NUCLEAR PHYSICS 19

Liquid drop model

&

Shell model

$\Rightarrow$

$\boxed{\text{RECAP}}$ → NUCLEAR PROPERTIES WE WANT TO DESCRIBE

1) Binding energy
   ($B/A \sim 8$ MeV/nucleon) } → basis of liquid drop model
2) Nuclear size ($\sim A^{1/3} r_0$, $r_0 \sim (1.1 - 1.2)$ fm)
3) J^P (important for nuclear structure)
4) EM moments
5) Stability & decay

NUCLEAR MODELS → Why do we need them?

1) $A \leq 10-12$ → calculate nuclei from nuclear potential & Schrödinger

(ab-initio calculations)

2) A larger

→ simplifications → nuclear models

MOST IMPORTANT MODELS:

1) LIQUID DROP MODEL (B/A)

2) SHELL MODEL (magic/closed shell nuclei + nuclei close to these N,Z)

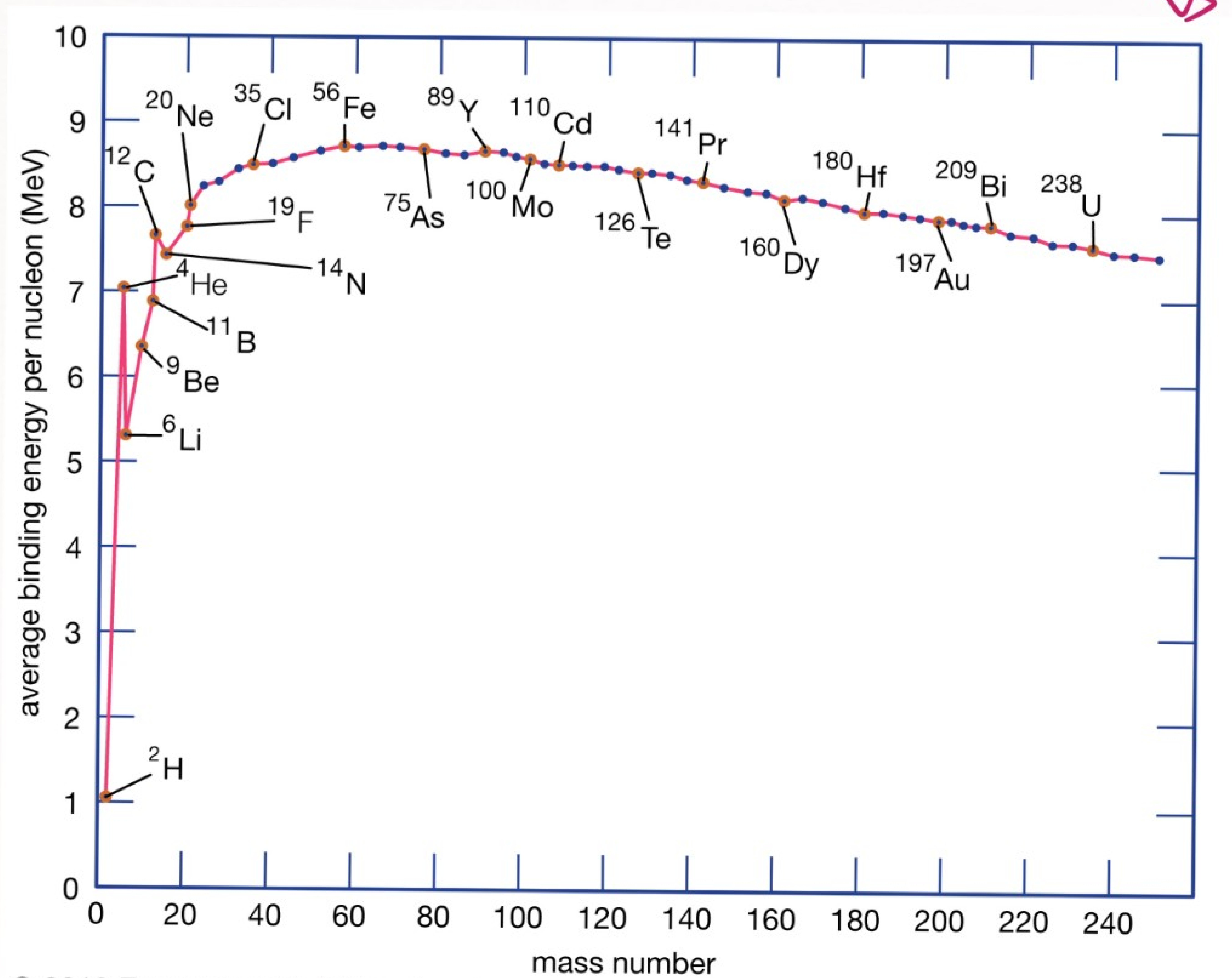
3) COLLECTIVE MODEL (larger nuclei far from magic N,Z)

   3.A) VIBRATIONAL

   3.B) ROTATIONAL

LIQUID DROP MODEL

(Explain this)



$B/A \sim 8$ MeV/nucleon

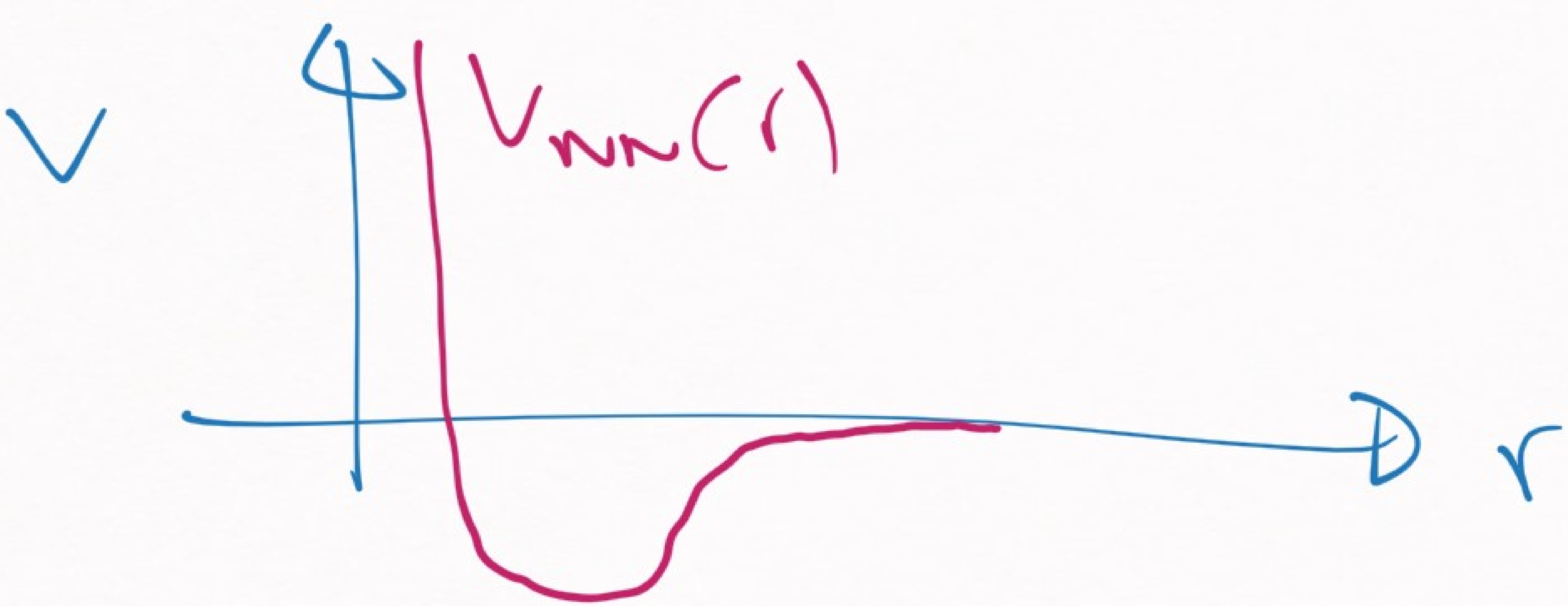
this is called saturation

SATURATION DEPENDS ON :

1) Nuclear forces have a finite range

2) Strong mid-range attraction

3) Stronger short-range repulsion



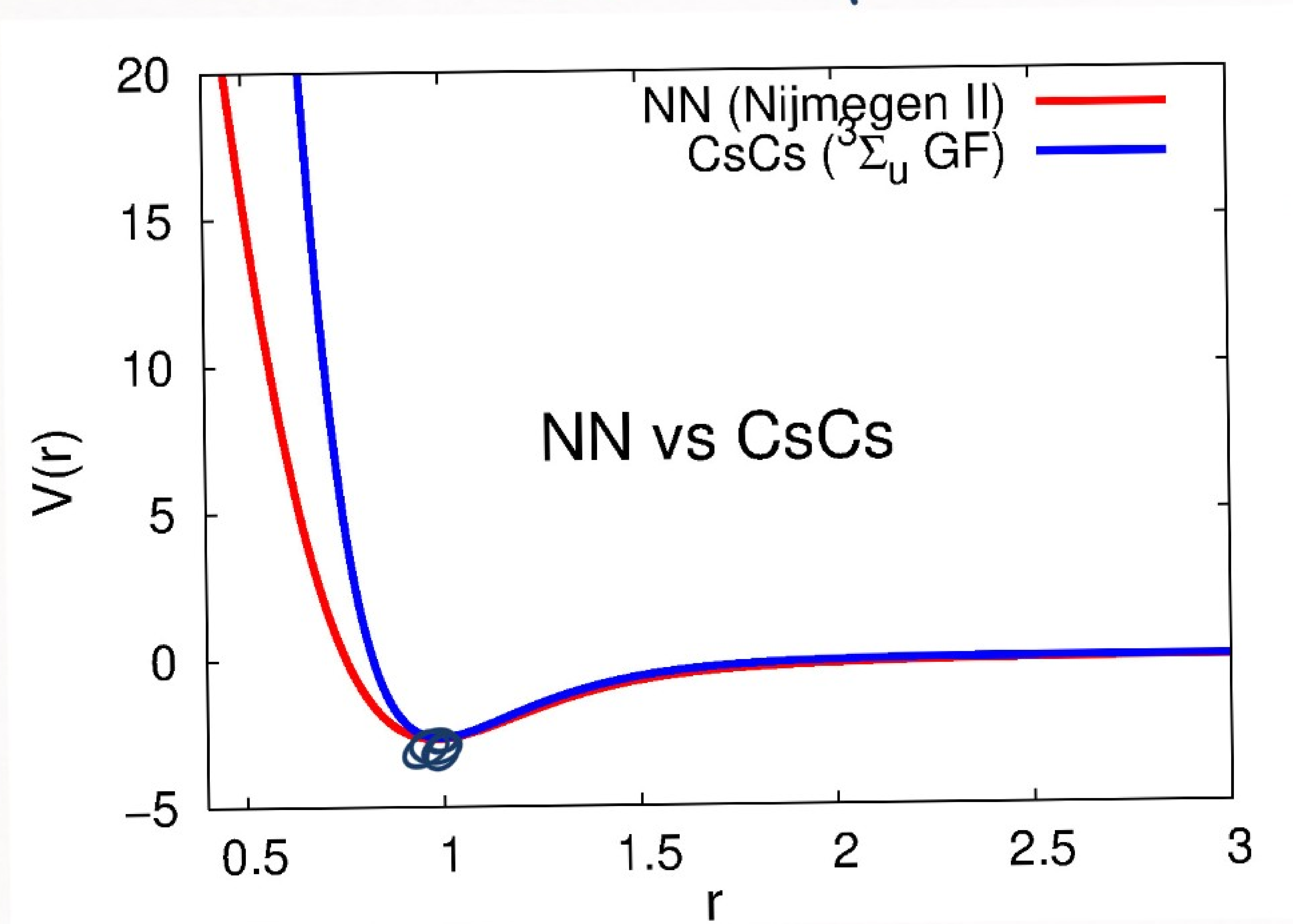
1+2+3) $\longrightarrow$ nuclear forces (nucleon-nucleon potential)

$\longrightarrow$ atomic forces (atom-atom potential)

$\exists$ ANALOGY BETWEEN THESE TWO

ANALOGY

→ units rescaled / minimum in same place



1) NN potential

2) Cesium-cesium potential

Qualitatively similar

BEHAVIOR OF $\begin{cases} \text{GROUP OF ATOMS} \\ \cong \\ \text{GROUP OF NUCLEONS} \end{cases}$

Atoms tend to form droplets (of liquid)

