

NUCLEAR PHYSICS (16)

Tensor forces



One boson exchange model



RECAP

Previous lesson \rightarrow T-matrix (Review)

$$(*) \quad f(\infty) = -\frac{M}{2\pi} \langle \vec{k}' | T(E) | \vec{k} \rangle, \quad \frac{d\sigma}{d\Omega} = |f(\infty)|^2$$

$$T(E) = V + V G_0(E) T(E)$$

Lippmann
- Schwinger
equation

⊛ T-matrix is easy to solve

for a particular type of potentials

(separable potentials)

$$\langle \vec{p}' | V | \vec{p} \rangle = \lambda g(\vec{p}) g(\vec{p}')$$

(definition of
a separable
potential)

$$T = V + V G_0 T$$

$$\langle \vec{p}' | T(E) | \vec{p} \rangle = Z(E) g(\vec{p}) g(\vec{p}')$$

$$Z(E) = \frac{1}{1 - J(E)}$$

$$J(E) = \int \frac{d^3 \vec{q}}{(2\pi)^3} \frac{g^2(\vec{q})}{E - \frac{q^2}{2\mu}}$$

Lippmann-Schwinger
equation
(for T-matrix)



Separable
potential



Simple
T-matrix

→ This is also useful

for contact-range theories

universal low-energy theories

Contact-range

$$\langle \vec{p}' | V | \vec{p} \rangle$$

$$= C_0(\Lambda) \overbrace{f\left(\frac{p'}{\Lambda}\right) f\left(\frac{p}{\Lambda}\right)}^{f(x)}$$

$$\rightarrow \left\{ \begin{array}{l} f(x) \rightarrow 1, x \rightarrow 0 \\ \hline f(x) \rightarrow 0, x \rightarrow \infty \end{array} \right. |$$

Kill the unwanted
high-energy modes



\forall depends on Λ

$$C_0(\Lambda) \rightarrow \frac{1}{\Lambda^2}$$

$$\left[\frac{d}{d\Lambda} \frac{1}{\Lambda^2} \right]$$

Example $\rightarrow \left. \begin{array}{l} \sigma \rightarrow 4\pi |a_0|^2 \\ E \rightarrow 0 \end{array} \right\} \tau(\epsilon) + \frac{d}{dN} \tau(\epsilon) = 0$

$$\Rightarrow \frac{1}{C_0(N)} = \frac{M}{2\pi} \left(\frac{1}{a_0} - \beta N \right)$$

Family of solutions that we will get

if $\rho(x) = \theta(1-x) \Rightarrow \beta = \frac{1}{N}$
 (sharp cutoff)

⊛ Bound states w/ T-matrix

$$T(E) \xrightarrow{E \rightarrow E_B} \frac{\text{Res } T(E_B)}{E - E_B}, \quad \text{Res } T = \langle B | V | B \rangle \langle B | V | B \rangle^{-1} \\ = G_0^{-1}(E_B) | B \rangle \langle B | G_0^{-1}(E_B)$$

$$\langle \vec{p}' | V | \vec{p} \rangle = \int d^3x \psi(\vec{x}) \psi(\vec{x}) \quad \text{if } \Lambda \rightarrow \infty \quad \text{if } a_0 \text{ fixed}$$

$$\psi_B(\vec{p}) = \langle \vec{p} | B \rangle = \frac{\sqrt{8\pi\gamma}}{p^2 + \gamma}, \quad \gamma = \frac{1}{a_0}$$

→ These were the most important points to remember about the T-matrix ↓

TENSOR FORCE

What is it?

3) Yukawa's idea to explain nuclear forces

$$V_Y(\vec{r}) = -g_Y^2 \frac{e^{-m_\pi r}}{4\pi r} \quad \left| \begin{array}{c} \text{meson} \\ \text{---} \end{array} \right| \quad (J^P = 0^+)$$

But \Rightarrow a problem \rightarrow deuteron quadrupole moment

Deuteron quadrupole moment $\rightarrow Q_d \approx 0.28 \text{ fm}^2 e$

$$Q_d \neq 0$$

\Rightarrow Deuteron contains

a D-wave component

$\Psi_d / S\text{-wave } (\Psi(\vec{r}) = \psi(r))$

$$\langle \psi | Q_d | \psi \rangle = 0$$

$$\left(\langle z^2 \rangle = \frac{1}{3} \langle r^2 \rangle \right)$$

$$Q_d \propto (3z^2 - r^2) \Rightarrow$$

\Rightarrow D-wave component

2) Usual explanation is a $\vec{J} \cdot \vec{\sigma}$ meson exchange

$OPE \rightarrow$ One pion exchange (pion)

$$V_{OPE}(\vec{q}) = - \frac{g_{\Delta}^2}{4f_{\pi}^2} \frac{1}{2, 2} \frac{1}{2, 2}$$

$$\frac{\vec{\sigma}_1 \cdot \vec{q} \vec{\sigma}_2 \cdot \vec{q}}{q^2 + m_{\pi}^2} \otimes (\sigma) \otimes (\sigma)$$

$$[g_{\Delta} \approx 1.26, f_{\pi} \approx 92.4 \text{ MeV}, m_{\pi} \approx 138 \text{ MeV}]$$

$$\textcircled{1} \Rightarrow \vec{g}_1 \cdot \vec{g}_2 = \sigma_{1i} \sigma_{2j} g_i g_j = \underline{\underline{g \cdot g}}$$

$$\textcircled{2} = \sigma_{1i} \sigma_{2j} \left[\frac{g^2}{3} \delta_{ij} \right] \textcircled{3}$$

$$1 \otimes 1 = 0_S \oplus 1_A \oplus 2_S \quad \textcircled{4}$$

$$g_i g_j = g_j g_i \rightarrow S$$

$$+ 2_S \left[(g_i g_j - \frac{1}{3} g^2 \delta_{ij}) \right] = \textcircled{5}$$

$$\textcircled{5} = \frac{1}{3} \vec{\sigma}_1 \cdot \vec{\sigma}_2 g^2 + \left(3 \vec{\sigma}_1 \cdot \frac{1}{4} \vec{\sigma}_2 \cdot \vec{g} - \frac{1}{3} \vec{\sigma}_1 \cdot \vec{\sigma}_2 g^2 \right)$$

$$\vec{\sigma}_1 \cdot \vec{\sigma}_2 = \frac{1}{3} \vec{\sigma}_1 \cdot \vec{\sigma}_2 \mathbb{I} + (\vec{\sigma}_1 \cdot \vec{e}_z \vec{\sigma}_2 \cdot \vec{e}_z - \frac{1}{3} \vec{\sigma}_1 \cdot \vec{\sigma}_2 \mathbb{I})$$

spin-spin
operation
(L=0)

tensor operator
(L=2)



