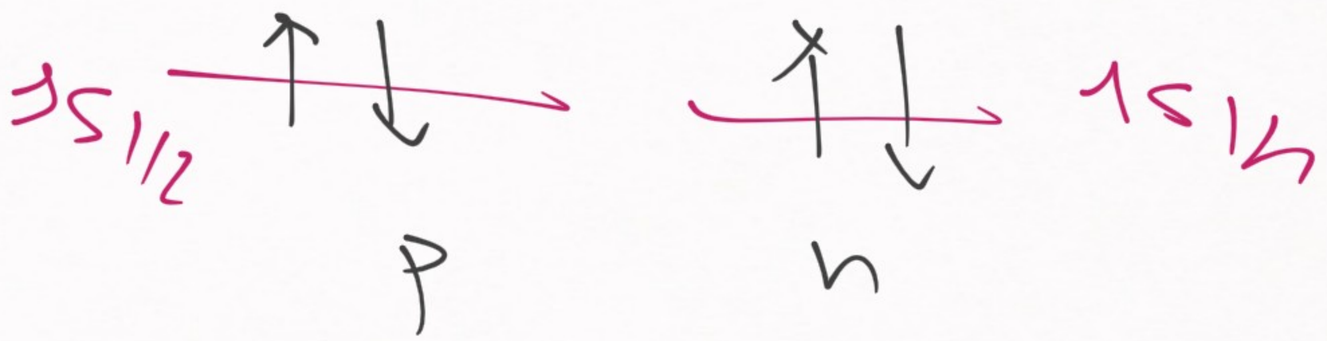
0 (D_0 states)

→ minimum energy state

⊜ ⇨ We begin by filling the Power shell, till we arrange all nucleons in shells

$^4He \rightarrow (2p, 2n)$

done !



→ But we could see this from a different perspective.

$$\mathcal{I}_a = \sum_{i=1}^{n} a_0^{(n_i)}$$

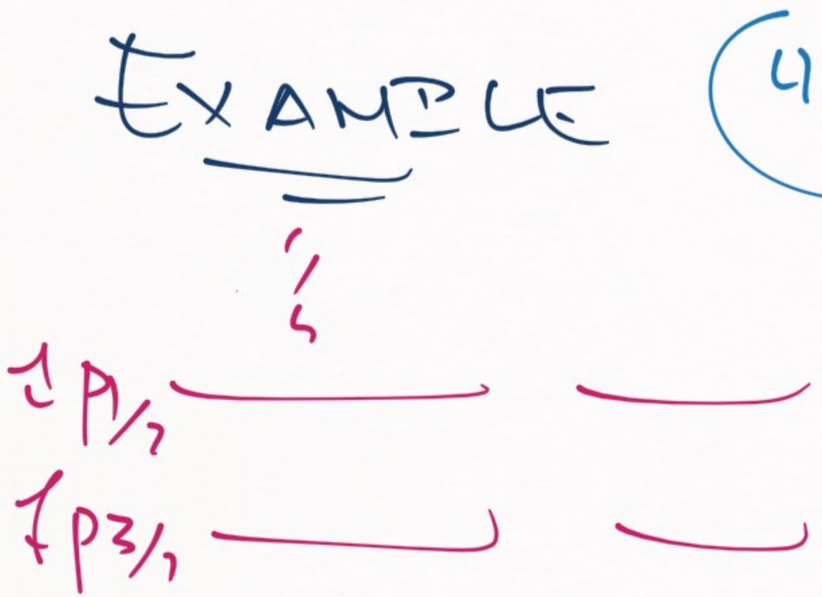
$\alpha = \{(z_0|n_0), (z_1|n_1), (z_2|n_2), \ldots\}$

↳ an arbitrary configuration

Hilbert space → $\{\mathcal{E}_\alpha\}$ with all possible α's

each of them represents a different nuclear state

EXAMPLE $^4He \rightarrow (2p, 2n)$



$\alpha \rightarrow$ { 2 protons in $1s_{1/2}$
2 neutron in $1s_{1/2}$

$\alpha' \rightarrow$ { 2 protons in $1s_{1/2}$,
1 neutron in $1s_{1/2}$,
1 neutron in $1p_{3/2}$