→ nucleons can also form a liquid droplet ↳

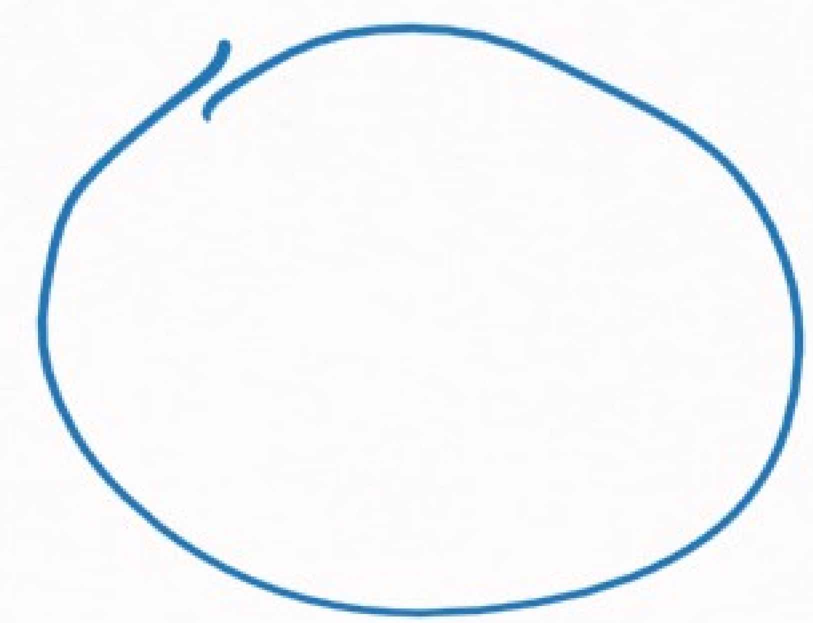
→ We can try to model the nucleus
as if it were a liquid

"Nuclear liquid" → nucleus is just
a drop of
this liquid

$\boxed{\text{LIQUID DROP MODEL}} \longrightarrow B(Z,A)$ (objective)

↳ several contributions to

1) VOLUME TERM



$V \propto A$

$B(Z,A) = a_V A + \ldots$

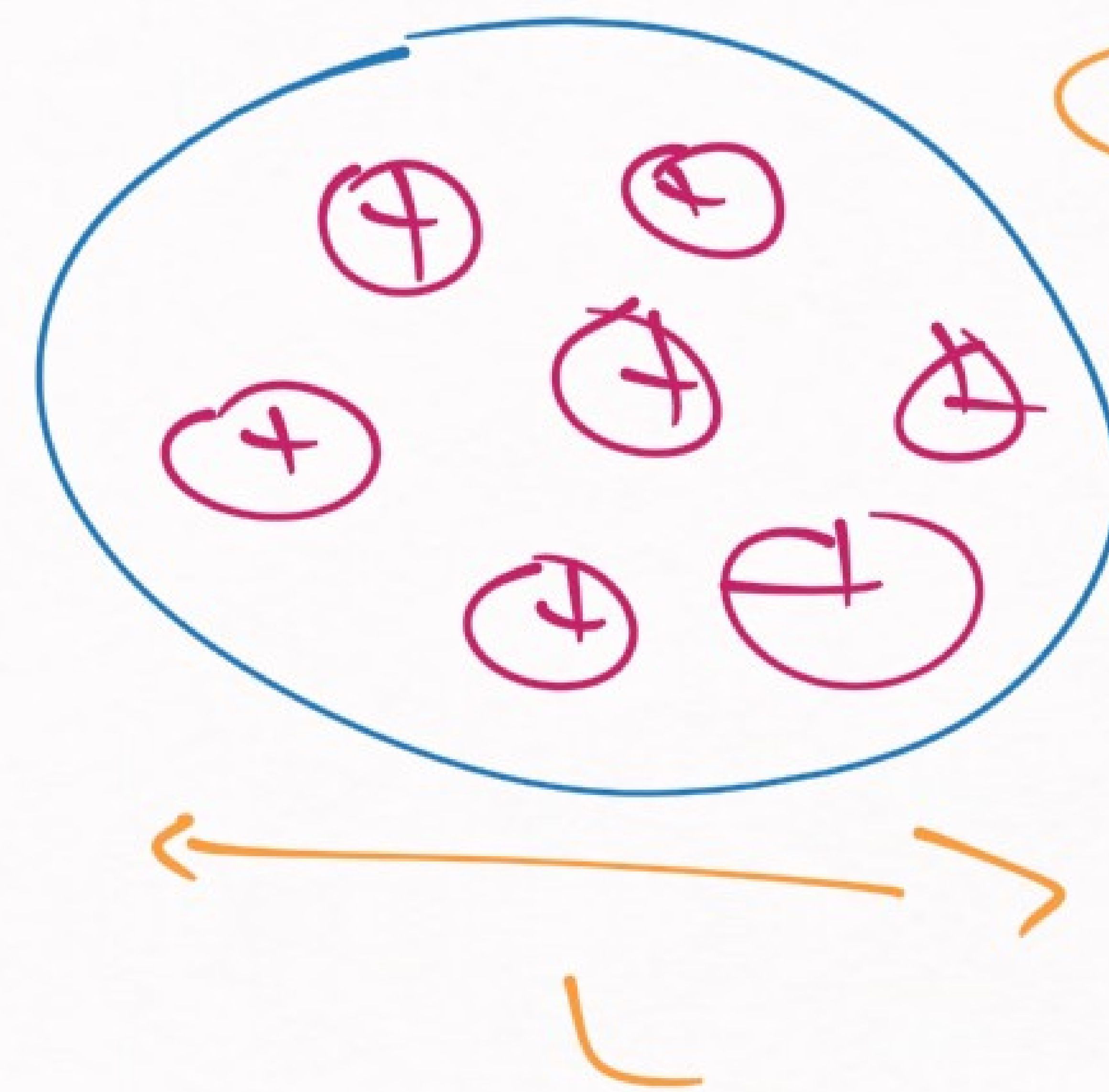
2) SURFACE TERM

$S \propto A^{2/3}$

$B(Z,A) = a_V A - a_S A^{2/3} + \ldots$

3) COULOMB TERM

$$B(Z,A) = a_v A - a_s A^{2/3} - a_c \frac{Z(Z-1)}{A^{1/3}} + \dots$$



Coulomb repulsion $\Rightarrow$ $\propto \frac{Z(Z-1)}{L}$

$L \propto A^{1/3}$

(1/2/3 $\rightarrow$ Easy to understand)

4) ASYMMETRY TERM $\propto \frac{(Z - A/2)^2}{A}$

$\rightarrow$ nuclei prefer to have $N = Z$ (ad-hoc)

$$B(Z, N) = a_V A - a_S A^{2/3} - a_C \frac{Z(Z-1)}{A^{1/3}} - a_A \frac{(Z - A/2)^2}{A} + \ldots$$

5) PAIRING TERM $\propto a_p \frac{((-1)^Z + (-1)^N)}{2A^{1/2}}$ (add. term)

→ even-even nuclei are the most common type of nuclei (previous lesson)

$$B(Z,N) = a_V A - a_S A^{2/3} - a_C \frac{Z(Z-1)}{A^{1/3}} - a_A \frac{(Z - A/2)^2}{A} + a_P \frac{((-1)^Z + (-1)^N)}{2A^{1/2}}$$

$(1+2+3+4+5) \longrightarrow$ LIQUID DROP MODEL

or

SEMI-EMPIRICAL MASS FORMULA

or

BETHE-WEISÄCKER FORMULA

(See previous page for the formula)

$B(Z,A) \to a_V, a_S, a_C, a_A, a_P$

$\exists$ a lot of fits, but most of them look similar to this:

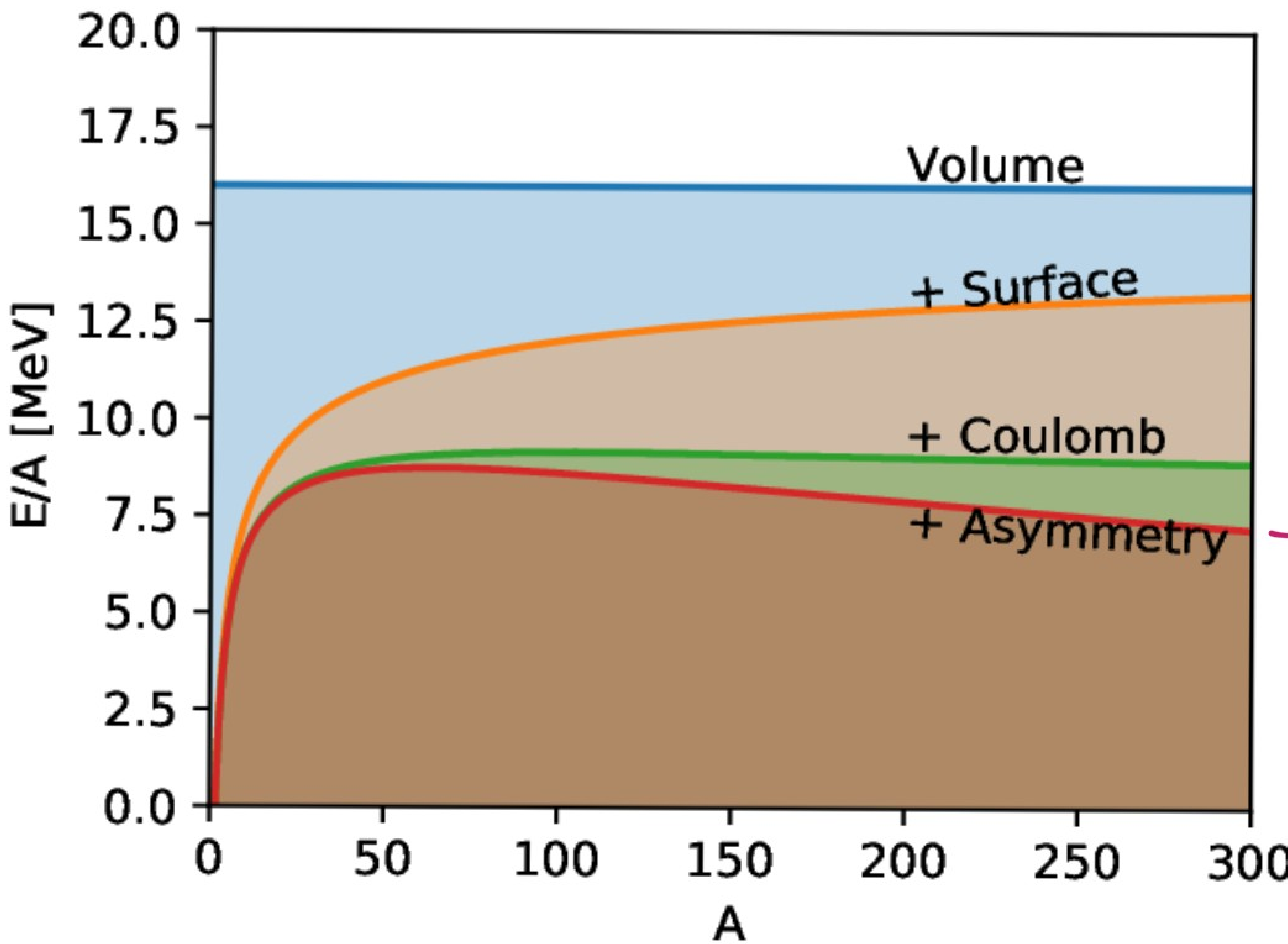
$a_V \simeq 16$ MeV

$a_S \simeq 18$ MeV

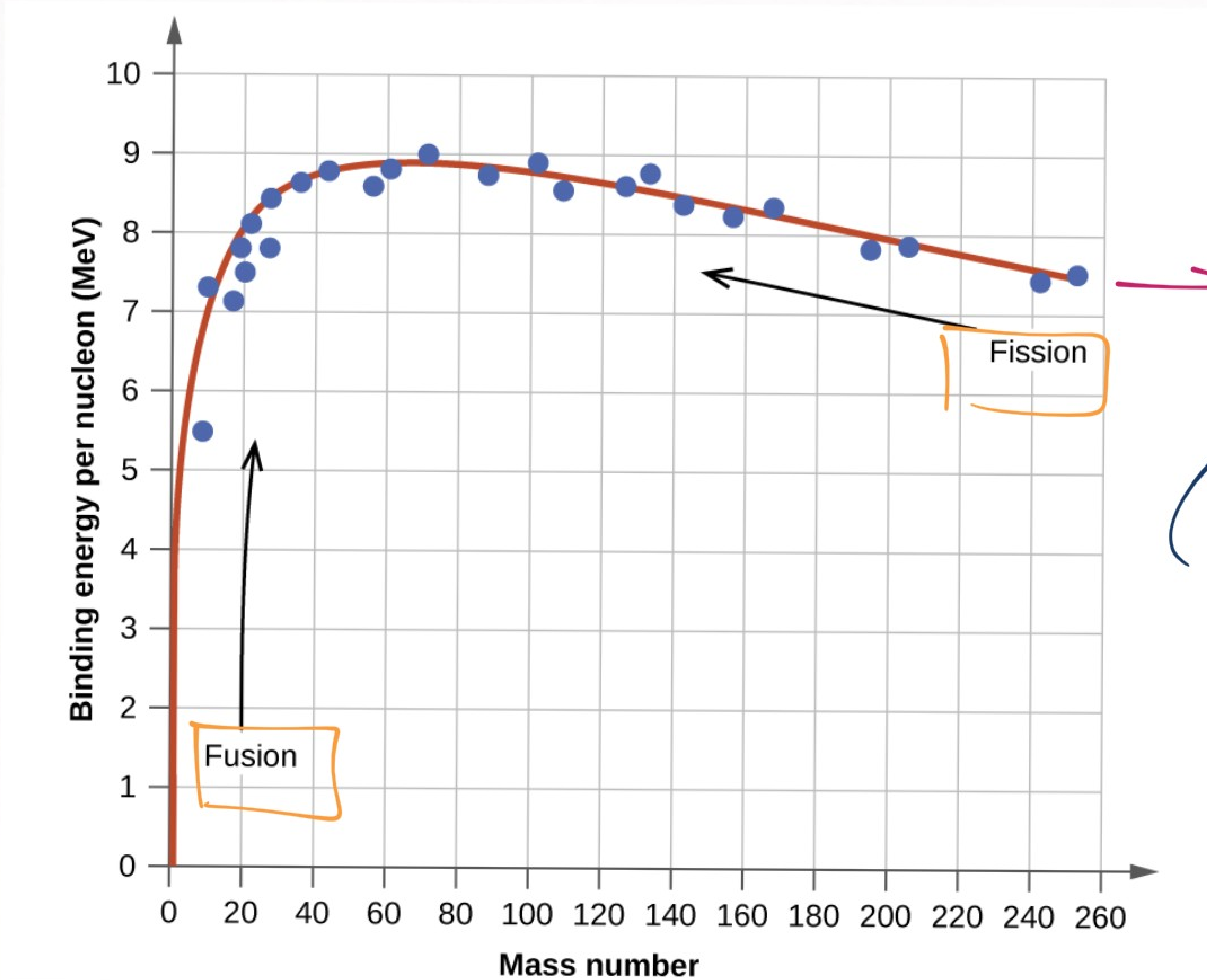
$a_C \simeq 0.7$ MeV

$a_A \sim 23$ MeV

$a_P \sim 11$ MeV



Result of adding the terms

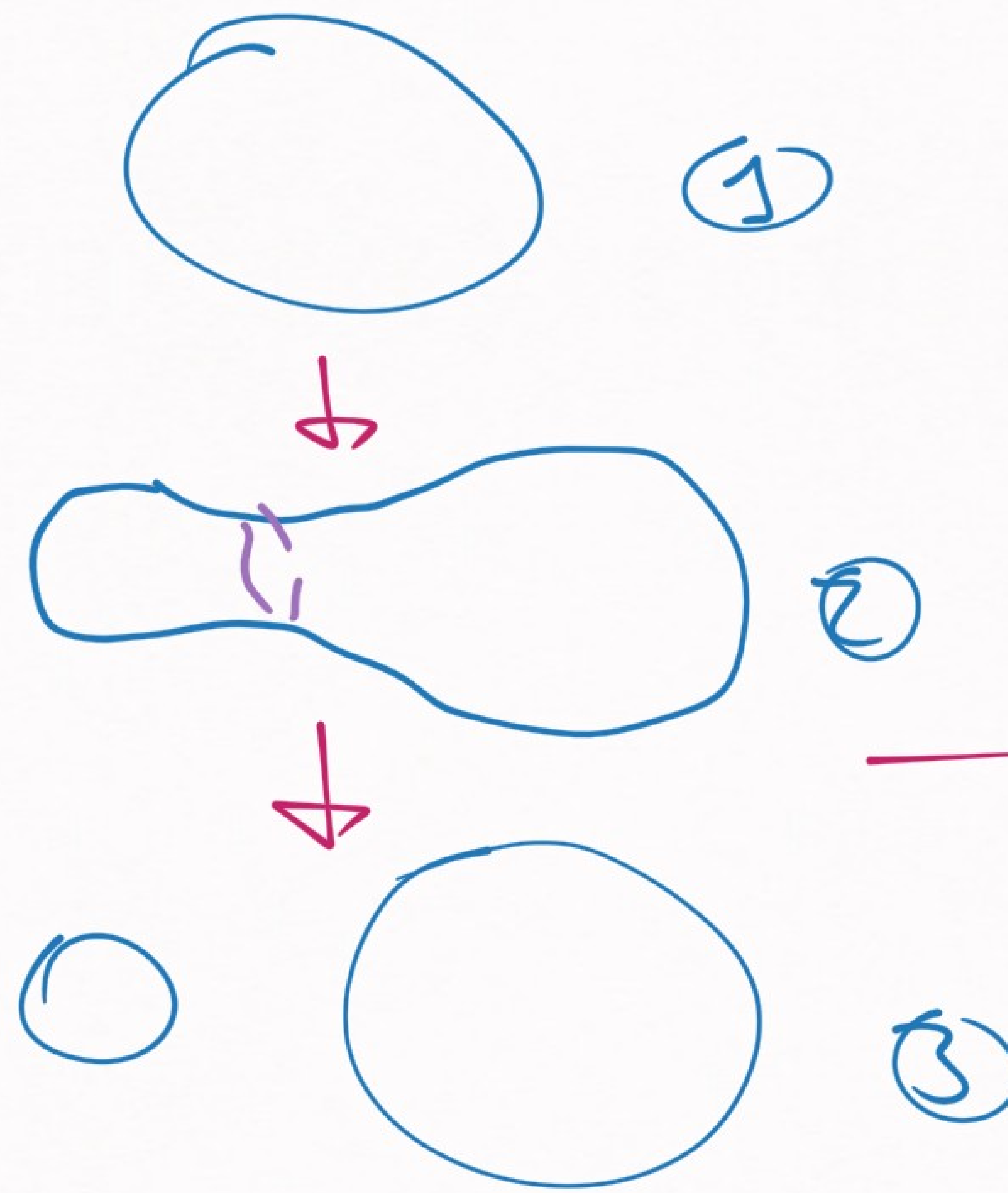


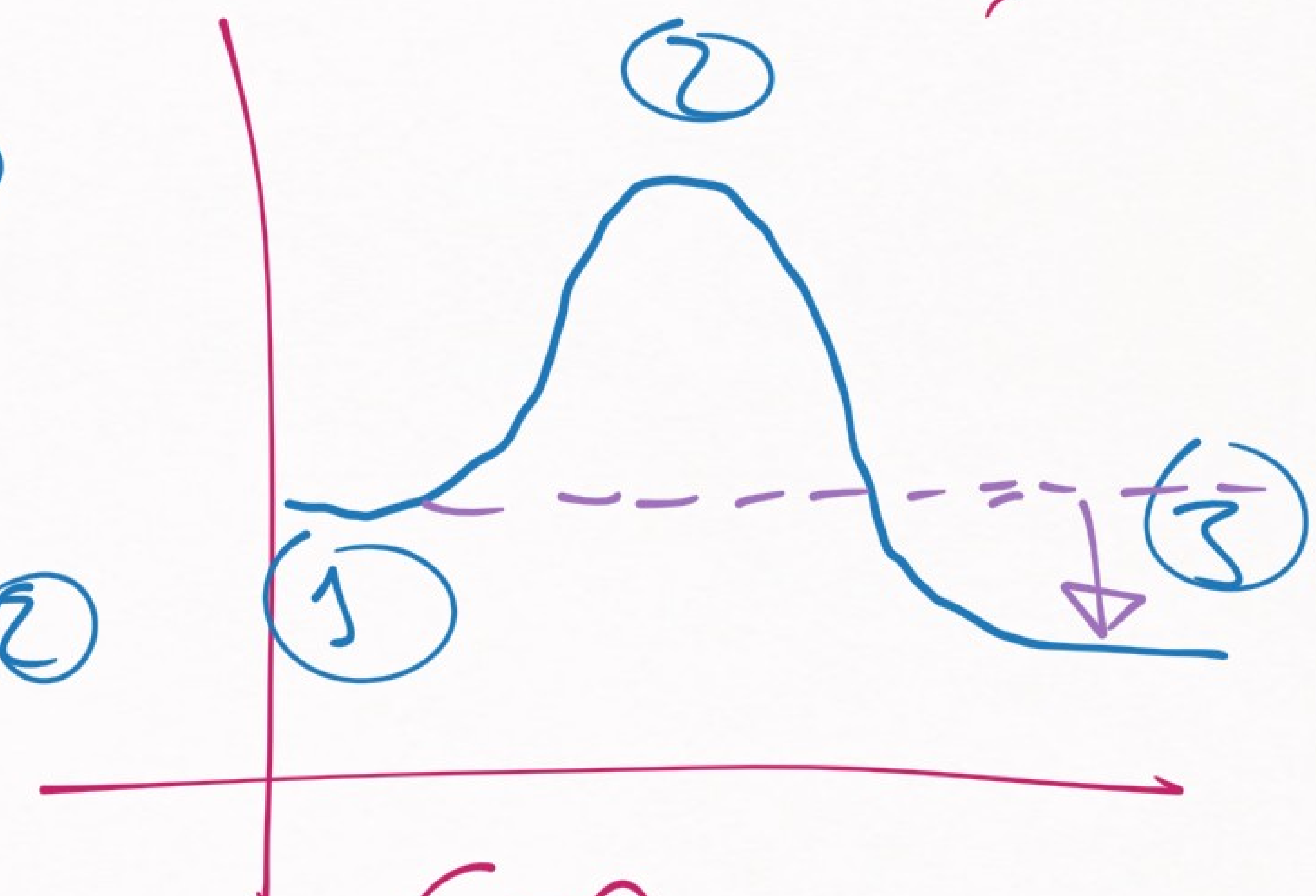
$\rightarrow \frac{B(Z,A)}{A}$

(it's really good at describing nuclei)

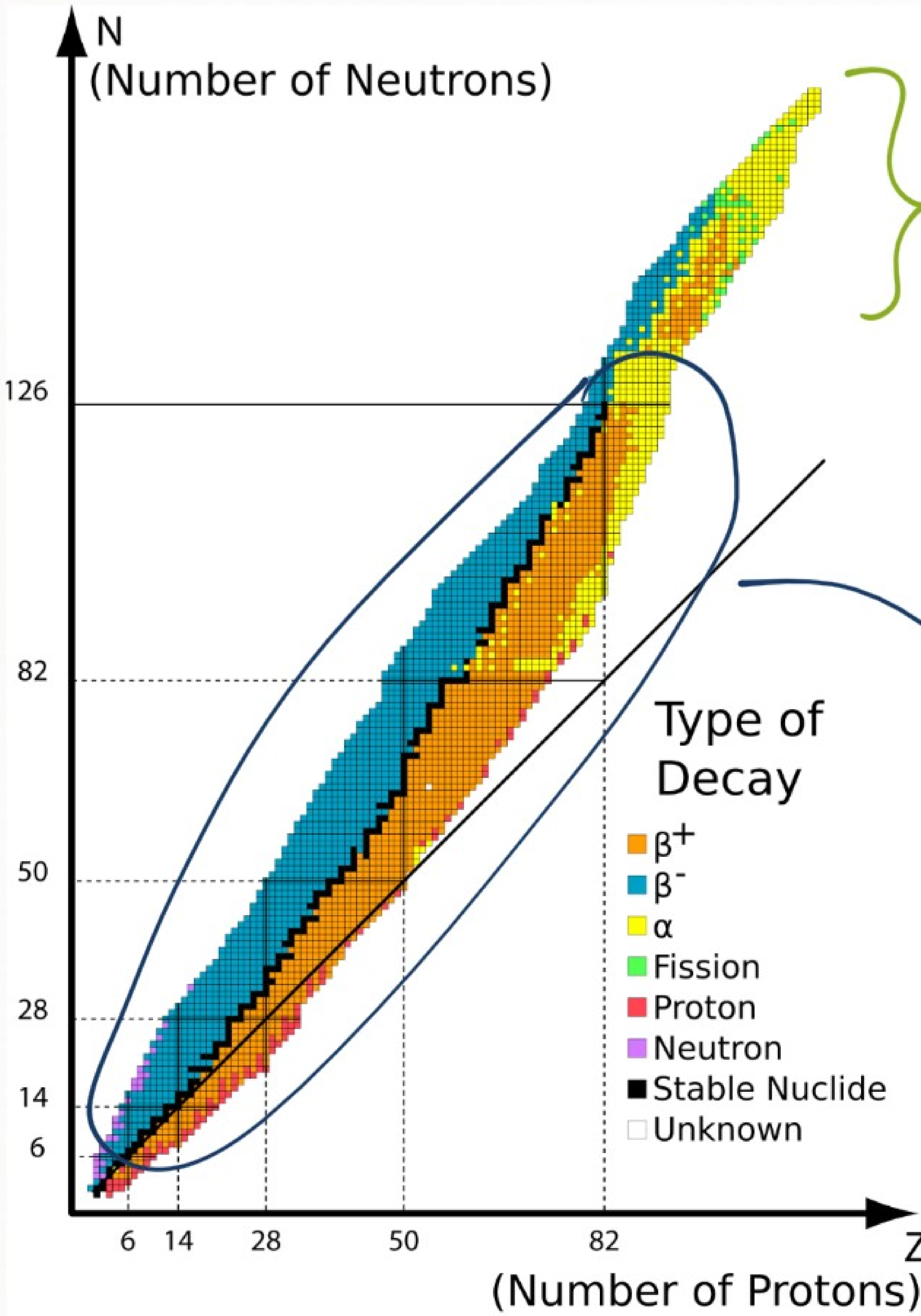
Explains when fusion/fission are allowed

Fission →





(slow process by potential barrier)



FISSION HERE WELL EXPLAINED IN LIQUID DROP MODEL

IT ALSO EXPLAINS THE STABLE NUCLEI

$\delta B(Z,A) = 0$

(local minima)

$\boxed{\text{SHELL MODEL}}$

1) Nucleons are fermions
   (antisymmetric wave function)

