NUCLEAR PHYSICS 13

THE TWO-NUCLEON SYSTEM

→ Review of the two-body problem
& of scattering theory

NUCLEAR PHYSICS $\longrightarrow$ $\boxed{\text{NUCLEI}}$

$\downarrow$

$\boxed{\text{Many-body systems}}$

$\downarrow$

$\boxed{\text{Many-body problem}}$

Many-body systems $\longrightarrow$ A nucleons

- $A = 2$ $\rightarrow$ Deuteron $\rightarrow$ well-known techniques (easy)
- $A = 3$ $\rightarrow$ ^3He, ^3H $\rightarrow$ Faddeev equations (more difficult)
- $A = 4$ $\rightarrow$ ^4He $\rightarrow$ Faddeev-Yakubovsky (serious stuff)

$A \geq 4$ $(A \leq 10)$ $\longrightarrow$ ab initio methods

(we can really try to solve Schrödinger)

Large A $\longrightarrow$ nuclear models

$\boxed{\text{TWO-BODY PROBLEM}}$ (brief review)

→ Schrödinger equation

$$\left[\frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + V(\vec{r}_1 - \vec{r}_2)\right]\Psi(\vec{r}_1, \vec{r}_2) = E_T\,\Psi(\vec{r}_1, \vec{r}_2)$$

p_i → momentum of particle $i = 1, 2$

m_i → mass of particle $i = 1, 2$

→ 6 variables not easy to solve

↳ change to center-of-mass coordinates

$$\vec{P} = \vec{p}_1 + \vec{p}_2 \quad \text{(total momentum)}$$

$$\vec{p} = \frac{m_1 \vec{p}_2 - m_2 \vec{p}_1}{m_1 + m_2} \quad \text{(relative momentum)}$$

$$\vec{R} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2} \qquad \text{(c.m. radius)}$$

$$\vec{r} = \vec{r}_2 - \vec{r}_1 \qquad \text{(relative radius)}$$

$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2} \qquad \text{(reduced mass)}$$

$$M = m_1 + m_2 \qquad \text{(total mass)}$$

$$\left[\frac{\vec{p}^{\,2}}{2\mu} + \frac{\vec{P}^{\,2}}{2M} + V(\vec{r})\right]\Psi(\vec{r},\vec{R}) = E_T\,\Psi(\vec{r},\vec{R})$$

→ we can factor out the cm motion

$$\Psi(\vec{r},\vec{R}) = \psi(\vec{r})\,e^{i\vec{K}\cdot\vec{R}}$$

$$\hookrightarrow \left[\frac{\vec{p}^{\,2}}{2M} + \frac{\vec{K}^{2}}{2M} + V(\vec{r})\right]\psi(\vec{r})\,\cancel{e^{i\vec{K}\cdot\vec{R}}} = E_T\,\psi(\vec{r})\,\cancel{e^{i\vec{K}\cdot\vec{R}}}$$

$$E_{cm} = E_T - \frac{\overline{K}^2}{2M}$$

$$\Rightarrow \left[\left[\frac{\vec{p}^{\,2}}{2\mu} + V(\vec{r}) \right] \psi(\vec{r}) = E_{cm}\, \psi(\vec{r}) \right]$$

→ still a 3-dimensional equation

(not easy to solve!)

→ try to reduce it to a set of 1-dim equations

[PARTIAL WAVE EXPANSION]

$$\psi(\vec{r}) = \sum_{lm} \frac{u_l(r)}{r} \underbrace{Y_{lm}(\hat{r})}_{\text{Spherical harmonics}}$$

$$\vec{r} \to (r, \hat{r})$$

$$\vec{L}^2 Y_{lm}(\hat{r}) = l(l+1) Y_{lm}(\hat{r})$$

$$\frac{\vec{p}^2}{2\mu} = -\frac{\vec{\nabla}^2}{2\mu} = -\frac{1}{2\mu}\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right) + \frac{\vec{L}^2}{2\mu}$$

Putting $\forall$ the pieces together:

$$-u_\ell''(r) + \left[2\mu V(r) + \frac{\ell(\ell+1)}{r^2}\right] u_\ell(r) = 2\mu E_{cm}\, u_\ell(r)$$

$\rightarrow$ Reduced Schrödinger equation

$$u_\ell''(r) = \frac{d^2}{dr^2} u_\ell(r)$$

(easy: 1-dim differential equation)

We can simplify further:

$$2\mu E_{cm} \begin{cases} E_{cm} > 0 \Rightarrow E_{cm} = \dfrac{k^2}{2\mu} & (1) \\ E_{cm} < 0 \Rightarrow E_{cm} = -\dfrac{\gamma^2}{2\mu} & (2) \end{cases}$$

$$(1) \quad -u_\ell'' + \left[2\mu U(r) + \frac{\ell(\ell+1)}{r^2}\right] u_\ell(r) = k^2 u_\ell(r)$$

$$(2) \quad -u_\ell'' + \left[2\mu U(r) + \frac{\ell(\ell+1)}{r^2}\right] u_\ell(r) = -\gamma^2 u_\ell(r)$$

→ A series of solutions
w/ known properties

① Asymptotic solutions ($r \to \infty$)

Assumption: $\boxed{\lim_{r \to \infty} r^n V(r) \to 0, \ \forall n \geq 0}$

Short-ranged (finite-range) potential

Finite-range potentials → easier

Examples → $V(r) \sim \frac{e^{-mr}}{r^{\alpha}}$ (Yukawa like)

$V(r) \sim V_0 \theta(R-r)$ (square well)

Infinite-range potentials → $V_C(r) = -\frac{\alpha}{r}$ → more complex solutions at $r \to \infty$

We concentrate on finite-range

1.a $E_{cm} < 0$

$$u_\ell(r) \xrightarrow[r\to\infty]{} A_\ell \, e^{-\gamma r}\left(1 + \mathcal{O}\left(\frac{1}{\gamma r}, \frac{1}{mr}\right)\right)$$

→ asymptotic normalization constant (ANC)

$\left[\ell = 0, \; u_0(r) \to D_s e^{-\gamma r}\right]$ → easiest case

1.b $E_{cm} > 0$

$$u_\ell(r) \xrightarrow[r \to \infty]{} a_\ell(k)\,\hat{j}_\ell(kr) + b_\ell(k)\,\hat{y}_\ell(kr)$$

$$\left.\begin{aligned} \hat{j}_\ell(x) &= x\, j_\ell(x) \\ \hat{y}_\ell(x) &= x\, y_\ell(x) \end{aligned}\right\} \quad \underbrace{j_\ell(x),\ y_\ell(x) \text{ spherical Bessel functions}}_{}$$

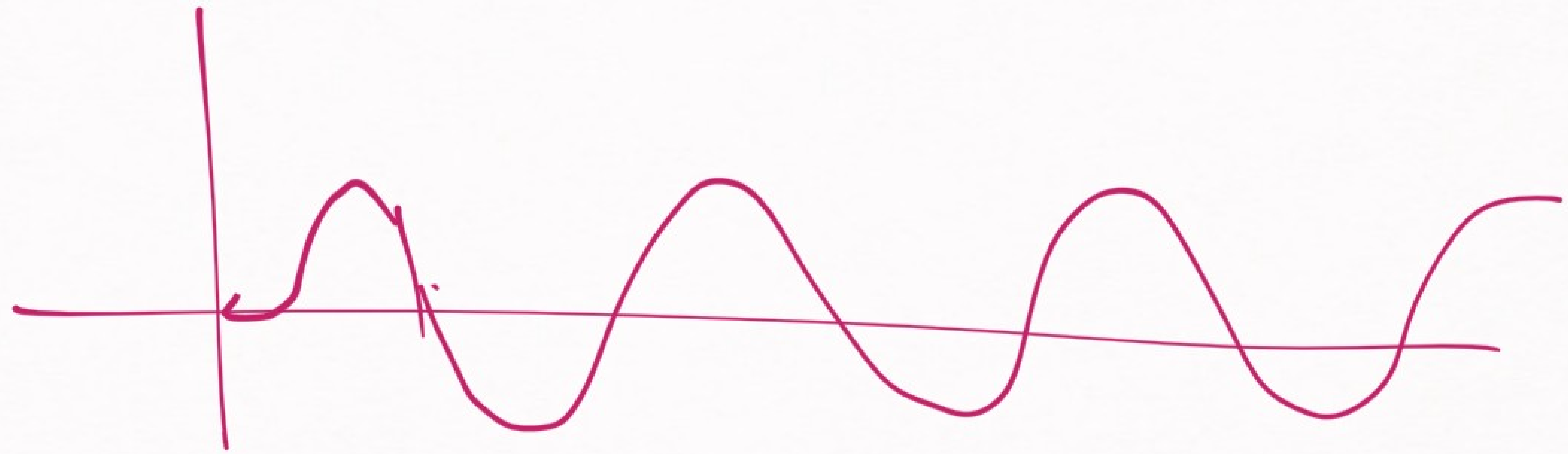
Reminder

$$\left.\begin{aligned} j_\ell(x) &\xrightarrow[x\to\infty]{} +\frac{1}{x}\sin\left(x-\ell\frac{\pi}{2}\right) \\ y_\ell(x) &\xrightarrow[x\to\infty]{} -\frac{1}{x}\cos\left(x-\ell\frac{\pi}{2}\right) \end{aligned}\right\} \Rightarrow$$