Decoupling will have
an L=2 component

[configuration space] (r-space)

$$V_{OPE}(\vec{r}) = \frac{g_A^2 m_\pi^3}{48\pi f_\pi^2} \vec{r}_1 \cdot \vec{r}_2 \left[\overset{\textcircled{1}}{\delta_1 \cdot \delta_2} W_C(r) \right]$$

$$\textcircled{2} \left[\underline{S_{12}(\hat{v})} W_T(r) \right]$$

$$[S_{12}(\hat{v}) = 3 \hat{\sigma}_1 \cdot \hat{r} \hat{\sigma}_2 \cdot \hat{r} - \delta_1 \cdot \delta_2]$$

usual definition
of
tensor
operator

① (Spin-spin) $\vec{\sigma}_1 \cdot \vec{\sigma}_2 = \begin{cases} 1 & S=1 \\ -3 & S=0 \end{cases}$

② (tensor) $S_{12}(\vec{r})$ a) $[S_{12}, \vec{L}^2] \neq 0$

a) $\Rightarrow S_{12}$ mixes different values of L

$[S_{12}, \vec{S}^2] = 0$ (*)

b/c) $\Rightarrow S_{12}$ conserves S, J

$[S_{12}, \vec{J}^2] = 0$

(*) only happens if $|S| < 2$ Always happens in NN

$$S_{12}(\hat{r}) \rightarrow \left[\begin{array}{l} \text{matrix in } e \\ \text{scalar in } s_{ij} \end{array} \right]$$

→ (when computing its matrix elements)

$$\langle (s'e')_{j'm'} | S_{12}(\hat{r}) | (se)_{jm} \rangle = \sum_{e'e'} \sum_{j'j} \sum_{m'm'} \delta_{jj'} \delta_{m'm'} \delta_{ss'}$$

↓
we can drop this, because $s=1$ for $\langle S_{12} \rangle \neq 0$

Additional simplification:

$$\underline{S=0} \Rightarrow \langle S_{12} \rangle = 0 \quad \text{Why?}$$

$$S_{12}(r) = 3 \vec{\sigma}_1 \cdot \hat{r} \vec{\sigma}_2 \cdot \hat{r} - \vec{\sigma}_1 \cdot \vec{\sigma}_2$$

if we exchange spins: $S_{12} \rightarrow S_{12}$

$$S=0 \rightarrow |00\rangle = \frac{1}{\sqrt{2}} (|+-\rangle - |-+\rangle) \quad (|00\rangle \rightarrow -|00\rangle)$$

$$S_{12} |00\rangle = -S_{12} |00\rangle \Rightarrow S_{12} |00\rangle = 0$$

(this explanation is a bit wacky)

$$S=1 \Rightarrow \langle S_{12} \rangle \neq 0 \quad (L \neq L' \neq 0)$$

$$L=0$$

$$L'=0$$

$$\int d^2r \left(\underbrace{3\sigma_1 \cdot r \sigma_2 \cdot r}_{\sigma_1 \cdot \sigma_2} - \sigma_1 \cdot \sigma_2 \right) = 0$$

$$\langle s=0 | S_{12} | s=0 \rangle = 0$$

$$\langle p=0 | S_{12} | p=0 \rangle = 0$$

How to calculate $S_{12}(\hat{r})$?

$[S_{12}(\hat{r}), \vec{J}] = 0 \rightarrow$ Define states w/

good J

$|(\ell) j m\rangle$

w/o tensor force \rightarrow

$$|l m_e\rangle |s m_s\rangle = \underbrace{\sum Y_{lm}(\hat{r})}_{\text{tensor force}} |s m_s\rangle$$

w/ tensor force $\rightarrow |(\ell s) j m\rangle$

$$|(\ell s) j m\rangle = \sum_{\substack{s m_s \\ l m_l}} \underbrace{\sum Y_{lm}(\hat{r})}_{\text{tensor force}} |s m_s\rangle$$

$$\times \langle l m_l s m_s | j m \rangle$$

Clebsch-Gordan
coefficient

(new part)

$$\mathcal{L}(s'e^{j\omega t} | S_{12}(s) | (sl)_{j\omega})$$

$$= S_{ee} \delta_{ss} \delta_{\omega} \delta_{nm}$$



assumes $s < 2$



(which is true for
the NN system)

How does this look for the deuteron?

wo/tensor force

$$\psi_d(\vec{r}) = \frac{u(r)}{r} \sum_{lm} c_{lm}(\hat{r}) |1\ m\rangle$$

$$|1\ 1\rangle = |+-\rangle$$

$$|1\ 0\rangle = \frac{1}{\sqrt{2}} (|-+\rangle + |-+\rangle)$$

$$|1\ -1\rangle = |--\rangle$$

(very simple
wave function)

w/ tensor force

$$\underbrace{|5m_s\rangle \langle 5m_s 00|}_{= |5m_s\rangle} |1m_d\rangle$$

$$\psi_d(\vec{r}) = \frac{U(r)}{r} \sum_{00} Y_{00}(\hat{r}) |1m_d\rangle$$

$$+ \left[\frac{W(r)}{r} \sum_{m_s, m_d} Y_{2m_s}(\hat{r}) |1m_s\rangle \right]$$

$$\langle 2m_1 1m_2 | 1m_d \rangle$$

D-wave component

$$\psi_D(\vec{r}) = \frac{u(r)}{r} |S(\downarrow m_d)\rangle + \frac{w(r)}{r} |D(\downarrow m_d)\rangle$$

$$\Rightarrow \langle S_{12}(\hat{r}) \rangle = \begin{pmatrix} \langle S | S_{12} | S \rangle & \langle S | S_{12} | D \rangle \\ \langle D | S_{12} | S \rangle & \langle D | S_{12} | D \rangle \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 2\sqrt{2} \\ 2\sqrt{2} & -2 \end{pmatrix}$$

