NUCLEAR PHYSICS 14

SCATTERING THEORY

- EFFECTIVE RANGE EXPANSION
- FORMAL SCATTERING THEORY

RECAP

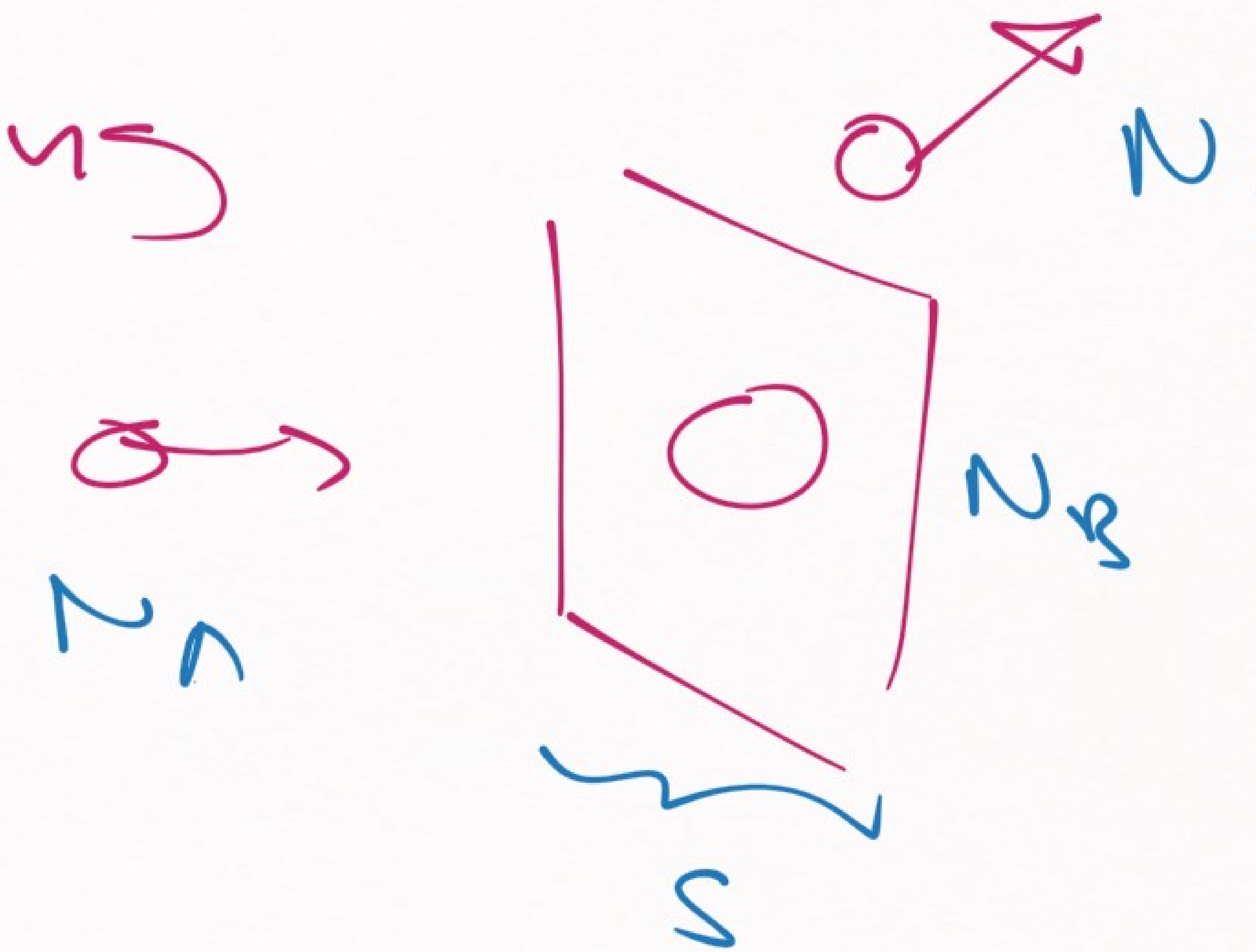
→ two-body system

1) Schrödinger equation
2) Partial wave expansion
3) Asymptotic solutions ($r \to \infty$)
4) Solutions near the origin ($r \to 0$)

(If you already know these things, do something else)

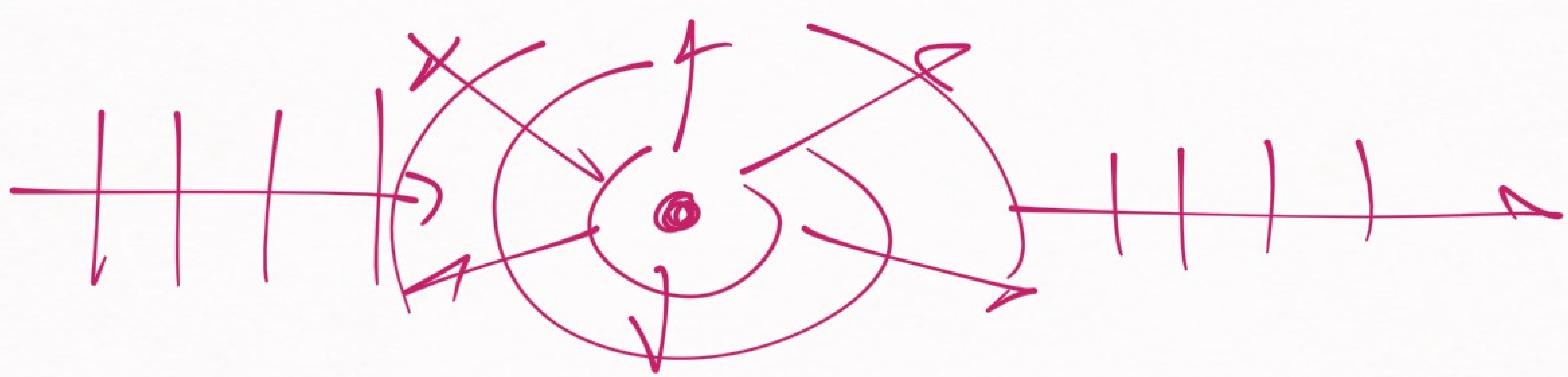
→ Two-body scattering

1) Classical scattering

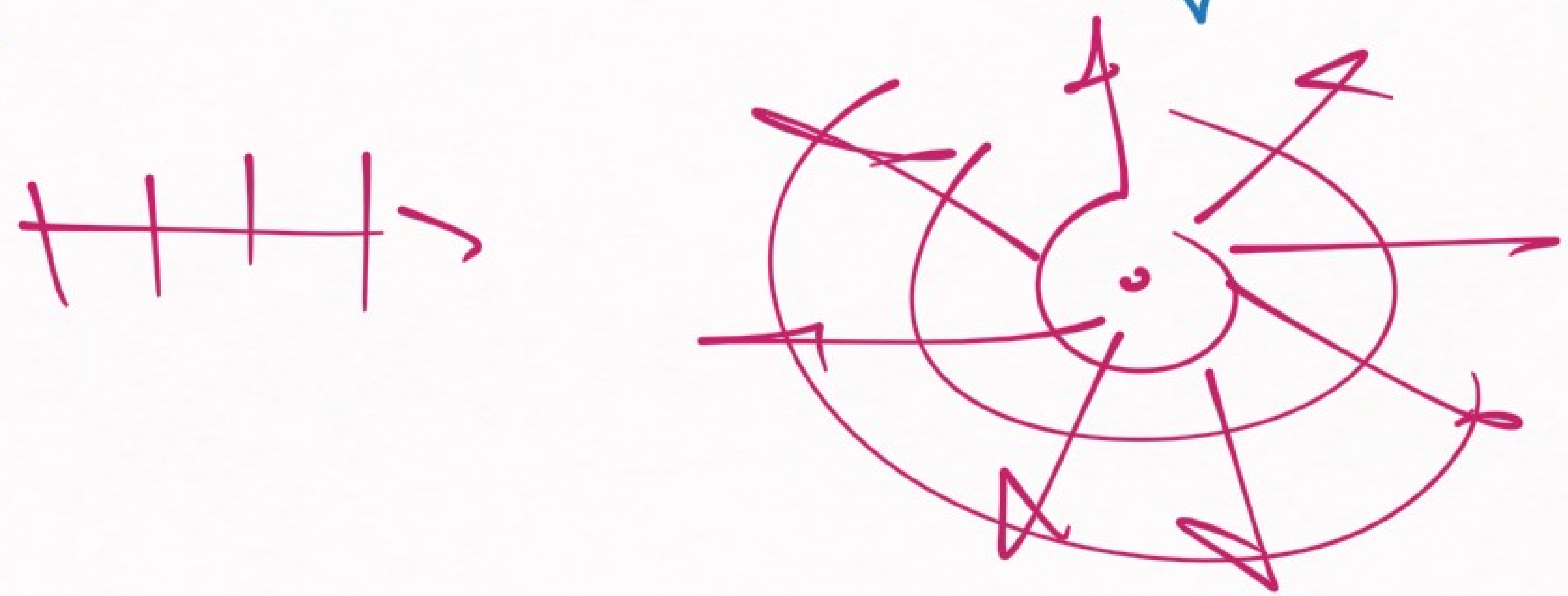


$$\sigma = \frac{N_S}{N_A N_B} S$$

2) Quantum mechanical scattering



$$\psi_k(\vec r) \xrightarrow[|\vec r|\to\infty]{} e^{i\vec k\cdot\vec r} + f(\Omega)\frac{e^{ikr}}{r}$$



$$\boxed{\sigma = \int |f(\Omega)|^2 \, d\Omega} \qquad \boxed{\frac{d\sigma}{d\Omega} = |f(\Omega)|^2}$$

Interesting connections:

$$u_\ell(r) \xrightarrow[r\to\infty]{} \sin\left(kr - \frac{\ell\pi}{2} + \delta_\ell(k)\right)$$

$$f(\theta) = \sum_\ell (2\ell+1) \times f_\ell(k) \times P_\ell(\cos\theta)$$

$$f_\ell(k) = \frac{1}{k\cot\delta_\ell - ik}$$

$$\boxed{\sigma = \frac{4\pi}{k^2} \sum_\ell \sin^2\delta_\ell(k)}$$

→ [LOW ENERGY BEHAVIOR OF $\delta_\ell(k)$]

$$\delta_\ell(k) \to -a_\ell k^{2\ell+1} + \mathcal{O}(k^{2\ell+3})$$

$$\boxed{\sigma \xrightarrow[k\to 0]{} 4\pi |a_0|^2}$$

Scattering length

1) $\sigma \xrightarrow[k \to 0]{} 4\pi |a_0|^2$ $\Rightarrow$ at low energies scattering is dominated by s-waves $\underline{\underline{(\ell = 0)}}$

2) $\boxed{\delta_0(k)}$

$\downarrow$

important phase shifts $\Big) \longrightarrow$ properties

3) $\boxed{NN}$ → isospin, spin, etc.