2) Nucleons will generate some sort of average potential

COMPLETE DESCRIPTION OF NUCLEI

$$H = \frac{1}{2m_N}\sum_{i=1}^{A} \vec{p}_i^{\,2} + \sum_{i<j} V_{2B}(\vec{r}_i - \vec{r}_j) + \sum_{i<j<k} V_{3B}(\vec{r}_i, \vec{r}_j, \vec{r}_k) + \dots$$

→ WE SOLVE IT $\boxed{H|\Psi_A\rangle = E|\Psi_A\rangle}$

$\circledcirc \to$ EXPONENTIALLY DIFFICULT TO SOLVE AS Δ GROWS

small Δ $\to$ solvable

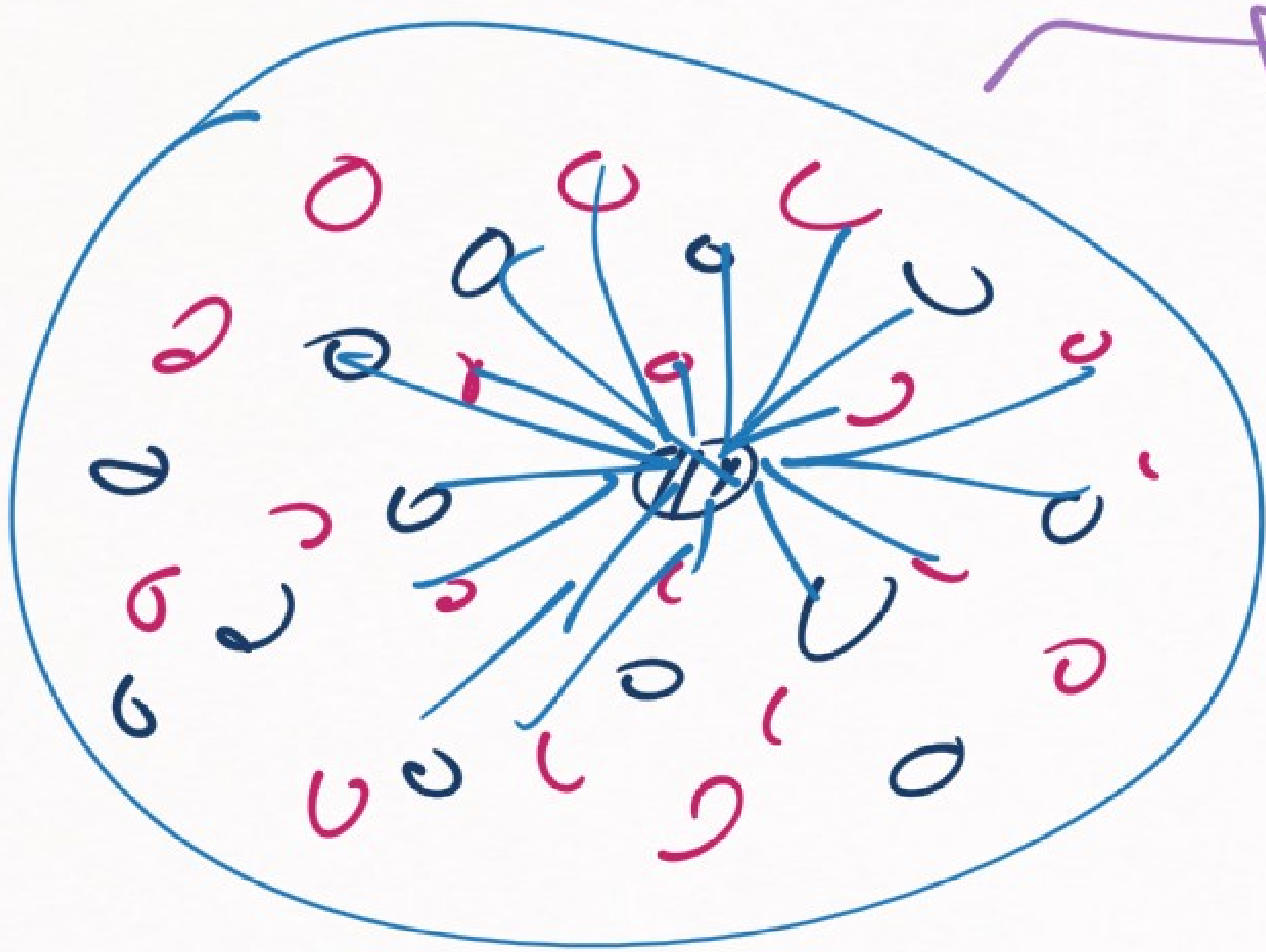
large Δ $\to$ unsolvable

$\to$ [NO GENERAL SOLUTION AVAILABLE]

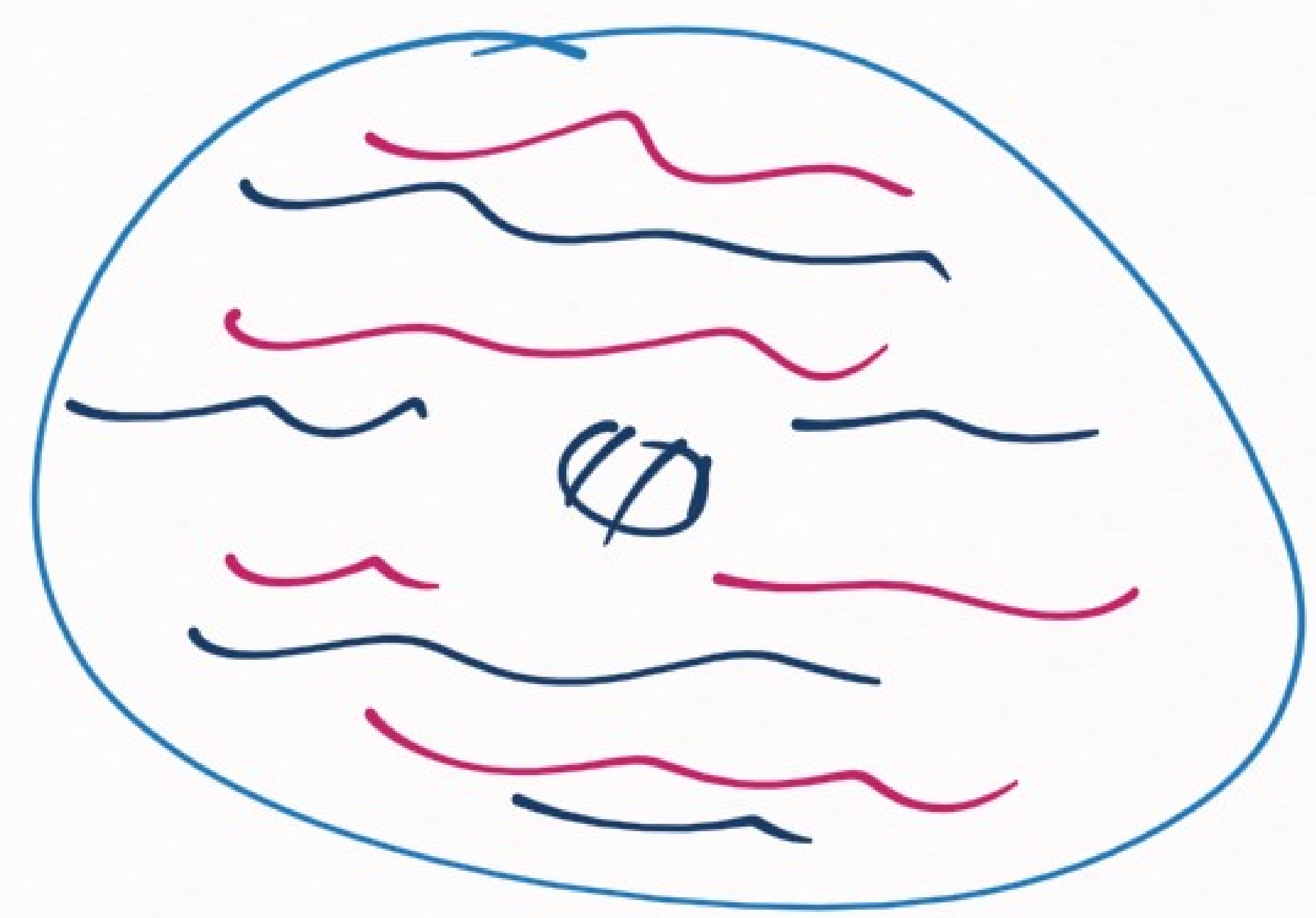
→ [INDIRECT SOLUTION]

MEAN FIELD POTENTIAL

we could try to average all this interactions



COMPLICATED



average potential

$$\sum_{i<j} V_{ij}^{2B} + \sum_{i<j<k} V_{ijk}^{3B} + \dots \Rightarrow \boxed{\sum_i V_i^{MF}}$$

approximation

$$\boxed{H = \frac{1}{2m_N}\sum_i \vec{p}_i^{\,2} + \sum_i V_i^{MF} + \Delta V}$$

diff. between full potential & mean field

$$\Delta V = \sum_{i<j} V_{ij}^{2B} + \sum_{i<j<k} V_{ijk}^{3B} + \ldots - \sum_{i} V_{i}^{MF}$$

residual interaction

IDEA → if V^{MF} is well chosen,

ΔV could be a perturbation

(i.e. small correction)

$$\left[H = \sum_i \left(\frac{p_i^2}{2m_N} + V_i^{MF} \right) \right] + \Delta V$$

small

First approximation for nuclear potential in a large nucleus

$\circledast \Rightarrow \boxed{H = \sum_{i=1}^{A} H_i}$ $\Rightarrow H_i \phi_i(r) = E_i \phi_i(r)$

easy to solve

(solve independently for each i)

$$\boxed{\begin{aligned} \Psi_A &= \prod_{i=1}^{A} \phi_i(r_i) \\ E_A &= \sum_i E_i \end{aligned}}$$

$\rightarrow$ Really simply (except for a detail)

$\circledast \rightarrow$ $\boxed{\text{NUCLEONS ARE FERMIONS}}$

$\Rightarrow$ Total wave function antisymmetric

Solution: antisymmetrize $\left[\Psi_A = \prod_{i=1}^{A} \phi_i(r_i) \right]$

$$\left[\Psi_A = \frac{1}{\sqrt{A!}} \sum_{\sigma} \prod_{i=1}^{A} (-1)^{\sigma} \phi_{\sigma(i)}(r_i) \right]$$

σ → permutation

sum over all permutations

→ this sum gives antisymmetric wf

Equivalently: (Slater determinant)

$$\Psi_A = \frac{1}{\sqrt{A!}} \begin{vmatrix} \phi_1(r_1) & \phi_1(r_2) & \cdots & \phi_1(r_A) \\ \phi_2(r_1) & \phi_2(r_2) & \cdots & \phi_2(r_A) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_A(r_1) & \phi_A(r_2) & \cdots & \phi_A(r_A) \end{vmatrix}$$

Idea $\rightarrow$ each nucleon is in a different state (bc they are fermions)

$\rightarrow$ Write down a few examples

The most simple mean-field possible:

$$V^{MF}(r) = \frac{1}{2} m_N \omega^2 r^2$$ (harmonic oscillator)

$$H_i \phi_i = \epsilon_i \phi_i$$

$$\epsilon_i(n,\ell) = \omega(2n + \ell + 3/2)$$

angular momentum

energy level (n=0,1,...)