For the detector:

$$S_{12} = \begin{pmatrix} 0 & 2\sqrt{2} \\ 2\sqrt{2} & -2 \end{pmatrix}$$

Knowing this, we can write a Schrödinger equation for $u(r)$, $w(r)$

wave tensor : $V(r) = V_c(r)$ (might include spin-spin)

$$\left[-u''(r) + 2\mu V(r)u(r) = -\gamma^2 u(r) \right]$$

it was a very simple differential equation

w/tensor

$$V(\vec{r}) = V_C(r) + \Sigma_{12}(\hat{r}) V_T(r)$$

$$-u''(r) + 2\mu[V_C(r)u + 2\sqrt{2}V_T(r)w] = -\gamma^2 u(r)$$

$$-w''(r) + \frac{6}{r^2}w(r) +$$

$$\begin{pmatrix} 0 & 2\sqrt{2} \\ 2\sqrt{2} & -2 \end{pmatrix}$$

$$2\mu[2\sqrt{2}V_T(r)u + (V_C - 2V_T)w(r)] = -\gamma^2 w(r)$$

More compact in matrix form.

$$-\begin{pmatrix} u \\ w \end{pmatrix}'' + 2\mu \begin{pmatrix} V_c & 2\sqrt{2}V_{-1} \\ 2\sqrt{2}V_{+1} & (V_c - 2V_{+1}) \end{pmatrix} \begin{pmatrix} u \\ w \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & 0 \\ 0 & \cancel{0} / r^2 \end{pmatrix} \begin{pmatrix} u \\ w \end{pmatrix} = -\gamma^2 \begin{pmatrix} u \\ w \end{pmatrix}$$



$\frac{l(l+1)}{r^2} \rightarrow$ centrifugal barrier

this also changes the asymptotic behavior

$$v(r) \rightarrow A_S e^{-\gamma r} \quad \left. \vphantom{v(r)} \right\} \rightarrow \text{dominant comp.}$$

$$w(r) \rightarrow A_D e^{-\gamma r} \left(1 + \frac{3}{\gamma r} + \frac{3}{(\gamma r)^2} \right)$$

$$r \rightarrow \infty$$

$$\frac{A_D}{A_S} \approx \underline{\underline{0.0256}}$$

Normalization \rightarrow

$$\int_0^{\infty} u^2(r) dr = 1 \quad \Rightarrow \int_0^{\infty} (u^2(r) + w^2(r)) dr = 1$$

Observables \rightarrow

$$\langle r_m^2 \rangle = \frac{1}{4} \langle r^2 \rangle$$

$$= \frac{1}{4} \int_0^{\infty} r^2 u^2(r) dr$$

$$\Rightarrow \langle r_m^2 \rangle = \frac{1}{4} \times$$

$$\int_0^{\infty} r^2 (u^2(r) + w^2(r)) dr$$

Quadrupole moment:

$$\langle Q_d \rangle = 0 \quad \text{if} \quad \langle Q_d \rangle = \frac{1}{20} \int_0^s r^2 w(r) \times (2\sqrt{2}U(r) - W(r)) dr$$

IF $w(1) = 0 \Rightarrow Q_d = 0$

D-wave probability:

$$P_D = \int_0^{\infty} W^2(r) dr$$

$$P_D \sim (3-5)\%$$

(only an observable until we include quantum corrections to e.w. observables)



Review of deuterium observables;

$$r_m = 1.9754(9) \text{ fm}$$

$$Q_d = 0.2859(3) \text{ fm}$$

$$P_D \sim (3.5)\%$$

$$\eta = \frac{\Delta_D}{AS} = 0.0256(4)$$

\equiv

[TENSOR FORCE]

- more complicated deuteron
(now it contains a D-wave)
- also changes scattering
amplitudes & phase shifts
(really complicated)