→ scattering will be more complicated because of these quantum number

$\boxed{\sigma \to 4\pi |a_0|^2}$ → w/o spin or anything else

$\boxed{NN}$

1) nucleons are fermions

$$(-1)^{L+S+T} = (-1)$$

2) S-waves ($L=0$)

$$S+T = \text{odd} \begin{cases} S=0,\ T=1 \\ S=1,\ T=0 \end{cases}$$

$S=0$, $J=1$ $\rightarrow$ singlet state that can be:

$$|11\rangle_J = |pp\rangle$$

$$|10\rangle_J = \frac{1}{\sqrt{2}}\left(|np\rangle + |pn\rangle\right)$$

$$|1\,{-1}\rangle_J = |nn\rangle$$

proton-proton

neutron-proton ($S=0$)

neutron-neutron

$S=1,\ I=0 \rightarrow |00\rangle_I = \frac{1}{\sqrt{2}}(|pn\rangle - |np\rangle)$

neutron-proton scattering w/ $S=1$

$$np \rightarrow \begin{cases} S=0 & \text{singlet} \quad (|00\rangle_S) \\ S=1 & \text{triplet} \quad (|1\,m_S\rangle_S, \\ & \qquad m_S = 1, 0, -1) \end{cases}$$

$$\left.\begin{array}{l} S=0 \\ S=1 \end{array}\right\} \longrightarrow \delta(k) \longrightarrow \left\{\begin{array}{l} \delta_s(k) \\ \delta_t(k) \end{array}\right.$$

(S-wave)

once we indicate L, S

$\Rightarrow$ isospin is determined

(we do not write it down explicitly)

NOTATION (IMPORTANT)

$$\boxed{{}^{2S+1}L_J}$$

$$^{2S+1}L_J$$

$2S+1$ → spin

L → orbital angular momentum

J → total angular momentum

$L = 0, 1, 2, 3, 4$

$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow$

$S \quad P \quad D \quad F \quad G$

singlet → $^1S_0$

triplet → $^3S_1$

usual names

No spin $\longrightarrow$ Spin

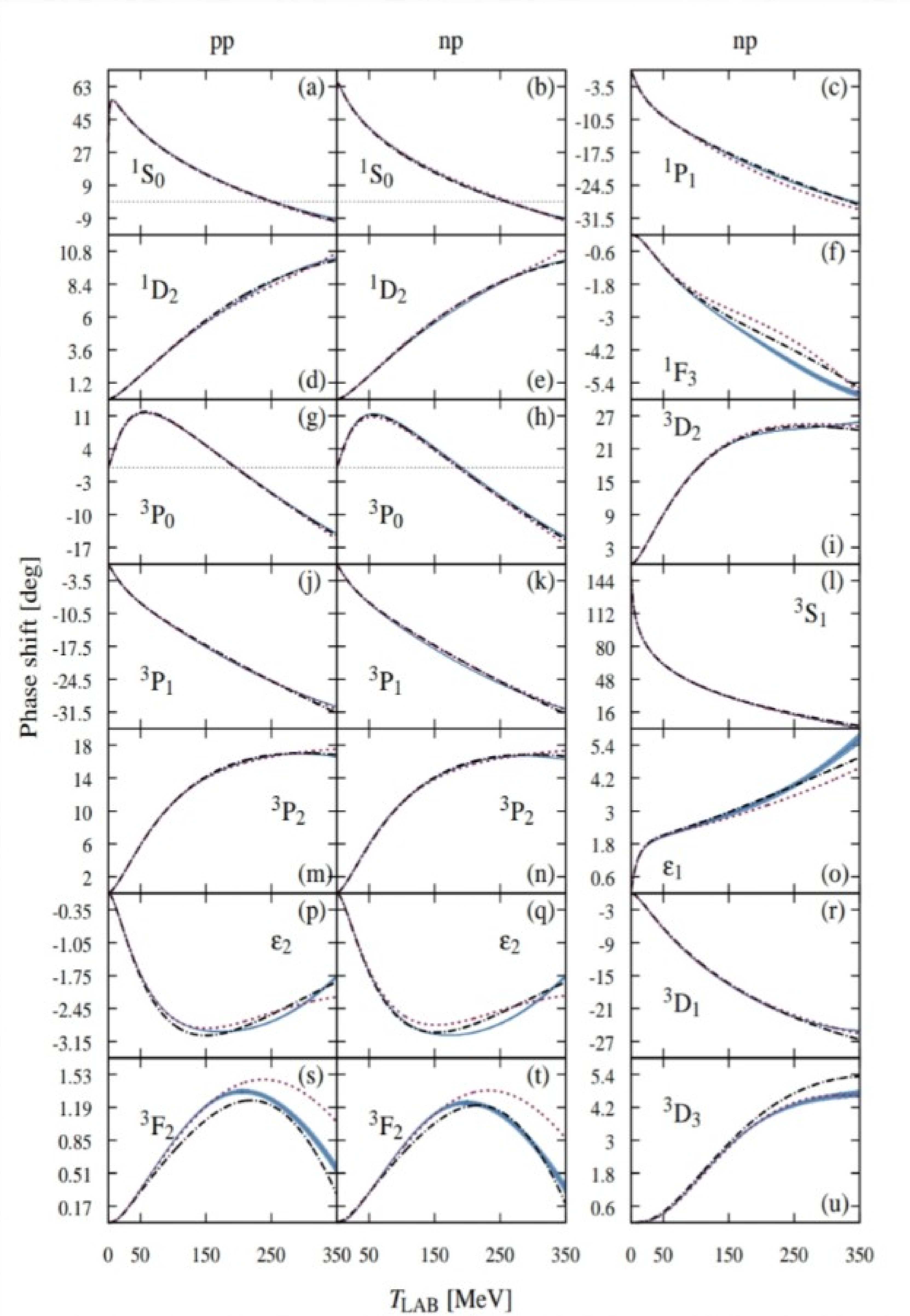
$\sigma \to 4\pi |a_0|^2$

$\sigma \to 4\pi\left(\frac{1}{4}|a_{0s}|^2 + \frac{3}{4}|a_{0t}|^2\right)$

(assumption: target, projectiles are unpolarized)

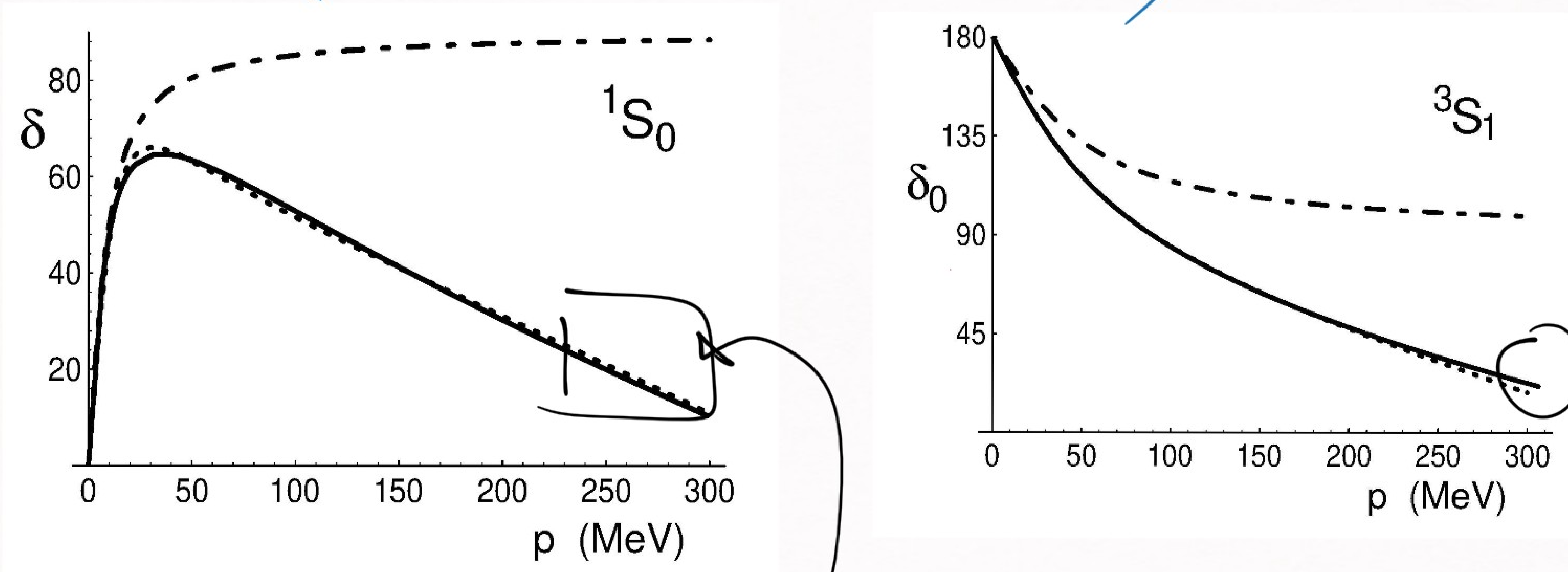
$$\delta(k) \longrightarrow \underbrace{\delta_{{}^1S_0}(k)\ ,\ \delta_{{}^3S_1}(k)}_{\text{look at their properties}}$$

$\to$ $\exists$ analyses of NN phase shifts



→ How the NN phase shifts look like

(dotted lines -> theory)



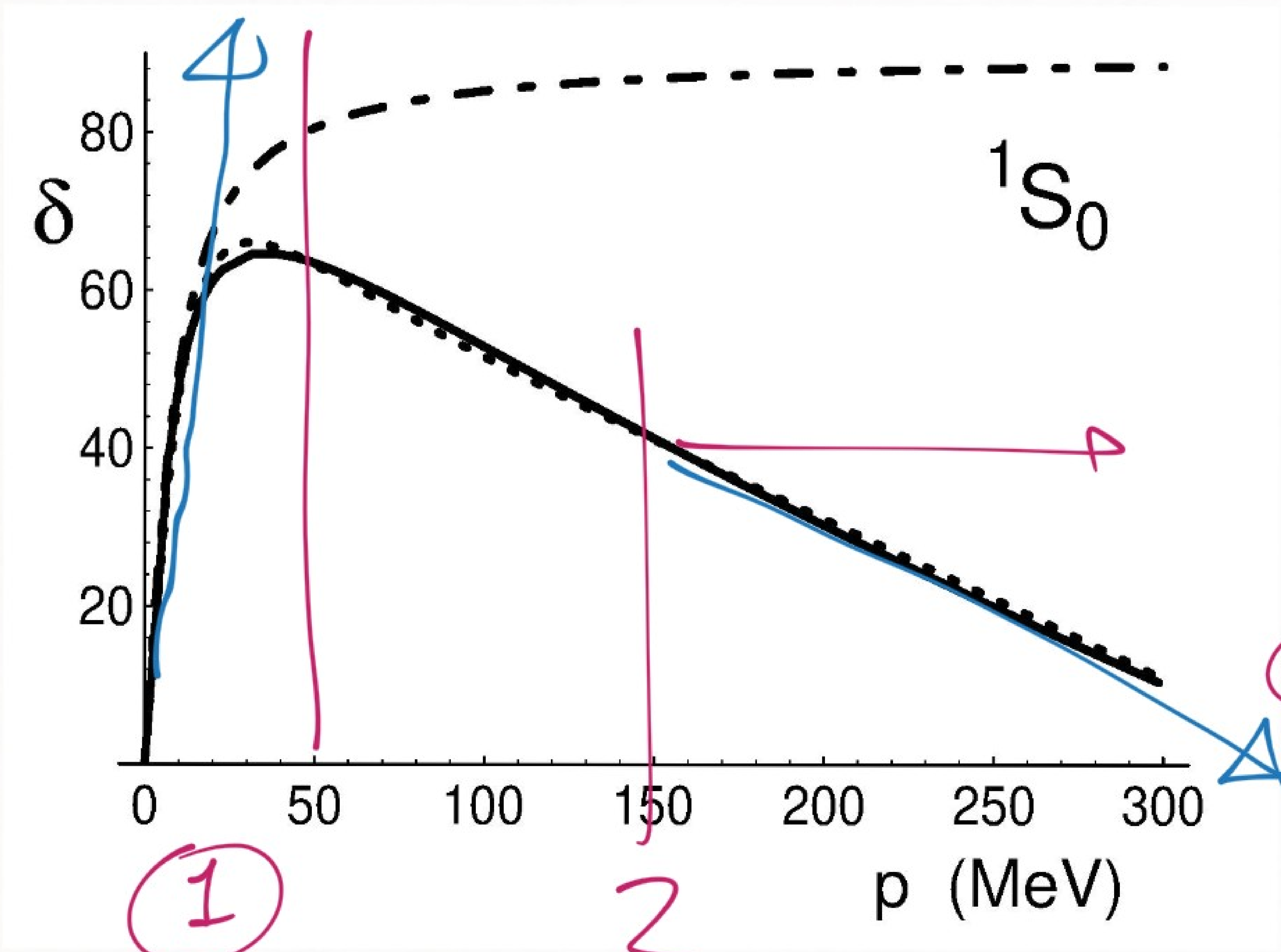
singlet

(solid line)

triplet

(solid line)

[PROPERTIES TO LOOK AT]