Explore the consequences of this $\underline{\underline{\text{HMF}}}$

0) GROUND STATE $n=0, \ell=0$

$\epsilon(0,0) = \frac{3}{2}\omega \rightarrow$ how many nucleons can I fit here?

$$\left.\begin{matrix} p \\ n \end{matrix}\right\} \otimes \uparrow, \downarrow \longrightarrow \left| \begin{matrix} p\downarrow & p\uparrow \\ n\downarrow & n\uparrow \end{matrix} \right|$$

(s-wave)

4 nucleons (2 neutrons, 2 protons)

1) FIRST EXCITED STATE $n=0, \ell=1$

$$\epsilon(0,1) = \frac{5}{2}\omega$$

$\ell = 1 \longrightarrow m_\ell = -1, 0, 1$ (3 states)

$\times$ spin $(\uparrow\downarrow)$ (6 states)

6 protons
6 neutrons $\Big\} \longrightarrow$ 12 $\underline{\underline{\text{nucleons}}}$

2) SECOND EXCITED STATE

$n=1,\ \ell=0$
$n=0,\ \ell=2$

$$\epsilon(1,0) = \epsilon(0,2) = \frac{7}{2}\omega$$

10 $\ell=2$ states, 2 $\ell=0$ states

12 neutrons
12 protons $\rightarrow$ 24 nucleons

→ We can go on forever w/ this

Shell-model ⇒ put nucleons within the low energy shells

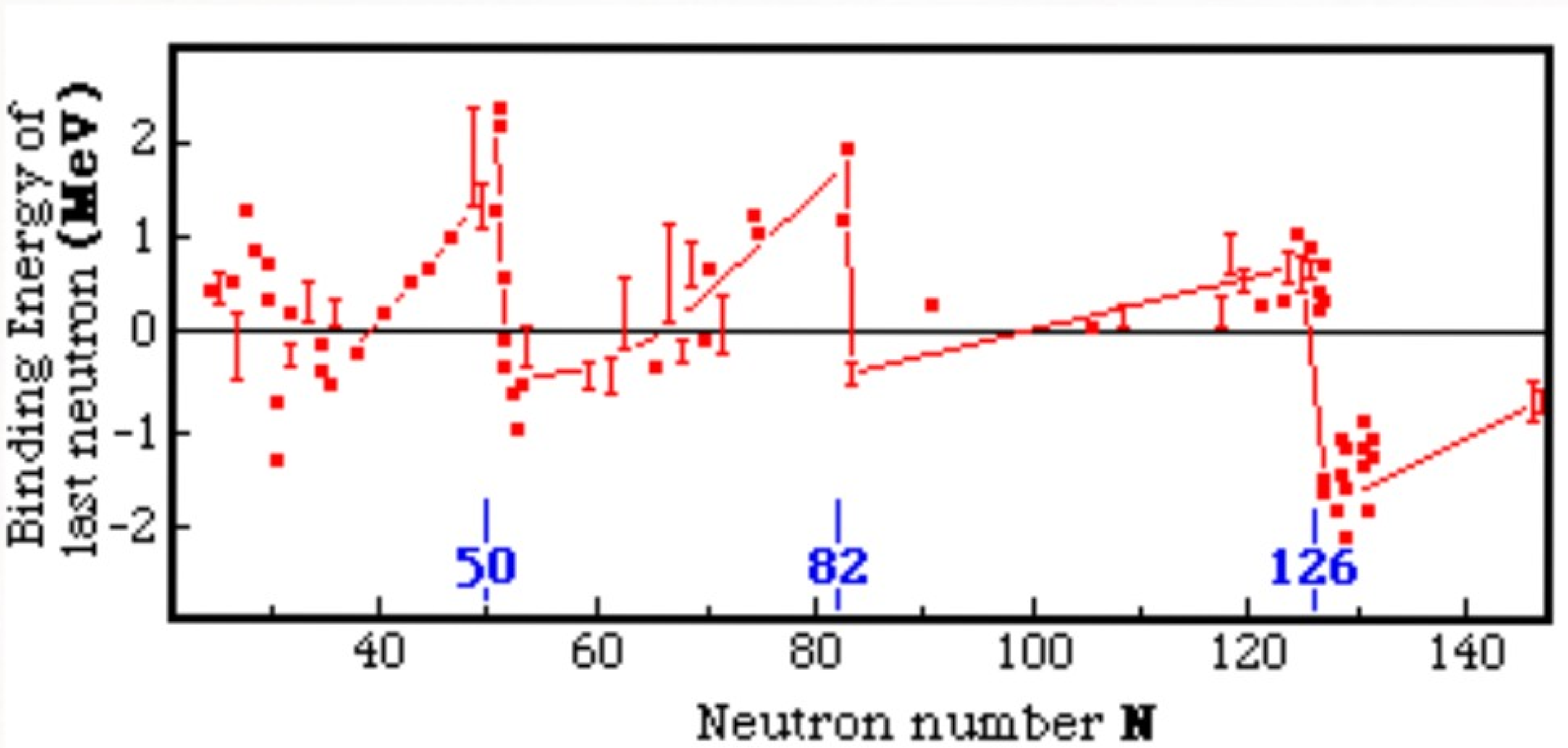
$$\boxed{{}^4He \to 2p + 2n}$$

p	n	
—	—	$5/2\,\omega$
$\uparrow\downarrow$	$\uparrow\downarrow$	$3/2\,\omega$

$$\boxed{{}^{16}O \to 8p / 8n}$$

p	n	
$\uparrow\downarrow\ \uparrow\downarrow\ \uparrow\downarrow$	$\uparrow\downarrow\ \uparrow\downarrow\ \uparrow\downarrow$	$5/2\,\omega$
$\uparrow\downarrow$	$\uparrow\downarrow$	$3/2\,\omega$

First observation → filling shells explain the magic number



N, Z = 2, 8, 20, 28, 50, 82, 126 ....

(magic number)

$$V^{MF} = \frac{1}{2} m_N \omega^2 r^2 \longrightarrow \exists \text{ magic numbers}$$

0) GROUND STATE $\longrightarrow$ 2 ✓

1) FIRST EXCITED $\longrightarrow$ $2+6=8$ ✓

2) SECOND EXCITED $\longrightarrow$ $2+6+12=20$ ✓

and we can continue ...

$$V_{HO} = \frac{1}{2} m_N (\omega r)^2$$

$$\Rightarrow \quad N, Z = \boxed{2, 8, 20}, 40, 70, \ldots$$

$$N, Z = \boxed{2, 8, 20}, 28, 50, 82, \ldots$$

→ Right direction, but something is missing

MISSING PIECE $\rightarrow$ $\boxed{\text{SPIN-ORBIT FORCE}}$

$$V^{MF} = \frac{1}{2} m_N \omega^2 r^2 - \zeta \vec{\ell} \cdot \vec{s}$$

$$(\zeta > 0)$$

this is what explains the other magic numbers

SHELL MODEL /w H.O. $+ \xi \vec{\ell}\cdot\vec{s}$

$\frac{9}{2}\omega$ 2p, 1f ~~(20p/20n)~~ → ?

1d 3/2 (4p/4n)

2s 1/2 (2p/2n)

$\frac{7}{2}\omega$ 1d, 2s (12p/12n) . . .

1d 5/2 (6p/6n)

1p 1/2 (2p/2p)

$\frac{5}{2}\omega$ 1p $n=0, \ell=1$ (6p/6n) . . .

1p 3/2 (4p/4n)

$\frac{3}{2}\omega$ 1s $n=0, \ell=0$ (2p/2n) . . . 1s 1/2 (J = 1/2)

$(n+1)\ell_J$

Reminder $\rightarrow$ $\vec{\ell}\cdot\vec{s}$ splits states w/ same ℓ but different j

$$\vec{\ell}\cdot\vec{s} = \frac{1}{2}\{j(j+1) - \ell(\ell+1) - s(s+1)\}$$

SHELL MODEL W/ H.O $\quad + \zeta\, \vec{\ell}\cdot\vec{s}$

new shell

$20 + 8 = 28$

$8p / 8n$

$\frac{9}{2}\omega$ 2p $(n=1, \ell=1)$

1f $(n=0, \ell=3)$

$1f_{7/2}$ $\frac{9}{2}\omega - \frac{3}{2}\zeta$

$1d_{3/2}$ $\frac{7}{2}\omega + \frac{3}{2}\zeta$

$\frac{7}{2}\omega$ 1d, 2s

$2s_{1/2}$ $\frac{7}{2}\omega$

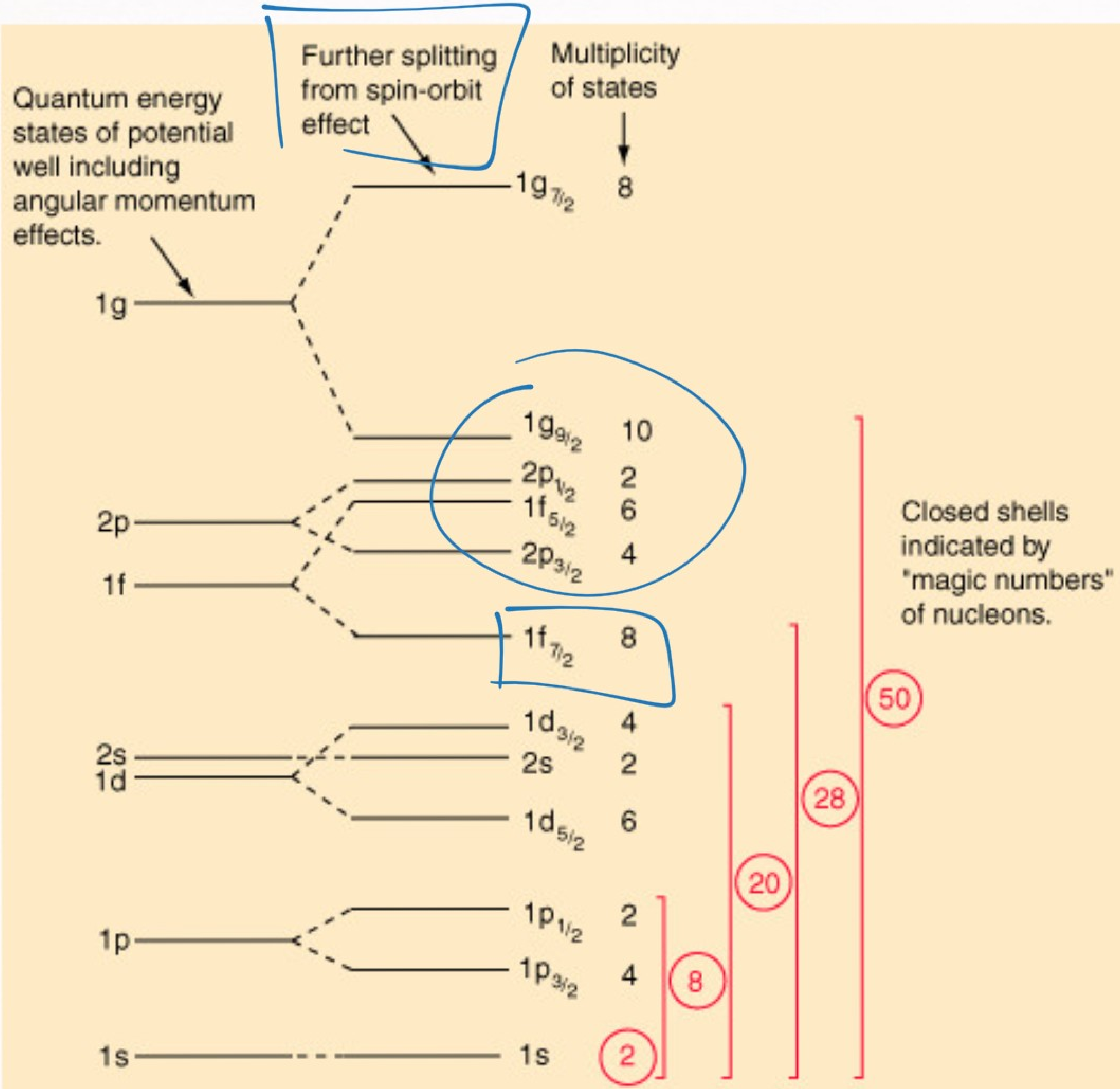
$1d_{5/2}$ $\frac{7}{2}\omega - \zeta$

1) $V_{MF} = \frac{1}{2} m \omega^2 r^2$

$$N, Z = \boxed{2, 8, 20}, 40, \cancel{70}, \ldots$$

2) $V_{MF} = \frac{1}{2} m \omega^2 r^2 - \xi\, \vec{\ell} \cdot \vec{s}$

$$N, Z = \boxed{2, 8, 20, 28, 50, 82, \ldots}$$



→ This is why spin-orbit is important

⇓

1) MAGIC NUMBERS

2) ORDER OF THE ENERGY LEVELS

→ [APPLICATIONS OF THE SHELL MODEL] ←

(full shells)

1) ^4He → $2p\,|\,2n$

$\vec{S}_{p1}+\vec{S}_{p2}=0$

$\vec{S}_{n1}+\vec{S}_{n2}=0$

∃ pairings

$1s_{1/2}$ ↑↓ protons

↑↓ neutrons $1s_{1/2}$

$J^P(^4He)=0^+$

CORRECT

From pairing → $\forall$ even-even nuclei have $J^P = 0^+$

(previous lesson)

