NUCLEAR PHYSICS 15

FORMAL SCATTERING THEORY (II)

$\rightarrow$ $\boxed{\text{T-MATRIX}}$

$\boxed{\text{RECAP}}$

(long chain of reasoning)

Basis: QM can be understood from very different points of view

→ we consider the following two viewpoints

⇓

$\Rightarrow$ $\boxed{\text{View 1}}$ (Standard one)

$\rightarrow$ Wave functions $\psi(\vec{r})$, differential equations (Schrödinger)

$$\left(-\frac{1}{2\mu}\vec{\nabla}^2\psi + V\psi\right) = E\psi(\vec{r})$$

$$\psi(\vec{r}) \xrightarrow[r \to \infty]{} e^{i\vec{k}\cdot\vec{r}} + f(\Omega)\frac{e^{ikr}}{r}$$

$$\frac{d\sigma}{d\Omega} = |f(\Omega)|^2$$

$\boxed{\text{View 2}}$ (Operator view)

Wave functions are vectors in a $\overbrace{\text{Hilbert space}}^{\infty\text{-dim vector spaces}}$

$\underline{\underline{|\psi\rangle}}$, $\underbrace{\text{operators}}_{\text{matrices (infinite dimensional)}}$ acting on $|\psi\rangle$

$$(H_0 + V)|\psi\rangle = E|\psi\rangle$$ (eigenvalue problem)

$$|\psi\rangle = |\vec{k}\rangle + G_0 T |\vec{k}\rangle$$

$$\left(\psi(\vec{r}) \to e^{i\vec{k}\cdot\vec{r}} + f(\Omega)\frac{e^{ikr}}{r} \right)$$

$$\frac{d\sigma}{d\Omega} = |f(\Omega)|^2, \quad f(\Omega) = -\frac{\mu}{2\pi}\langle \vec{k}|T|\vec{k}'\rangle$$

$T \to$ T-matrix (information about scatterings)

$\boxed{\text{View 2}}$ → Objective : to recast ∀ that we knew about scattering in this operator language.

→ To find a definition for the T-matrix & some equation for it

You already this → relax now

You don't know this → relax & take notes

(unfortunately, this will be complex)

↓

this takes time

$\boxed{\text{Derivation}}$ (continuing from past lesson)

1) $H|\phi\rangle = E|\phi\rangle$ (Schrödinger eq.)

2) $H = H_0 + V$ ($H_0 \rightarrow$ kinetic term,
$V \rightarrow$ potential term)

3) $(E - H_0)|\phi\rangle = V|\phi\rangle$ (we might use Green's function methods to solve it)

4) $|\phi\rangle = |\vec{k}\rangle + G_0(E) V |\phi\rangle$

$$G_0(E) = \frac{1}{E - H_0}$$ (prep. for Green's function definition)

4') $\langle \vec{r} | \phi \rangle = \phi(\vec{r})$

$$\phi(\vec{r}) = e^{i\vec{k}\cdot\vec{r}} + \int d^3\vec{r}\,' G_0(\vec{r}-\vec{r}\,') V(\vec{r}\,') \phi(\vec{r}\,')$$

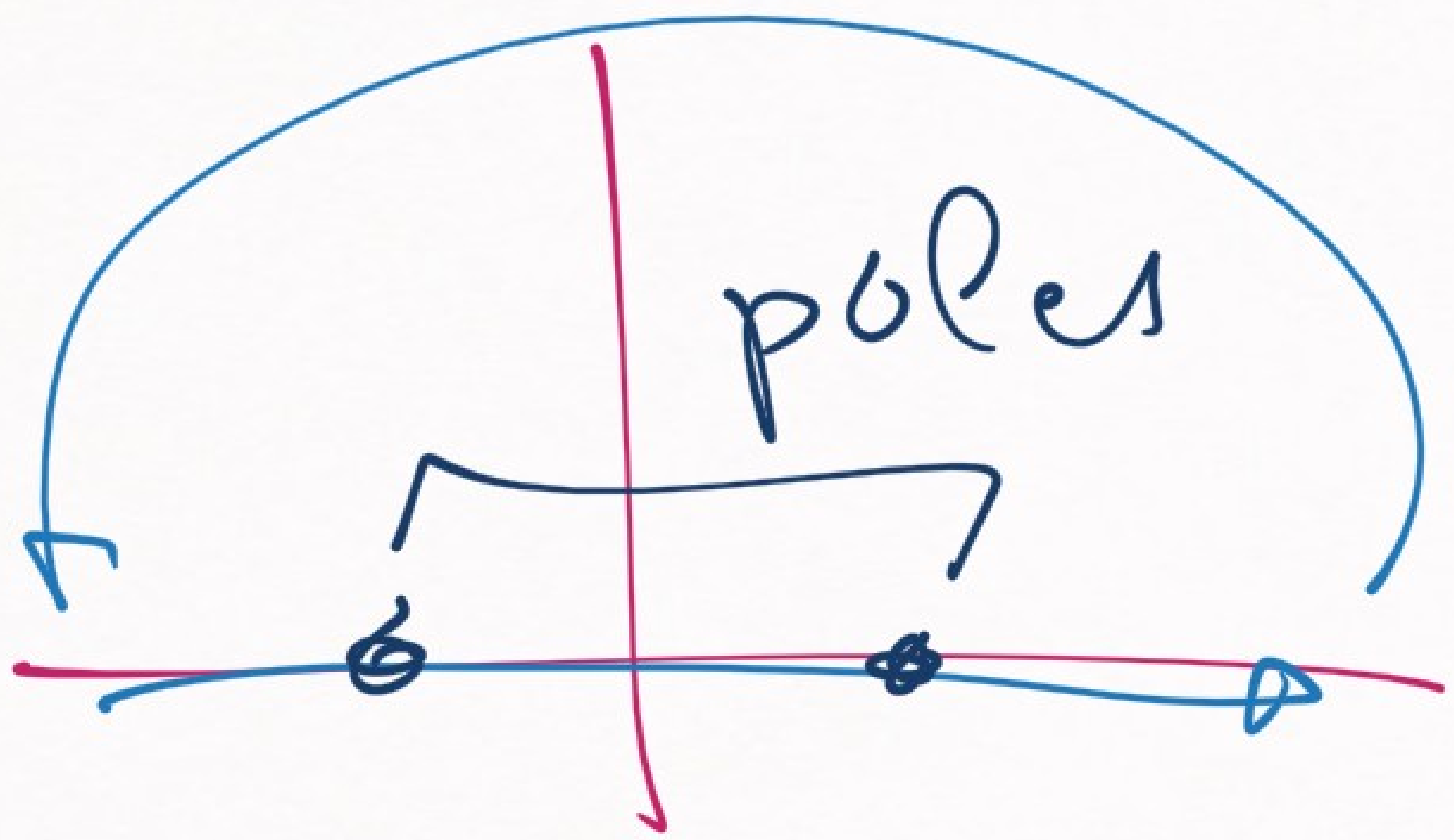
$(E - H_0)|\phi\rangle = V|\phi\rangle \Rightarrow \left(-\frac{\nabla^2}{2\mu} - E\right) G_0(\vec{r}) = \delta^{(3)}(\vec{r})$

?