1. Low momentum behavior

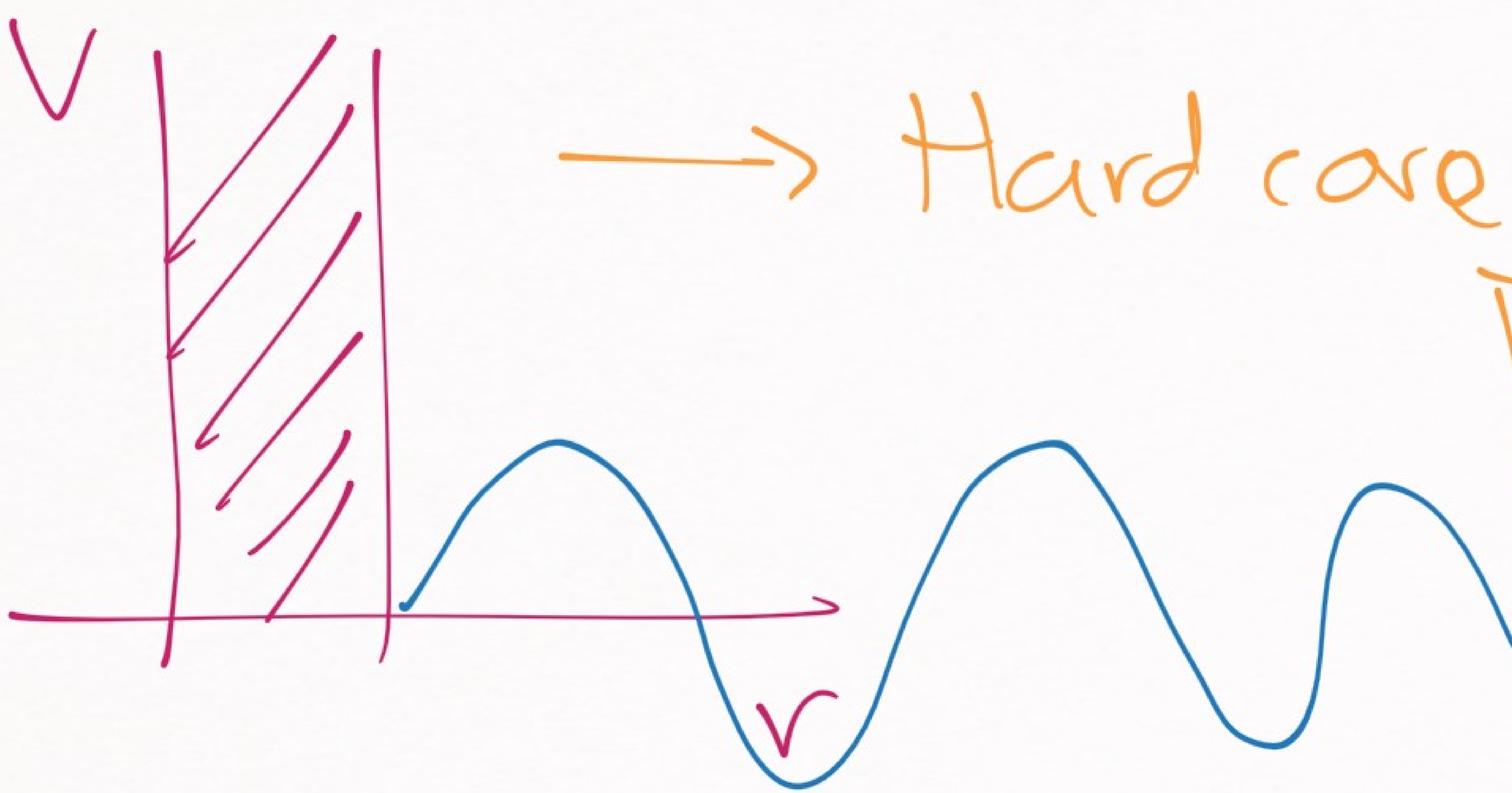
$\delta(p) \to -a_0 p + \ldots$

$[a_0 \simeq -27.3\,\mathrm{fm}]$

2. Intermediate momentum behavior

straight line

(2) straight line → interesting because it points towards a property of the nuclear force

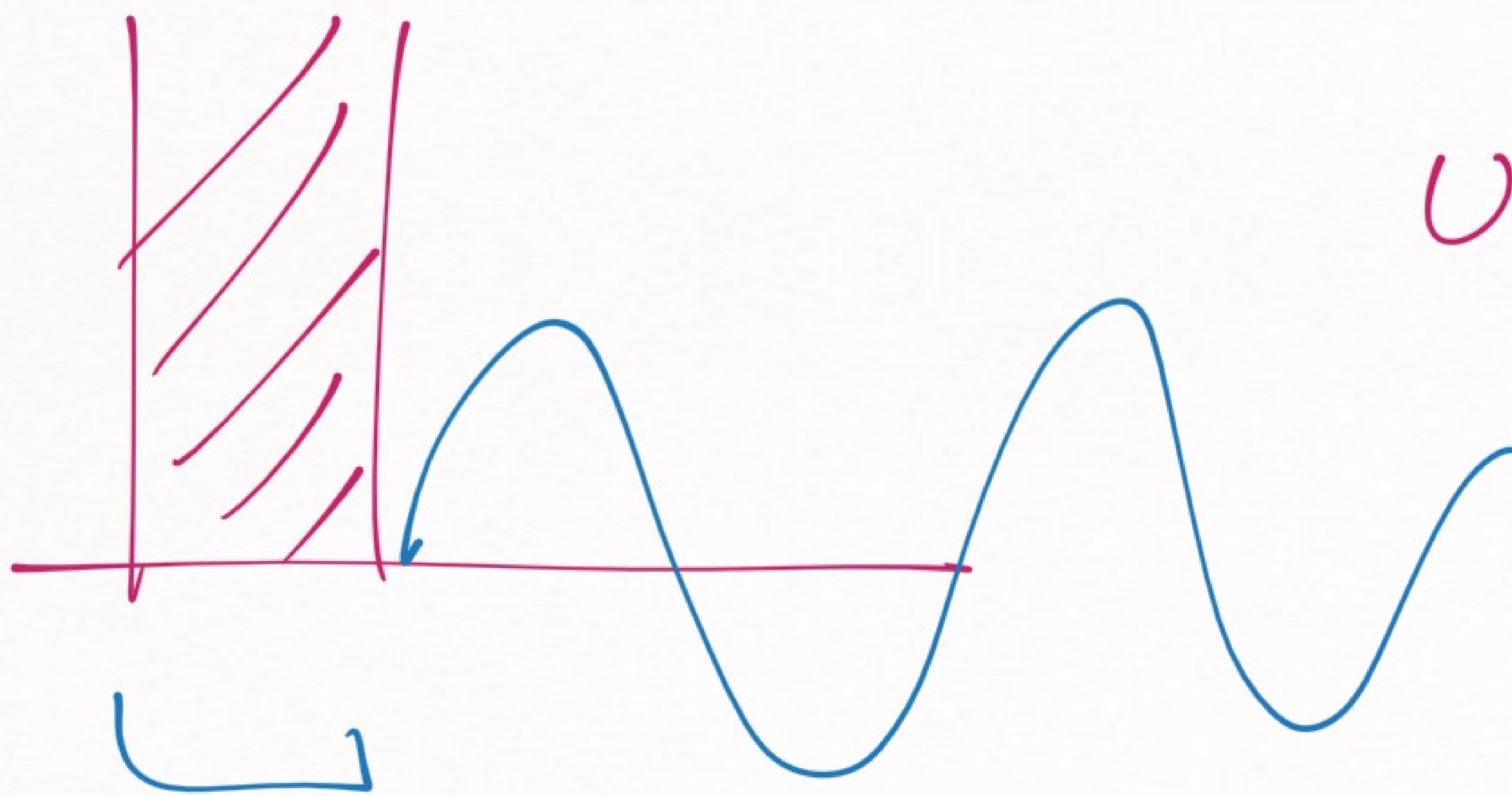


→ Hard core potential:

$$\begin{cases} V(r) = \infty \; , r < R_c \\ V(r) = 0 \; , r > R_c \end{cases}$$

Hard core potential

$u(R_s) = 0$



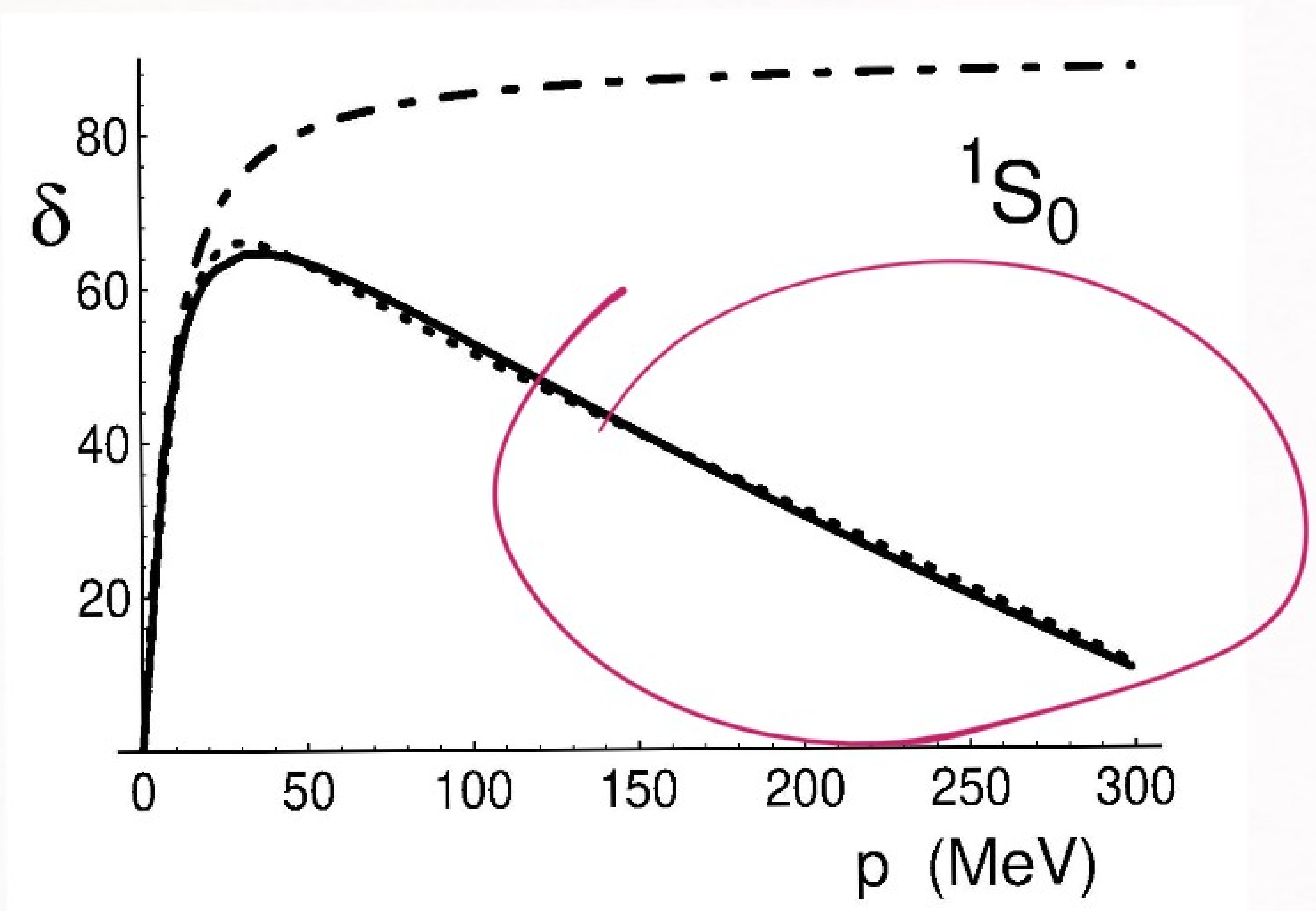
$$u_k(r) = \sin(k(R - R_s))$$
$$= \sin(kR + \delta(k))$$
$$\boxed{\delta(k) = -kR_s}$$

particles can't enter the core
(because is infinitely repulsive)