$\Rightarrow \quad u_\ell(r) \xrightarrow[r\to\infty]{} N_\ell \times \underbrace{\sin\left(kr - \ell\frac{\pi}{2} + \boxed{\delta_\ell(k)}\right)}_{\text{very simple}}$

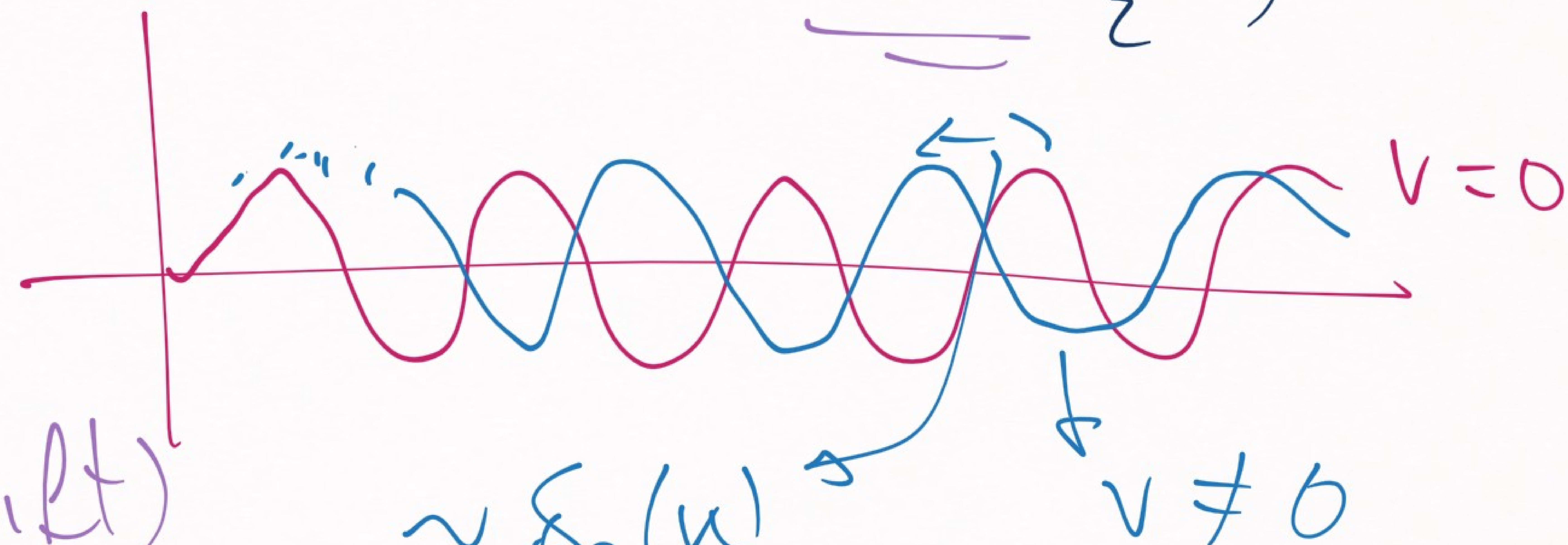
$\rightarrow$ $\delta_\ell(k)$ is called the phase shift

$V(r) = 0$



$u_\ell(r) \sim \sin\left(kr - \frac{\ell\pi}{2}\right)$

$V(r) \neq 0$



(phase shift)

$$\ell = 0 \longrightarrow u_0(r) \underset{r\to\infty}{\longrightarrow} \sin(kr + \delta_0(k))$$

$$\text{l.c.} \quad E_{cm} \to 0$$

"scattering length /volume"

$$\delta_\ell(k) \to -a_\ell k^{2\ell+1} + \mathcal{O}(k^{2\ell+3})$$

$$[a_\ell] = \underline{[\text{Length}]}^{2\ell+1}$$

$\ell = 0$, $\delta_0(k) \to \underbrace{-a_0}k + \mathcal{O}(k^3)$

scattering length $\to$ ⊛

$\Downarrow$

$u_0(r) \to 1 - \frac{r}{a_0}$ $\Big]$ $\to$ simple form

$r \to \infty$

⊛ $\to$ related to scattering at low energies

$\left(\sigma \xrightarrow[k \to 0]{} 4\pi |a_0|^2 \right) \to$ we check later

(2) Solutions near the origin ($r \to 0$)

$\to$ Regularity condition : $u_\ell(0) = 0$ $\leftarrow$ a), b)

$$\left[u_\ell(r) \xrightarrow[r \to 0]{} 0 \right]$$

$\boxed{\text{WHY?}}$ $\to$ a) $\langle \psi | \psi \rangle = \int d^3\vec{r}\, |\psi(\vec{r})|^2$

b) $\underline{\langle \psi | T | \psi \rangle < \infty}$ $\quad = \int dr\, |u_\ell(r)|^2 < \infty$

Consequences of $u_\ell(0) = 0$

2.a REGULAR POTENTIAL

$$\lim_{r \to 0} r^2 V(r) = 0$$

$$\Downarrow$$

$$|V(r)| \ll \frac{\ell(\ell+1)}{r^2}$$

the centrifugal barrier overcomes $V(r)$ at short distances

$r \to 0$ $\Rightarrow$ $u_\ell(r)$ is determined by $\frac{\ell(\ell+1)}{r^2}$

(don't need to know the form of the potential)

$\Downarrow$

$$\boxed{\lim_{r \to 0} u_\ell(r) \sim a_\ell r^{\ell+1} + \frac{b_\ell}{r^\ell}} + \boxed{u_\ell(0) = 0}$$

a

b

$\to$ $\emptyset$

$\circledast \Rightarrow \boxed{\begin{array}{c} u_\ell(r) \longrightarrow a_\ell \, r^{\ell+1} \\ r \to 0 \end{array}}$

Solution of $u_\ell(r)$ near the origin
for any regular potential

Σ

w/this condition, we determine the solution
of Schrödinger Eq

PROBLEM $\Rightarrow$ in many cases we have to deal w/ potentials that are not regular

Imagine we only know the expansion of $V(r)$ for $r \to \infty$:

$$V(r) = \sum_n c_n \frac{e^{-mr}}{r^n}$$

if $n \geq 2$, V is singular

2.b → SINGULAR POTENTIALS

$$\lim_{r \to 0} r^2 V(r) \neq 0$$

EASIEST CASE → $V(r) = \pm \frac{1}{2\mu} \frac{a^{n-2}}{r^n}$

$(n > 2)$

We know $r \to 0$ solutions

[First case] → $V(r) > 0$ (repulsive)

→ we can calculate the wave function for $r \to 0$ (e.g. using WKB approx.)

$$u_\ell(r) = C_+\left(\frac{r}{a}\right)^{n/4}\exp\left[+\frac{2}{n-1}\left(\frac{a}{r}\right)^{\frac{n-2}{2}}\right] \to ①$$

$$+\,C_-\left(\frac{r}{a}\right)^{n/2}\exp\left[-\frac{2}{n-1}\left(\frac{a}{r}\right)^{\frac{n-2}{2}}\right] \to ②$$

(1) → $u_e(r) \to \infty$ for $r \to 0$

(2) → $u_e(r) \to 0$ for $r \to 0$

RECAP → $\boxed{u_e(0) = 0}$ ⟹ (1) $= 0$

$\boxed{C_+ = 0}$

DETERMINES THE SHORT-RANGE SOLUTION

Repulsive singular potential $\simeq$ regular potential

[Second case] → Attractive singular potential

$$V(r) = -\frac{1}{2\mu}\frac{a^{n-2}}{r^n} \longrightarrow \text{(WKB approx)} \rightarrow$$

$$u_e(r) \xrightarrow[r \to 0]{} c_1 \left(\frac{r}{a}\right)^{n/4} \sin\left(\frac{2}{n-2}\left(\frac{a}{r}\right)^{\frac{n-2}{2}}\right) + c_2 \left(\frac{r}{a}\right)^{n/4} \sin\left(\frac{2}{n-2}\left(\frac{a}{r}\right)^{\frac{n-2}{2}}\right)$$

$$\Downarrow$$

$\boxed{u_e(0) = 0}$ always $\rightarrow$ for any combination of c_1, c_2

→ Attractive singular potentials don't have a unique solution

PROBLEM → You solve this by adding an extra short-range potential (e.g. $\delta^{(3)}(\vec{r})$)

Other interesting cases:

1) $n=1$ $\rightarrow$ limit case (lecture notes)

2) $V(\vec{r}) = C\,\delta^{(3)}(\vec{r})$

$$\rightarrow \frac{C(R_c)}{4\pi R_c^2}\,\delta(r-R_c)$$ or other regularization

A few quick remarks about singular potentials:

$\exists$ a physical meaning of why the solution of V singular & attractive is not well determined $\longrightarrow \circledast$

⊗ → These are "incomplete potentials"

a long-range approx.
to the true potential

⇓

→ You are missing somethings → more than one solution

1. (108) **Singular potentials and limit cycles**

S.R. Beane, Paulo F. Bedaque, L. Childress, A. Kryjevski, J. McGuire (Washington U., Seattle),
U. van Kolck (Arizona U. & RIKEN BNL & Caltech, Kellogg Lab). Oct 2000. 8 pp.