(Solve 4')

5) $G_0(\vec{r}, \underbrace{E \pm i\epsilon}) = -\frac{\mu}{2\pi}\frac{e^{\pm ikr}}{r}$ } $E \pm i\epsilon$ chooses how to deal w/ the poles

what is this? → our choices of how to solve $G_0(\vec{r})$



$$G_0(\vec{r}) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{e^{i\vec{p}\cdot\vec{r}}}{E - \frac{\vec{p}^2}{2\mu}}$$

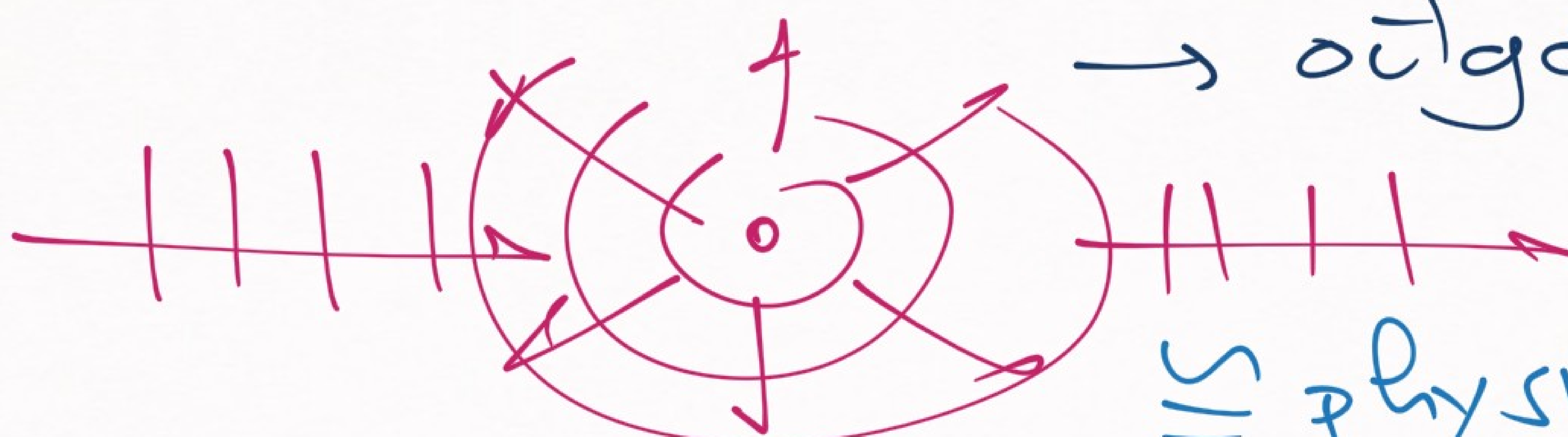
$p = \pm\sqrt{2\mu E}$

6) $\phi^{\pm}(\vec{r}) = e^{i\vec{k}\cdot\vec{r}} + \int d^3\vec{r}\,' G_0(\vec{r}-\vec{r}\,')V(\vec{r}\,')\phi^{\pm}(\vec{r}\,')$

$E \pm i\epsilon$

Last lesson we said that:

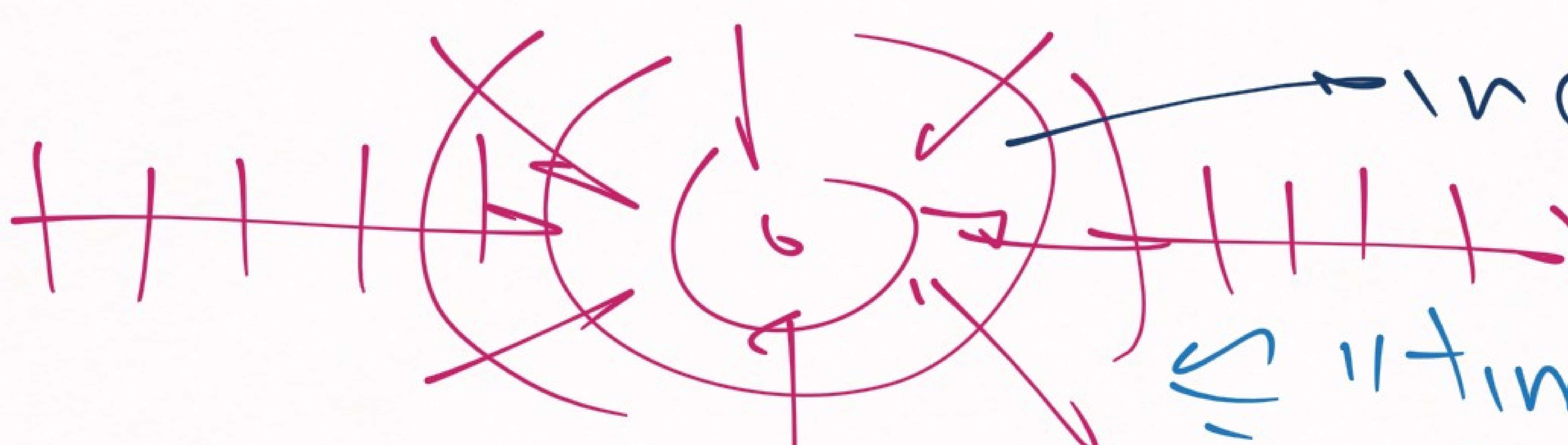
ϕ^+



$\rightarrow$ outgoing spherical wave

$\Leftrightarrow$ physical scatterings

ϕ^-



incoming wave

$\Leftrightarrow$ "time-inverted" scatterings

Why exactly do we choose G^+ &
what type of wf we get?

$$\Rightarrow) \quad \psi^+(\vec{r}) = e^{i\vec{k}\cdot\vec{r}} + \int d^3\vec{r}\, G_0(\vec{r}-\vec{r}\,')\, V(\vec{r}\,')\, \psi(\vec{r}\,')$$

$$\psi(\vec{r}) \xrightarrow[|\vec{r}| \to \infty]{} e^{i\vec{k}\cdot\vec{r}} + \boxed{f(\Omega)\frac{e^{ikr}}{r}}$$

Relation? =D

7. a) $G_0(\vec{r}; E+i\epsilon) = -\frac{\mu}{2\pi}\frac{e^{ikr}}{r}$

$\Downarrow$

$$\phi^+(\vec{r}) = e^{i\vec{k}\cdot\vec{r}} - \frac{\mu}{2\pi}\int d^3\vec{r}\,' \boxed{\frac{e^{ik|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|}} V(\vec{r}\,')\phi^+(\vec{r}\,')$$

lim $r \to \infty$ of this

$$|\vec{r}-\vec{r}\,'| = r\sqrt{1+\left(\frac{r'}{r}\right)^2 - 2\frac{\vec{r}\cdot\vec{r}\,'}{r^2}} = \left\{\sqrt{1+x} \simeq 1+\frac{x}{2}\right\}$$

$$= r\left(1 - \frac{\vec{r}\,'\cdot\vec{r}}{r^2} + O\left(\frac{1}{r^2}\right)\right) \quad O\left(\frac{1}{r}\right)$$

$$\psi^{+}(\vec{r}) \underset{|\vec{r}|\to\infty}{\longrightarrow} e^{i\vec{k}\cdot\vec{r}} \boxed{- \frac{\mu}{2\pi} \frac{e^{ikr}}{r} \int d^3\vec{r}\,' e^{-i\vec{k}'\cdot\vec{r}\,'} V(\vec{r}\,') \psi^{+}(\vec{r}\,')}$$

$$\psi(\vec{r}) \underset{|\vec{r}|\to\infty}{\longrightarrow} e^{i\vec{k}\cdot\vec{r}} + \boxed{f(\Omega) \frac{e^{ikr}}{r}}$$

$$f(\Omega) = -\frac{\mu}{2\pi} \int d^3\vec{r}\,' e^{-i\vec{k}'\cdot\vec{r}} V(\vec{r}\,') \psi^{+}_{\vec{k}}(\vec{r})$$

$$= -\frac{\mu}{2\pi} \langle \vec{k}' | V | \psi^{+} \rangle$$

$$\boxed{f(\Omega) = -\frac{\mu}{2\pi}\langle \vec{k}' | V | \phi^+ \rangle} \longrightarrow$$ closer to the idea for the T-matrix

3) $$\boxed{V|\phi^+\rangle = T|\vec{k}\rangle} \longrightarrow$$ definition

Where does this come from?