HARD CORE POTENTIAL:

$$\boxed{\delta(k) \to -kR_s}$$

similar to $^1S_0$ $\delta(k)$

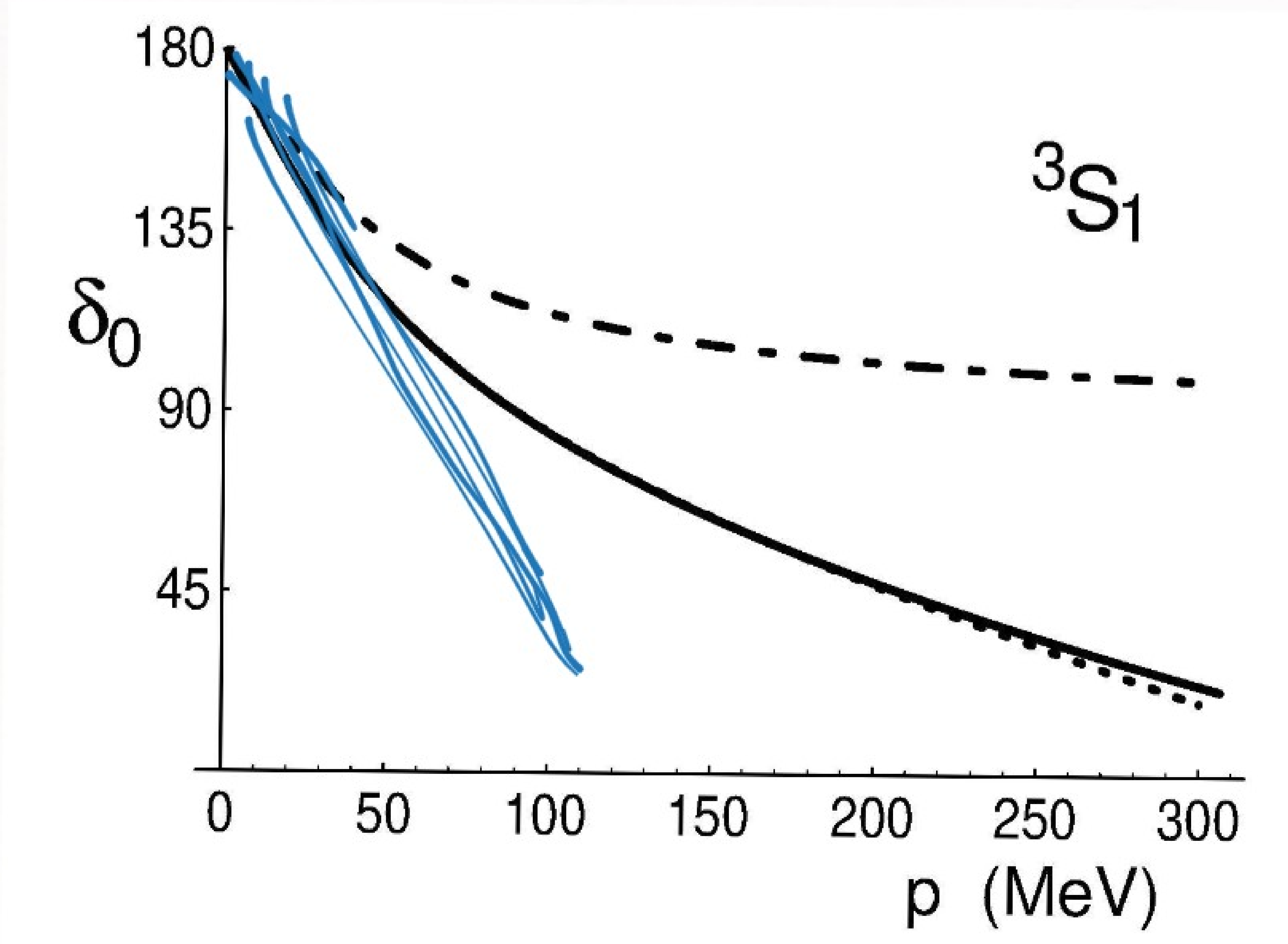


$\delta(k) \to -k(R_s - R^*)$

only difference → offset

$$2 \to \underbrace{\delta_{{}^1S_0}(k) \longrightarrow -k\,(R_s - R^*)}_{\text{indicates the } \exists \text{ of short-range repulsion in the two-nucleon system}}$$

TRIPLET



(1) LOW MOMENTUM BEHAVIOR

$$\delta(p) \to \pi - a_0 p + \dots$$

Why this?

$a_0 \simeq 5.4$ fm

$\exists$ theorem in scattering:

$$\delta(p=0) - \delta(p\to\infty) = n\pi \,, \quad n \text{ \# bound states}$$

$\delta(p) \to 0$ (choice)
$p \to \infty$

$\Rightarrow$ ${}^3S_1$ $\quad \delta(0) \to \pi - a_0 p + \dots$
$p \to 0$

The deuteron is a ${}^3S_1$ state

$$\delta_{^3S_1}(0) - \delta_{^3S_1}(\infty) = \pi$$

$\boxed{\text{NEXT STEP}}$ (consider S-waves only)

$$\delta(k) \longrightarrow -a_0 k + \underbrace{\mathcal{O}(k^3)}_{}$$

what goes next?

This is solved w/ the effective range expansion (ERE) =D

$\Rightarrow$ $\underbrace{k\cot\delta(k)} = -\frac{1}{a_0} + \mathcal{O}(k^2)$

why? Better expansion properties

$\delta(k) \to -a_0 k$ ($\cot x \to \frac{1}{x}$ for $x \to 0$)

$k\cot\delta \to \frac{k}{\delta(k)} \to -\frac{1}{a_0}$

[EFFECTIVE RANGE EXPANSION]

$$k\cot\delta(k) = -\frac{1}{a_0} + \frac{1}{2} r_0 k^2 + \sum_{n=2}^{\infty} v_n k^{2n}$$

↳ a complete description of $\delta(k)$ at low energies

$\Rightarrow$

→ $\boxed{\text{DERIVATION}}$

DERIVATION OF ERE

$$\delta(k) \to -a_0 k + \underbrace{\text{(corrections)}}_{\text{which are these corrections?}}$$

ERE

$$k\cot\delta = -\frac{1}{a_0} + \frac{1}{2} r_0 k^2 + \sum_{n=2}^{\infty} v_n k^{2n}$$

(Taylor expansion)

→ Trick : Wronskian identity

(Really useful in QM)

⇝ a comparison of Schrödinger equations

↳

Eq.(1) $\rightarrow$ $-u_k'' + 2\mu V(r)\, u_k(r) = k^2 u_k(r)$

$\left(u_k(r) \rightarrow \sin(kr+\delta)/\underline{\underline{\sin\delta}} \;,\; r \rightarrow \infty \right)$

Eq.(2) $\rightarrow$ $-u_0'' + 2\mu V(r)\, u_0(r) = 0$

$\left(u_0(r) \rightarrow 1 - \frac{r}{a_0} \;,\; r \rightarrow \infty \right)$

(assumption $\rightarrow$ finite range potential)

What is the idea behind the Wronskian identity?

$$(\text{Eq. }(1)) \times u_0(r) - (\text{Eq. }(1)) \times u_k(r)$$

$$\Downarrow \quad 1$$

$$-(u_k'' u_0 - u_k u_0'') = k^2 u_k u_0 \overset{2}{\Longrightarrow}$$

(2)

$\Rightarrow \quad (u_k'' u_0 - u_k u_0'') = (u_k' u_0 - u_k u_0')'$

(Exact derivative)

$$-(u_k' u_0 - u_k u_0')' = k^2 u_k u_0$$

$\Downarrow$ Integrate $\int_{r_c}^{R} dr$

$$-(u_k' u_0 - u_k u_0')\Big|_{r_c}^{R} = k^2 \int_{r_c}^{R} u_k(r) u_0(r)\, dr$$

Second Wronskian identity;

Eq (3) $\to$ $-v_k'' = k^2 v_k$ $\qquad v_k = \dfrac{\sin(kr+\delta)}{\sin\delta}$

Eq (4) $\to$ $-v_0'' = 0$ $\qquad v_0 = 1 - r/a_0$

v_k, v_0 / $\left.\begin{array}{l} u_k \xrightarrow[r\to\infty]{} v_k \\ u_0 \xrightarrow[r\to\infty]{} v_0 \end{array}\right\}$ useful property

→ Repeating the same steps:

$$-\left(v_n' v_0 - v_n v_0'\right)\Big|_{r_c}^{R_c} = k^2 \int_{r_c}^{R} v_n(r)\, v_0(r)\, dr$$

Next step $\longrightarrow$ difference of the two Wronskians

$$-\left(u_k' u_0 - u_k u_0'\right)\Big|_{r_c}^{R} = k^2 \int_{r_c}^{R} u_k(r)\, u_0(r)\, dr$$

$$-\left(v_k' v_0 - v_k v_0'\right)\Big|_{r_c}^{R} = k^2 \int_{r_c}^{R} v_k(r)\, v_0(r)\, dr$$

$= D$

$$(u_k' u_0 - u_0' u_k)\Big|_{r_c}^{R} - (v_k' v_0 - v_0' v_k)\Big|_{r_c}^{R}$$

$$= k^2 \int_{r_c}^{R} (v_k v_0 - u_k u_0)\, dr$$

$$\left.\begin{aligned} &\lim R \to \infty \\ &\lim r_c \to 0 \end{aligned}\right\} \to \text{easy to take} \Rightarrow$$

$$k\cot\delta + \frac{1}{a_0} = k^2\int_0^\infty (v_k v_0 - u_k u_0)\,dr$$

$$\Rightarrow$$

$$\left[\; k\cot\delta = -\frac{1}{a_0} + k^2\int_0^\infty (v_k v_0 - u_k u_0)\,dr \;\right]$$

$\longrightarrow$ few comments

most important observation:

$$\left.\begin{aligned} u_k &= u_0 + k^2 u_2 + k^4 u_4 + \dots \\ v_k &= v_0 + k^2 v_2 + k^4 v_4 + \dots \end{aligned}\right\} \text{good } k^2 \text{ expansion}$$

$$k \cot \delta = -\frac{1}{a_0} + k^2 \boxed{\int_0^\infty (v_k v_0 - u_k u_0)\, dr}$$

good k^2-expansion

$$k\cot\delta = -\frac{1}{a_0} + k^2 \int_0^\infty (v_k v_0 - u_k u_0)\,dr$$

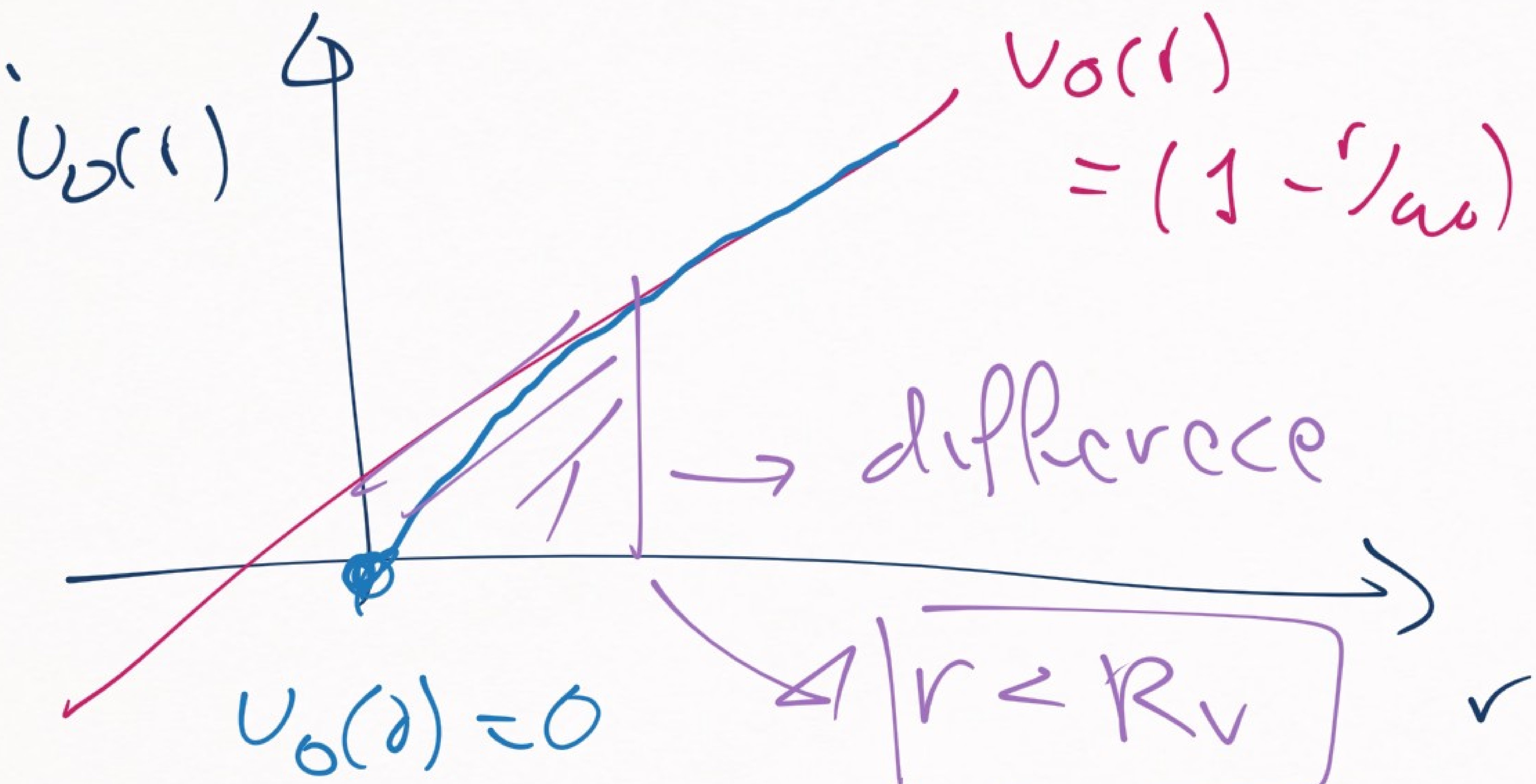
k^2-expansion

$$\Updownarrow$$

$$k\cot\delta = -\frac{1}{a_0} + \frac{1}{2}\underbrace{r_0}_{\text{effective range}} k^2 + \sum_{n=2}^{\infty} \underbrace{v_n}_{\text{shape parameters}} k^{2n}$$