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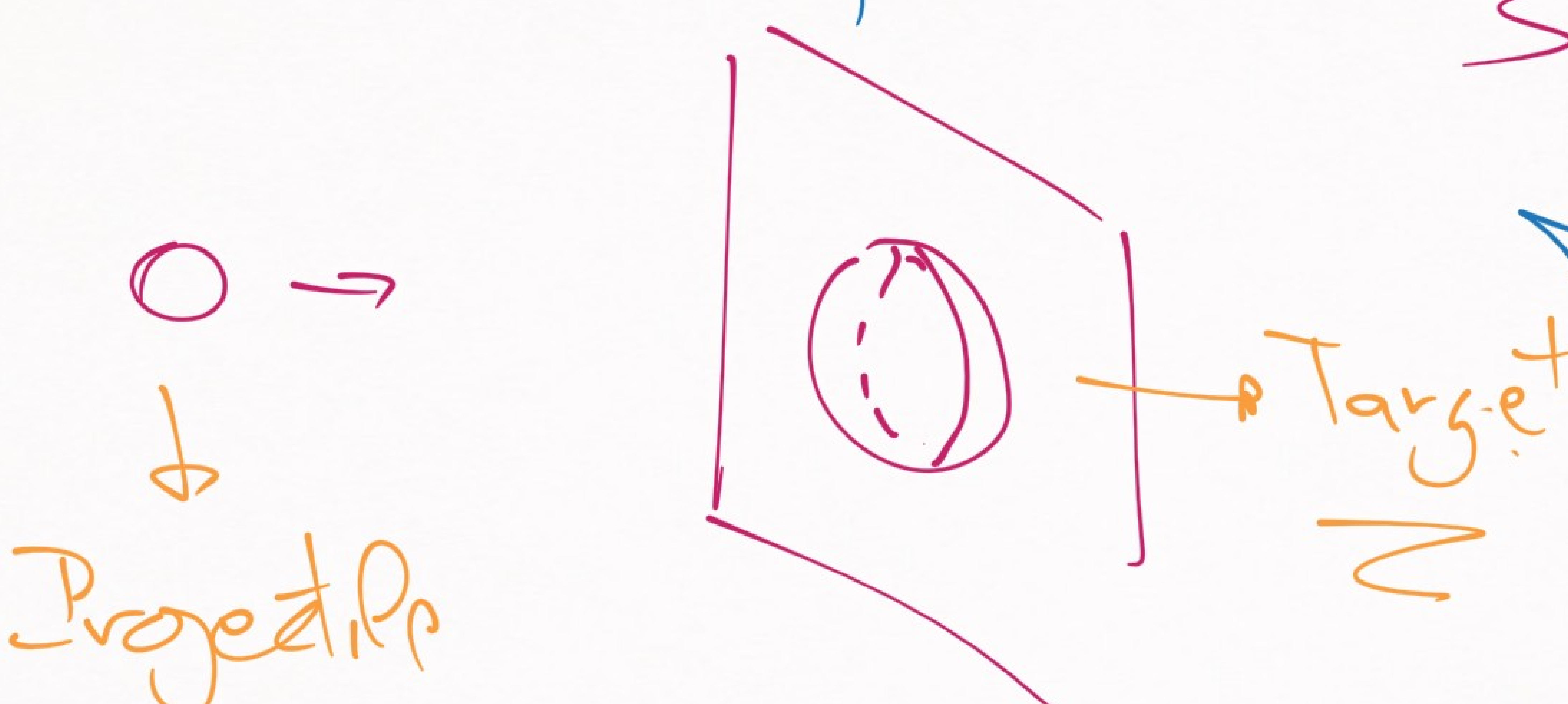
NT-UW-00-023, DOE-ER-41132-102-INT-00, RBRC-140, KRL-MAP-271
DOI: 10.1103/PhysRevA.64.042103
e-Print: **quant-ph/0010073** | **PDF**

↳ good introductory reading for understanding singular potentials

Review of $\boxed{\text{SCATTERING THEORY}}$

Classical example ($\to$ cross section)

σ



Effective area that the target gives to the projectile

cross section will be related to this area

CA will be a little big

bigger)

More concrete definition :

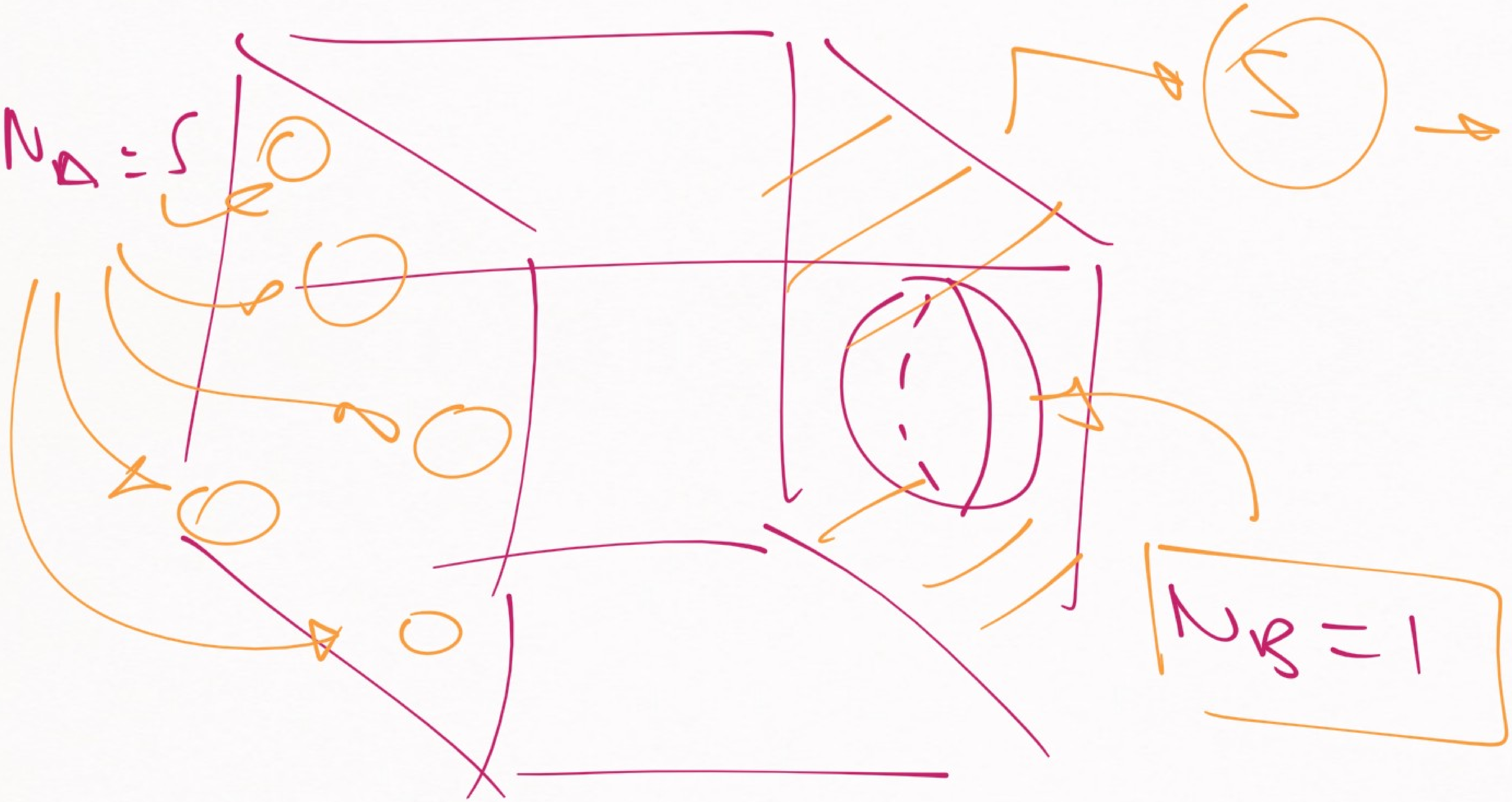
$$\boxed{\sigma = \frac{N_s}{N_A N_B}\, S}$$