$$|\phi^+\rangle = |\vec{k}\rangle + G_0 V |\phi^+\rangle$$

$$= |\vec{k}\rangle + G_0 V \left(|\vec{k}\rangle + G_0 V |\phi^+\rangle \right) = \ldots$$

$$\circledast = |\vec{k}\rangle + G_0 V|\vec{k}\rangle + (G_0 V)^2|\vec{k}\rangle + (G_0 V)^3|\vec{k}\rangle + \ldots$$

$$\Rightarrow \quad |\phi^+\rangle = (1 + G_0 V + (G_0 V)^2 + \ldots)|\vec{k}\rangle$$

(Iterative equation) $\Longrightarrow \circledast$

$\hookrightarrow \quad f(\omega) = -\frac{\mu}{2\pi}\langle \vec{k}'|V|\phi^+\rangle$

$\oplus \Rightarrow \langle \vec{k}' | V | \phi^+ \rangle =$

$$\langle \vec{k}' | V \left(1 + (G_0 V) + (G_0 V)^2 + \dots\right) | \vec{k} \rangle =$$

$$\langle \vec{k}' | \underbrace{V + V G_0 V + V G_0 V G_0 V + \dots}_{T} | \vec{k} \rangle$$

$$= \langle \vec{k}' | T | \vec{k} \rangle$$

$$V | \phi^+ \rangle = T | \vec{k} \rangle$$

Consider the series we have:

$$\langle \vec{k}' | T | \vec{k} \rangle = \langle \vec{k}' | V + VG_0V + VG_0VG_0V + \dots | \vec{k} \rangle$$

$$T = V + VG_0V + VG_0VG_0V + \dots$$

$$= V + VG_0 [\underbrace{V + VG_0V + VG_0VG_0V + \dots}_{T}]$$

$$= V + VG_0T$$

$$\boxed{T = V + VG_0T}$$

LIPPMANN-SCHWINGER EQUATION

$$\boxed{T = V + V G_0 T}$$

T

$$\boxed{f(\Omega) = -\frac{\mu}{2\pi}\langle \vec{k}' | T | \vec{k} \rangle}$$

$$\boxed{\frac{d\sigma}{d\Omega} = |f(\Omega)|^2}$$

How to deal with the LSE?

$\longrightarrow$ take matrix elements $\langle \vec{k}'|$ and $|\vec{k}\rangle$

$$\langle k'| \; T = V + V G_0 T \; |\vec{k}\rangle \quad , \quad \mathbb{1} = \int \frac{d^3\vec{q}}{(2\pi)^3} |\vec{q}\rangle\langle \vec{q}|$$

$$\boxed{\langle \vec{k}'|T(E)|\vec{k}\rangle = \langle \vec{k}'|V|\vec{k}\rangle + \int \frac{d^3\vec{q}}{(2\pi)^3} \frac{\langle \vec{k}'|V|\vec{q}\rangle\langle \vec{q}|T|\vec{k}'\rangle}{E - \frac{q^2}{2\mu}}}$$

Most common representation

$$G_0(E)|\vec{k}\rangle = \frac{1}{E-H_0}|\vec{k}\rangle = \frac{1}{E-\frac{k^2}{2\mu}}|\vec{k}\rangle \quad \text{(diagonal)}$$

$$\langle \vec{k}' | \vec{k} \rangle = (2\pi)^3 \delta^{(3)}(\vec{k}' - \vec{k})$$

$\rightarrow$ these are used in getting the LSE as $\langle \vec{k}' | T(E) | \vec{k} \rangle$

[T-matrix] $\rightarrow$ Very abstract object

$\rightarrow$ find connections w/ things we know

1) T-matrix & the perturbative expansion

2) T-matrix & Feynman diagrams

$\sum$

[T-matrix as a perturbative series]

$$f(\omega) = -\frac{\mu}{2\pi} \langle \vec{k}' | T(E+i\epsilon) | \vec{k} \rangle$$

$$T = V + V G_0 T$$

$$= \underbrace{V}_{\mathcal{O}(V)} + \underbrace{V G_0 V}_{\mathcal{O}(V^2)} + \underbrace{V G_0 V G_0 V}_{\mathcal{O}(V^3)} + \dots \Rightarrow$$

If potential is weak...

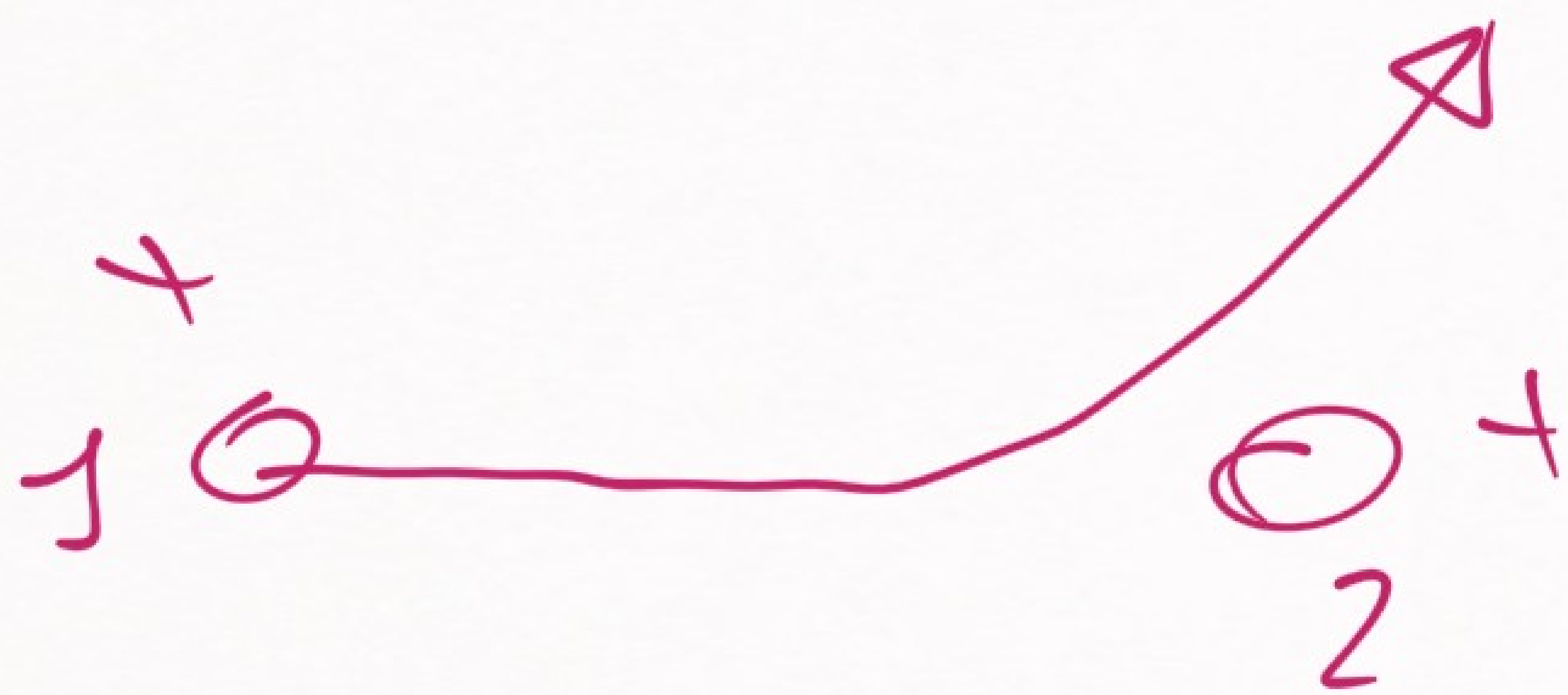
$$\boxed{T \simeq V + \mathcal{O}(V^2)} \rightarrow \text{BORN APPROX.}$$

$$\boxed{f(\Omega) = -\frac{\mu}{2\pi} \underbrace{\int d^3\vec{r}\, e^{i(\vec{k}-\vec{k}')\cdot\vec{r}}\, V(\vec{r})}_{\text{Fourier transform}} + \mathcal{O}(V^2)}$$