$\alpha'', \alpha''', \alpha'''', \ldots$

→ this generates the full Hilbert space for (Z,N) nucleus

We can understand this better w/ a 1-Body example →

1 BODY CASE + 1 dimension

1) Standar Schrödinger equation

$$H|\Psi\rangle = E\psi \rightarrow -\frac{1}{2m}\psi''(x) + V(x)\psi(x) = E\psi(x)$$

Basis $\rightarrow \{|x\rangle\} \Rightarrow \langle\psi|x\rangle$

2) Momentum space Basis $\rightarrow \{|p\rangle\}$

$$\langle\psi|p\rangle = \int d^3\vec{x}\, \langle\psi|x\rangle\langle x|p\rangle$$

$V(p,p') = \langle p'|V|p\rangle$

3) → can be done in any basis

$\{|x\rangle\}$, $\{|p\rangle\}$ not the only ones

4) Oscillator basis

$$V^{MF}(x) \simeq \frac{1}{2} m\omega^2 x^2$$

$$V(x) \simeq \underbrace{\frac{1}{2} m\omega^2 x^2}_{\text{mean-field}} + \underbrace{\left(V(x) - \frac{1}{2} m\omega^2 x^2\right)}_{\Delta V \text{ (residual)}}$$

$H^{MF}|n\rangle = \hbar\omega\left(n+\frac{1}{2}\right)|n\rangle \rightarrow$ 1-dimensional harmonic oscillator functions

4') We can solve in the new basis

$\langle\psi|n\rangle \Rightarrow \boxed{|\psi\rangle = \sum_n |n\rangle\langle n|\psi\rangle}$

$\int dx \langle\psi|x\rangle\langle x|n\rangle$

$\rightarrow$ infinite dimensional vector

$$|\psi\rangle = \sum_n |n\rangle\langle n|\psi\rangle = \sum_n |n\rangle \psi_n$$

$$\hat{H}|\psi\rangle = \sum_n \hat{H}|n\rangle \psi_n$$

→ if you want to obtain (1)

$$H_{mn} = \langle m|\hat{H}|n\rangle$$

$$\boxed{H_{mn}\psi_n = E\psi_m}$$

$$\hat{H}|\psi\rangle = E|\psi\rangle$$

$$\hat{H}\psi(x) = \left(-\frac{1}{2m}\frac{d^2}{dx^2} + V(x)\right)\psi(x)$$

$$\int dx' \, \delta(x-x')$$

$$\int dx' \, \langle x|\hat{H}|x'\rangle \psi(x')$$

$$= E\,\psi(x)$$

$$|\psi\rangle \longrightarrow \psi(x) = \langle x|\psi\rangle$$

$$-\frac{1}{2m}\psi''(x) + V(x)\psi(x) = E\psi(x)$$

2) $\{|n\rangle\} \rightarrow H_{mn}\psi_n = E\psi_m$

$H_{mn} = \langle m|H|n\rangle$

$\psi_n = \langle n|\psi\rangle$

$\Longrightarrow$

→ Oscillator basis is just another representation for Schrödinger

1) $H\psi(x) = \left[-\frac{1}{2m}\frac{d^2}{dx^2} + V(x)\right]\psi(x) = E\psi(x)$

(differential equation)

2) $H_{mn}\psi_n = E\psi_m$ (matrix equation)

3) $\langle 1 \Rightarrow 2 \rangle$

We can always choose a give representation

1-dim → Using the oscillator basis is straightforward

Shell-model → Same idea, but in 3-dim

(w/ spin-orbit, etc.)