scattered particles

area in which A & B can collide

# of projectiles

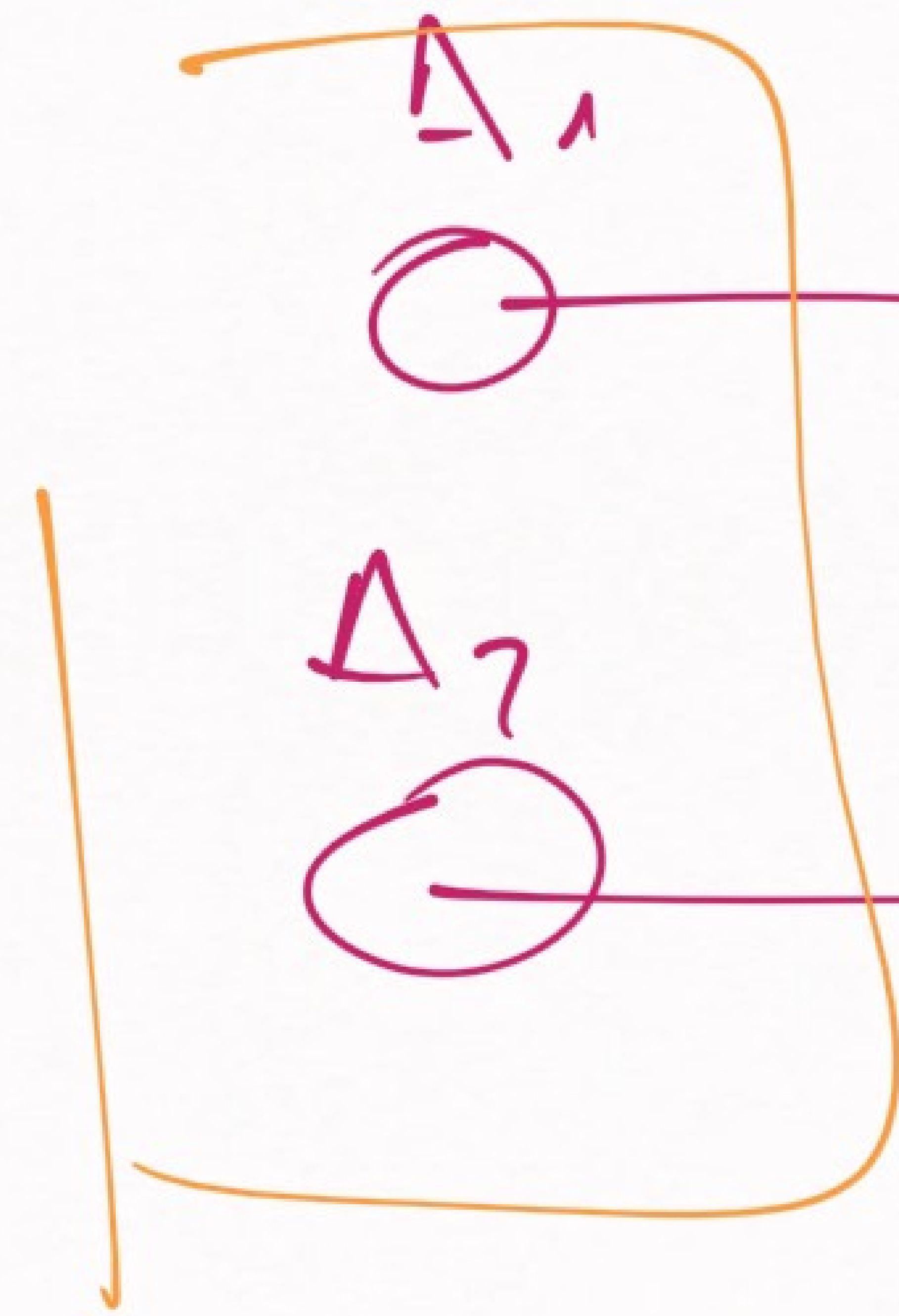
# of targets

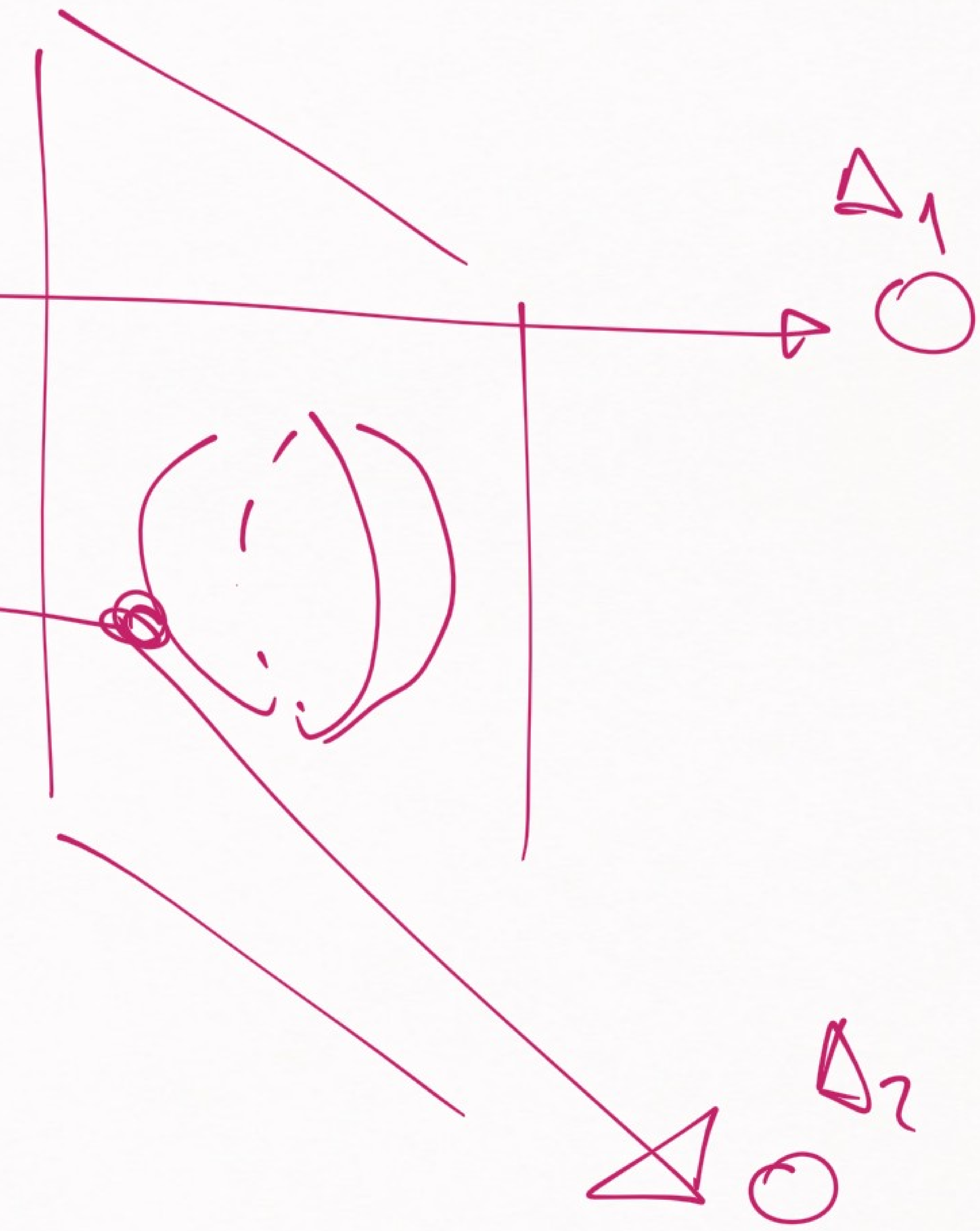


area where A are thrown

S

$$r = \frac{N_S}{N_\Delta N_S} S$$



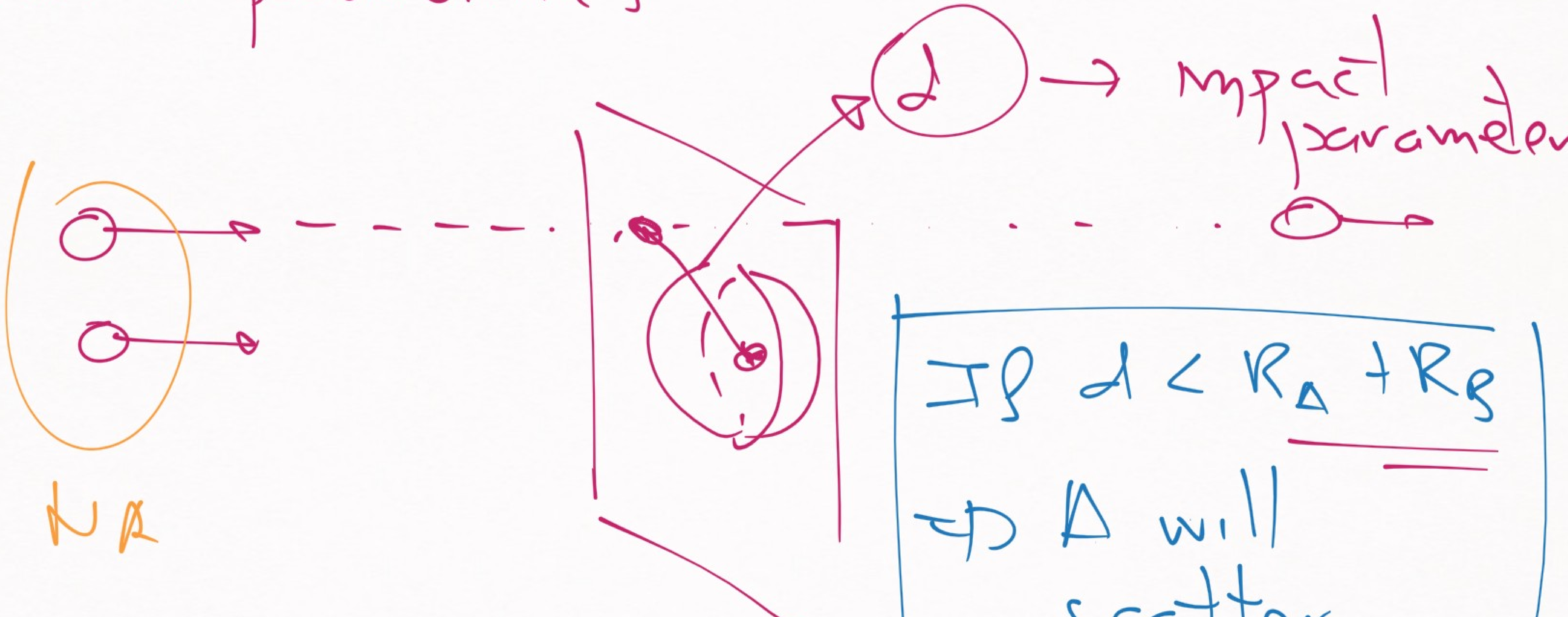


$N_S = 1$

$N_\Delta = 2$

(just an example)