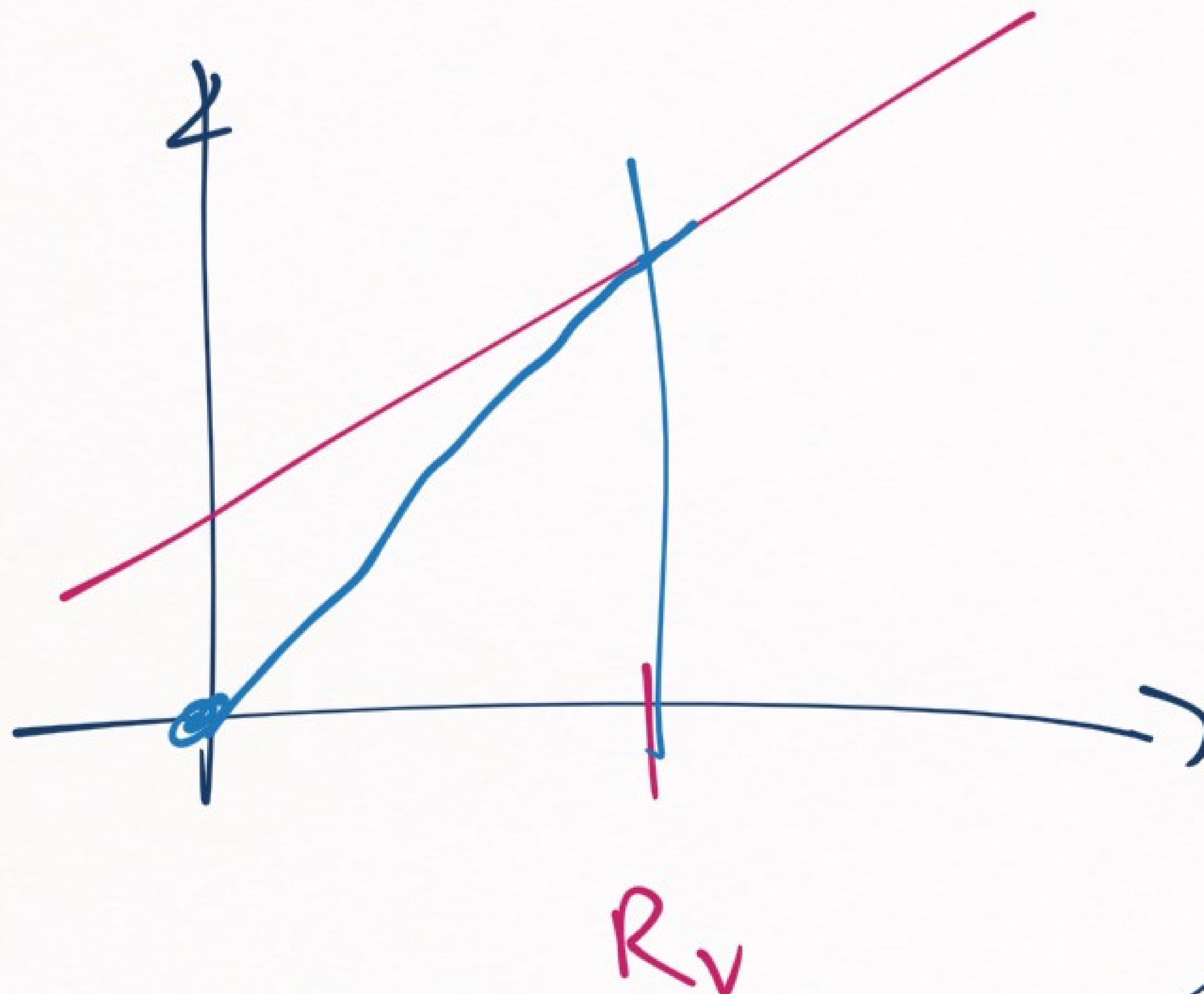
Why it's called ERE?

$$r_0 = 2\int_0^\infty (v_0^2 - u_0^2)\, dr = 2\int_0^\infty \left[\left(1 - \frac{r}{a_0}\right)^2 - u_0^2(r)\right] dr$$



$u_0(r) \longrightarrow 1 - \frac{r}{a_0}$

$r \gg$ range of $V(r)$



$$r_0 = 2\int_0^{\infty} \left[\left(1 - \frac{r}{a_0}\right)^2 - u_0^2(r) \right] dr$$

$$\simeq 2\int_0^{R_V} \left[\left(1 - \frac{r}{a_0}\right)^2 - u_0^2(r) \right] dr$$

$$+ \underbrace{2\int_{R_V}^{\infty} \left[\left(1 - r/a_0\right)^2 - u_0^2(r) \right] dr}_{\simeq 0}$$

$$\boxed{r_0 \propto R_V}$$

$$k\cot\delta = -\frac{1}{a_0} + \frac{1}{2} r_0 k^2 + \sum_{n=2}^{\infty} v_n k^{2n}$$

proportional to the range of the potential

depend on shape of potential

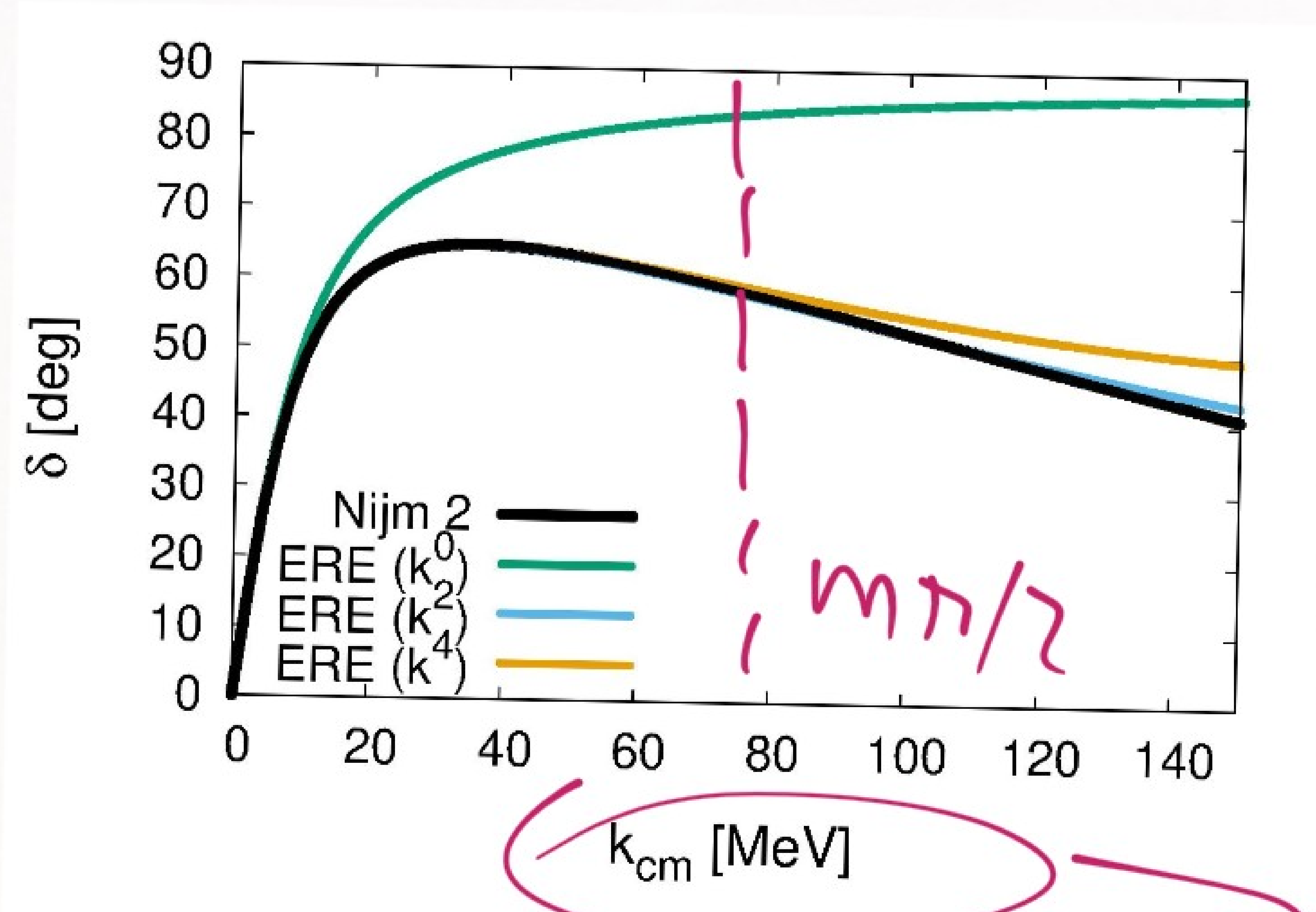
ERE

Another important property:

If $V(r) = f(r)\, e^{-mr}$ → exponential fall

$k \cot \delta = -\frac{1}{a_0} + \frac{1}{2} r_0 k^2 + \sum v_n k^2$ only converges for $|k| < \frac{m}{2}$

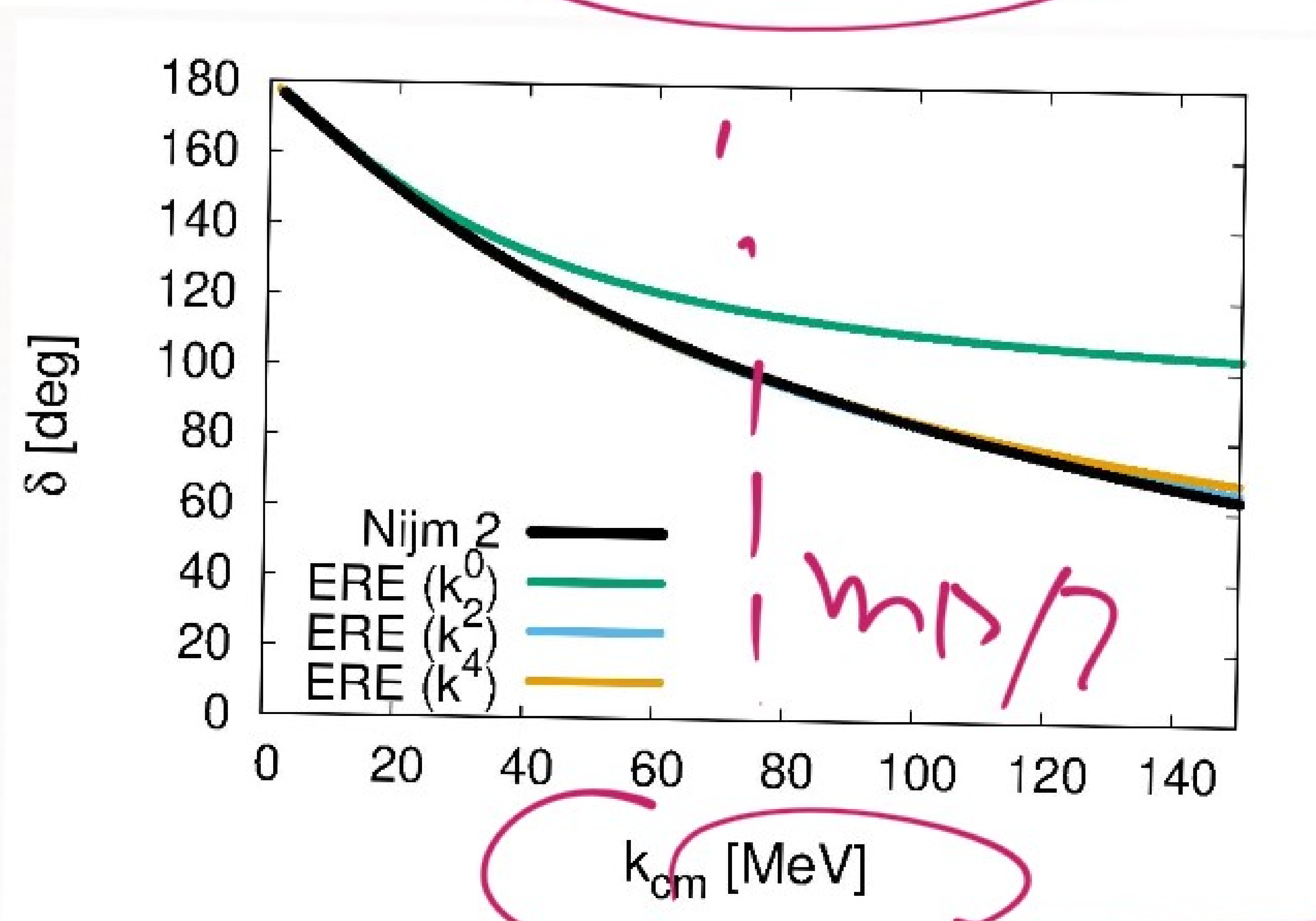
ERE only converges at small k



$$k\cot\delta = -\frac{1}{a_0} + \frac{1}{2} r_0 k_{cm}^2$$

$$+ v_2 k_{cm}^4$$

(first 3 terms)



