NUCLEAR PHYSICS 9

RENORMALIZATION

$$\Sigma$$

$\boxed{\text{RECAP}}$ (1) Renormalization means:

"Physics at long-distances does not depend on short-distance details"

Example $\rightarrow$ $\boxed{V(\vec{q}), m} \xrightarrow{|\vec{q}|^2 \ll m^2} V_{eff}(\vec{q}) = C_0 + C_2 \vec{q}^{\,2} + C_4 \vec{q}^{\,4} + \dots$

$$\Big| \dots \Big| \; -\frac{g^2}{\vec{q}^{\,2}+m^2} \xrightarrow{|\vec{q}|^2 \ll m^2} -\frac{g^2}{m^2} + \underline{\text{corrections}}$$

(Non-relativistic two-body problem)

$$\boxed{V_{true}(\vec{q})} \longrightarrow \left[V_{eff}(\vec{q}) = c_0 + c_2 \vec{q}^{\,2} + c_4 \vec{q}^{\,4} + \dots \right]$$

Characteristic scale m

(e.g Yukawa)

$\rightarrow$ In most cases, I always obtain the same effective potential at low energies.

→ Insensitivity to short-range details

$$\boxed{V_{eff}(\vec{q}) = C_0 + C_2 \vec{q}^{\,2} + \dots}$$

$$\Psi_{true}(\vec{r}) \xrightarrow[mr \gg 1]{} \boxed{\Psi_{eff}(\vec{r}) = \frac{A_s}{\sqrt{4\pi}} \frac{e^{-\gamma r}}{r}}$$

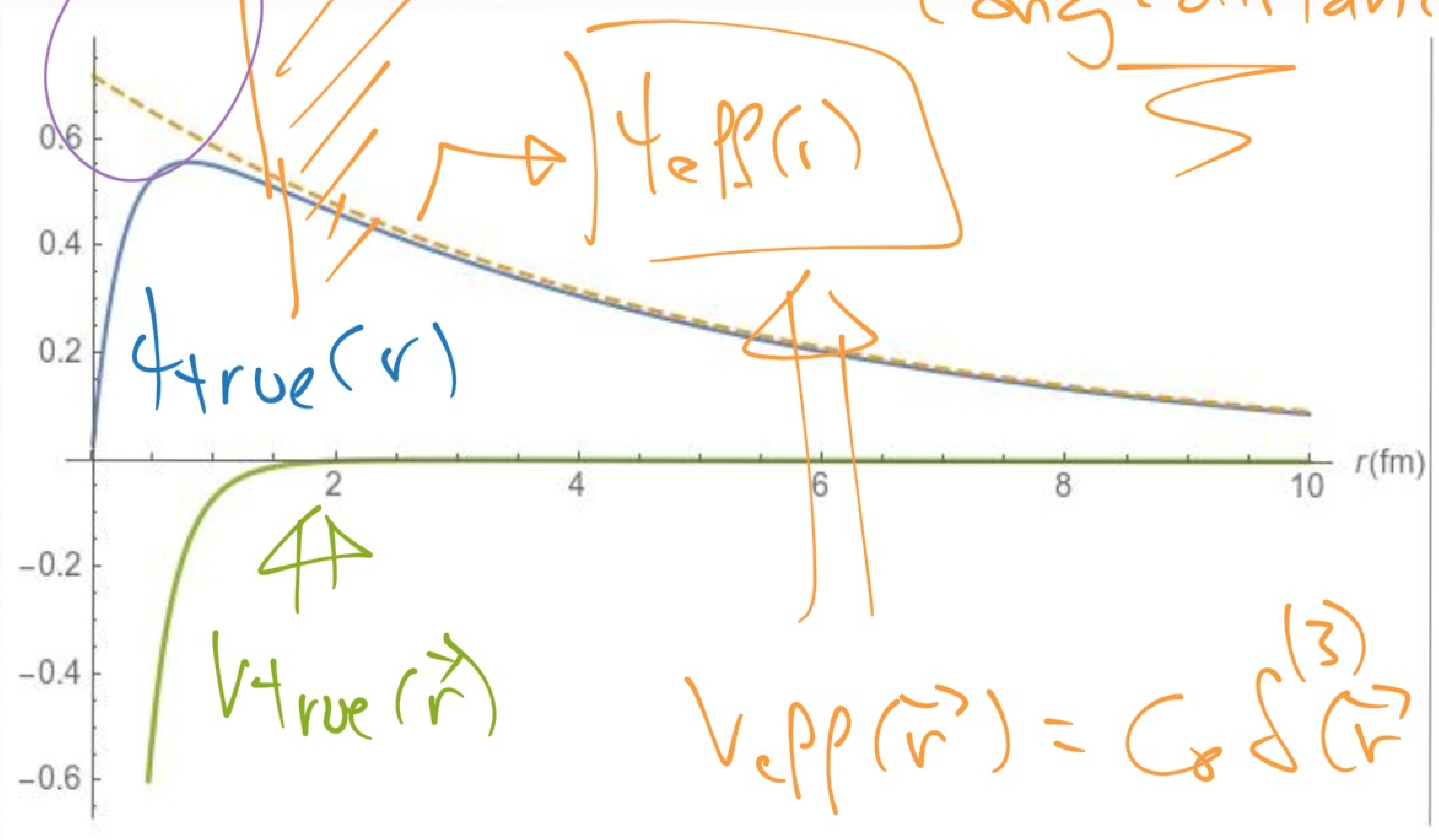
For most situations, at low energies I can approximate the two-body system with