Think in probabilities



If $d < R_A + R_B$

$\Rightarrow$ A will scatter

→ N_A, N_B are known

$$N_S = N_A \frac{1}{S} \underbrace{S(A+B)}_{\pi (R_A+R_B)^2} N_B \Rightarrow$$

$$\Rightarrow \quad \sigma = \frac{N_S}{N_A N_B} S = \frac{\left(\cancel{N_A} \cancel{N_B} \cancel{\frac{1}{S}} S(A+B)\right)}{\cancel{N_A} \cancel{N_B}} \cancel{S}$$

$$= S_{A+B} = \pi (R_A + R_B)^2$$

$$\boxed{\sigma = \pi (R_A + R_B)^2} \rightarrow$$

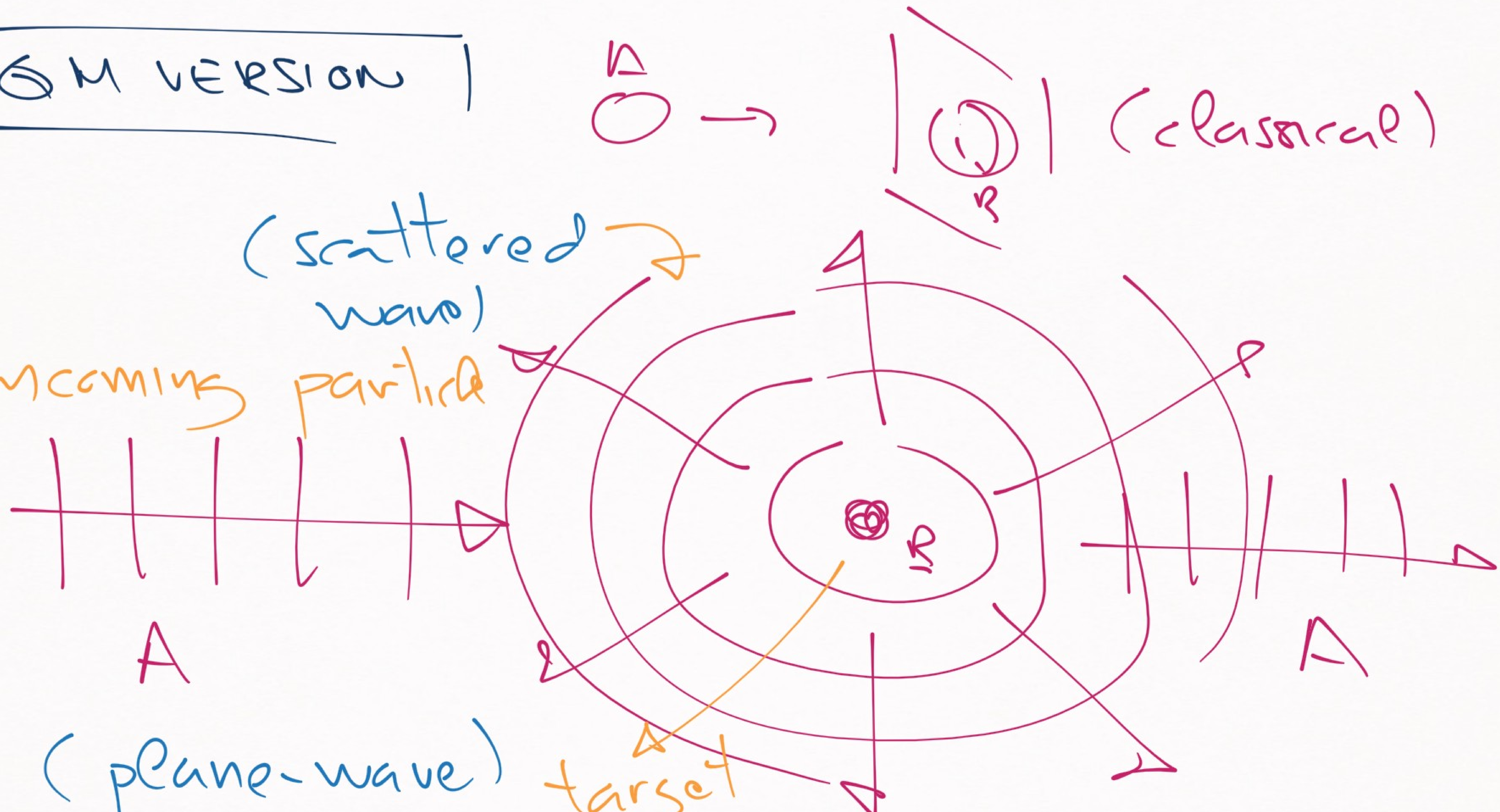
for the scattering of classical balls

→ Overall, σ is a simple concept

BUT WE NEED THE QUANTUM MECHANICAL EQUIVALENT OF THIS

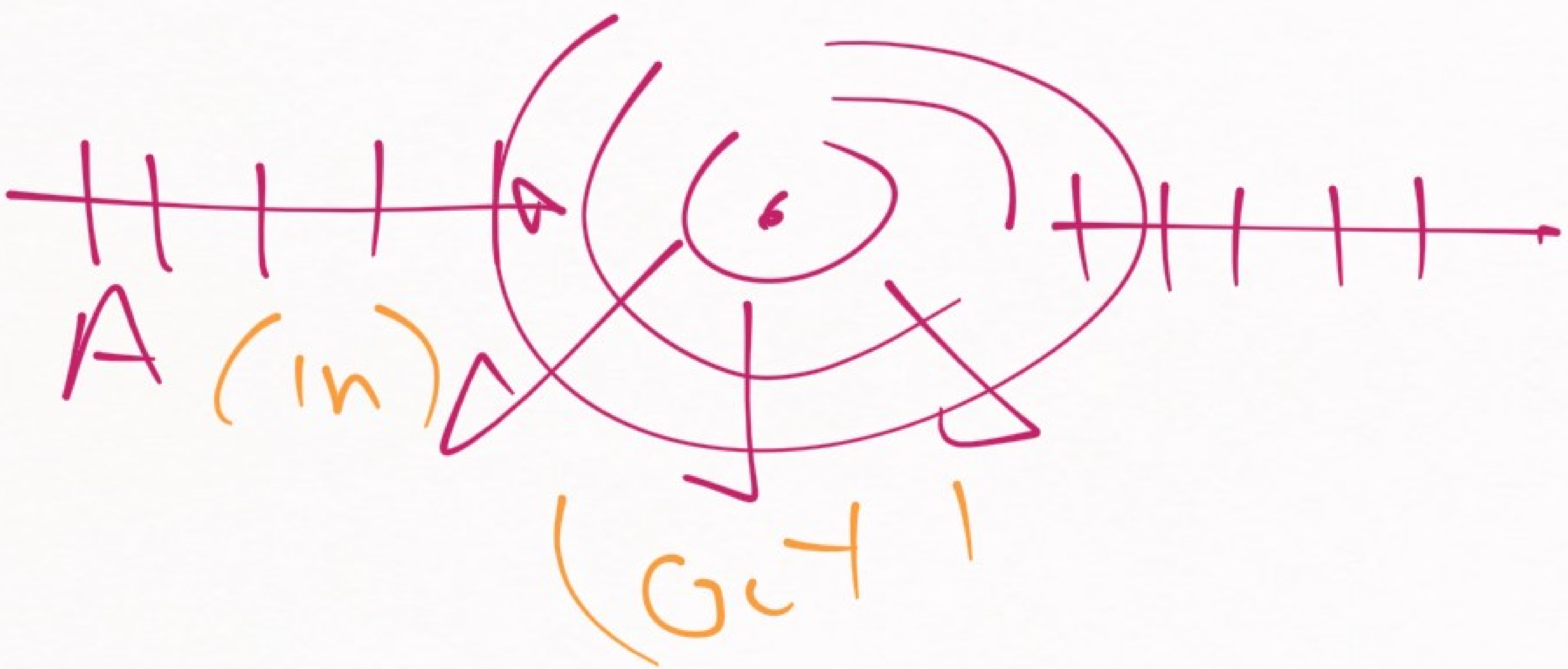
→ not trivial

QM VERSION



A
B
(classical)
(scattered wave)
incoming particle
A
(plane-wave)
B
target
A

How to describe this? $\rightarrow$ w/ a wave function



$$\psi_{in} \rightarrow e^{i\vec{k}\cdot\vec{r}}$$

$$\psi_{out} \rightarrow \underline{f(\theta)}\, \frac{e^{ikr}}{r}$$

$$\psi(\vec{r}) = \psi_{in}(\vec{r}) + \psi_{out}(\vec{r})$$

angle dependence

spherical outgoing wave

$$\psi(\vec{r}) \to e^{i\vec{k}\cdot\vec{r}} + f(\Omega)\frac{e^{ikr}}{r}$$

$$(\psi_{in} + \psi_{out})$$

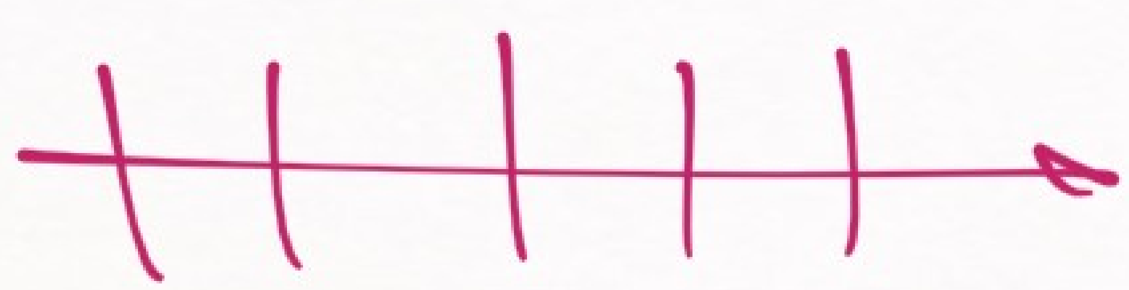
deduce the equivalents of

$$\sigma = \frac{N_S}{N_A N_B} S$$

$$\boxed{\begin{array}{l} N_A N_B \sim \psi_{in} \\ N_S S \sim \psi_{out} \end{array}}$$