BASIC IDEA → We can solve Schrödinger in any basis we like (including the VMF basis)

→ If we use the oscillator basis, then we end up with an infinite dimensional matrix equation

1) $H = \sum_i T_i + \sum_{ij} V_{ij}^{2B} + \sum_{ijk} V_{ijk}^{3B} + \dots$

$\rightarrow H = \sum_i H_i + \Delta V$

2) $H_{\alpha\beta} \psi_\beta = E \psi_\alpha$

$\psi_\alpha = A\left[\prod_{i=1}^{A} \phi_i\right]$

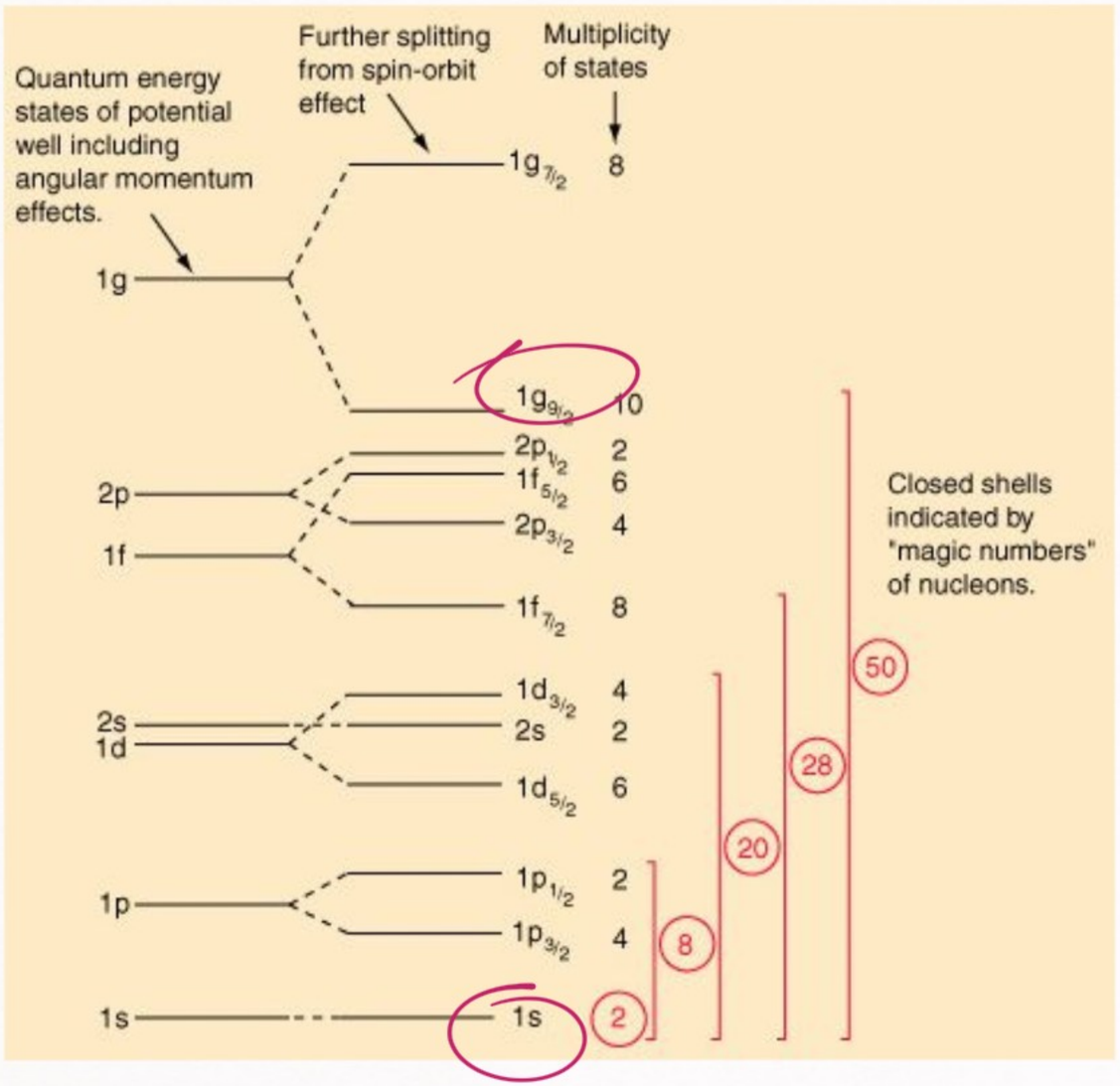
} → interactions in the shell model

PROBLEM → [INFINITE DIMENSIONAL PROBLEM]