$$\boxed{V_{eff}(\vec{s}) \;\&\; J_{eff}(\vec{s})}$$

Short-distances

Long-distance

$$\psi_{eff} \simeq \psi_{true}$$



$\psi_{eff}(r)$

$\psi_{true}(r)$

$V_{true}(\vec{r})$

$$V_{eff}(\vec{r}) = C_0\,\delta^{(3)}(\vec{r})$$

This contains the basic ideas

[Renormalization] → easy to understand at the conceptual level

the difficult part is implementation

(e.g. check previous calculation)

Why is difficult to implement renormalization at the mathematical level?

$$V_{eff}(\vec{q}) = C_0 + C_2 \vec{q}^{\,2} + C_4 \vec{q}^{\,4} + \ldots$$

$$V_{eff}(\vec{r}) = C_0 \delta^{(3)}(\vec{r}) + C_2 \vec{\nabla}^2 \delta^{(3)}(\vec{r}) + C_4 \vec{\nabla}^4 \delta^{(4)}(\vec{r}) + \ldots$$

how to use them?

$C_0 \boxed{\delta^{(3)}(\vec{r})} \rightarrow$ PROBLEM

VARIATIONAL PRINCIPLE

$H \rightarrow \Psi_{\text{solution}}$ or Ψ_{sol}

$$\boxed{\langle \Psi_{sol} | H | \Psi_{sol} \rangle \leq \langle \Psi_t | H | \Psi_t \rangle}$$

↓

true wf of a system minimizes the energy

$\Psi_t \rightarrow$ trial wave function (anything we want)

$$\left.\begin{array}{l} V_{eff} = C_0\, \delta^{(3)}(\vec{r}) \\[1em] \psi_{trial} = \dfrac{\Delta_s}{\sqrt{4\pi}} \dfrac{e^{-\gamma r}}{r} \end{array}\right\} \langle \psi_{trial} | V_{eff} | \psi_{trial} \rangle \propto C_0 \times (\infty)$$

If $C_0 < 0 \Rightarrow \langle \psi_{trial} | H | \psi_{trial} \rangle \to -\infty$

$\to -\infty$

$$\langle \psi_{sol} | H | \psi_{sol} \rangle \leq \langle \psi_{trial} | H | \psi_{trial} \rangle$$