$f(\Omega) \rightarrow$ PROP. TO FOURIER TRANSFORM OF $V(\vec{r})$ IN $\vec{q} = \vec{k}' - \vec{k}$

EXAMPLE $\rightarrow$ RUTHERFORD SCATTERING

(^4He nucleus or two heavy particles that are electrically charged)



$V_C(\vec{r}) = Z\,\frac{\alpha}{r}$

(Coulomb potential)

$f(\omega) = -\frac{\mu}{2\pi} V_C(\vec{q}) + O(\alpha^2)$ (Z=1 for simplicity)

$V_C(\vec{q}) = \int d^3\vec{r}\, e^{-i\vec{q}\cdot\vec{r}}\, V_C(\vec{r}) = 4\pi \frac{\alpha}{|\vec{q}|^2}$

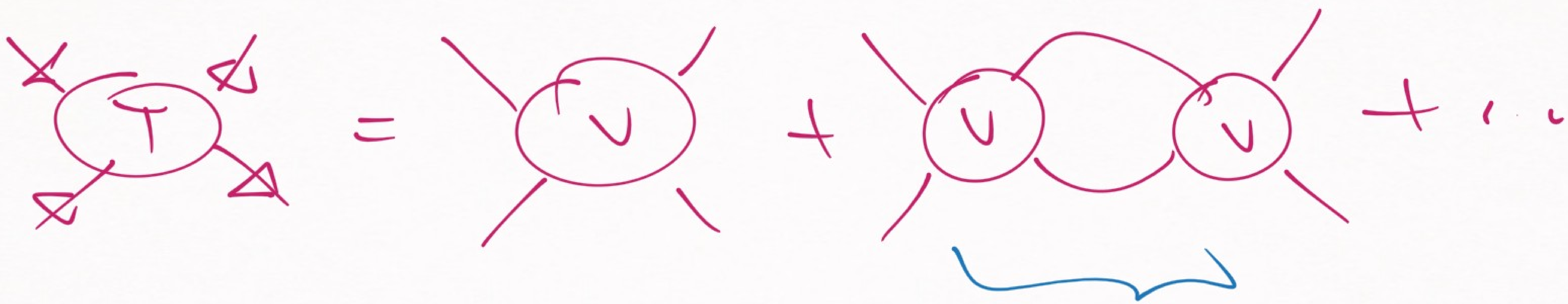
$$\Rightarrow \left[f(\Omega) = -\frac{2\mu\alpha}{|\vec{k}'-\vec{k}|^2} + \mathcal{O}(\alpha^2) \right]$$

$$\left[\frac{d\sigma}{d\Omega} = \frac{4\mu^2\alpha^2}{|\vec{k}'-\vec{k}|^4} + \mathcal{O}(\alpha^4) \right]$$

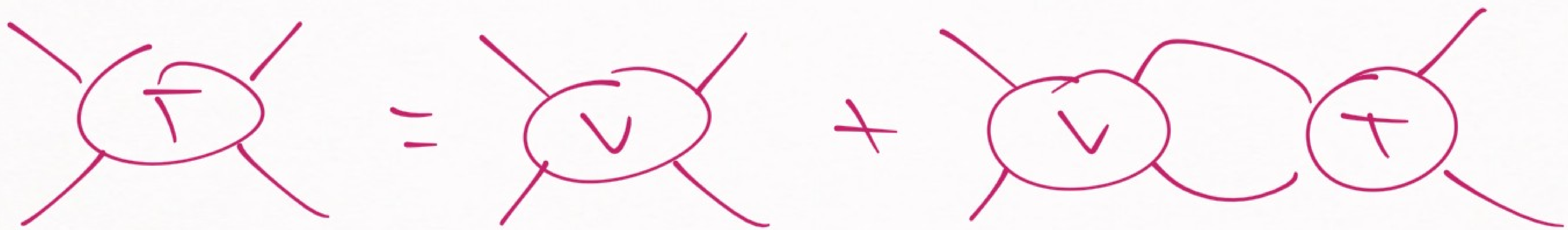
Rutherford scattering

2) Relation to Feynman diagram language

$$T = V + V G_0 V + V G_0 V G_0 V + \ldots$$



loop $\int \frac{d^3\vec{q}}{(2\pi)^3} G_0(E)$



(Lippmann-Schwinger equation
in diagrammatic form)

[WHY THE T-MATRIX?] $\rightarrow$ overcomplicated way of solving Schrödinger

$\downarrow$

T-matrix is easier in cases where we have non-local potentials

(↔ LSE becomes easier than Schrödinger)

T-matrix useful for non-local potentials

$$\langle \vec{k}' | V | \vec{k} \rangle \neq V(\vec{k}' - \vec{k})$$

1) $H|\psi\rangle = E|\psi\rangle$ $\quad (V(\vec{k}'-\vec{k}) \Rightarrow V(\vec{r}))$

$\hookrightarrow$ if V non-local $\Rightarrow$ very difficult

2) $T = V + VG T$ $\rightarrow$ equally difficult for local & non-local

[SOLUTIONS OF THE T-MATRIX]

$$\langle \vec{k}' | T(E) | \vec{k} \rangle = \langle \vec{k}' | V | \vec{k} \rangle + \int \frac{d^3 \vec{e}}{(2\pi)^3} \frac{\langle \vec{k}' | V | \vec{e} \rangle \langle \vec{e} | T(E) | \vec{k} \rangle}{E - \frac{\vec{e}^2}{2\mu}}$$

We will try $\rightarrow$ $\langle \vec{k}' | V | \vec{k} \rangle = \lambda\, g(k')\, g(k)$

$k' = |\vec{k}'|\,,\ k = |\vec{k}|$

$$\langle \vec{k}' | V | \vec{k} \rangle = \lambda\, g(k)\, g(k')$$

Separable potential (Yamaguchi potential)

Loop

$$\int \frac{d^3\vec{\ell}}{(2\pi)^3} \frac{\langle \vec{k}' | V | \vec{\ell} \rangle \langle \vec{\ell} | T | \vec{k} \rangle}{E - \frac{\vec{\ell}^2}{2\mu}} = \lambda\, g(k') \int \frac{d^3\vec{\ell}}{(2\pi)^3} \frac{g(\ell) \langle \vec{\ell} | T | \vec{k} \rangle}{E - \frac{\ell^2}{2\mu}}$$

goes outside

Ansatz
(proposal of a solution) $\longrightarrow$

$$\left[\begin{array}{l} \text{for } \langle \vec{k}' | V | \vec{k} \rangle = \lambda\, g(k)\, g(k') \\ \text{then } \langle \vec{k}' | T(E) | \vec{k} \rangle \\ \qquad = \tau(E)\, g(k)\, g(k') \end{array}\right]$$

$$\tau(E)\, g(k) \cancel{g(k')} = \lambda\, g(k) \cancel{g(k')} + \lambda \tau(E)\, g(k) \cancel{g(k')} \times \int \frac{d^3\vec{l}}{(2\pi)^3} \frac{g^2(l)}{E - \frac{l^2}{2M}}$$

$$\left[\tau(E) = \frac{\lambda}{1 - \lambda \int \frac{d^3 \vec{p}'}{(2\pi)^3} \frac{g^2(p')}{E - \frac{p'^2}{2m}}}\right]$$

$\rightarrow$ solution for a separable potential

$\Rightarrow |\vec{k}'| = |\vec{k}|$

$$f(\omega) = -\frac{m}{2\pi} \tau(E) g(k) g(k') = -\frac{m}{2\pi} \tau(E) g^2(k)$$

Separable potential $\Rightarrow$ $$\underbrace{\frac{d\sigma}{d\Omega} = \left(\frac{\mu \tau(E)}{2\pi} g^2(k) \right)^2}$$