We need to simplify thr

↓

reduce the number of configurations that we use



→ not all the shells are equally important for all nuclei

→ we can 'ignore' shells that are unlikely to be filled

Example → ^4He $1s_{1/2}$ ↑↓ ↑↓ $1p_{3/2}$

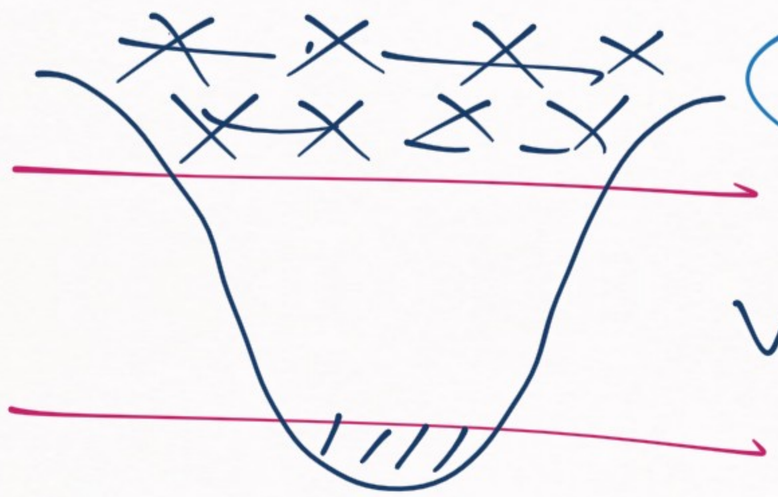
For $\Delta V \neq 0$ (for $\Delta V = 0$)

⇓

Mixture of a lot of shell

→ but, it's very unlikely that there will be a large contribution from $1g_{9/2}$ shell

[TYPICAL SET-UP]



always empty

EXTERNAL

division of shells

VALENCE

where things happen

CORE

always full

DIVIDE STUFF INTO 3 CATEGORIES:

1) CORE
2) VALENCE → what matters
3) EXTERNAL

EXAMPLE → 180 = 160 (CORE) + two redrivers

IBO

EXTERNAL

1p

1s

1d

1p

1s

VALENCE

EXTERNAL

CORE

hectares

protons

^{18}O → 1) CORE → ^{16}O

2) VALENCE → $\{1s_{1/2}, 1s_{1/2}, 1d_{3/2}\}$ orbitals

3) EXTERNAL → $\{1f_{7/2}, \ldots\}$ all higher energy orbitals

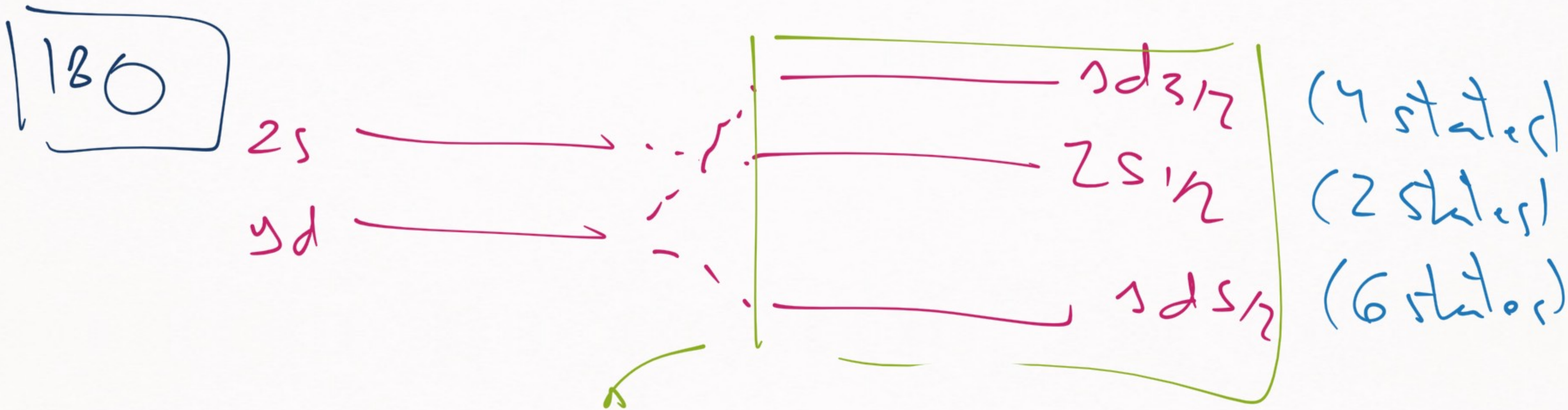
TYPICAL EXAMPLE OF THE INTERACTING SHELL MODEL

How do we continue? solve it?

1) We have to write down the basis for the wave functions in the valence

$$\Psi_\alpha = \underbrace{\Psi_\alpha^{CORE}}_{\text{fixed}} \times \underbrace{\Psi_\alpha^{VALENCE}}_{\text{non-fixed}}$$

2) We find the dimension of $\Psi^{VALENCE}$



∃ 12 different levels in the valence

↳ Fill the valence w/ two fermions

What is the dimension of the valence space?

1) $^{170} \rightarrow$ 1 neutron in valence

$\rightarrow$ Dim = 12

2) $^{180} \rightarrow$ 2 neutrons in valence space

$$\text{Dim} = \frac{12 \cdot 11}{2} = \binom{12}{2} = 66 \text{ states}$$

^{18}O in this interacting shell model

$\hookrightarrow \binom{12}{2} = 66$ dimensional basis

$\rightarrow$ Solve a 66x66 matrix to obtain the eigenvalues

=

66 dim-space : → we do all combi

$|1\rangle = |{}^1d_{5/2}(+5/2), {}^1d_{5/2}(+3/2)\rangle$

$|2\rangle = |{}^1d_{5/2}(+5/2), {}^1d_{5/2}(+1/2)\rangle$

⋮

$|n\rangle = |{}^1d_{5/2}(+5/2), {}^2S_{1/2}(+1/2)\rangle$

$|n+1\rangle = |{}^1d_{5/2}(+5/2), {}^2S_{1/2}(-1/2)\rangle$

NEXT STEP → $|1\rangle, \ldots, |66\rangle$

$$H = \langle \alpha | H | \beta \rangle = \begin{pmatrix} H_{1,1} & H_{1,2} & \cdots & H_{1,66} \\ H_{2,1} & H_{2,1} & \cdots & \\ \vdots & \vdots & & \vdots \\ & & & \end{pmatrix}$$

66×66 matrix

↓

find the eigenvalues/eigenvectors

→ the problem is the size of the matrices

$B_0 \rightarrow 16\,\mathrm{O} + (1d, 2s)$ valence $\rightarrow \binom{12}{2} = 66$ ✓