$\boxed{\text{PROBLEM}}$

$V_{eff} = C_0\, \delta^{(3)}(\vec{r})$ w/ $C_0 < 0$

→ no ground state

→ describes a two-body system that collapses

→ $\boxed{\text{not what we want}}$

SOLUTION $\longrightarrow$ Two steps

1) REGULARIZATION

2) RENORMALIZATION

1) REGULARIZATION → "make things regular"

$$V_{eff}(\vec{r}) = C_0 \boxed{\delta^{(3)}(\vec{r})}$$

singular, $|\delta^{(3)}(\vec{0})| \to \infty$

REGULARIZ.

$$= C_0 \boxed{\delta^{(3)}(\vec{r}; R_c)}$$

$|\delta^{(3)}(\vec{0}; R_c)| < \infty$

→ Regularization can be done in many ways

Σ

$\delta^{(3)}(\vec{r}; R_c)$

1.a) finite at $|\vec{r}| = 0$

1.b) $R_c \to 0$, $\delta^{(3)}(\vec{r}; R_c) \to \delta^{(3)}(\vec{r})$

$$\boxed{\int d^3\vec{r}\, \delta^{(3)}(\vec{r}; R_c) = 1}$$

Σ

Gaussian regulator $\longrightarrow$ $$\delta^{(3)}(\vec{r};R_c) = \frac{e^{-(r/R_c)^2}}{\pi^{3/2} R_c^3}$$

Delta-shell regulator $\longrightarrow$ $$\delta^{(3)}(\vec{r};R_c) = \frac{\delta(r-R_c)}{4\pi R_c^2}$$

(Infinite possibilities)

(δ is not singular in one dimension)

You can also do it in p-space

$$1 \xrightarrow{(\Lambda \text{ cutoff})} \frac{\Lambda^2}{\Lambda^2+g^2} \begin{cases} \Lambda \to \infty & \\ \quad \to 1 & \checkmark \\ g \to \infty & \\ \quad \to 0 & \checkmark \end{cases}$$

($\delta^{(3)}$ in p-space)

(regular)

$$1 \longrightarrow e^{-\left(g^2/\Lambda^2\right)}$$

(same features)

REGULARIZATION → Infinite possibilities

↓

PROBLEM → The cutoff is not a physical parameter

⇒ ⊛

$\otimes \Rightarrow$ The cutoff is a parameter of our theory (not of nature)

$\Rightarrow$ Results should not depend on the cutoff $\Rightarrow \otimes$

$\circledast$ ⟹ We need to achieve cutoff independence

2) RENORMALIZATION

$$V_{eff}(\vec{r}) = C_0 \delta^{(3)}(\vec{r}) \xrightarrow{①} C_0 \delta^{(3)}(\vec{r}, R_c)$$

$$\xrightarrow{②} \boxed{C_0(R_c)} \delta^{(3)}(\vec{r}, R_c) + \text{something else} \Rightarrow \circledast$$