$\exists$ same relation

[FIGURE OUT WHAT THIS RELATION IS]

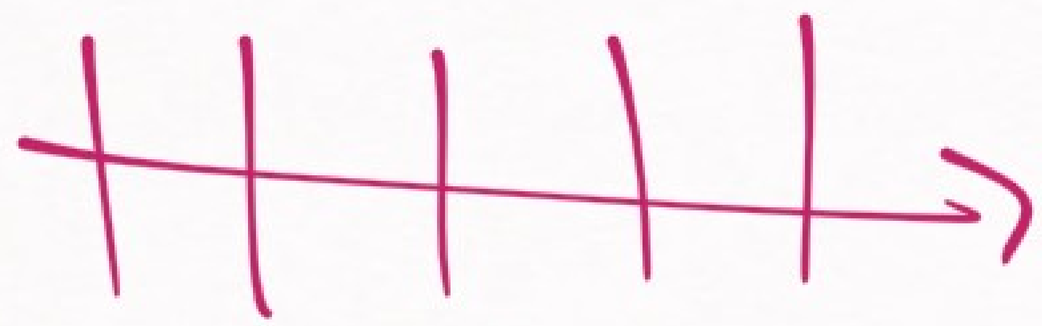


$\psi_{in}(\vec{r})$ $\Big\} \longrightarrow$ incoming flux/current

$$\vec{J} = -\frac{i}{2m}\left[\psi^* \vec{\nabla} \psi - \psi \vec{\nabla} \psi^*\right]$$

or $\vec{\mathcal{J}}$

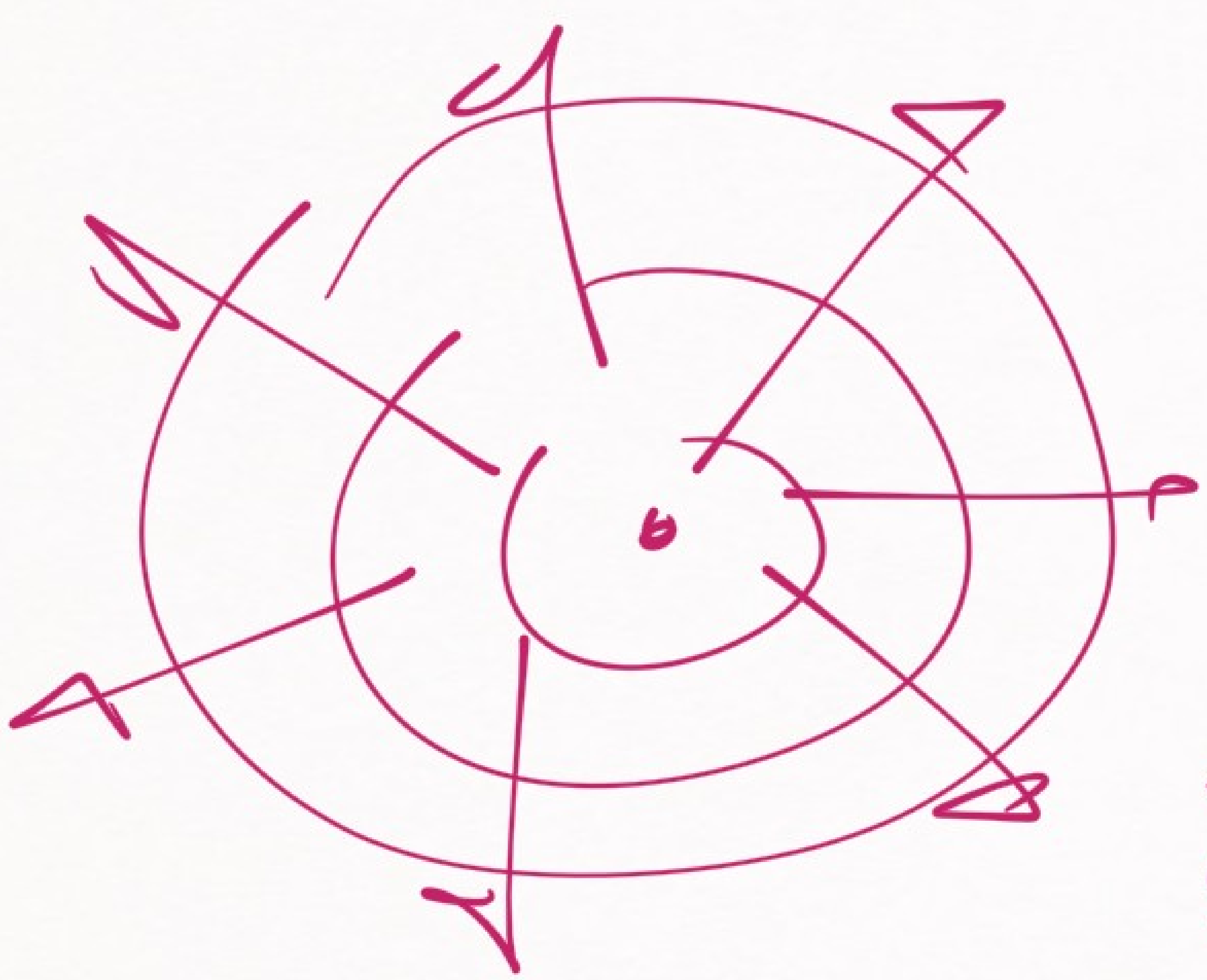
$\vec{\mathcal{J}} \sim$ number of particles per unit time



Φ_{in}

$\vec{k}$

$$\boxed{N_B = 1} \Rightarrow \frac{N_A}{T} \propto \Phi_{in} = \vec{\Phi}_{in} \cdot \hat{k}$$



Φ_{out}

$$\Rightarrow \sigma \frac{N_r}{T} \propto \int d\vec{S} \cdot \vec{\Phi}_{out}$$

integration for $R \to \infty$

[IDENTIFICATION]

$$\frac{N_A N_B}{T} \propto |\vec{\overline{\underline{\Phi}}}_{in}|$$

$$\boxed{\text{CALCULATE} \quad \vec{\overline{\underline{\Phi}}}_{in} , \overline{\underline{\Phi}}_{out}}$$

$$\frac{N_S S}{T} \propto \lim_{R \to \infty} \int d\vec{S} \cdot \vec{\overline{\underline{\Phi}}}_{out}$$

$\rightarrow \quad j_{in} = |\vec{j}_{in}|$

$$\vec{j}_m = -\frac{i}{2m}\left[\psi_{in}^{*}\vec{\nabla}\psi_{in} - \psi_{in}\vec{\nabla}\psi_{in}^{*}\right]$$

$$\psi_m = e^{i\vec{k}\cdot\vec{r}} \Rightarrow \vec{j}_{in} = \frac{\vec{k}}{m}, \quad |\vec{j}_{in}| = \frac{k}{m}$$

 J_{out}

$$\vec{J}_{out} = -\frac{i}{2m}\left[\psi^*_{out}\vec{\nabla}\psi_{out} - \psi_{out}\vec{\nabla}\psi^*_{out}\right]$$

$$\psi_{out} = f(\omega)\frac{e^{ikr}}{r} \longrightarrow \vec{J}_{out} = \frac{k}{m}|f(\omega)|^2\frac{\hat{r}}{R^2}$$

$$\int d\vec{S}\cdot\vec{\Phi}_{out} \longrightarrow \quad d\vec{S} = R^2\, \underline{\underline{d\Omega}}\, \hat{r}$$

$$\lim_{R\to\infty} \int d\vec{S}\cdot\vec{\Phi}_{out} = \frac{k}{m}\int |f(\Omega)|^2\, d\Omega$$

$$\left.\begin{array}{l} \dfrac{1}{T} N_A N_B \longrightarrow \dfrac{k}{m} \\[2ex] \dfrac{1}{T} N_S S \longrightarrow \dfrac{k}{m}\displaystyle\int |f(\Omega)|^2\, d\Omega \end{array}\right] \quad \sigma = \frac{N_S S}{N_A N_B}$$

$\circledast \rightarrow \boxed{\sigma = \int |f(\Omega)|^2 d\Omega}$

$\rightarrow$ Quantum-mechanical cross section

Differential cross-section $\sum$ angular distributions

$$d\sigma = |f(\Omega)|^2 d\Omega \rightarrow \boxed{\frac{d\sigma}{d\Omega} = |f(\Omega)|^2}$$