$^{60}_{30}Zn_{30} \longrightarrow {}^{40}_{20}Ca_{20}$ + p/f shells as valence

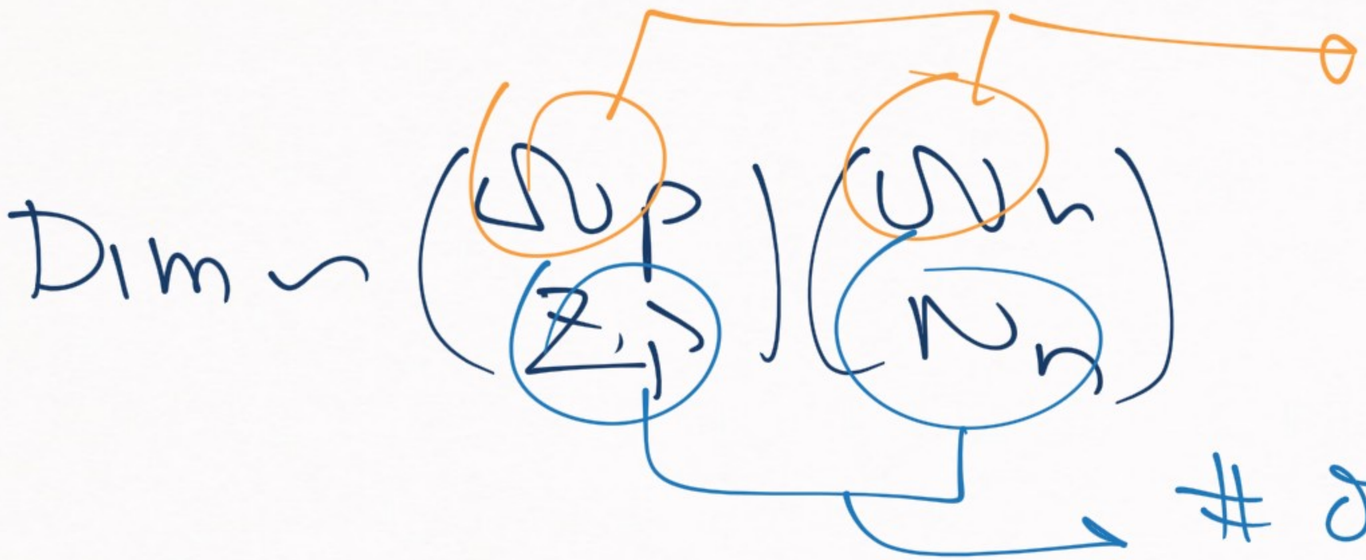
(10) (10) Cores

$\mathrm{Dim}\left(^{60}_{30}Zn_{30}\right) = \binom{20}{10}\binom{20}{10}$ (up to 20 nucleons)