$\Rightarrow$ this something else is a condition

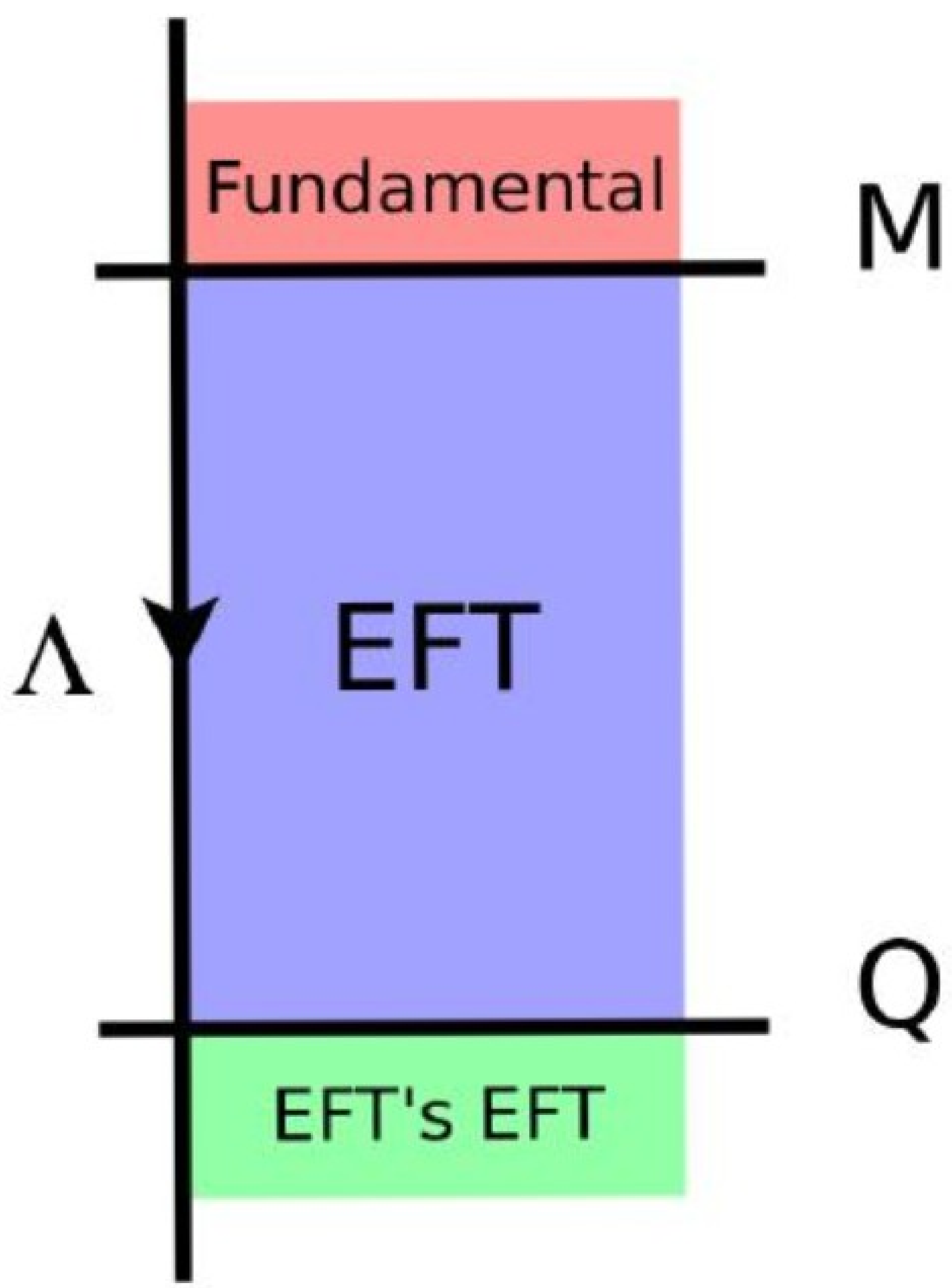
→ $\boxed{\text{Results should not depend on the cutoff } R_c / \Lambda}$

$$\langle \psi | \hat{O} | \psi \rangle = \langle \hat{O}(R_c) \rangle \quad , \quad \boxed{\frac{d}{dR_c} \langle \hat{O}(R_c) \rangle = 0}$$

$$\Rightarrow \boxed{\frac{d}{dR_c}\langle \hat{O}(R_c)\rangle = 0}$$

$\Rightarrow$ $\hat{O}(R_c)$ must have a specific dependence on R_c

It's abstract, but simple



Physics is unique, but choice of theory depends on resolution Λ:

- $\Lambda \geq M$: Fundamental
- $M \geq \Lambda \geq Q$: EFT

For equivalent descriptions:

$$\boxed{\frac{d}{d\,\Lambda}\langle\Psi|\mathcal{O}|\Psi\rangle = 0}$$

Renormalization group invariance

$$\frac{d}{dR_c}\langle \hat{O}(R_c)\rangle = 0$$

$$\frac{d}{d\Lambda}\langle \hat{O}(\Lambda)\rangle = 0$$

$\boxed{\text{How to do this?}}$

→ Solve $V_{eff}(\vec{r}, R_c) = C_0(R_c)\, \delta^{(3)}(\vec{r}; R_c)$

→ Obtain some observable as a function of $C_0(R_c)$

→ We force $\frac{d}{dR_c}(\;) = 0$

$$\boxed{\text{SIMPLE EXAMPLE!}}$$

$$V_{eff}(r;R_c) = \frac{C_0(R_c)}{4\pi R_c^2}\,\delta(r-R_c) \to \text{bound state}$$