$\Downarrow$

Easy

$\nexists$ dependence on angles

$\Rightarrow$ S-wave scattering

$\Rightarrow$

[CONTACT INTERACTIONS] $\leftarrow$ separable potentials $\neq$

$$\underbrace{V_c(\vec{r}) = C_0\, \delta^{(3)}(\vec{r})}$$

$$\Big\downarrow \mathcal{F}$$

$$\boxed{V_c(\vec{q}) = C_0}$$

$\Rightarrow$ like separable w/ $\underline{g(k) = 1}$

If $g(k) = 1 \Rightarrow$

$$\tau(E) = \frac{C_0}{1 - C_0 \int \frac{d^3 \vec{\ell}}{(2\pi)^3} \frac{1}{E - \frac{\ell^2}{2\mu}}}$$

$$\int^{\infty} \ell^2 d\ell \frac{1}{\ell^2} \sim \int^{\infty} d\ell \longrightarrow \infty \Rightarrow \text{Have to regularize}$$

$$\langle \vec{k}' | V_0 | \vec{k} \rangle = C_0 \longrightarrow C_0\, f\left(\frac{k}{\Lambda}\right) f\left(\frac{k'}{\Lambda}\right)$$

$$\boxed{\begin{aligned} f(x) &\to 1 \quad x \to 0 \\ f(x) &\to 0 \quad x \to \infty \end{aligned}} \Rightarrow T(E) = \frac{1}{\frac{1}{C_0(\Lambda)} - I(E;\Lambda)}$$

$$I(E;\Lambda) = \int \frac{d^3\vec{\ell}}{(2\pi)^3} \frac{f^2(\ell/\Lambda)}{E - \frac{\ell^2}{2\mu}} \longrightarrow \underline{\underline{\text{finite}}}$$

$$f(x) = \Theta(1-x) \quad \Leftrightarrow \quad \langle \vec{k}' | V_c | \vec{k} \rangle = 0$$

if $k, k' > \Lambda$

$$I(E \pm i\epsilon; \Lambda) = \frac{\mu}{\pi^2}\left[\mp i\frac{\pi}{2}k - \Lambda + \frac{k}{2}\log\left|\frac{\Lambda+k}{\Lambda-k}\right|\right]$$

$$k = \sqrt{2\mu E}$$

$\rightarrow$ end up w/ finite theory

$\leadsto$ how to fix $C_0(\Lambda)$?

$$\boxed{\begin{aligned}&\sigma \xrightarrow[E\to 0]{} 4\pi |a_0|^2\end{aligned}}$$

$$\sigma = 4\pi \left| \frac{\mu T(E)}{2\pi} f^2\left(\frac{k}{\Lambda}\right) \right|^2$$

$$\xrightarrow[E\to 0]{} 4\pi \left| \frac{\mu T(0)}{2\pi} \right|^2$$

$$T(E+i\epsilon) \xrightarrow[E\to 0]{} + \frac{2\pi}{\mu} a_0$$

$$I(E=0,\Lambda) = -\frac{\mu}{\pi^2}\Lambda$$

$\rightarrow$ (Put all the pieces together)

$$\boxed{\frac{1}{C_0(\Lambda)} = \frac{M}{2\pi}\left(\frac{1}{a_0} - \frac{2}{\pi}\Lambda\right)}$$

Running of $C_0(\Lambda)$

Interesting because of this:

$$f_\ell(k) = \frac{1}{k\cot\delta_\ell - ik} \rightarrow T(E+i\epsilon) = -\frac{2\pi}{\mu} \frac{1}{k\cot\delta_0 - ik}$$

$$T(E+i\epsilon) \xrightarrow[E\to 0]{} \frac{2\pi}{\mu} \frac{1}{1/a_0 + ik}$$

$$\langle \vec{k}' | V_c | \vec{k} \rangle = C_0(\Lambda)\, f\left(\frac{k}{\Lambda}\right) f\left(\frac{k'}{\Lambda}\right)$$

$C_0(\Lambda) / a_0$ fixed

$$\boxed{\Lambda \to \infty}$$

$$k \cot\delta = -\frac{1}{a_0} + \cancelto{0}{\frac{1}{2} r_0 k^2}$$

$\delta^3(\vec{r}) \to$ has no range

$$\boxed{r_0 = 0}$$

[BOUND STATES & T-MATRIX]

→ AIM: what is the relation among bound states & the T-matrix

$$T = V + V G_0 T = V + V G V$$

$$G = G_0 + G_0 V G_0 + G_0 V G_0 V G_0 + \cdots = G_0 + G_0 V G$$

$$\boxed{T = V + VGV}$$

$$G_0(E) = \frac{1}{E - H_0} \longrightarrow G(E) = \frac{1}{E - H}$$

Kinetic term

full hamiltonian

$N \times N$

$A = A^+$ (A matrix)

$\det A \neq 0$

$+$ $A|\lambda_i\rangle = \lambda_i|\lambda_i\rangle$

$i = 1, \ldots, N$

$\Rightarrow$

$\Rightarrow \quad \mathbb{1}_{n \times n} = \sum_i |\lambda_i\rangle\langle\lambda_i| \rightarrow$ also valid for ∞-dim Hilbert states

$H_0 \rightarrow \mathbb{1} = \int \frac{d^3\vec{k}}{(2\pi)^3} |\vec{k}\rangle\langle\vec{k}|$

$H \longrightarrow \mathbb{1} = \sum_{i=1}^{n_B} |B_i\rangle\langle B_i| + \int \frac{d^3\vec{k}}{(2\pi)^3} |\phi^+_{\vec{k}}\rangle\langle\phi^+_{\vec{k}}|$

Identity for the spectrum of $\mathcal{H}$

→ two contributions

1) bound states (discrete)

2) scattering states (continuum)

Σ

$$G(E) = \frac{1}{E-H} \times \mathbb{1} = \sum_{i=1}^{n_B} \frac{|B_i\rangle\langle B_i|}{E-E_i} + \underbrace{\int \frac{d^3\vec{k}}{(2\pi)^3} \frac{|\phi_{\vec{k}}\rangle\langle \phi_{\vec{k}}|}{E - \frac{\vec{k}^2}{2\mu}}}_{\text{Branch cut}}$$

contain poles

$\longrightarrow$ $\circledast$ (next page)

$\circledast \Rightarrow \quad T(E) = V + V\,G(E)\,V$

$$G(E) \xrightarrow[E \to E_i]{} \frac{|B_i\rangle\langle B_i|}{E - E_i}$$

$$\Rightarrow \boxed{T(E) \xrightarrow[E \to E_i]{} V \frac{|B_i\rangle\langle B_i|}{E - E_i} V} \Rightarrow \circledast$$

$\ominus \Rightarrow$ The T-matrix has poles at the energies $(E<0)$ at which we have bound states

Σ

Res $T(E) = V|B_i\rangle\langle B_i|V$

$E \to E_i$

$\rightarrow$ we will be able to calculate bound state wave functions

$\Downarrow$

$\rightarrow \exists$ a complication w/ the energy & momentum

$\Rightarrow$ [$\sqrt{E}$ has two branches in the complex plane]

(will give an example)

$\boxed{\text{CONTACT THEORY}} \Rightarrow \boxed{T(E+i\epsilon) = \frac{2\pi}{\mu} \frac{1}{\frac{1}{a_0} + ik}}$