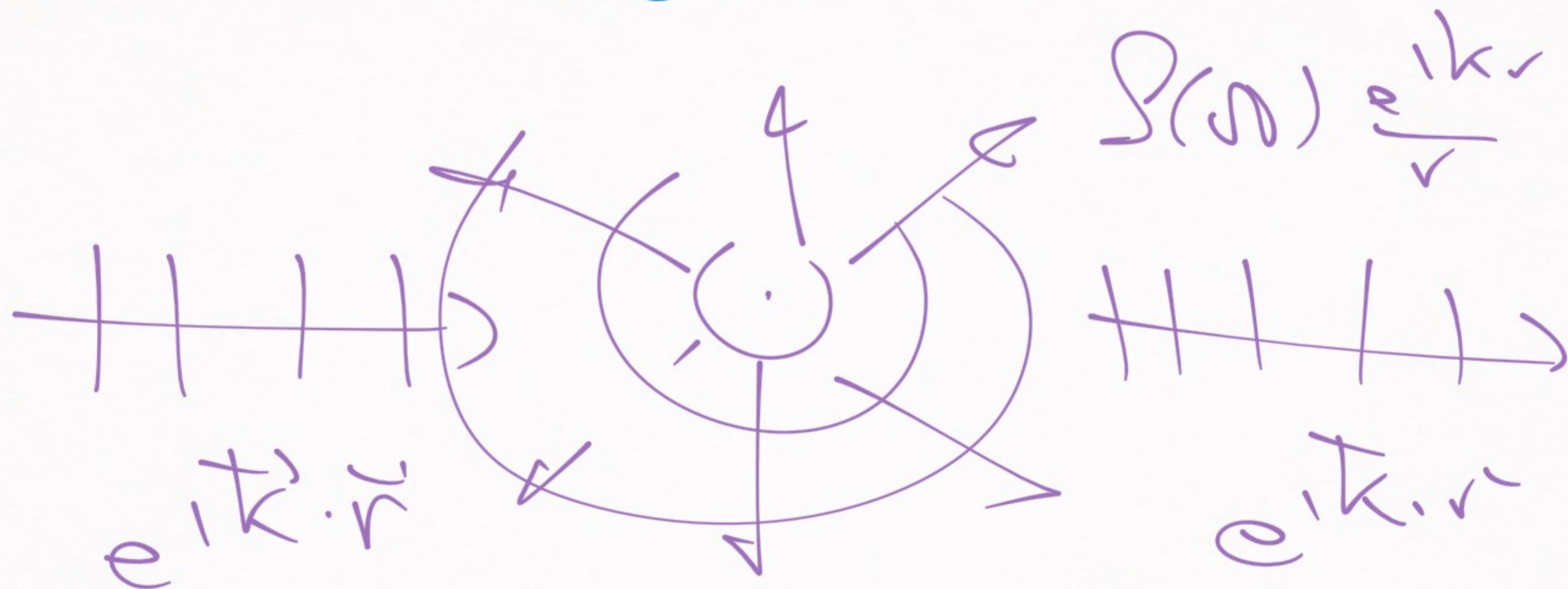
Tensor Force + Scattering

→ very simplified overview



Effect of including spin in scattering

w/o spin $\rightarrow \psi(\vec{r}) \xrightarrow{r \rightarrow \infty} e^{i\vec{k} \cdot \vec{r}} + f(\theta) \frac{e^{ikr}}{r}$



w/spin

$$\psi(\vec{r}) \rightarrow e^{i\vec{k}\cdot\vec{r}} |s m_s\rangle + \int m_s' m_s(\theta) \frac{e^{i\vec{k}\cdot\vec{r}}}{r} |s m_s'\rangle$$

(assumes $s' = s$)



w/o spin \rightarrow $P(N) = \sum_e (2e+1) \rho_e P_e(\cos\theta)$

$P_e(x) = \frac{e^{2i} \rho_{e-1}}{2ik}$ phase shift

w/spin \rightarrow $\int_{msms'}(N) = \sum_{je} \underbrace{(\dots)}_{m_j, m_j'} \rho_{je}^j$

$$f_{es}^j(\omega) = \frac{e^{z_l \sum_{s=0}^j(\omega)} - 1}{z_l \omega}$$

↳

↳

this assumes $j'=j$, $e'=l$, $s'=s$

✗

If I break any of these assumptions

⇒ f will become a matrix

Tensor Parca \rightarrow $e \neq e'$

$$f_{es}(k) = \frac{e^{2i\delta_{es}^0(k)} - 1}{2ik} \rightarrow S_{ee} =$$

$$\frac{1}{2ik} (S_{ee}^0(k) - \delta_{ee})$$

\rightarrow ignore s because $s = s' = 1$

$$\int_{ee'}^J = \frac{1}{2ik} (S_{ee'}^J(k) - \delta_e e')$$

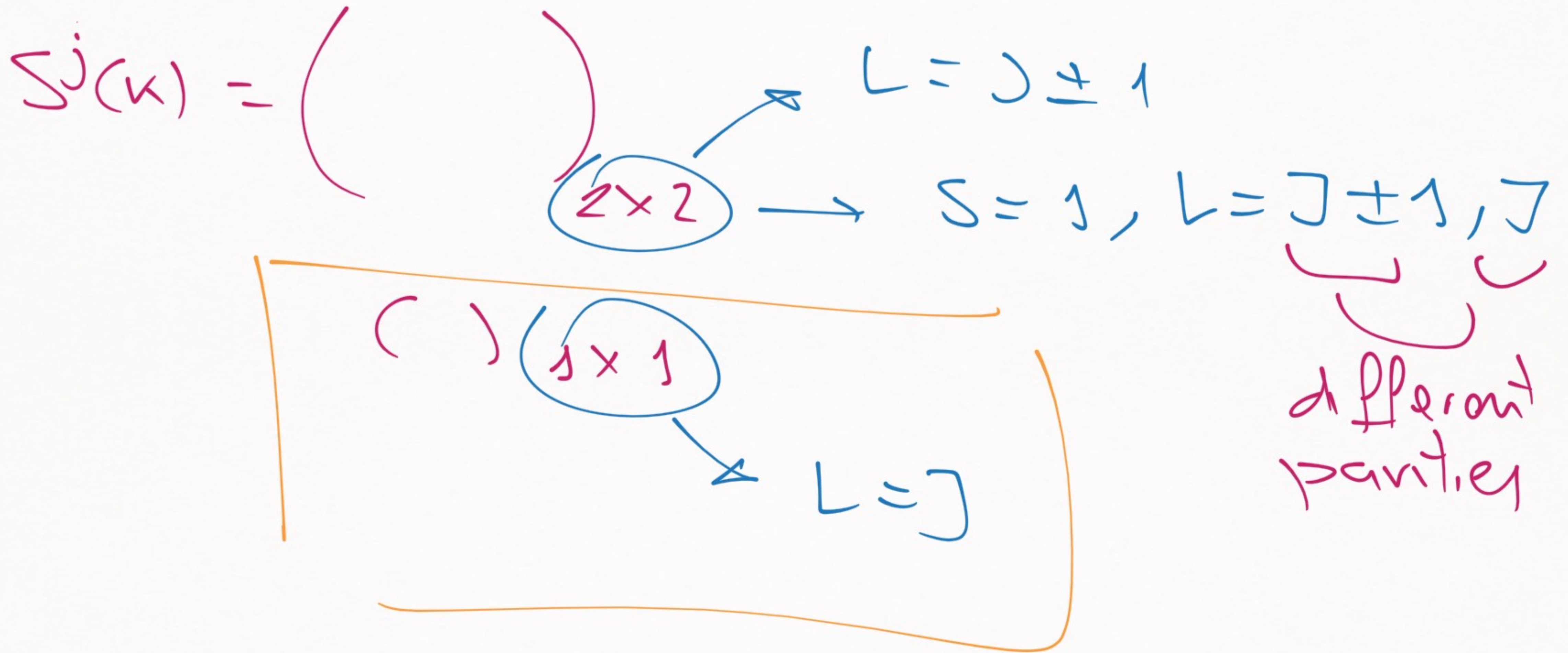
$$\rho_e(k) = \frac{1}{2ik} (S_e(k) - 1)$$