$$-\gamma - \frac{1}{R_c} + \cancel{O(\gamma R_c)} = 2\mu\frac{C_0(R_c)}{4\pi R_c^2} \Rightarrow$$

$$\Rightarrow \quad \gamma = -\frac{\lambda}{R_c} - \frac{2\mu}{4\pi R_c^2} G(R_c) \qquad \left(\frac{\gamma}{2\mu} = B \right)$$

$$\boxed{\frac{d\gamma}{dR_c} = 0} \Rightarrow \frac{d}{dR_c}\left(\frac{\lambda}{R_c} + \frac{2\mu}{4\pi R_c^2} G(R_c) \right) = 0$$

$$\Rightarrow \boxed{G(R_c) \sim R_c}$$

EXAMPLE

1) $V_{eff}(\vec{r};R_c) = C_0(R_c)\frac{\delta(r-R_c)}{4\pi R_c^2}$

2) Binding energy/wave number is known ($\underline{B}$ or γ)

3) This forces $C_0(R_c)$ to depend on R_c

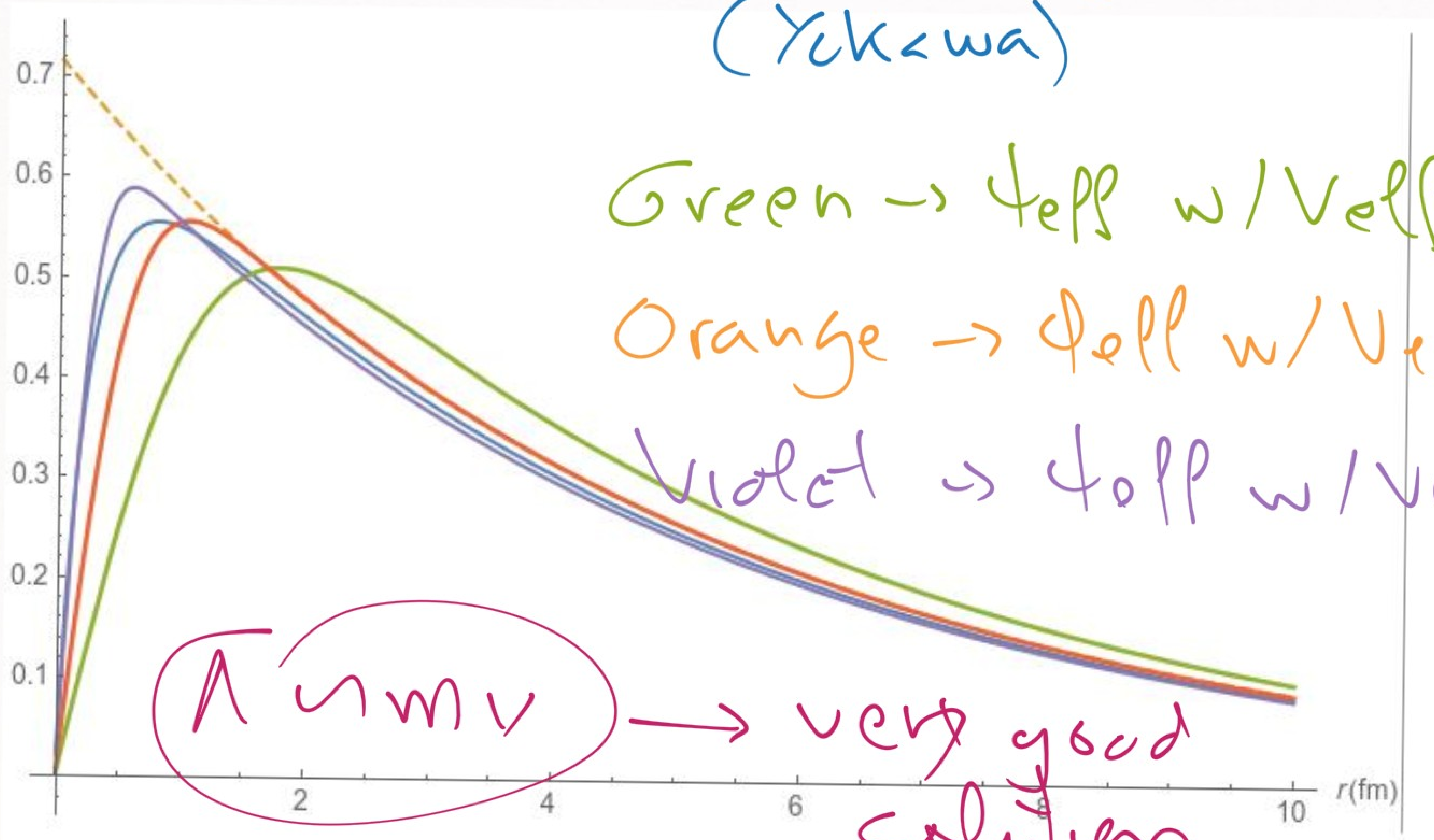
We can show more examples

$$V_{eff}(\vec{q};\Lambda) = C_0(\Lambda)\, e^{-(\vec{q}^2/\Lambda^2)}$$

w/ $C_0(\Lambda)$ such that it reproduces a given bound state

(Example → Yukawa toy model from past lesson)

Blue → ϕ_{true} w/ V_{true}, $\gamma \simeq 0.075 m_\pi$ (Yukawa)



Green → ϕ_{eff} w/ V_{eff} and $\Lambda = \frac{m_\pi}{2}$

Orange → ϕ_{eff} w/ V_{eff} and $\Lambda = m_\pi$

Violet → ϕ_{eff} w/ V_{eff} and $\Lambda = \frac{2m_\pi}{2}$

$\Lambda \sim m_\pi$ → very good solution

$\boxed{\text{TAKE-HOME MESSAGE}}$

$$C_0(\Lambda)\, e^{-(g^2\Lambda^2)} \leftrightsquigarrow -\frac{g_Y^2}{4\pi}\frac{e^{-m_Y r}}{r}$$