$\sim 3.4 \cdot 10^{10}$ dimensional

INTERACTING SHELL MODEL

↪ DIMENSION OF VALENCE SPACE CAN GROW REALLY QUICKLY

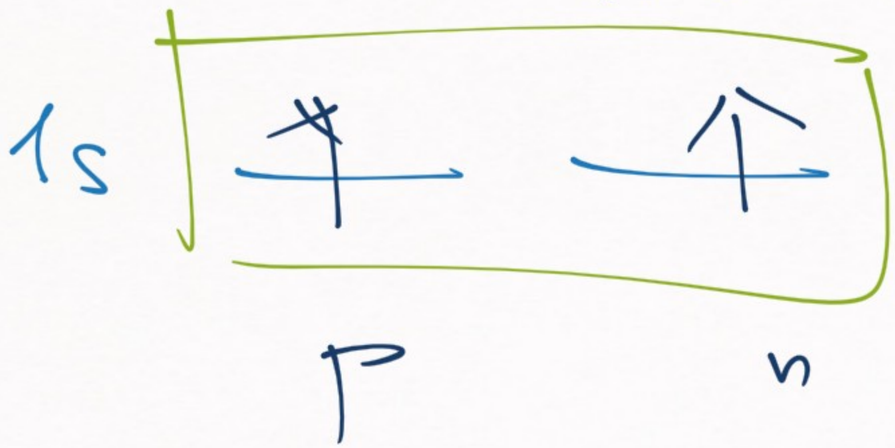


→ it quickly becomes unwieldy

[GOOD NEWS] → we can reduce the dimensionality by considering specific JT states

Example → deuteron

VALENCE → $Dim = \binom{2}{1}\binom{2}{1} = 4$



But deuteron $J^P = 1^+$

$Dim(y+) = 3$

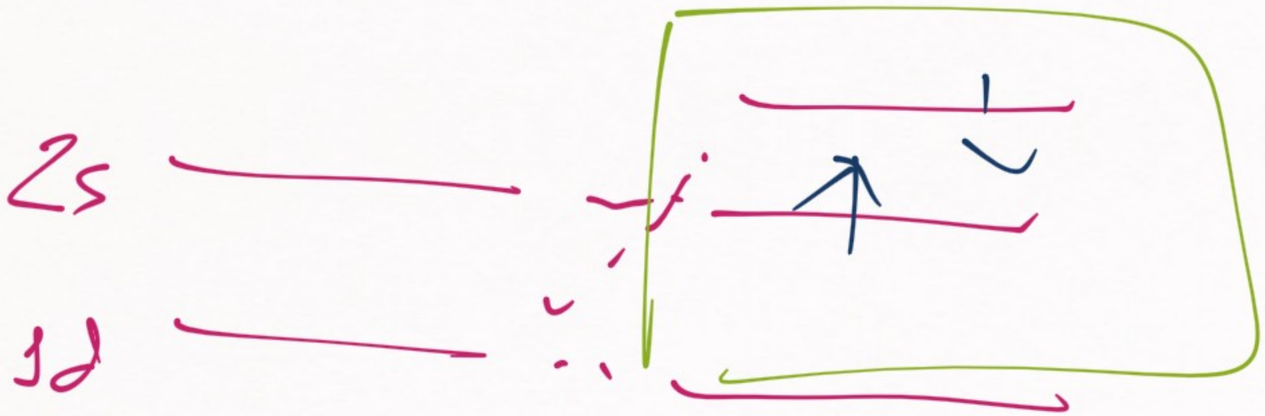
$Dim(0+) = 1$

$J(J+1)$

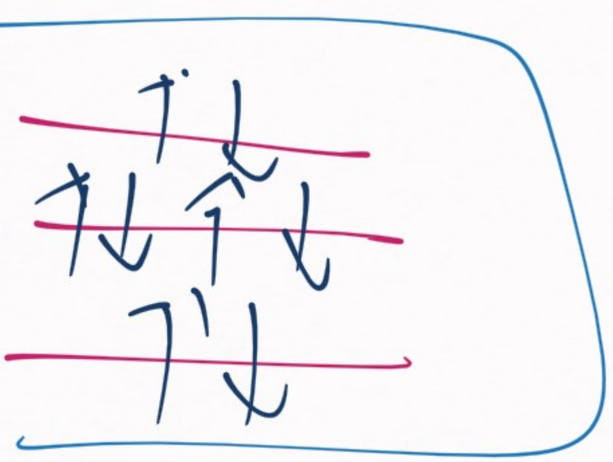
$Dim(1^+) < Dim(\forall J^P)$

→ By considering specific T we will
be able to reduce the dimension
of the valence space

EXAMPLE → Mg, but only Mg²⁺ ion states



VALENCE → 0+



→ 0+

CORE (FULL)

$^{18}O\ (0^+) \rightarrow {}^{16}O\ (0^+) +$ 2 electrons coupled to 0^+

Count possible valence configurations that give $J^\pi = 0^+$ valence

→ EASY

$d_{5/2}$

$s_{1/2}$

$d_{5/2}$

$J^P \to J_1^{P_1} \otimes J_2^{P_2}$

$3/2 \otimes 1/2 \simeq 1 \oplus 2$ ~~$\to J^P = 0^+$~~

we don't count this one

10^+

$(11, 2+1)$

$3/2 \otimes 3/2 = 0 \oplus 1 \oplus 2 \oplus 3$

$1/2 \otimes 1/2 = 0 \oplus 1$ → SYMMETRIC

$5/2 \otimes 5/2 = 0 \oplus 1 \oplus 2 \oplus 3 \oplus 4 \oplus 5$

64 → only 3 options:

→ only 3 possibilities → dim(α) = 3

$$\dim(\mathcal{O}^t_{\text{valence}}) \ll \dim(\text{valence})$$

$$3 \ll 66$$

$\Rightarrow$

→ wonderful simplification

$\dim = 3$ → calculation $\ll$ easy $\gg$

How to do it?