$\rightarrow$ small k (convergence)

$^1S_0$

$a_0 \simeq -23.7\ \text{fm}$

$r_0 \simeq 2.7\ \text{fm}$

$v_2 \simeq -0.5\ \text{fm}^3$

$^3S_1$

$a_0 \simeq 5.4\ \text{fm}$

$r_0 \simeq 1.7\ \text{fm}$

$v_2 \simeq 0.65\ \text{fm}^3$

$\Rightarrow$ Important over

indeed we have the correct range

similar to

$\frac{1}{m_\pi} - \frac{2}{m_\pi}$

$R_\pi \simeq 1/m_\pi \simeq 1.4\ \text{fm}$

FORMAL SCATTERING THEORY

$\rightsquigarrow$ (T-MATRIX)

(objective)

idea $\rightsquigarrow$ recast scattering theory in a more abstract language

(good in specific situations)

Motivation → TWO POSSIBLE VIEWS OF QM

1) Traditional view of wave functions & differential equations

$$\left[-\frac{\nabla^2}{2\mu} + V(\vec{r})\right]\psi(\vec{r}) = E\,\psi(\vec{r})$$

2) Operators acting on vectors on a Hilbert space

$$\hat{H}|\psi\rangle = E|\psi\rangle$$

Reconsider scattering $\longrightarrow$ rewrite everything

1) $\psi(\vec{r}) \longrightarrow e^{i\vec{k}\cdot\vec{r}} + f(\Omega)\dfrac{e^{ikr}}{r}$ $\qquad \dfrac{d\sigma}{d\Omega} = |f(\Omega)|^2$

2) $|\psi_{\vec{k}}\rangle = |\vec{k}\rangle + G_0 T |\vec{k}\rangle$ $\qquad$ T-matrix

$$\frac{d\sigma}{d\Omega} = |\langle\vec{k}'|\hat{f}|\vec{k}\rangle|^2 \qquad f(\Omega) = -\frac{\mu}{2\pi}\langle\vec{k}'|T|\vec{k}\rangle$$

$$= \left|\frac{\mu}{2\pi}\langle\vec{k}'|\hat{T}|\vec{k}\rangle\right|^2$$

Objective $\rightarrow$ construct an operator $\hat{T}$ / its matrix elements are proportional to the scattering amplitude

T-matrix

(long process w/ many steps)

Warm up:

1) $H|\phi\rangle = E|\phi\rangle$

Hamiltonian

wave function (now a vector)

2) $H = H_0 + V$

Kinetic energy

potential

3) $$\underbrace{(E - H_0)|\psi\rangle = V|\psi\rangle}_{\text{amenable to a perturbative expansion}}$$

→ Green function methods
as a way to solve it $\Rightarrow$

$|\phi\rangle \rightarrow \langle\vec{r}|\phi\rangle = \phi(\vec{r})$

ANSATZ

$$\left[\phi(\vec{r}) = e^{i\vec{k}\cdot\vec{r}} + \int d^3\vec{r}\, G_0(\vec{r}-\vec{r}\,')V(\vec{r}\,')\,\phi(\vec{r}\,')\right]$$

A way of writing the solution (a trick)

We will try this trick with:

$$(E - H_0)|\phi\rangle = V|\phi\rangle$$

$\Downarrow$ we try the ansatz (in wf language)

(this happens after a few calculations)

$$(E - H_0)\,G_0(\vec{r} - \vec{r}\,') = \delta^{(3)}(\vec{r} - \vec{r}\,')$$

$H_0 = -\frac{\hbar^2 \nabla^2}{2m}$ $\longrightarrow$ Green function

Little RECAP

Step 1 $\rightarrow$ $\psi(\vec{r}) = e^{i\vec{k}_i\cdot\vec{r}} + \int d^3\vec{r}\,' G_0(\vec{r}-\vec{r}\,')V(\vec{r}\,')\psi(\vec{r}\,')$

Step 2 $\longrightarrow$ $(E - H_0)\,G_0(\vec{r}) = \delta^{(3)}(\vec{r})$

G_0 is a Green function of H_0

Step 3 $\longrightarrow$ find a solution for G_0

(our objective for today)

How to find this solution? → go to p-space

$$G_0(\vec{r}) = \int \frac{d^3\vec{\ell}}{(2\pi)^3} G_0(\vec{\ell})\, e^{i\vec{\ell}\cdot\vec{r}}$$

$$(E - H_0)\, G_0(\vec{r}) = \delta^{(3)}(\vec{r})$$

$$\int d^3\vec{r}\; \delta^{(3)}(\vec{r})\, e^{-i\vec{\ell}'\cdot\vec{r}} = 1$$

$\ominus \rightsquigarrow$ $$\boxed{\begin{aligned}(E - H_0)\, G_0(\vec{p}) &= 1 \\ H_0 &= \frac{\vec{p}^{\,2}}{2\mu}\end{aligned}}$$

$$\Downarrow$$

$$\boxed{G_0(\vec{p}) = \frac{1}{E - \frac{\vec{p}^{\,2}}{2\mu}}} \Rightarrow$$

$$\Rightarrow \boxed{G_0(E) = \frac{1}{E - H_0}}$$

$\rightarrow$ G_0 is the propagator (or the resolvent operator)

$\rightarrow \boxed{G_0(\vec{r})} \longrightarrow$ we want to analyze $\phi(\vec{r})$

$$|\phi\rangle = |\vec{k}\rangle + G_0 V |\phi\rangle$$

(a way of writing

$$\phi(\vec{r}) = e^{i\vec{k}\cdot\vec{r}} + \int \cdots)$$

$\downarrow$

Fourier transform

Σ

$$G_0(\vec{r}) = \int \frac{d^3\vec{l}}{(2\pi)^3} G_0(\vec{l})\, e^{i\vec{l}\cdot\vec{r}} \Rightarrow \left\{ G_0(\vec{l}) = \frac{1}{E - \frac{l^2}{2m}} \right\}$$

$$= \frac{1}{2\pi r^2} \int_0^\infty dl\, \frac{l \sin(lr)}{E - \frac{l^2}{2m}} = \frac{1}{4\pi r^2} \operatorname{Im}\left[\int_{-\infty}^{\infty} dl\, \frac{l e^{ilr}}{E - \frac{l^2}{2m}} \right]$$

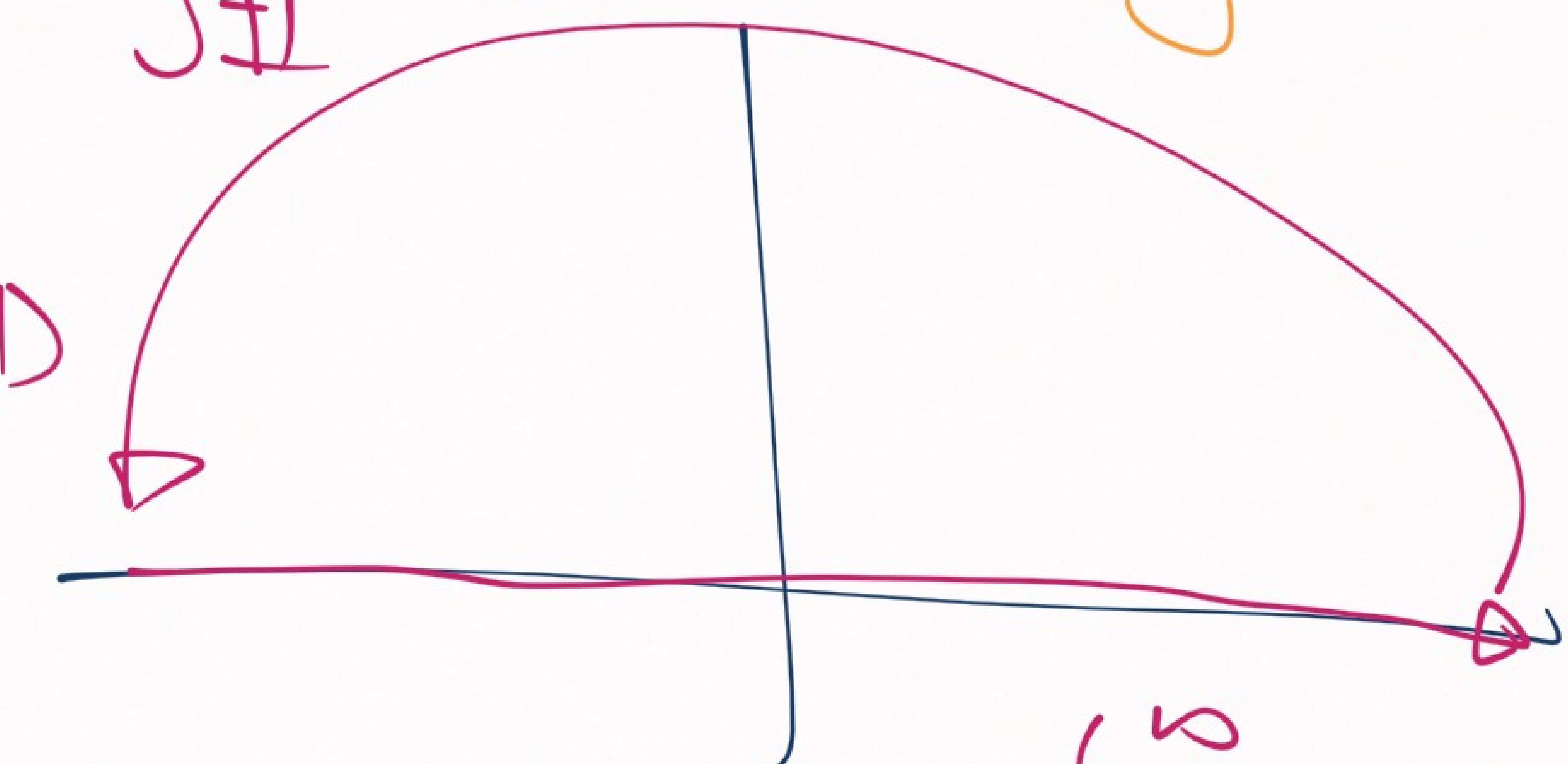
$$= \frac{1}{4\pi r^2} \int_{-\infty}^{\infty} dl\, \frac{l e^{ilr}}{E - \frac{l^2}{2m}}$$

$$G_0(\vec{r}) = \int_{-\infty}^{\infty} d\rho \, \rho \, \frac{e^{i\rho r}}{E - \frac{\rho^2}{2\mu}}$$