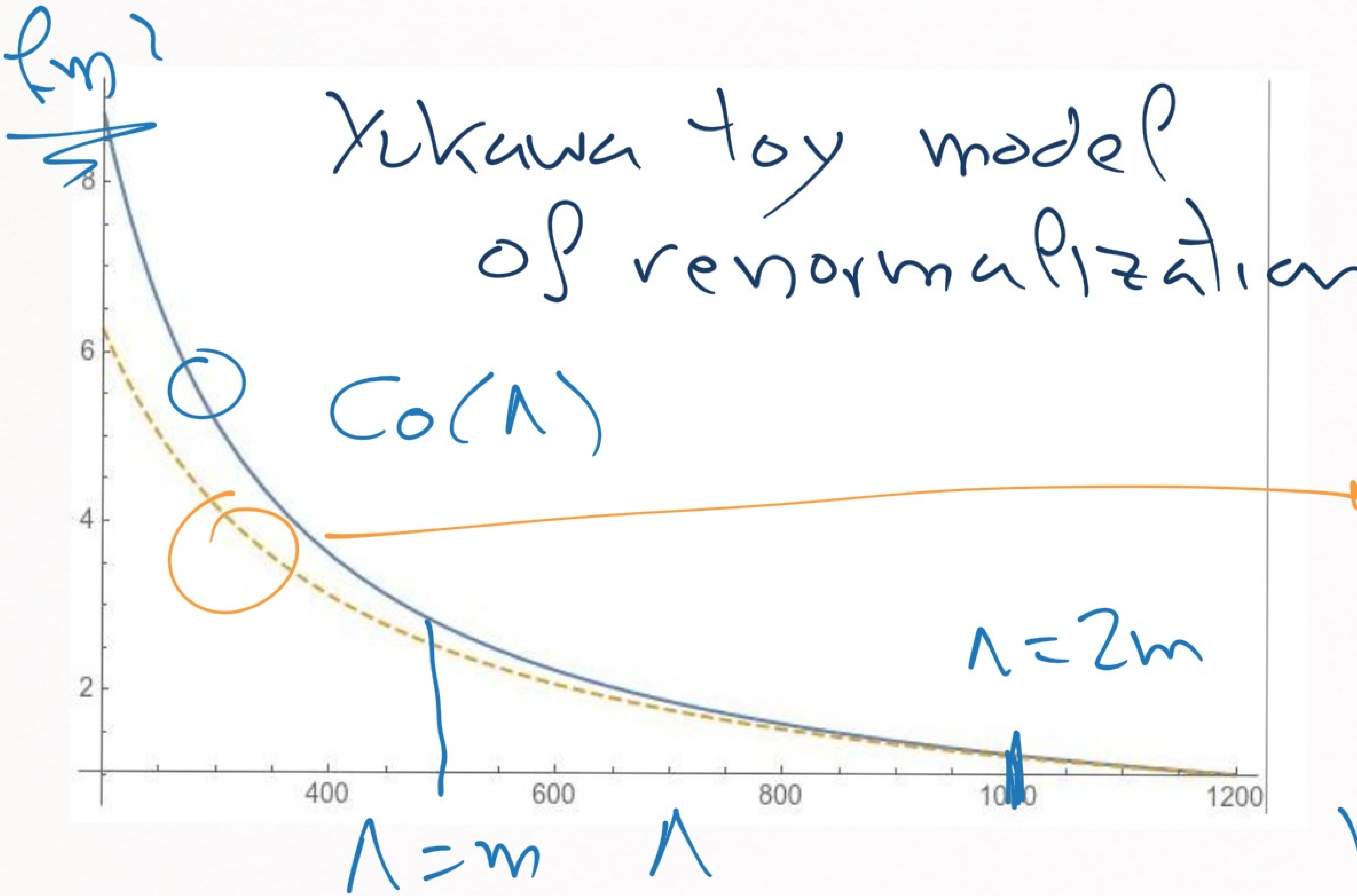
SAME RESULTS IF $\gamma \ll m_Y$

$\rightarrow$ Insensitivity to short distance (m_Y)

→ For most problems, it does not matter if we know the "true potential"

REASON → RENORMALIZATION

Renormalization $\Rightarrow$ $C_0(R_c)$ depends on the cutoff



Approx

$C_0(\Lambda) \sim \frac{1250}{\Lambda}$,

$m = 500\ \text{MeV}$

$\rightarrow$ $\boxed{\begin{array}{l} C_\sigma(R_c) \sim R_c \\ C_\sigma(n) \sim \dfrac{1}{n} \end{array}}$ Fourier-transform

(inverse of each other)

$\boxed{\text{RGE}}$

[Renormalization group evolution] of the effective description

RGE $\rightarrow$ $C_0(R_c) \sim R_c$ / $C_0(\Lambda) \sim \frac{1}{\Lambda}$

RGE
(Renormalization group equation) $\rightarrow$

$$\frac{d}{dR_c}\left[\frac{C_0(R_c)}{R_c}\right] \sim 0$$

$$\frac{d}{d\Lambda}\left[\Lambda C_0(\Lambda)\right] \sim 0$$

(so you know the language people use)

RECAP $\longrightarrow$ A brief review of renormalization

1) Consider some physical system

1.a) "true potential" $\rightarrow$ not always necessary

1.b) Check the scales, see if an effective description is possible $\rightarrow$ if ok

2) Consider an effective interaction

$$V_c(\vec{r}) = C_0 \, \delta^{(3)}(\vec{r}) \quad , \quad V_c(\vec{q}) = C_0$$

3) Regularize & Renormalize

3.a) Regularize (include a cutoff)

$$V = C_0 \delta^{(3)}(\vec{r}) \longrightarrow V = C_0(R_c) \frac{\delta(r-R_c)}{4\pi R_c^2}$$