1) FIND THE SINGLE PARTICLE LEVELS

$$h_i^{MF} \phi_i = \epsilon_i \phi_i$$

$$\epsilon(1d_{5/2}), \epsilon(2s_{1/2}), \epsilon(1d_{3/2})$$

+ wave functions

→ this doesn't need a VMF

(because these energies can be extracted from exp. data) → 17O

$\epsilon(1d_{5/2}) = E_B(^{17}O) - E_B(^{16}O)$ ‖ $k_{\ell=2}$

$\epsilon(2s_{1/2}) = \epsilon(1d_{5/2})$

$+ \left(E_B^{*}(^{17}O, 1/2^+) - E_B(^{17}O)\right)$

etc

$$\epsilon(3d_{3/2}) - \epsilon(1d_{5/2}) = E_B^{\alpha}(17O, {}^3\!/_2{}^+) - E_B(17O, {}^5\!/_2{}^+)$$

→ only requires a bit of work

(2 days)

$\epsilon(1d_{5/2}) = -4.1\ MeV$

$\epsilon(2s_{1/2}) = -3.7\ MeV$

$\epsilon(1d_{3/2}) = +0.9\ MeV$

} Monoparticular energies

$^{13}O\ (0^+) \rightarrow |\alpha_R\rangle = Z\delta_{\alpha\beta}\epsilon_\alpha + (\Delta V)_{\alpha\beta}$

$\alpha, \beta = 1, 2, 3$

$1 \rightarrow 2n$ in $1d_{5/2}$

$2 \rightarrow 2n$ in $1s_{1/2}$

$3 \rightarrow 2n$ in $1d_{3/2}$

(AU) a_R → depends on choice of V^{MF}, V^{2B}, V^{3B}, ...

Let's assume we obtain this:

$$\underline{\underline{\Delta V}} = \begin{pmatrix} -2.8 & -1.3 & -3.2 \\ & -2.1 & -1.1 \\ \text{sym} & & -2.7 \end{pmatrix}$$

$H + \Delta V \rightarrow 3 \times 3$ matrix $\rightarrow$ find eigenvalues

$\rightarrow$ -12.6, -8.1, $+0.6$

$IBO(O_1^+)$ ground state

O_2^+, O_2^- excited states

Exp that usually we use energy differences

$\Delta E(0_1^+) = 4.4\ MeV$

$\Delta E(0_2^+) = 13.1\ MeV$

} → this is clear
this is dam

(∃ problem of the calculation
of Vap)
(not covered here)

For the full spectrum →

Repeat the calculation for:

$J^P = 1^+, 2^+, 3^+, 4^+$

(1st forbidden, bc/symmetric)

FURTHER EXAMPLE → ^{16}O, $J^P = 3^+$

What are the options?

$1d_{3/2}$ ——

$1s_{1/2}$ ——

$1d_{5/2}$ ——

$3/2 \otimes 3/2 = 0 \oplus 1 \oplus 2 \oplus 3$ ✗

$1/2 \otimes 5/2 = 2 \oplus 3$ ✓

$3/2 \otimes 5/2 = 1 \oplus 2 \oplus 3 \oplus 4$ ✓

$5/2 \otimes 5/2 = 0 \oplus \dots \oplus 4 \oplus 5$

Sym

Sym

Sym

Sym

Asym

1) Build the basis

$$\alpha = \{|1d_{5/2}\rangle|_3 , |1d_{5/2}\rangle|_2\}$$

$$= \{|3,2\rangle$$ Simplification → we don't need to worry about m_l (7 states)

2) Hamiltonian

$$H_{11} = \epsilon(1d_{5/2}) + \epsilon(2s_{1/2}) + \Delta V_{11}$$

$$H_{12} = H_{21} = \Delta V_{21} = \Delta V_{12}$$

$$H_{22} = \epsilon(1d_{5/2}) + \epsilon(1d_{3/2}) + \Delta V_{22}$$

Imagine that $\Delta V = \begin{pmatrix} 0.2 & 0.7 \\ 0.7 & 0.6 \end{pmatrix}$ MeV

↙

Diagonalize $H + \Delta V \rightarrow$ $(-6.6, -7.5$ MeV$)$

→ Exactly same process

$\Delta E(^{18}O(3_0^+)) = (-6.6) - (-17.6) = +6$ MeV

→ [General pattern for interacting shell model is easy]

(Repeating the previous process for any nucleus of interest)

NO CORE SHELL MODEL → REMOVE THE CORE (+COMPUTING POWER)

→ NEXT LESSON:

How to obtain the mean field potential.