→ Let's see how to get bound states from the contact theory

$$\tau(E+i\epsilon) = \frac{2\pi}{\mu} \frac{1}{1/a_0 + ik}$$

k (not the energy)

rewrite this in term of E

$K = \sqrt{2\mu E}$

→ $E < 0$

need some extension

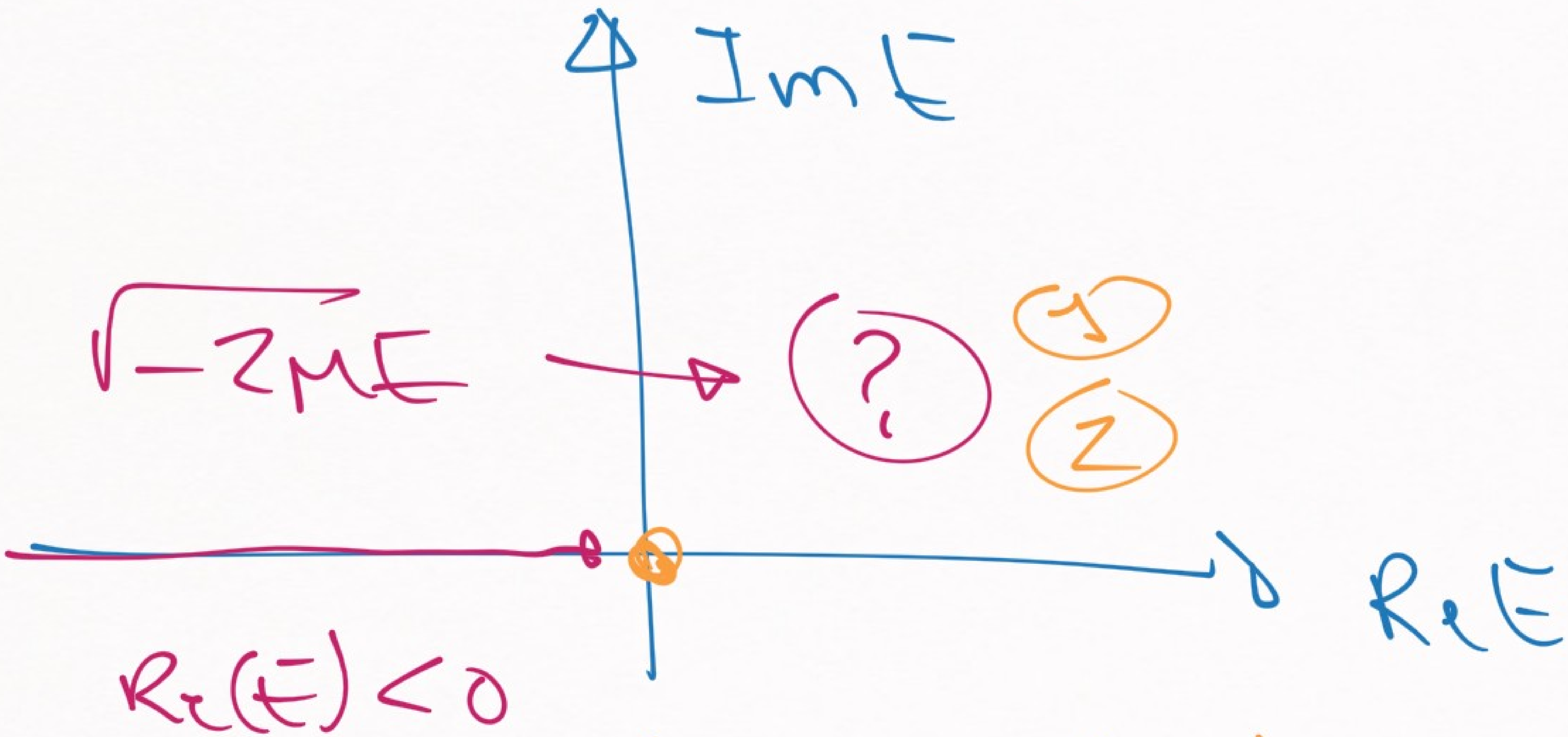
$\tau(E+i\epsilon)$ $\Rightarrow$ $\tau(E<0)$ $\rightarrow$ bound states
$E>0$

$$\frac{1}{1/a_0 + ik} \simeq \frac{1}{1/a_0 \mp \sqrt{-2\mu E}}$$

$\hookrightarrow$ $-$ or $+$?