$$S_e S_e^\dagger = 1$$

$$\Rightarrow \underline{\underline{S_e(k) = e^{2i\phi_e(k)}}}$$

$$S(S^\dagger)^+ = 1$$

$S_{op}^J(k)$
unitary
in L^2 -space



→ exactly identical to standard scattering

$$L = J \pm 1$$

$$S J(\kappa) = R D R^{-1} \rightarrow \text{rotation matrix}$$

→ diagonal matrix (unitary)

$$= \begin{pmatrix} \cos \epsilon_j & -\sin \epsilon_j \\ \sin \epsilon_j & \cos \epsilon_j \end{pmatrix} \begin{pmatrix} e^{z_i \beta_1} & 0 \\ 0 & e^{z_i \beta_2} \end{pmatrix} \begin{pmatrix} \cos \epsilon_j & +\sin \epsilon_j \\ -\sin \epsilon_j & \cos \epsilon_j \end{pmatrix}$$

→ Eigen phase shifts

$$S_j(\omega) = R(\epsilon) D R^T(\epsilon)$$

(see previous page)

→ intuitive parametrization

→ Nuclear bar

$$\underline{S}(k) = \begin{pmatrix} e^{i\sqrt{1}} & 0 \\ 0 & e^{i\sqrt{2}} \end{pmatrix} \begin{pmatrix} \cos Z\bar{E}_J & i \sin Z\bar{E}_J \\ i \sin Z\bar{E}_J & \cos Z\bar{E}_J \end{pmatrix}$$

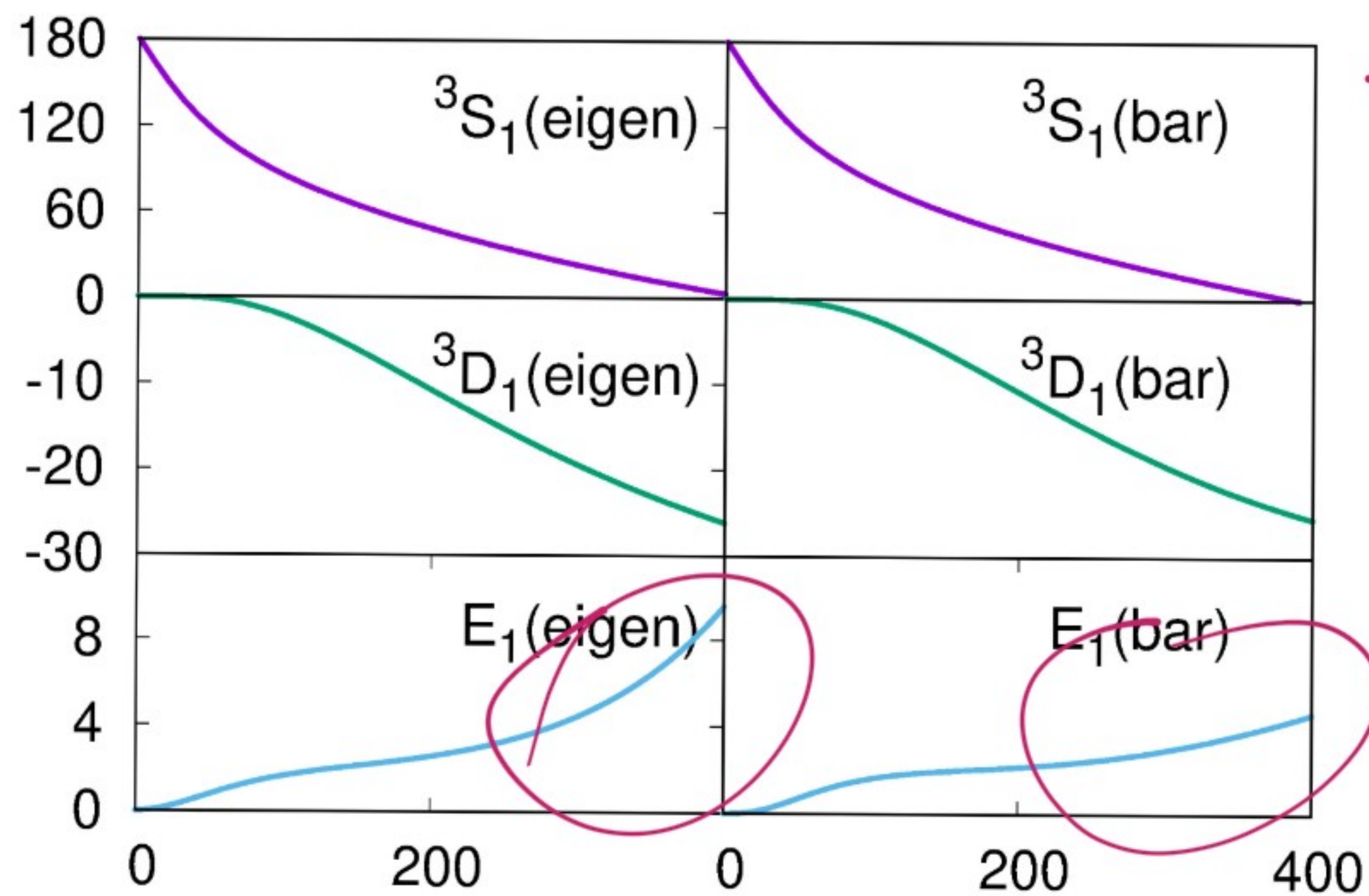
↓

↳ the same,

$$\begin{pmatrix} e^{i\sqrt{1}} & 0 \\ 0 & e^{i\sqrt{2}} \end{pmatrix}$$

but can be written in many forms

→ these are the most used parametrization
 (→ as many as you want)