2) J^P for nuclei that are just 1p/1n away from a magic number

$\Rightarrow$

2.a) $\boxed{+\ 1p/1n} \rightarrow \boxed{{}^{17}O,\ {}^{17}F}$

$^{17}O \rightarrow {}^{16}O + n$

$^{17}F \rightarrow {}^{16}O + p$

$\boxed{^{16}O} \rightarrow 8p + 8n$

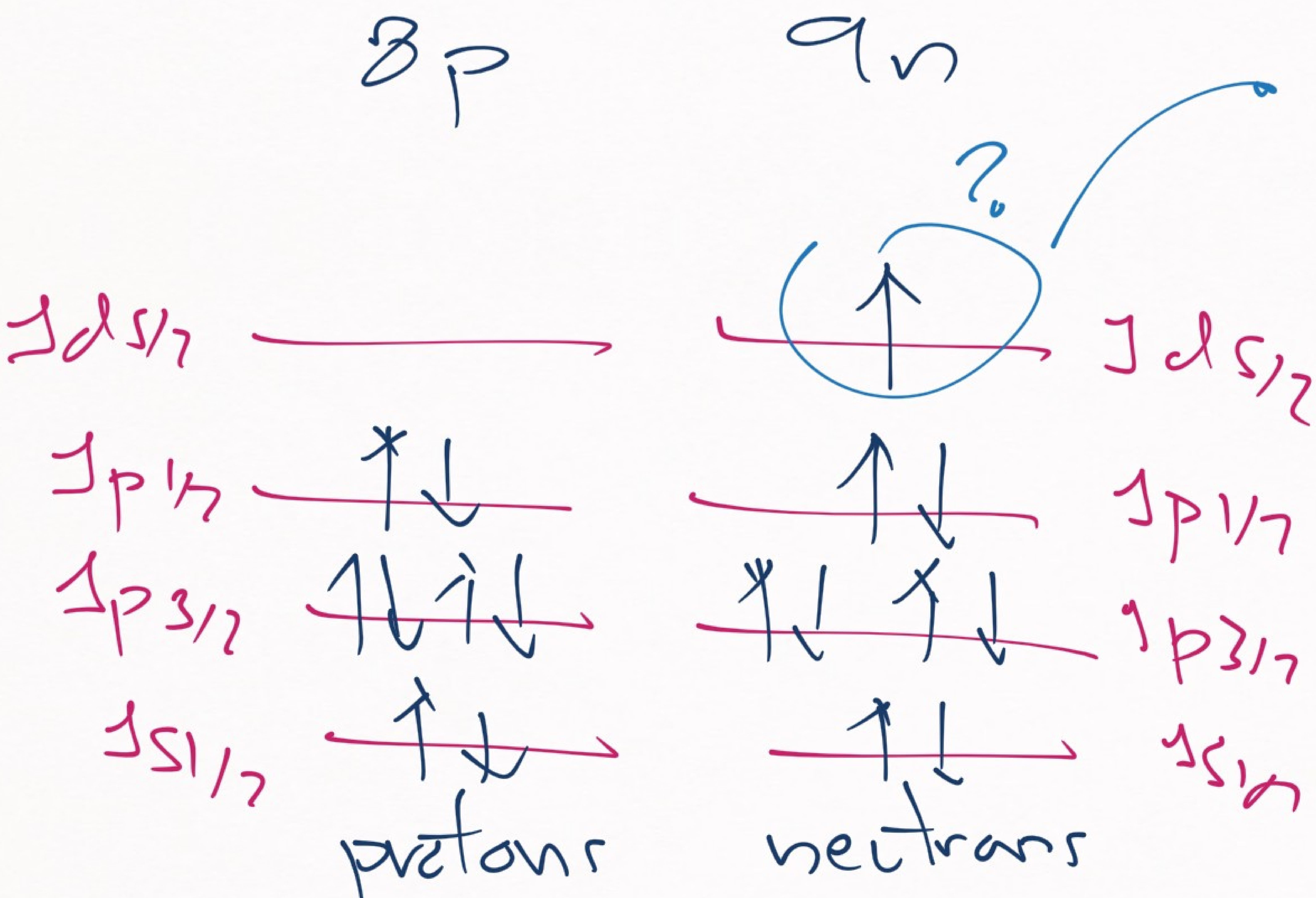
(doubly magical)

$J^P({}^{16}O) = 0$

$J^P({}^{17}O) \rightarrow J^P(n)$

$J^P({}^{17}F) \rightarrow J^P(p)$

$\boxed{^{17}O}$



$$J^P(^{17}O) = J^P(1d_{5/2})$$

$$= \frac{5}{2}^{+}$$

$5/2$

$l = 2$

$P = (-)^l$

$\boxed{{}^{17}F}$ → the same w/ protons ⟷ neutrons

$$\left.\begin{aligned} J^P({}^{17}O) &= \frac{5}{2}^+ \\ J^P({}^{17}F) &= \frac{5}{2}^+ \end{aligned}\right\}$$

this happens to be correct

2.b) we are 1p/1n away from a magic number

$\boxed{160}$ − 1p/1n → $\boxed{159\text{N} / 159\text{O}}$

What are their JP?

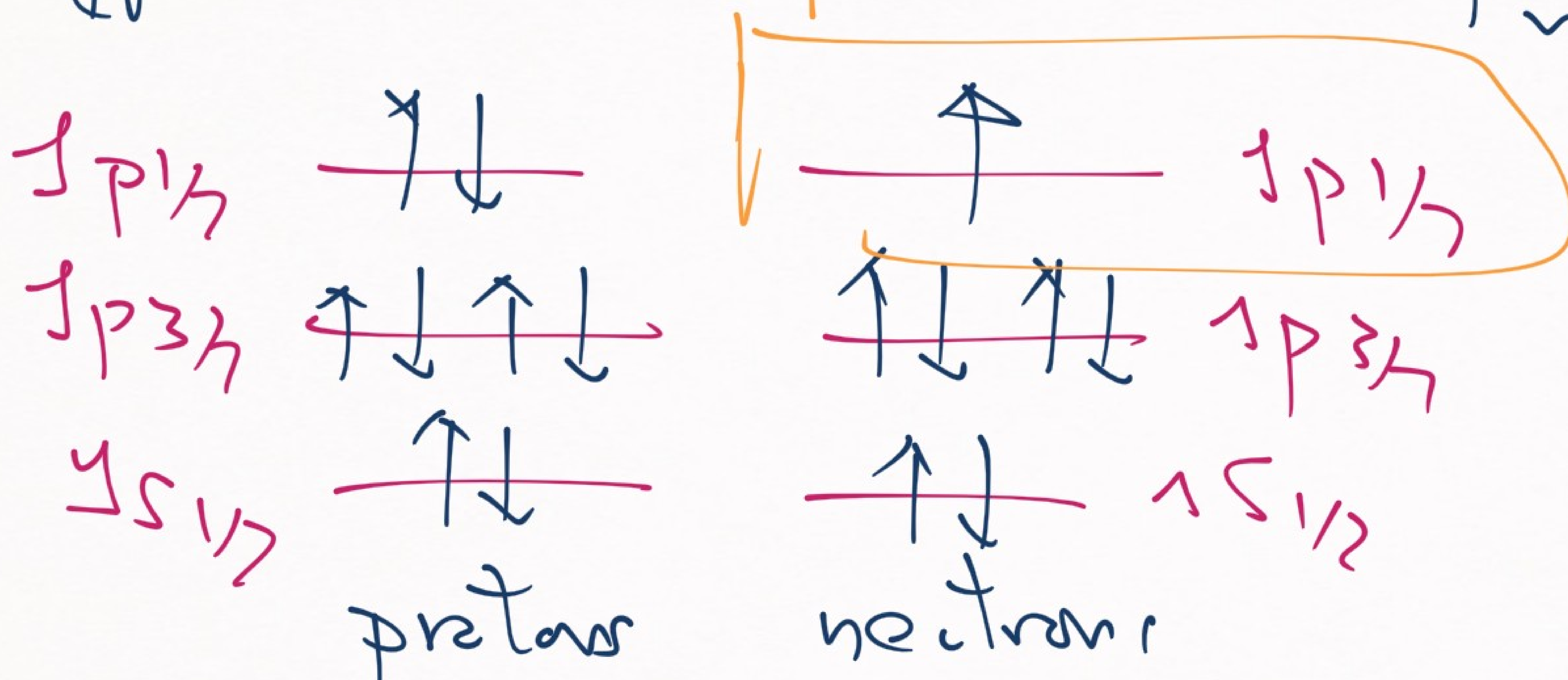
^{15}O → we fill shells

p-wave ($\ell = 1$)

unpaired neutron

J^P (protons) = 0

J^P (unpaired neutron) = $\frac{1}{2}^{-}$



Prediction of the shell model

$$J^P(^{15}O) = J^P(^{15}N) = \frac{1}{2}^-$$

$\rightarrow$ this is correct

3) EXCITED STATES IN SHELL MODEL

$^{41}Ca \rightarrow {}^{40}Ca + 1n$ (like ^{17}O
(20p, 20n) $= {}^{16}O + 1n$)

Experiment

Excited $\left\{ \begin{array}{l} \text{———— } 3/2^{+} \\ \text{———— } 3/2^{-} \end{array} \right.$

Ground ———— $7/2^{-}$

Can we explain this?

GROUND STATE

$1f_{5/2}$

$2p_{3/2}$

$1f_{7/2}$

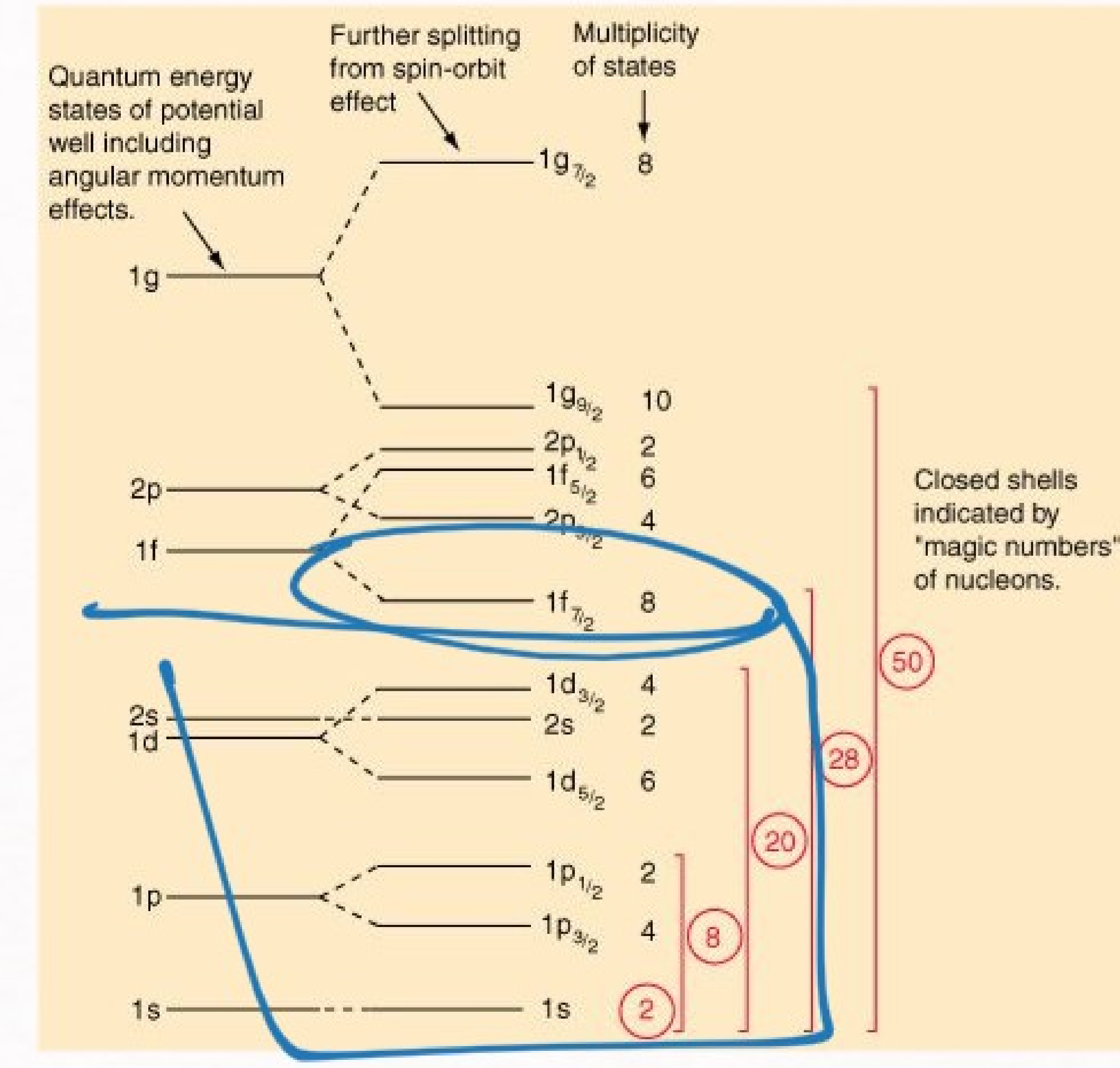
$1d_{3/2}$

neutrons

CORRECT

$$J^P({}^{41}Ca) = \frac{7}{2}^{(-)}$$

f-wave: $\ell = 3$



21 neutrons (only the last few shell are shown)