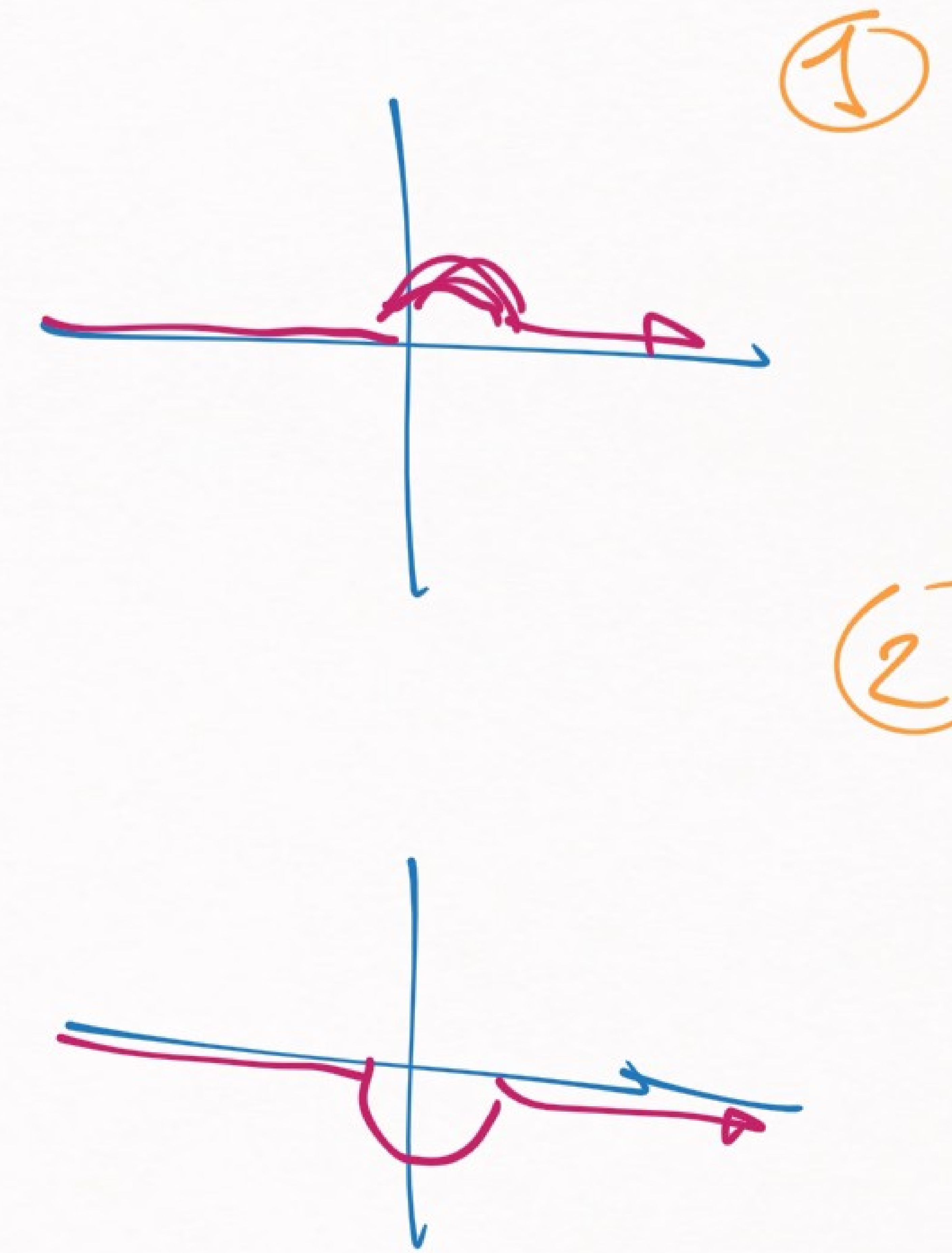
we have to check this

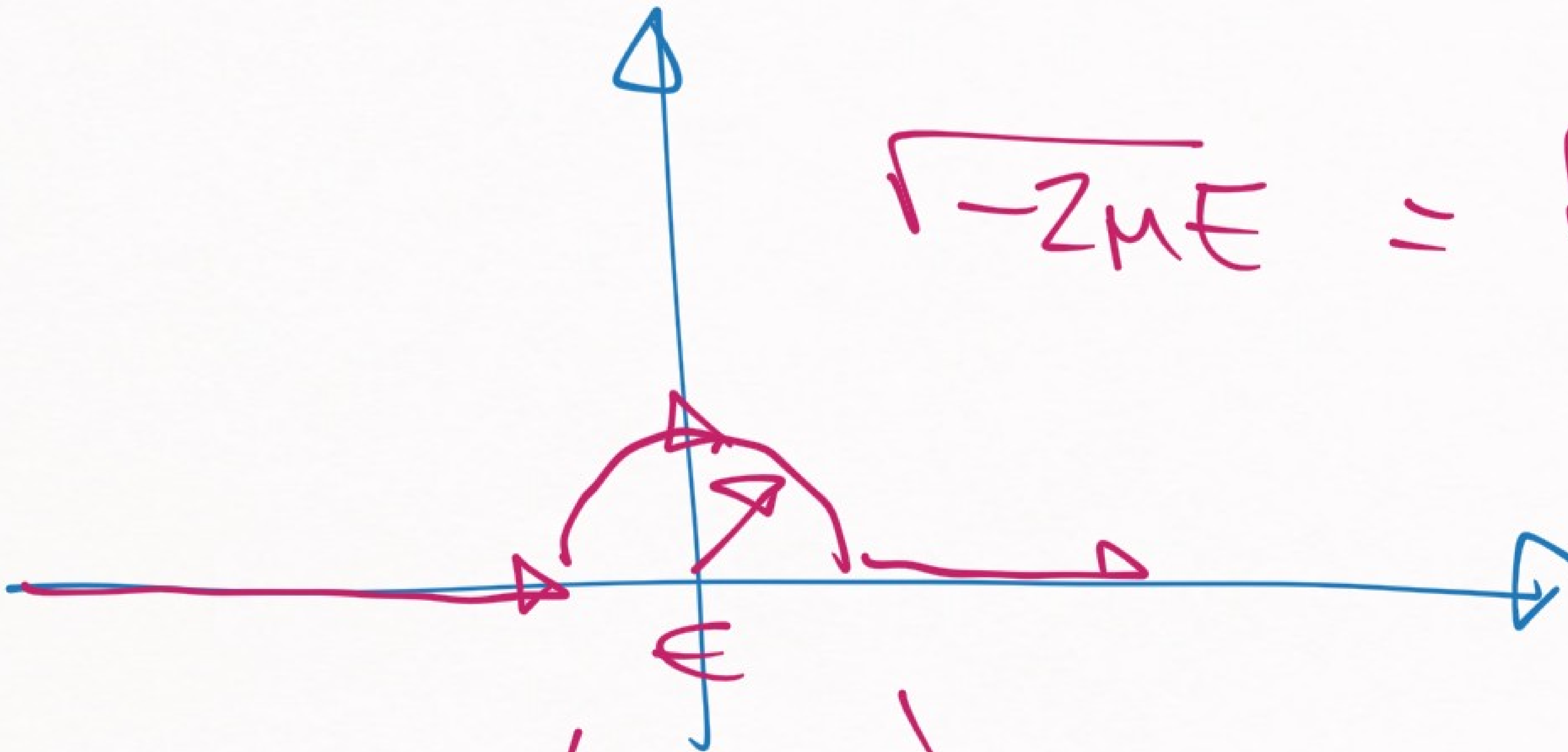
We will work the problem backwards:



two possibilities

1 & 2





$$\sqrt{-2\mu E} = \sqrt{|2\mu E| e^{i\theta}}$$

$$= e^{i\theta/2}\sqrt{|2\mu E|}$$

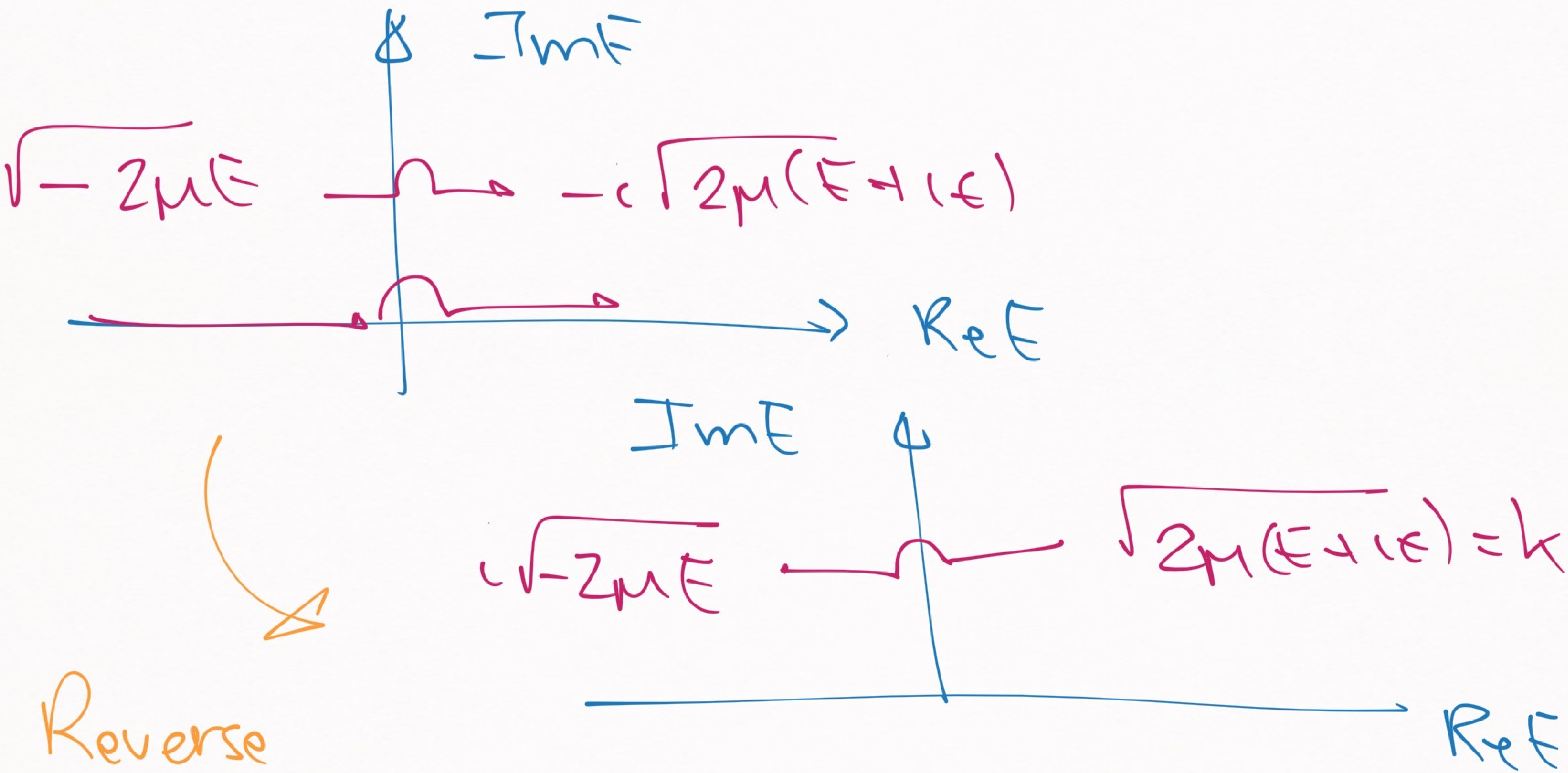
$$=$$

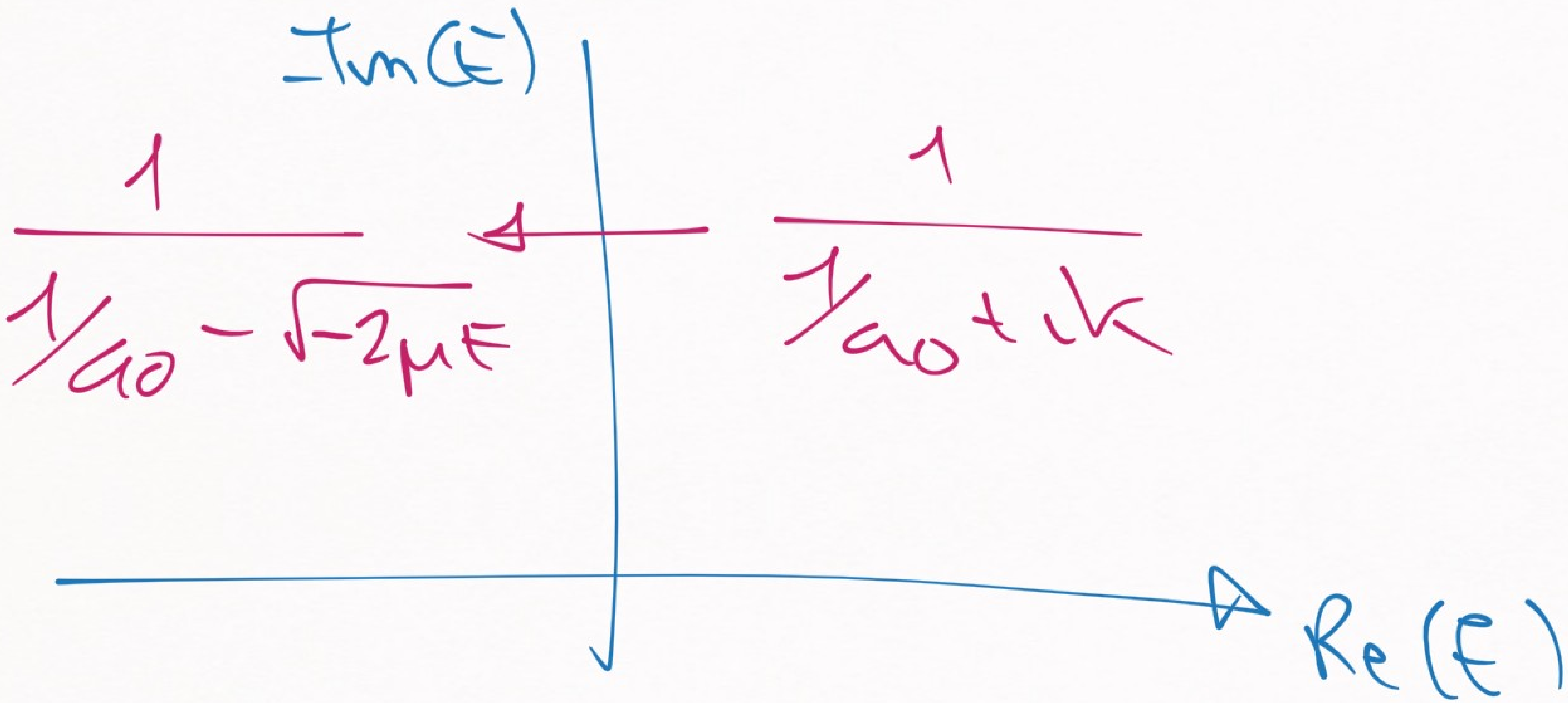
$$-i\sqrt{2\mu(E+i\epsilon)}$$

$$\|$$

$\epsilon = 0$ $\qquad$ $\epsilon = -\pi + \epsilon$

$$\sqrt{-2\mu E} \longrightarrow e^{-\frac{i\pi}{2}} e^{\frac{i\epsilon}{2}} \sqrt{|2\mu E|} = -i e^{i\epsilon/2} \sqrt{2\mu E}$$


Im E
$\sqrt{-2\mu E}$
$-i\sqrt{2\mu(E+i\epsilon)}$
Re E
Reverse
Im E
$i\sqrt{-2\mu E}$
$\sqrt{2\mu(E+i\epsilon)}=k$
Re E



Im(E)
$\frac{1}{1/a_0 - \sqrt{-2\mu E}}$
$\frac{1}{1/a_0 + ik}$
Re(E)

$\boxed{\text{IN SHORT}}$

$$T(E+i\epsilon) = \frac{2\pi}{\mu}\frac{1}{1/a_0 + ik}$$

$\Downarrow$ (w/ previous choices)

$$T(E<0) = \frac{2\pi}{\mu}\frac{1}{1/a_0 - \underline{\sqrt{-2\mu E}}} \rightarrow \text{contains a pole}$$

$$\tau(E<0) = \frac{2\pi}{\mu} \frac{1}{1/a_0 - \sqrt{-2\mu E}} = \frac{2\pi}{\mu} \frac{1/a_0 + \sqrt{-2\mu E}}{1/a_0^2 + 2\mu E}$$

$$\boxed{E_B = -\frac{1}{2\mu}\left(\frac{1}{a_0}\right)^2}$$

$\Rightarrow$ [$\tau(E)$ contains a pole]

$$\left[\begin{array}{l} \operatorname{Res} \tau(E) \\ E = -\frac{1}{2\mu}\left(\frac{1}{a_0}\right)^2 \end{array} = \frac{\pi}{\mu^2}\frac{2}{a_0}\right] \rightarrow \text{wave } \underline{\underline{\text{function}}}$$

How do we do this?

$$\operatorname{Res} T = V|B\rangle\langle B|V$$

$$T = V + V G_0 T \qquad \text{Res}\,T = V|B\rangle\langle B|V$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\text{Res}\,T = V G_0\, \text{Res}\,T \qquad \text{Res}\,T =$$

$$(= (\text{Res}\,T)\, G_0 V) \qquad V G_0 V|B\rangle\langle B|V G_0 V$$

$$\Rightarrow \boxed{|B\rangle = G_0 V |B\rangle}$$

Bound state equation

$$|B\rangle = G_0(E_B)\,V\,|B\rangle$$

$$\downarrow$$

$$G_0^{-1}(E_B)\,|B\rangle = V\,|B\rangle$$

$$\downarrow$$

$$\langle p|\left(E_B - \frac{p^2}{2\mu}\right)|B\rangle = \langle p|V|B\rangle$$

$$\mathrm{Rest} = V|B\rangle\langle B|V$$

$$\downarrow$$

$$\langle \vec{p}\,'|\mathrm{Rest}|\vec{p}\rangle = \left(E_B - \frac{p^2}{2\mu}\right)\left(E_B - \frac{p'^2}{2\mu}\right)$$

$$\times\, \psi_B(\vec{p})\,\psi_B^\dagger(\vec{p}\,')$$

[CONTACT-RANGE THEORY]

$$\langle \vec{p} | R_{EST} | \vec{p} \rangle = \frac{\pi}{\mu^2} \frac{2}{a_0} = \left(E_B - \frac{p^2}{2\mu} \right)^2 |\Psi_B(\vec{p})|^2$$

$$\Rightarrow \quad \Psi_B(\vec{p}) = \pm \frac{\sqrt{8\pi/a_0}}{p^2 + 1/a_0^2} \rightarrow \frac{1}{a_0} = \gamma \rightarrow$$

$$\psi_B(\vec{p}) = \frac{\sqrt{8\pi\gamma}}{p^2+\gamma^2}$$

$$\mathcal{F}\{$$

$$\boxed{\psi_B(\vec{r}) = \frac{1}{\sqrt{4\pi}}\,\frac{\sqrt{2\gamma}\,e^{-\gamma r}}{r}}$$

Wave function for contact theory

SEE YOU
ON FRIDAY