→ Deuteron channel
 ${}^3S_1 - {}^3D_1$

→ only important
 difference

TO SUMMARIZE

→ tensor force creates a few
technical complications
for the phase shifts

(important if you study NN in detail)



ORE MODEL

We have seen a lot about the generic description of the two nuclear systems

[One boson exchange (OBE) model]

first qualitatively successful description
of the nuclear force



→ very intuitive (direct extension of
Yukawa's idea of meson
exchange)

Historical notes

1) Before QCD

1.a) Dirac theories of the SD's → failed

1.b) OBE model → work, ← easy

2) After QCD

2.a) QCD-inspired models

2.b) EFT methods (ongoing, seems to work)

difficult

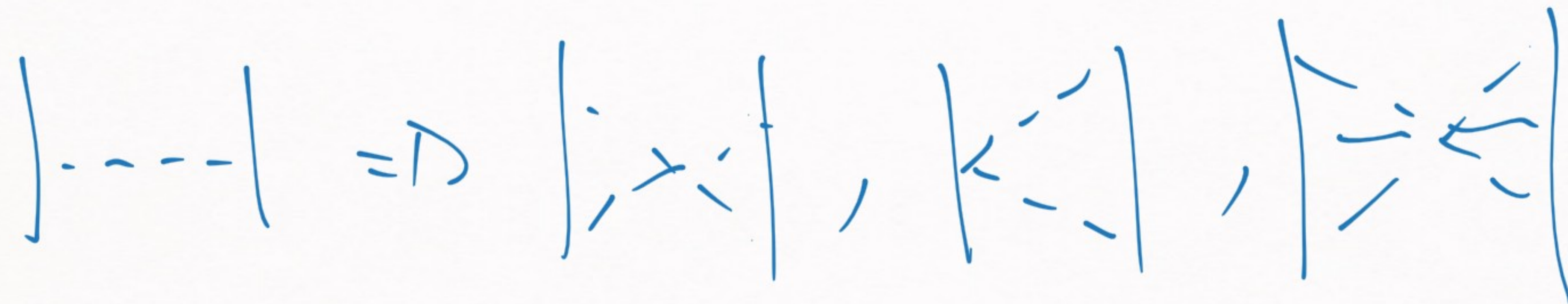
OBE model → easy-to-understand description

↳ Why was it proposed?

1) Yukawa's idea: $|\pi\rangle$ → first step is NN description

2) How to go beyond this?