$\boxed{\text{RECAP}}$

1) TWO-BODY SYSTEM $\left(-\frac{\nabla^2}{2\mu} + V\right)\psi = \frac{k^2}{2m}\psi$

2) SOLUTION LIKE THIS:

$$\psi(\vec{r}) \xrightarrow[r \to \infty]{} e^{i\vec{k}_i \cdot \vec{r}} + f(\Omega)\frac{e^{ikr}}{r}$$

3) DIFFERENTIAL CROSS SECTION

$$\frac{d\sigma}{d\Omega} = |f(\Omega)|^2 \qquad \sigma = \int |f(\Omega)|^2 d\Omega$$

→ We can combine 1, 2, 3) with what we already know about two body

$\longrightarrow \mathcal{E}$

→ [PARTIAL WAVE EXPANSION]

$$\frac{d\sigma}{d\Omega} \quad / \quad \psi(\vec{r}) = \sum_{\ell m} \frac{u_\ell(r)}{r} Y_{\ell m}(\hat{r})$$

$$u_\ell(r) \propto \sin\left(kr - \frac{\ell\pi}{2} + \delta_\ell\right)$$

$\rightsquigarrow$ requires elaboration (but it's easy)

$$\psi(\vec{r}) = \sum_{\ell m} \psi_\ell(r) Y_{\ell m}(\hat{r}) \quad , \quad \psi_\ell(r) = \frac{u_\ell(r)}{r}$$

$$e^{i\vec{k}\cdot\vec{r}} = 4\pi \sum_{\ell m} i^\ell j_\ell(kr) Y^*_{\ell m}(\hat{k}) Y_{\ell m}(\hat{r})$$

$$= \sum_\ell (2\ell+1) i^\ell j_\ell(kr) \underbrace{P_\ell(\hat{k}\cdot\hat{r})}_{\text{Legendre pol}}$$

→ Expand $f(\Omega)$ in terms of $P_\ell(\hat{k}\cdot\hat{r})$

$$f(\Omega) = \sum_{\ell} (2\ell+1)\, f_\ell(k)\, P_\ell(\hat{k}\cdot\hat{r})$$

Partial wave expansion of the scattering amplitude

$$\psi_{\vec{k}}(\vec{r}) = \sum_{l} (2l+1)\, i^{l}\, \frac{u_l(r)}{r}\, P_l(\hat{k}\cdot\hat{r})$$

$$\frac{u_l(r)}{r} \to e^{i\delta_l}\big(\cos\delta_l(k)\, j_l(kr) - \sin\delta_l(k)\, y_l(kr)\big)$$

+ a bit of elaboration →

You get the following:

$$f_\ell(k) = \frac{e^{i\delta_\ell}\sin\delta_\ell}{k} = \frac{1}{k\cot\delta_\ell - ik}$$

$$\sigma = \int |f(\Omega)|^2 d\Omega = \frac{4\pi}{k^2}\sum_\ell \sin^2\delta_\ell$$

$(k \to 0)$

RECAP

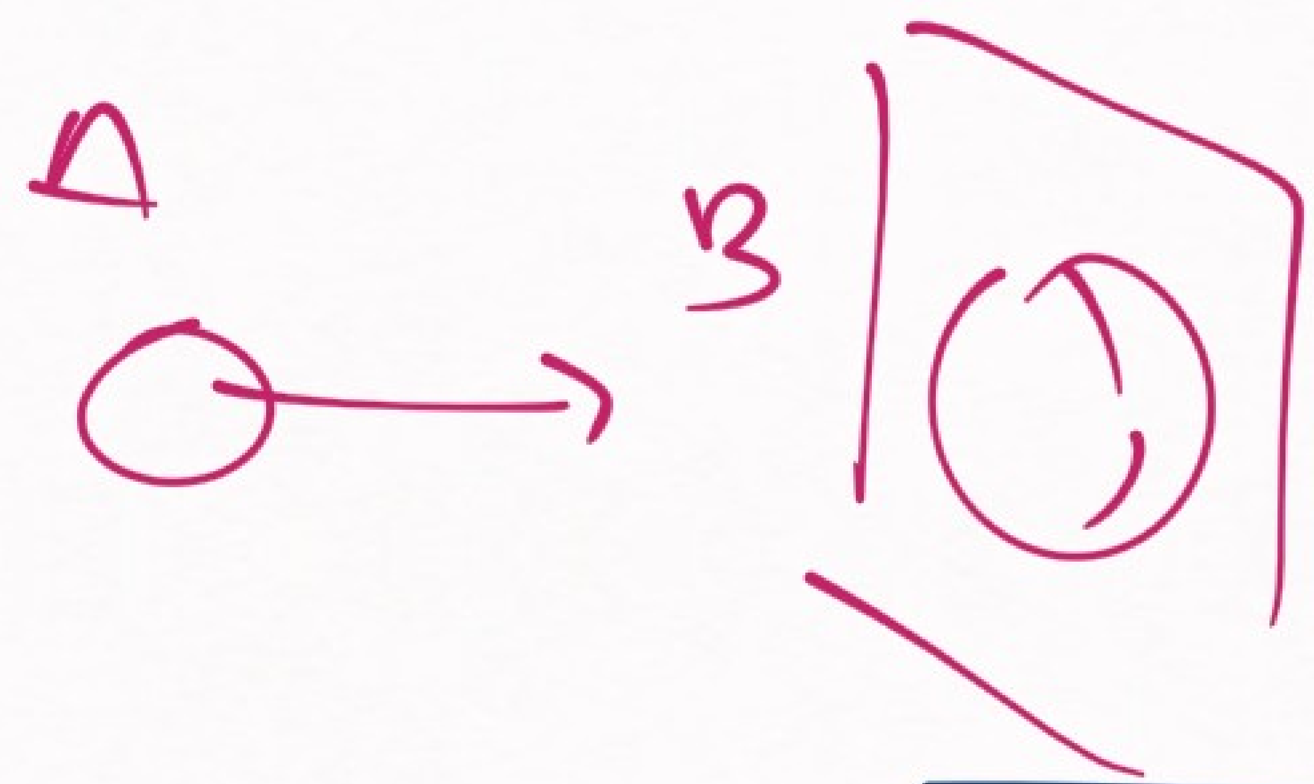
$k \to 0, \quad \delta_\ell(k) \to -a_\ell k^{2\ell+1} + \ldots$

$$\sigma = \frac{4\pi}{k^2} \sum_\ell \sin^2\delta_\ell \xrightarrow[k \to 0]{} \underline{\underline{4\pi |a_0|^2}}$$

$$\boxed{\lim_{k \to 0} \sigma = 4\pi |a_0|^2} \longrightarrow \text{scattering length}$$

→ Interpretation of the scattering length

CLASSICAL:



$$\sigma = \pi (R_A + R_B)^2$$

If $R_A = R_B = R \Rightarrow$ $\boxed{\sigma = 4\pi R^2}$

QUANTUM: $\boxed{\sigma = 4\pi |a_0|^2}$

→ SCATTERING LENGTH IS LIKE THE HARD-BALL RADIUS OF A QUANTUM PARTICLE WHEN $K \to 0$

EFFECTIVE

$$\boxed{\sigma = 4\pi |a_0|^2} \longrightarrow \text{nucleon-nucleon case}$$

$S=0$ (singlet) $\rightarrow$ np, pp, nn

$$\underbrace{a_0 \simeq -23.7\ \text{fm}}_{\text{really large}}$$

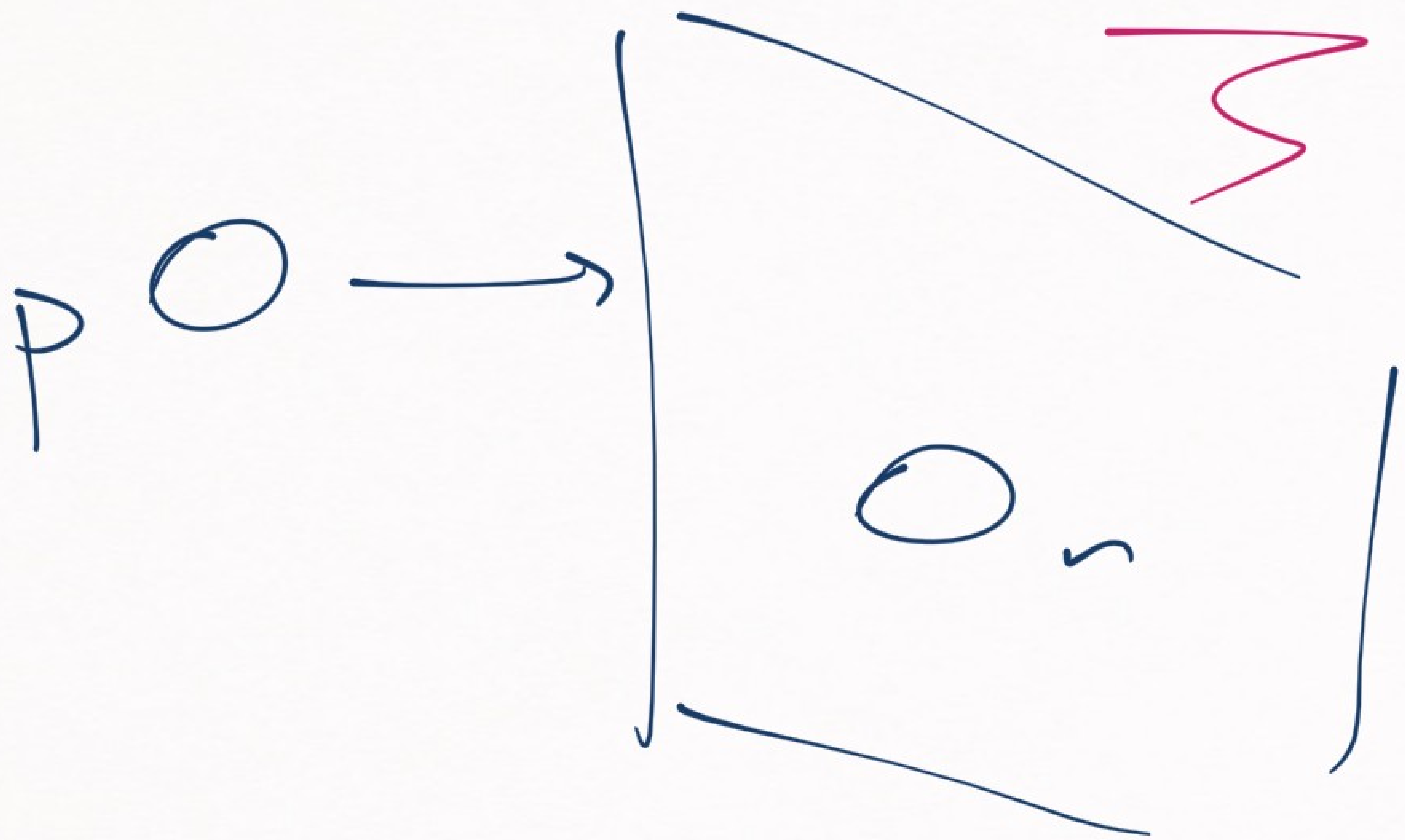
 $N = n, p$ size $\rightarrow (0.5 - 1.0)$ fm