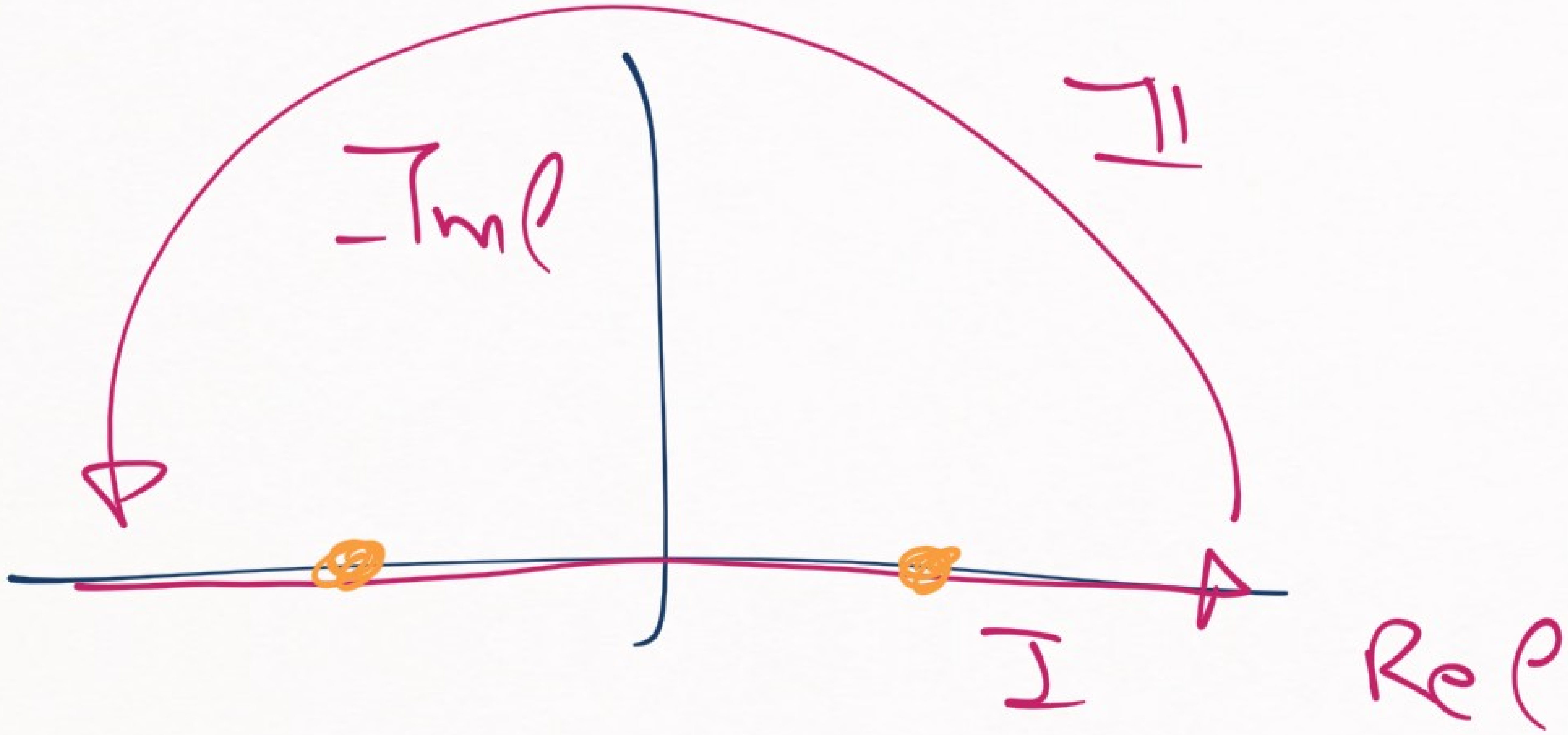
$\longrightarrow$ Residues (Contour Integration)

$$\int_{II} \to 0$$

$$\int_{-\infty}^{\infty} = \oint \Rightarrow$$



$$\int_{-\infty}^{\infty}$$



$$\int d\rho \, \frac{\rho \, e^{i\rho r}}{E - \frac{\rho^2}{2m}}$$

Two poles

$$\rho = \pm k$$

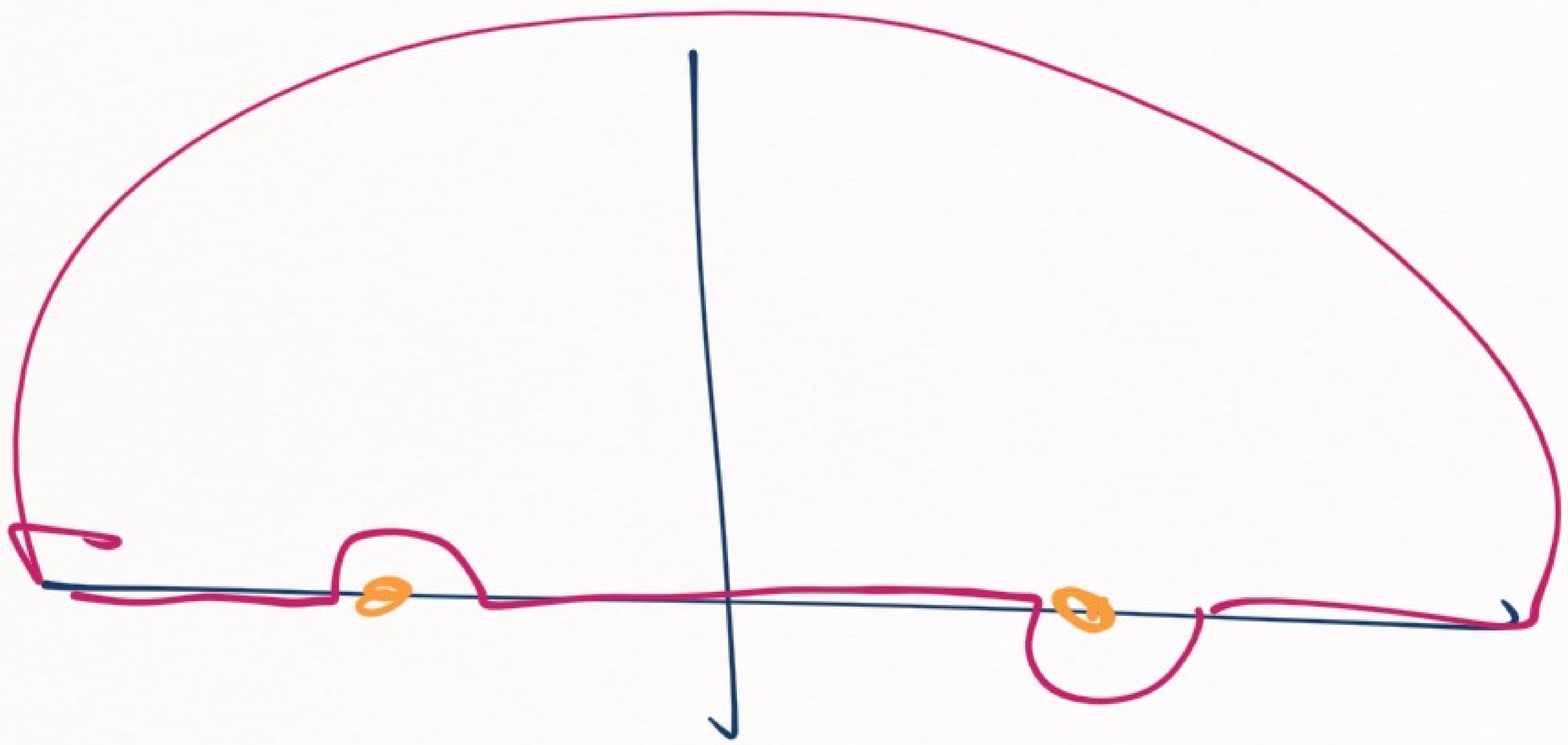
$$k = \sqrt{2mE}$$

$$\oint = \int_{I} + \int_{II}$$

$$\int_{I} = \int_{-\infty}^{+\infty} \quad ✓$$

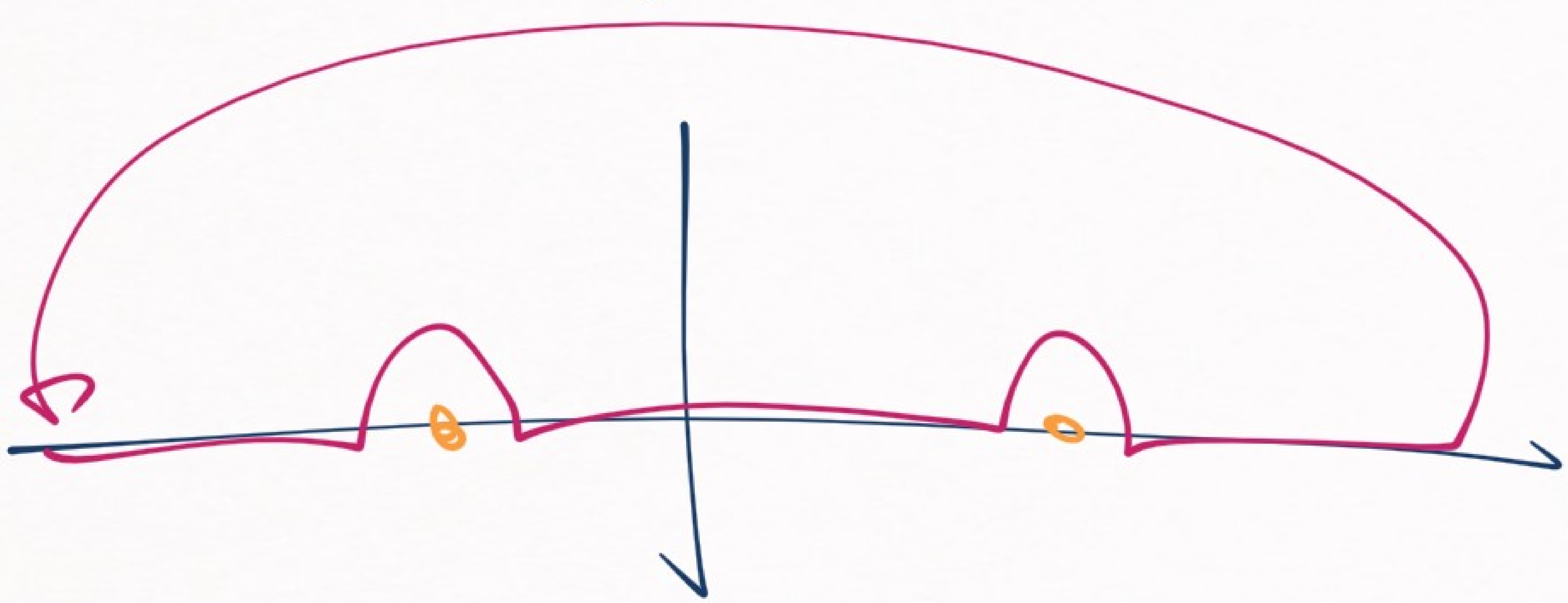
$$\int_{II} \to 0 \quad \text{if } \operatorname{Im}(\rho) > 0 \quad ✓$$

Decide how to avoid the poles:

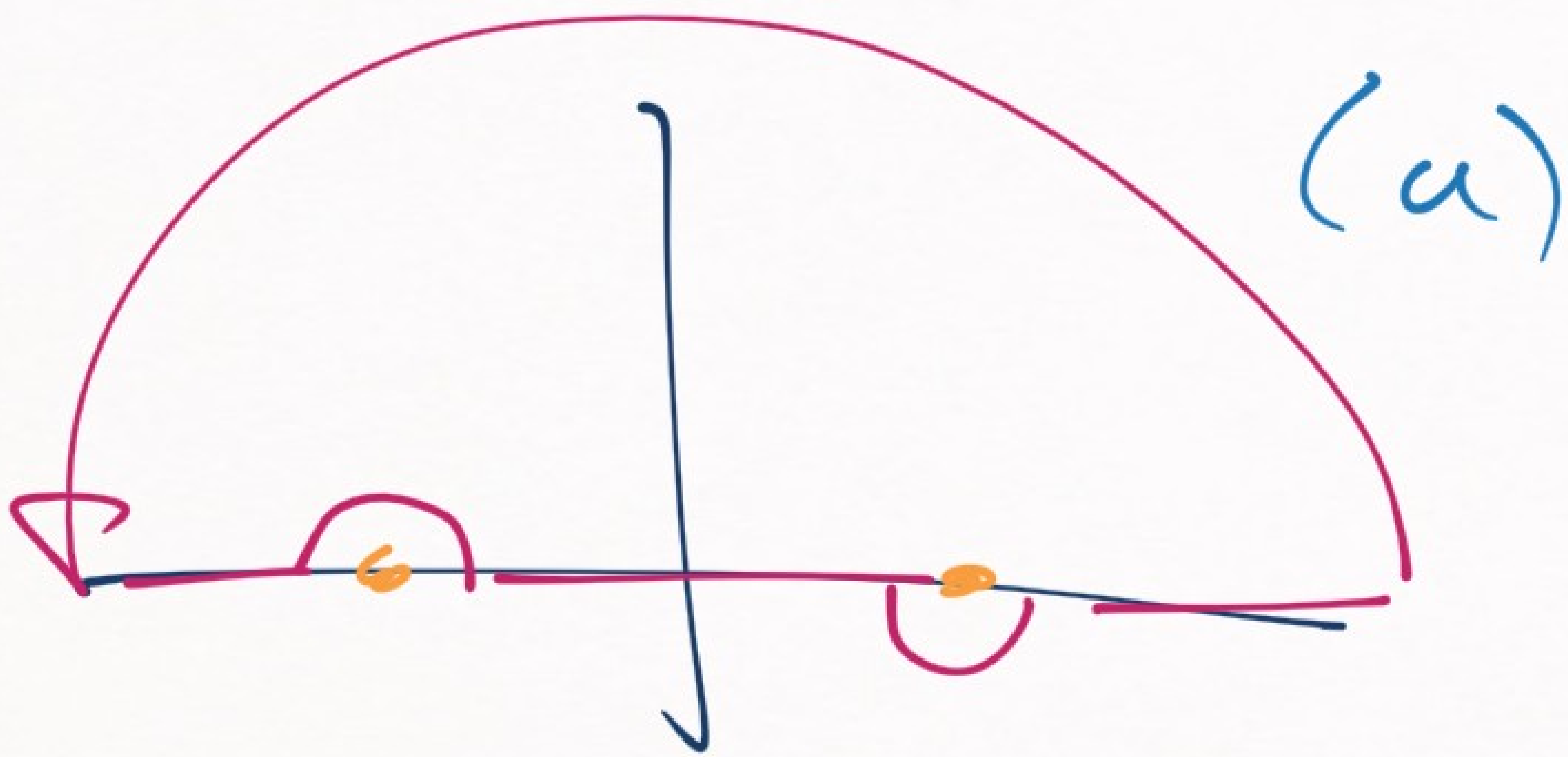


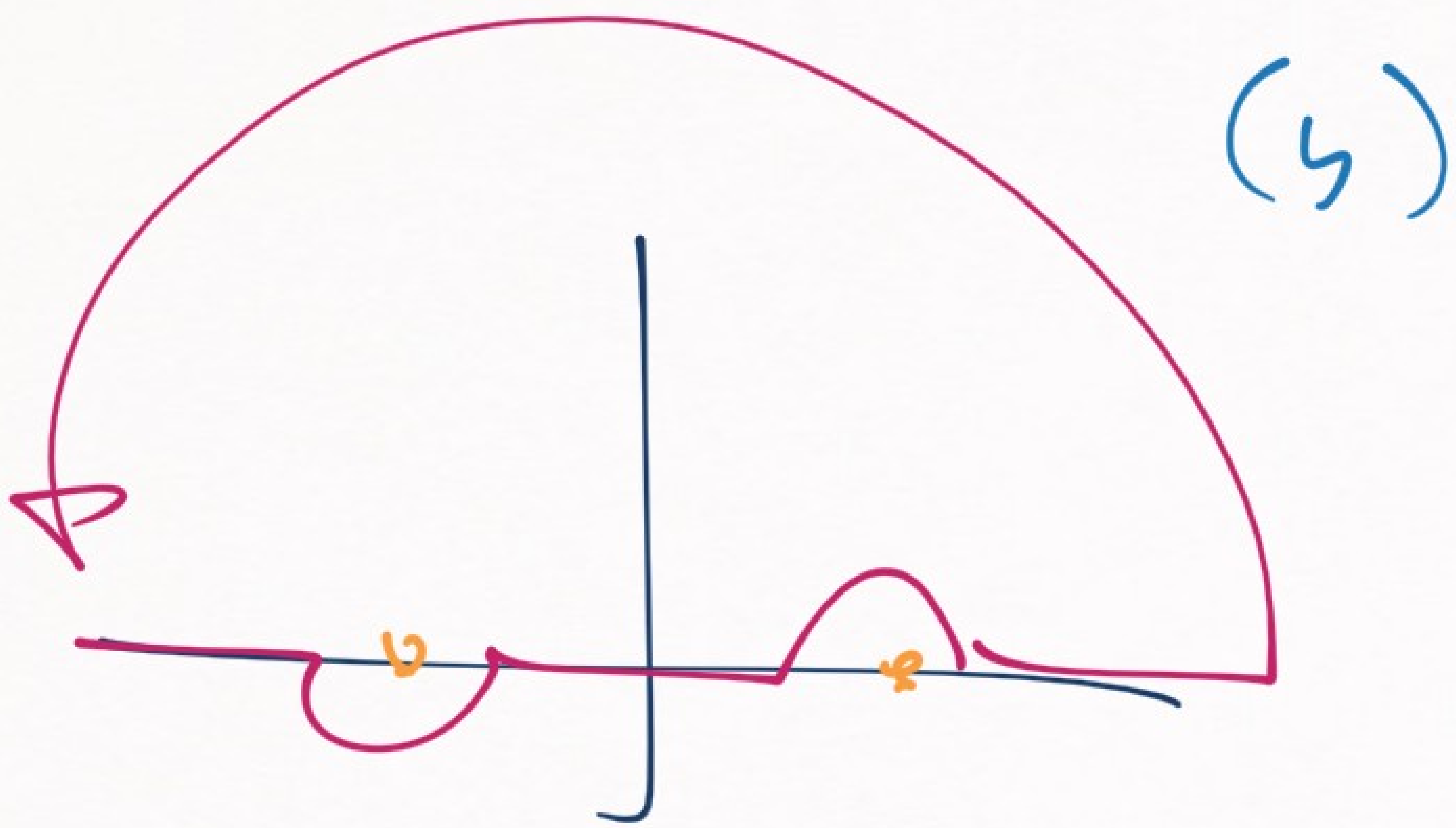
Four possibilities

(two poles
+ two decisions
per pole)



$\to 0 \qquad \oint(\text{no poles}) = 0$





$$p = \pm\sqrt{2\mu E} = \pm k$$

$$E \to E + i\epsilon \quad (a)$$

$$E - i\epsilon \quad (b)$$

$$p \to \pm\left(\sqrt{2\mu E} \pm i\epsilon\right)$$

(not difficult to check)

Integrate for contours (a) and (b):

Contour (a) $\rightarrow$ $G_0(\vec{r}) = -\frac{\mu}{2\pi}\frac{e^{+ikr}}{r}$

$E + i\epsilon$

Contour (b) $\rightarrow$ $G_0(\vec{r}) = -\frac{\mu}{2\pi}\frac{e^{-ikr}}{r}$

$E - i\epsilon$

$\rightarrow$ $\boxed{\text{+ or - sign}}$

$$\left[G(\vec{r}; E \pm i\epsilon) = -\frac{\mu}{2\pi} \frac{e^{\pm ikr}}{r} \right]$$

→ two possibilites

→ Physical interpretation of each one

$E \pm i\epsilon$

$$\phi(\vec{r}) = e^{i\vec{k}\cdot\vec{r}} + \int d^3\vec{r}\, G_0(\vec{r}-\vec{r}')\, V(\vec{r}')\, \phi(\vec{r}')$$

$E \pm i\epsilon \rightarrow \phi^{(\pm)}(\vec{r})$ to be compare

with: $\psi_k(\vec{r}) \xrightarrow[|\vec{r}| \to \infty]{} e^{ikr} + f(\Omega)\dfrac{e^{ikr}}{r}$

$\boxed{\text{Big clue}}$

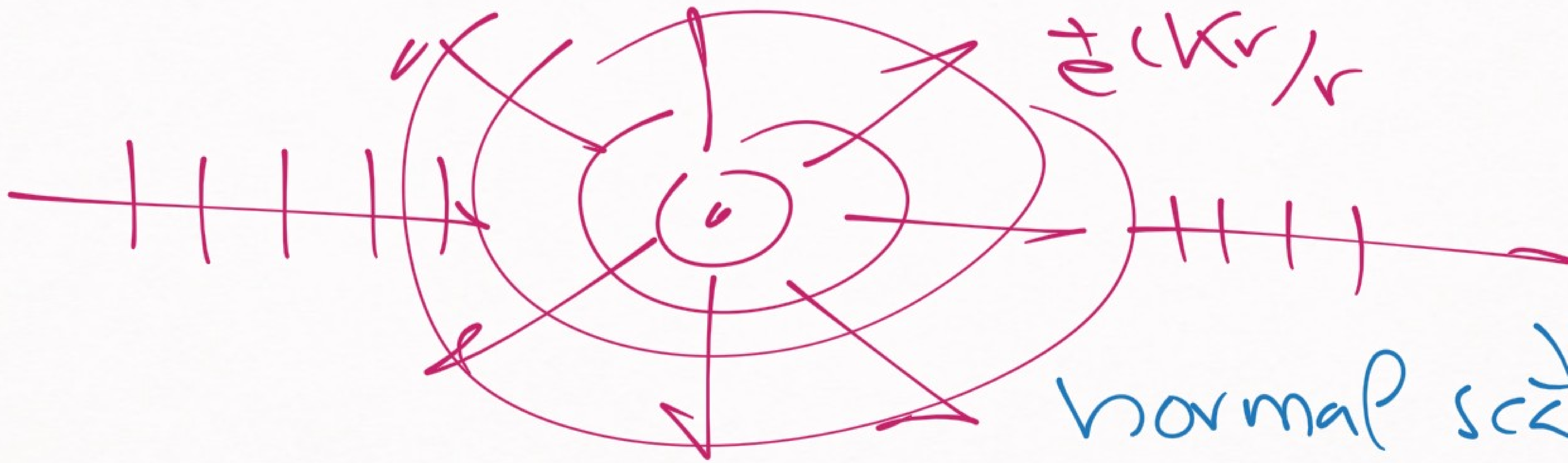
$$\psi^{(\pm)}(\vec{r}) \xrightarrow[|\vec{r}| \to \infty]{} e^{i\vec{k}\cdot\vec{r}} + \frac{e^{\pm ikr}}{r}(\ldots)$$

Integral

$E + i\epsilon \rightarrow$ scatterings in QM

$E - i\epsilon \rightarrow$ not going to be our solution

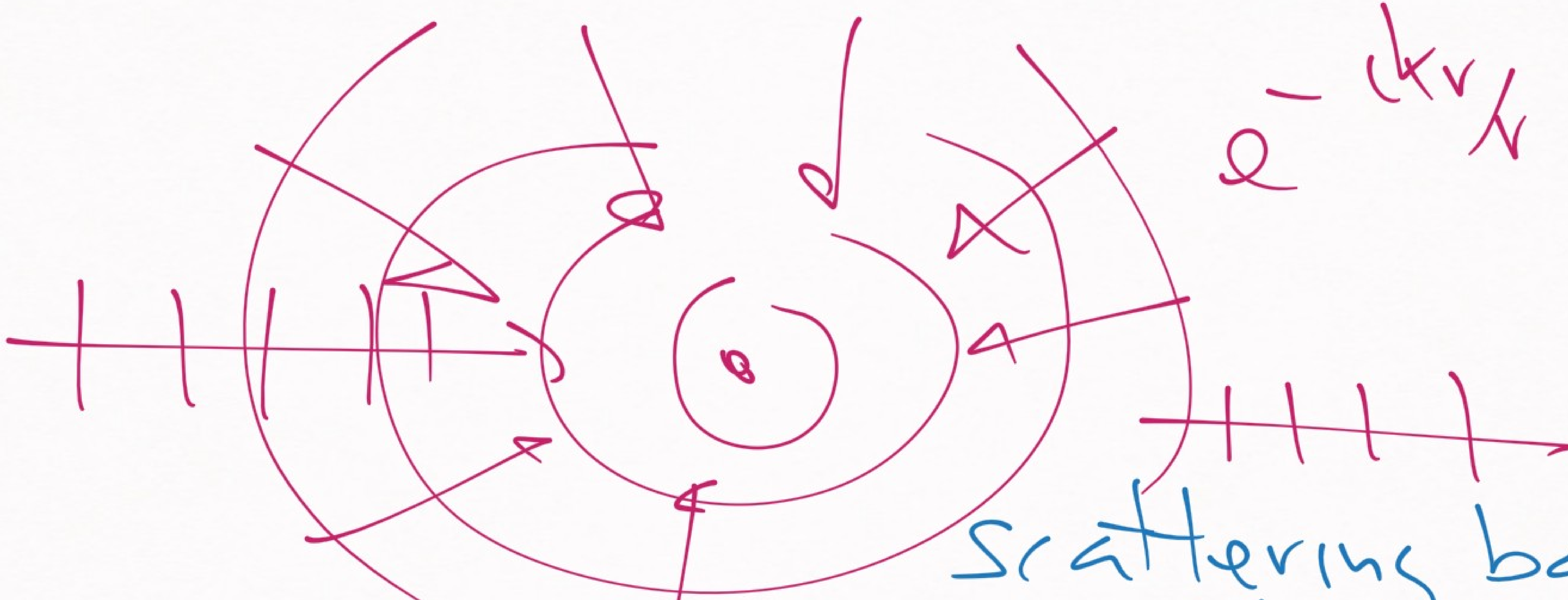
$\phi^{(+)}$



e^{ikr}/r

normal scattering

$\phi^{(-)}$



e^{-ikr}/r

scattering backwards in time

$\phi^{(4)} \longrightarrow$ physical solution for scattering

$$\boxed{\text{CONTINUE ON TUESDAY}}$$