2.a) Multi-plan theories

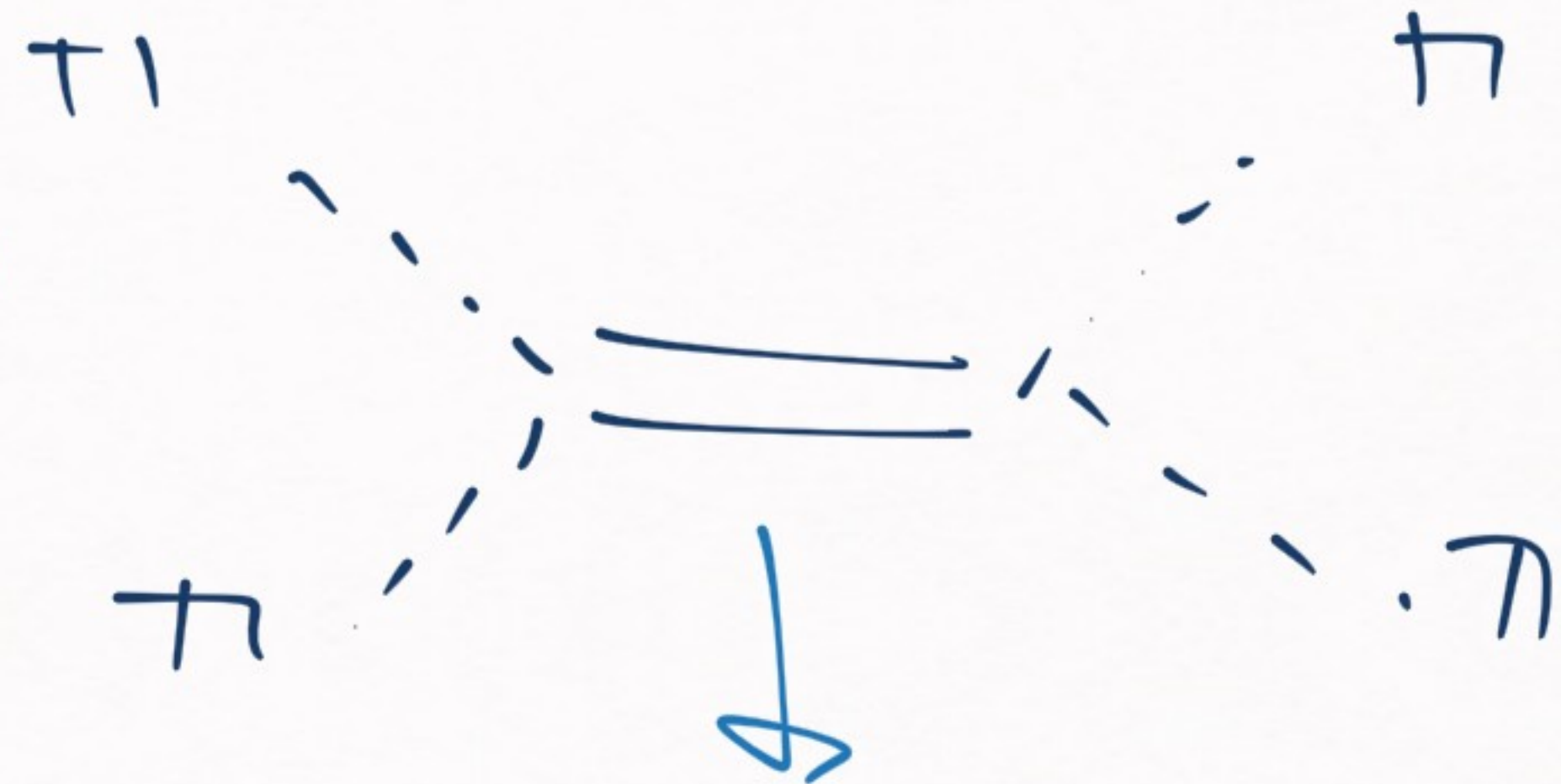


more plans \rightarrow conceptually straightforward

- In the 50's people didn't understand
plan dynamics / renormalization

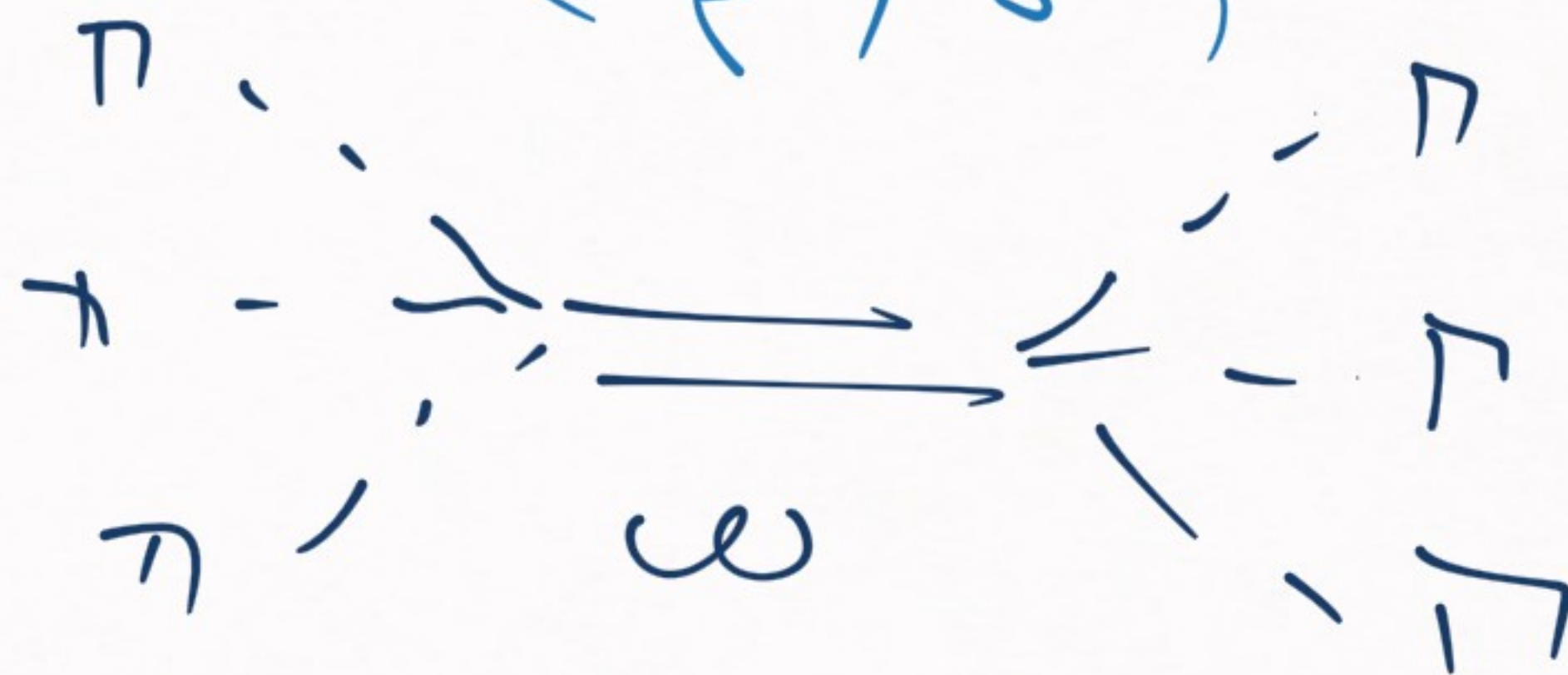
→ Failed

2.b) ORF model



resonance

(ρ, σ)





IDEA

reduce multi-pion exchange

→ exchange of a
single meson



RESULT OF THIS IDEA IS REALLY EASY:

| π |

\Rightarrow

| $\pi, \sigma, \rho, \omega$ |

(+ sometimes a few more)

Who are the important mesons?

1) The pion (π): $J^P = 0^-$, $I = 1$, $m_\pi = 140 \text{ MeV}$

(Job \rightarrow explain Q_d in the deuteron)

2) The sigma (σ): $J^P = 0^+$, $I = 0$, m_σ
 $\approx 800 \text{ MeV}$

(Job \rightarrow strong mid-range attraction)

3) The rho(ρ): $J^P = 1^-, I = 1, m_\rho \approx 770 \text{ MeV}$

(Job \rightarrow the tensor force of the pion is too strong & the rho counters it)

4) The omega(ω): $J^P = 1^-, I = 0, m_\omega \approx 780 \text{ MeV}$

(Job \rightarrow provide short-range repulsion)

GBE Potential:

$$V_{\text{GBE}} = V_{\pi} + V_{\zeta} + V_{\omega} + V_{\sigma} + \dots$$

→ sum of individual contributions

→ Very simple description
of NN forces

$V_H(\vec{g}) \rightarrow$ (we have shown it before)

$$V_S(\vec{g}) = -\frac{g_0^2}{|\vec{g}|^2 - m_0^2}$$

$$V_W(\vec{g}) = \frac{g_W^2}{|\vec{g}|^2 + m_W^2} + \frac{(p_W + g_W)^2}{4M_W^2} \frac{(\vec{\tau}_1 \wedge \vec{g}) \cdot (\vec{\tau}_2 \wedge \vec{g})}{|\vec{g}|^2 + m_W^2}$$

$$V_P(\vec{g}) = \vec{\tau}_1 \cdot \vec{\tau}_2 \cdot \left[V_W / g_W \rightarrow g_e, \right. \\ \left. p_W \rightarrow p, m_W \rightarrow m_e \right]$$