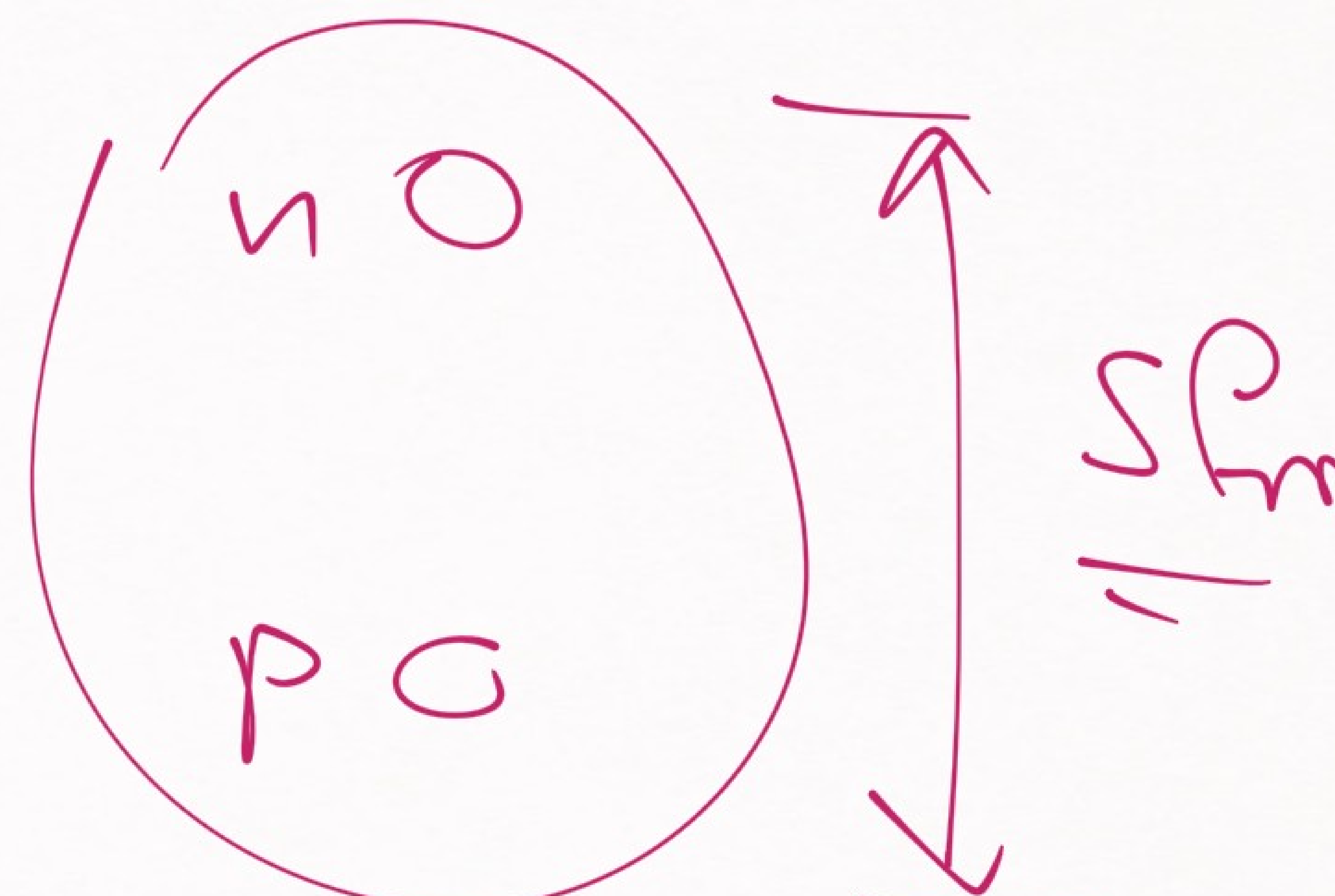
BUT, EFFECTIVE SIZE IN SCATTERING IS $(25-50)$ TIME THIS

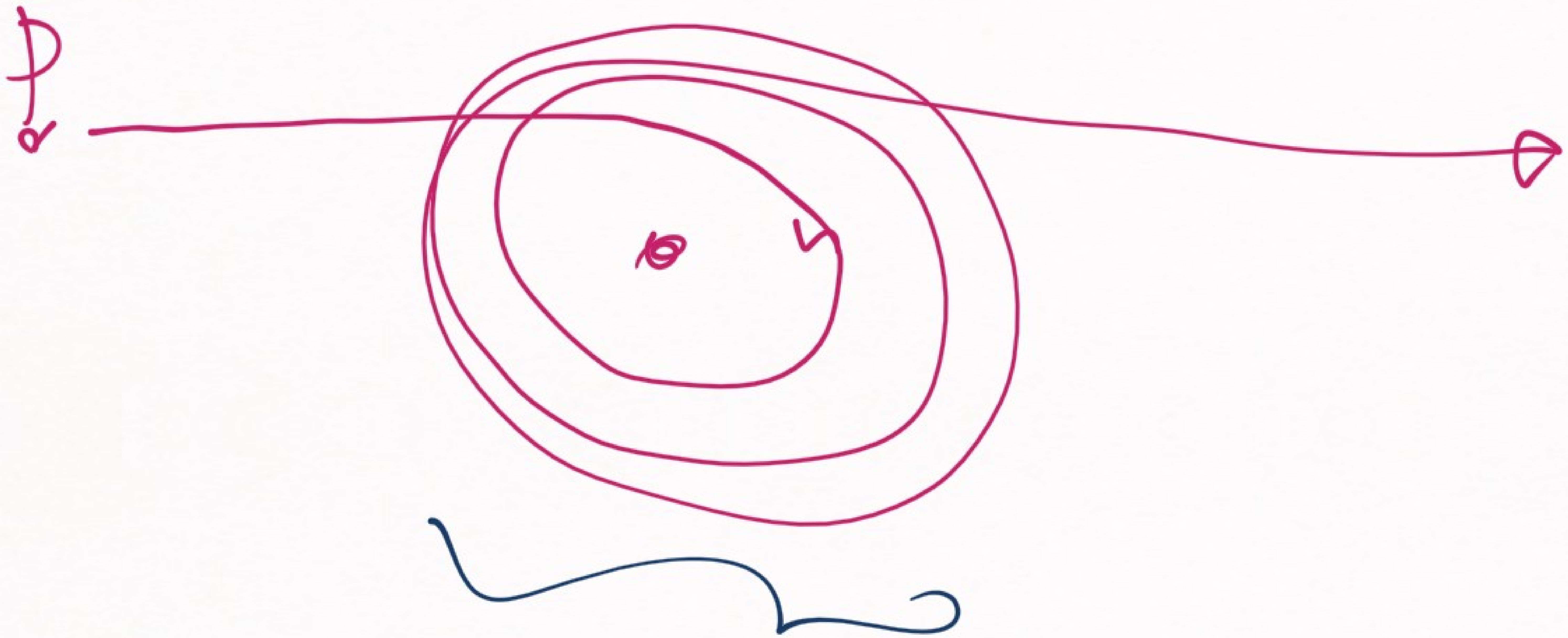
$$\boxed{a_0 \simeq -23.7 \text{ fm}} \rightarrow \text{very strong interaction}$$

$S=1 \rightarrow$ deuteron $\rightarrow$ $a_0 \simeq 5.4$ fm

(5-10 times larger than expected!)







very large cross section

(why? → they could potentially bind)

→ FINISH FOR TODAY

SEE YOU NEXT DAY

(TUESDAY OR FRIDAY)?