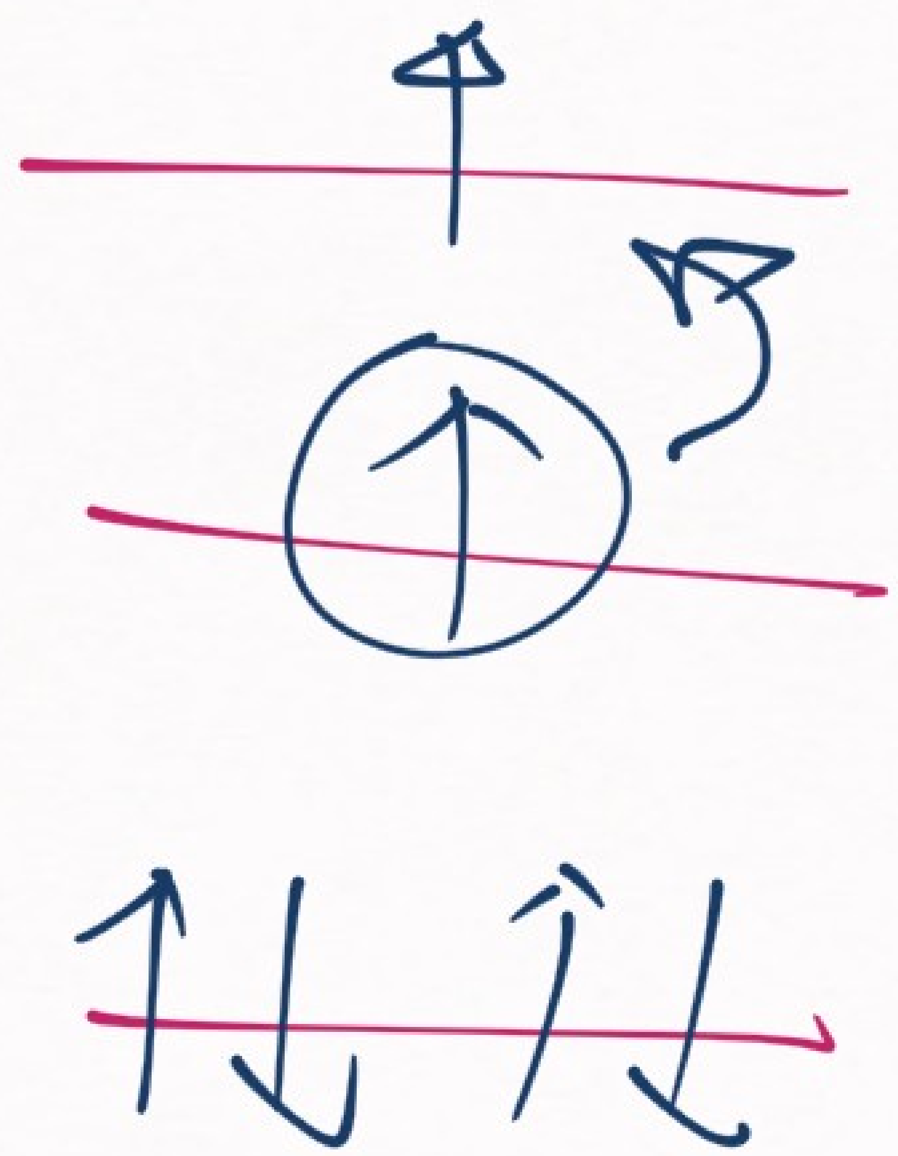
FIRST EXCITED STATE

$J^P_{exp} = \frac{3}{2}^-$

$2p_{3/2}$ → excited

$1p_{1/2}$ → Ground

$1p_{3/2}$



neutrons

$$\boxed{J^P(^{41}Ca^*) = \frac{3}{2}^-}$$

CORRECT

SECOND EXCITED: $J^P({}^{41}Ca^{**}) = \frac{3}{2}^+$

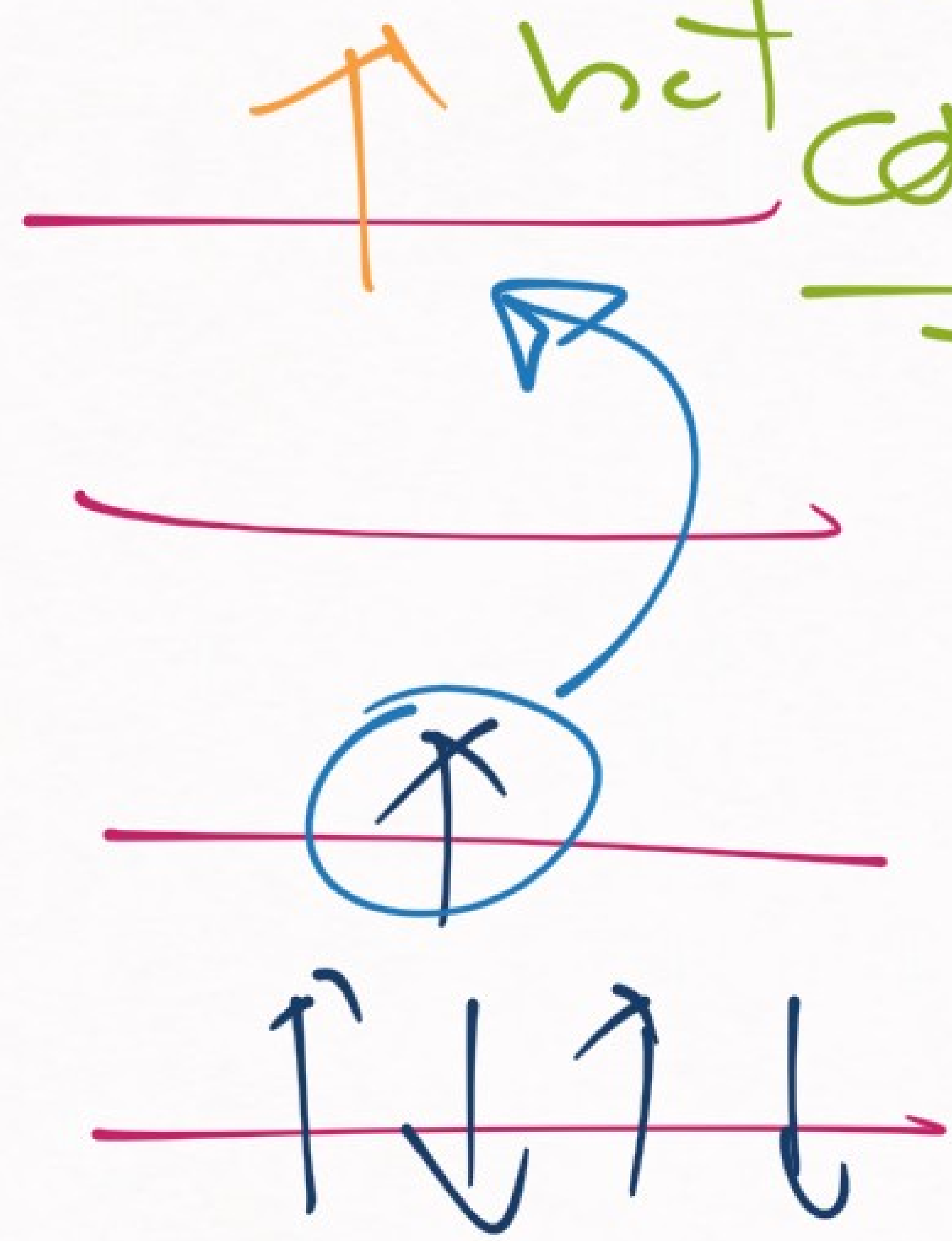
(A) $J^P = \frac{5}{2}^-$ not correct

$1f_{5/2}$

$2p_{3/2}$

$1f_{7/2}$

$1d_{3/2}$



(neutrons)

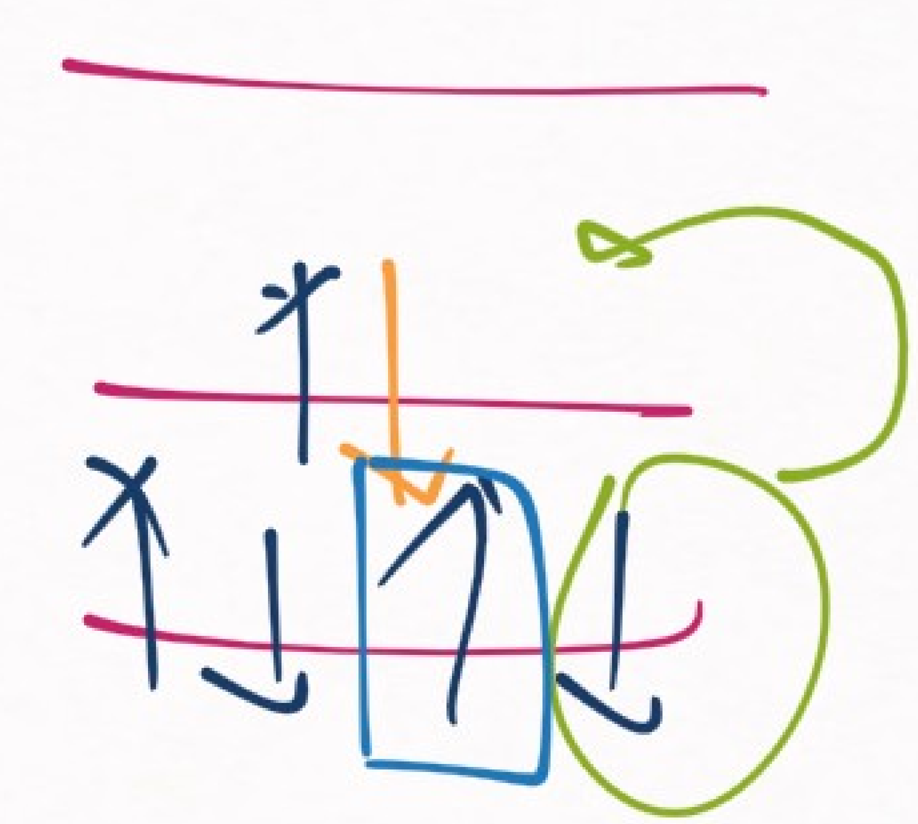
(B) $J^P = \frac{3}{2}^+$ correct

$1f_{5/2}$

$2p_{3/2}$

$1f_{7/2}$

$1d_{3/2}$



(neutrons)

^{41}Ca second excited

→ two ways of building it

(A) $J^P = 5/2^-$

(B) $\boxed{J^P = 3/2^+}$ → chosen by nature

3.b) $\boxed{{}^{38}Ar}$

Exp:

———— 2^+ 1st excited

———— 0^+ ground

this happens in almost all even-even nuclei

$$\left.\begin{array}{l} J^P(\text{even-even}) = 0^+ \\ J^P(\text{even-even}^*) = 2^+ \end{array}\right\}$$ can we understand it?

GROUND STATE $^{38}A = 18p \mid 20n$

$$\boxed{J^P(^{38}Ar) = 0^+}$$

CORRECT

$1f_{7/2}$

$1d_{3/2}$ $\uparrow\downarrow$

$2s_{1/2}$ $\uparrow\downarrow$

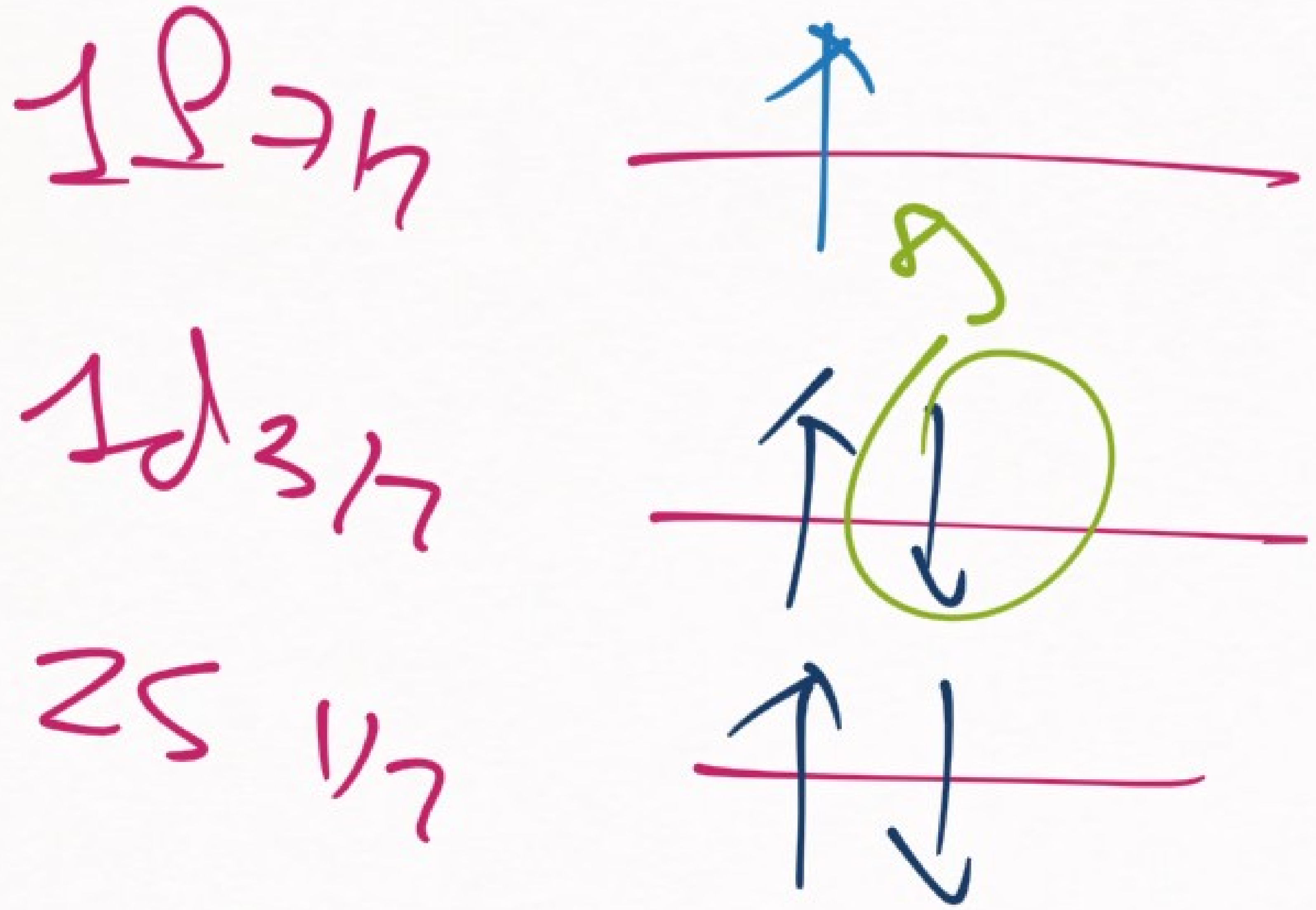
protons

magical (no need to worry → super stable!)