Some simplifications

$$\rightarrow \rho_e \gg g_r \Rightarrow g_r \approx 0$$

$$f_w \ll g_w \Rightarrow g_w \approx 0$$

→ Very common simplifications

[OBE potential] \rightarrow some contributions
to the potential
are singular

v_H, v_P, v_W
(tensor forces) } $\sim \frac{1}{r^3}$ at short distances

Regularization \rightarrow Form factors

FORM FACTORS | (avoid $1/v^3$ singular
behavior)



Since hadrons have a finite-size,

we can include this effect

in the potential

$$V_M(\vec{q}) \rightarrow V_M(\vec{q}) F_M^2(\vec{q})$$

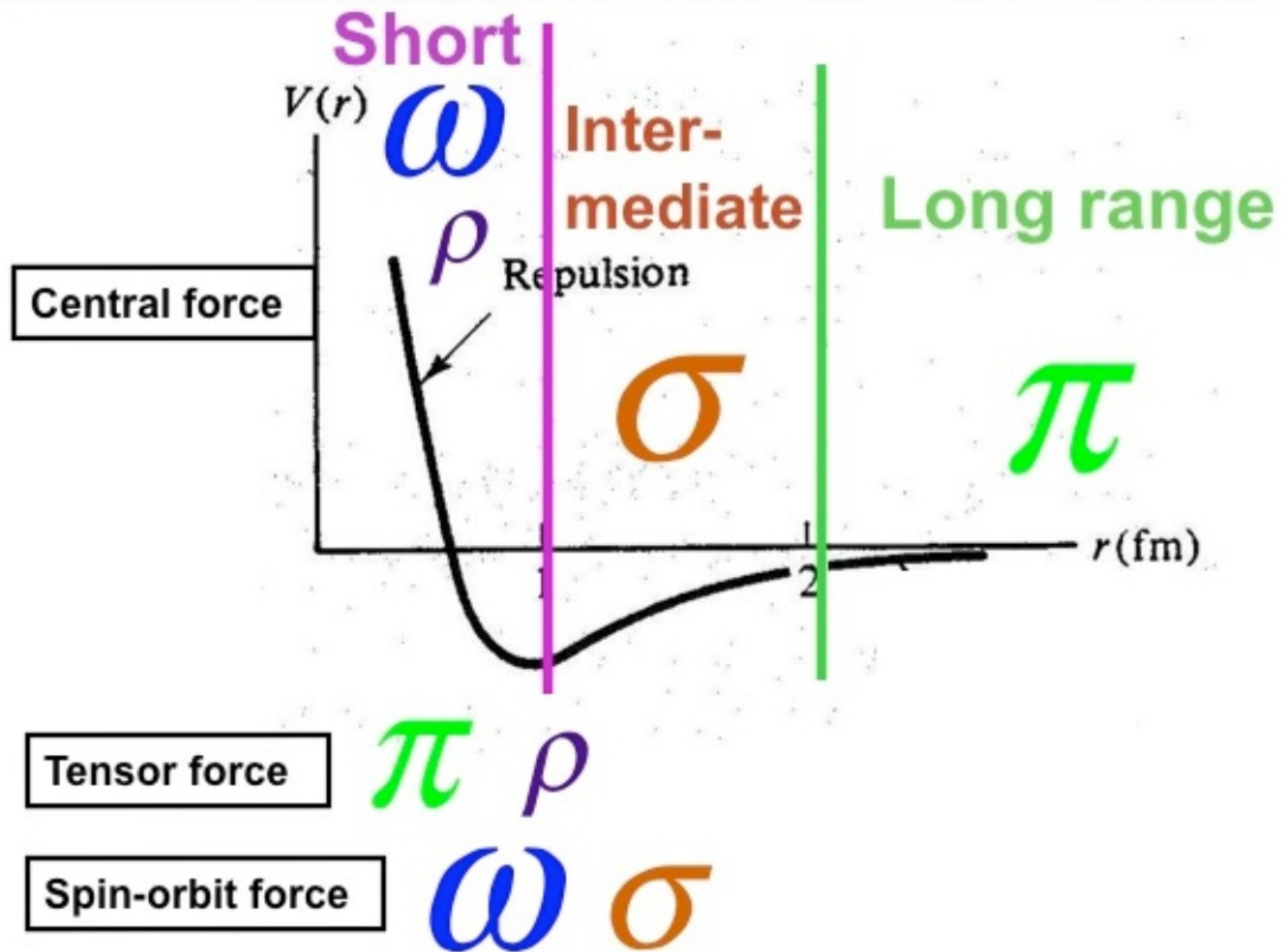
to account for
hadron finite-size

$$F_M(\vec{q}) = \left(\frac{K^2 - m^2}{K^2 + \frac{1}{2}q^2} \right) \rightarrow \text{multipole form-factor}$$

→ also relativistic corrections

$|\sigma, \omega\rangle \rightarrow$ rel. corrections generate
a spin orbit force

→ important for a few
partial waves



Summary
of OBE
model
~

Scholarpedia → Nuclear Force by MedQeist

Table 1: The four most important mesons and the main characteristics of their contributions to components of the nuclear force.

Meson	Central	Spin-Spin	Tensor	Spin-Orbit
$\pi(138)$	---	weak, long-ranged	strong , long-ranged	---
$\sigma(500)$	strong, attractive , intermediate-ranged	---	---	moderate, intermediate-ranged
$\omega(782)$	strong, repulsive , short-ranged	---	---	strong , short-ranged, coherent with σ
$\rho(770)$	---	weak, short-ranged, coherent with π	moderate, short-ranged, opposite to π	---

→ Summary of the job done by every meson //

→ In summary, this is an easy
and intuitive mode!



→ SEE YOU ON TUESDAY