We have to check the protons

EXCITED STATE: (2+)

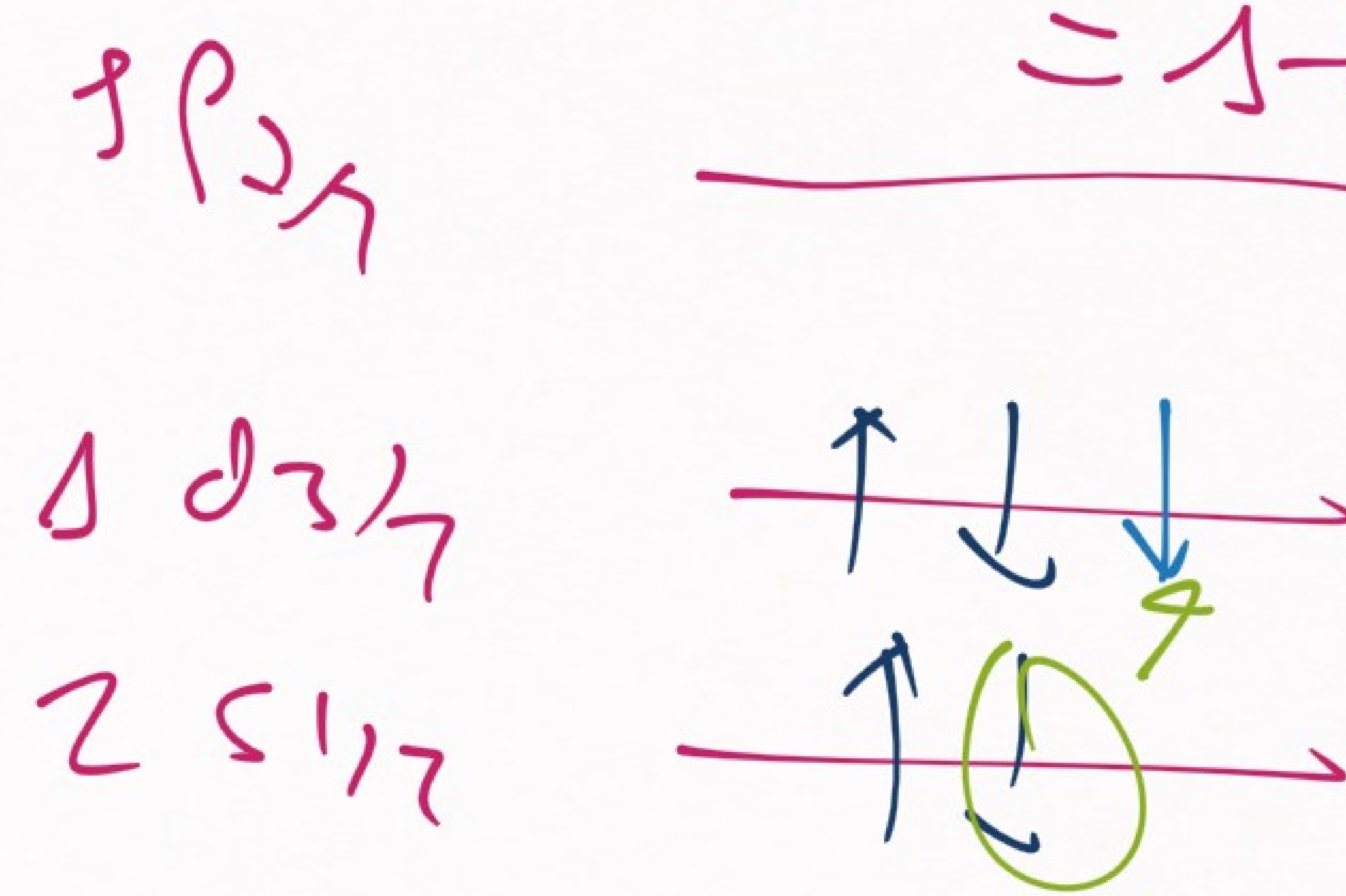
(A) → does not happen



$J^P_A = \left(\frac{3}{2} \otimes \frac{7}{2}\right)^- = 2^-, 3^-, 4^-, 5^-$

(B)

$J^P_B = \left(\frac{1}{2} \otimes \frac{3}{2}\right)^+ = 1^+, 2^+$



Why nature chooses B here?

→ [Residual interactions in the shell model]

$$H = \sum_i H_i + \Delta V$$

Residual interactions

The easiest residual interaction is

$\boxed{\text{PAIRING INTERACTION}}$

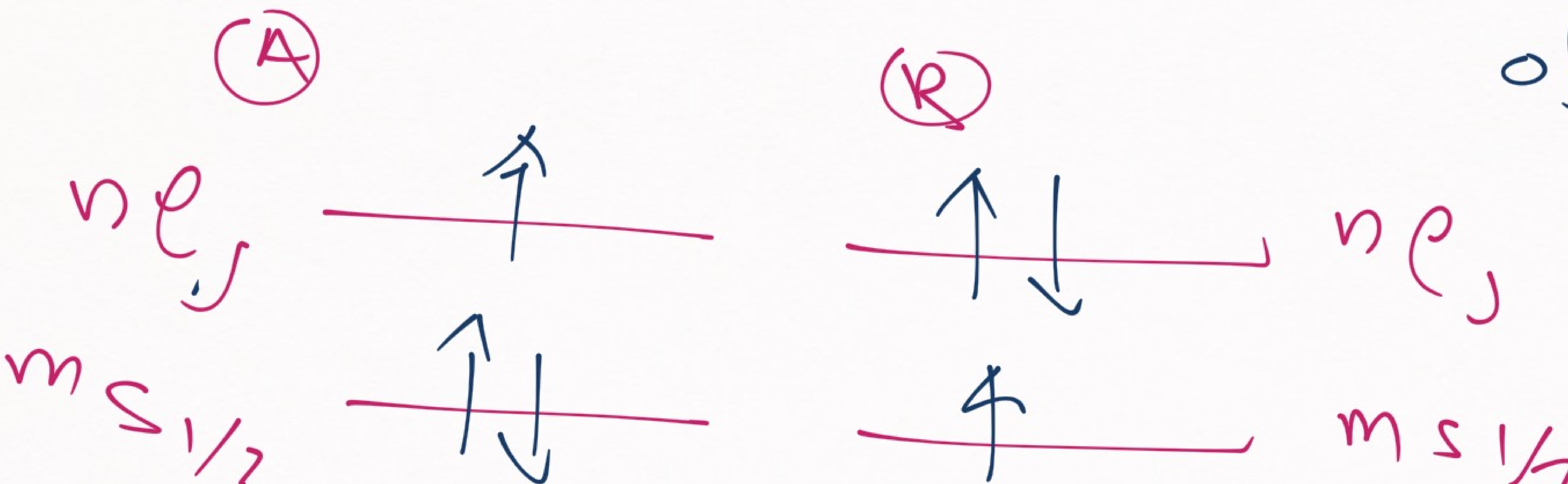
$$\langle jj(JM)|V_{pairing}|jj(JM)\rangle =$$

$$-\underline{\tfrac{1}{2}g(j+1)}\,\underline{\underline{\delta_{J0}}}\,\delta_{M0}$$

strength increases w/ j

0^+ is preferred

For some nuclei, pairing is a larger effect than the usual ordering of shells



If j large $\Rightarrow$ $\boxed{E_B < E_A}$ $\Rightarrow$ $\boxed{\text{B wins}}$

EXAMPLES → NUCLEI CLOSE TO SHELL WITH HIGH J

$J^P({}^{203}Tl) \to \underline{\underline{J^P(p)}}$

${}^{203}Tl$, ${}^{205}Tl$

$(\boxed{81p}, \underline{\underline{122n}})$ $(\boxed{81p}, \underline{\underline{124n}})$

$J^P = 0^+$ $J^P = 0^+$

A $\quad$ 10 proton $\qquad$ B $\quad$ 12 proton

$1h_{11/2}$ $\qquad\qquad$ $1h_{11/2}$

$3s_{1/2}$ $\qquad\qquad$ $3s_{1/2}$

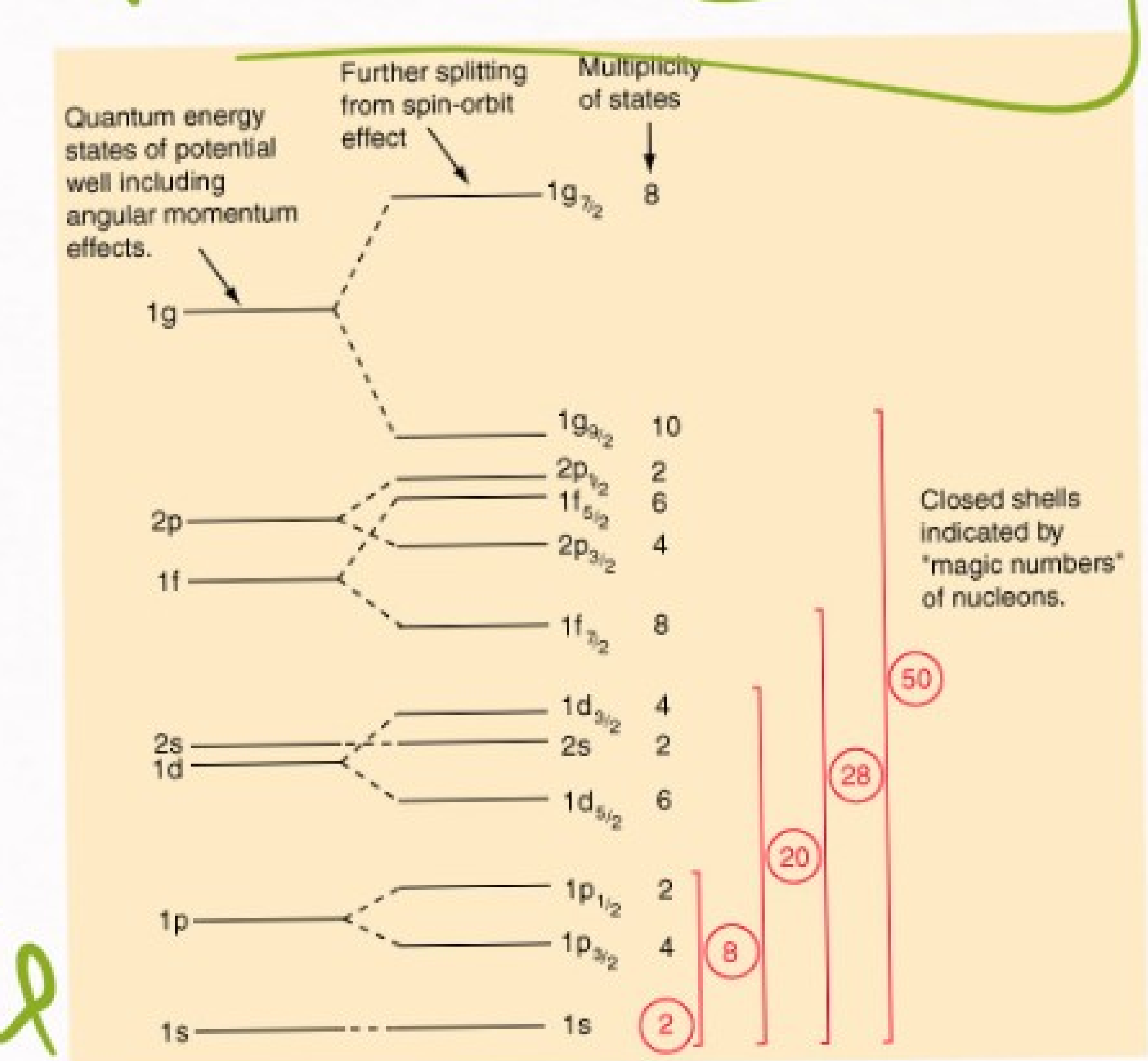
protons

without pairing $\rightarrow$ $E_A < E_B$

with pairing $\rightarrow$ $E_A > \boxed{E_B}$

ground state

$\boxed{J^P(^{203}Tl) = 1/2^+}$



^{203}Tl

^{205}Tl

$\to$

(A) no pairing $\to J^P = \frac{11}{2}^-$

(B) pairing $\to \boxed{J^P = \frac{1}{2}^+}$

experimentally observed

More complicated example

$^{207}Pb \rightarrow 82p, 125n$

$J^P_A = 13/2^+$ full

JP comes from here

(A) 12n

(B) 14n

$1i_{13/2}$

$i_{13/2}$

unpaired

$3p_{1/2}$

$3p_{1/2}$

neutrons

$\boxed{J^P_B = 1/2^-}$

$^{207}\underline{Pb} \longrightarrow \begin{cases} \text{no pairing} & J^P = \frac{13}{2}^+ \\ \text{pairing} & \boxed{J^P = \frac{1}{2}^-} \end{cases}$

this is what happens

This shows the importance of pairing interactions

Shell model ✓

Next Person → COLLECTIVE MODEL

TILL FRIDAY