3.b) Renormalize (determine $C_0(R_c)$ from some observable quantity, e.g. the binding energy)

4) Check if the description works

(if 1.b is OK, if $\exists$ a good scale separation $\Rightarrow$ it will work)

$\rightarrow$ $\boxed{\text{STEPS WE HAVE TO FOLLOW}}$

$\rightsquigarrow$ [$\exists$ several ways to formulate renormalization] $\to$ $\underline{\underline{C_0(R_c)}}$

a) $C_0(R_c)$ from R_c & some observable
(like binding energy)

b) $C_0(R_c)$ from the RGE
$\left(\frac{d}{dR_c} \langle\psi|\hat{O}|\psi\rangle = 0 \right)$

Let's see this one

b) → more abstract, but can be useful in many situations

$$C_0(R_c)\,\frac{\delta(r-R_c)}{4\pi R_c^2}\;\begin{cases} \text{a) } & 2\mu\,\dfrac{C_0(R_c)}{4\pi R_c^2} = -\gamma - \dfrac{1}{R_c} \\[2ex] \text{b) } & \dfrac{d}{dR_c}\langle\psi|H|\psi\rangle = 0 \;\leftarrow\; \text{second view} \\[2ex] & \hookrightarrow\; \dfrac{d}{dR_c}\left[R_c^{a}\, C_0(R_c)\right] = 0 \end{cases}$$

$\boxed{\text{RGE}}$ → Problem is usually to transform

$$\frac{d}{dR_c}\langle\psi|\hat{O}|\psi\rangle = 0 \quad \text{into} \quad \frac{d}{dR_c}\left[R_c^{a}\, C_O(R_c)\right] = 0$$

(a is some power)

$$\sum$$

(Simplification) →

$$\left[\frac{d}{dR_c}\underbrace{\langle\psi|H|\psi\rangle}=0\right] \longrightarrow \frac{d}{dR_c}\langle\psi|V|\psi\rangle=0$$

$$E=\langle\psi|H|\psi\rangle$$

$$\parallel$$

$$-B$$

$$V_c(\vec{r}) = C_0(R_c)\,\frac{\delta(r-R_c)}{4\pi R_c^2}$$

$$\frac{d}{dR_c}\langle\psi|V|\psi\rangle = 0$$

$$\langle\vec{r}|\psi\rangle = \frac{A_s}{\sqrt{4\pi}}\,\frac{e^{-\gamma r}}{r} \quad (\text{for } r > R_c)$$

$$\langle\psi|V_c|\psi\rangle = C_0(R_c)\int d^3\vec{r}\;\frac{\delta(r-R_c)}{4\pi R_c^2}\,\left|\langle\vec{r}|\psi\rangle\right|^2$$

$$= \circledast$$

$$\Theta \simeq C_0(R_c)|\psi(R_c)|^2$$

$$\langle\psi|V|\psi\rangle \sim C_0(R_c)|\psi(R_c)|^2$$

$$\frac{d}{dR_c}\langle\psi|V|\psi\rangle = \frac{d}{dR_c}\left[C_0(R_c)|\psi(R_c)|^2\right] \leq 0$$

$$\psi(R_c) \sim \frac{1}{R_c}$$ (check previous slide)

With this simplification:

$$\frac{d}{dR_c}\left[\frac{C_0(R_c)}{R_c^2}\right] = 0 \Rightarrow C_0(R_c) \sim R_c^2$$

Not completely correct (previously we saw that $C_0(R_c) \sim \dot{R}_c$)

(we are ignoring T)

but it's quite good

$$\boxed{\frac{d}{dR_c}\langle\psi|U|\psi\rangle \simeq 0}$$ $\longrightarrow$ quick & dirty trick to derive the RGE of $C_0(R_c)$

CAVEAT: IT WILL FAIL FOR NON-PERTURBATIVE PROBLEMS (BECAUSE IT IGNORES $\langle T\rangle$)

→ We will use this trick to explore the existence of different EFTs for the same problem

"Cup" power counting

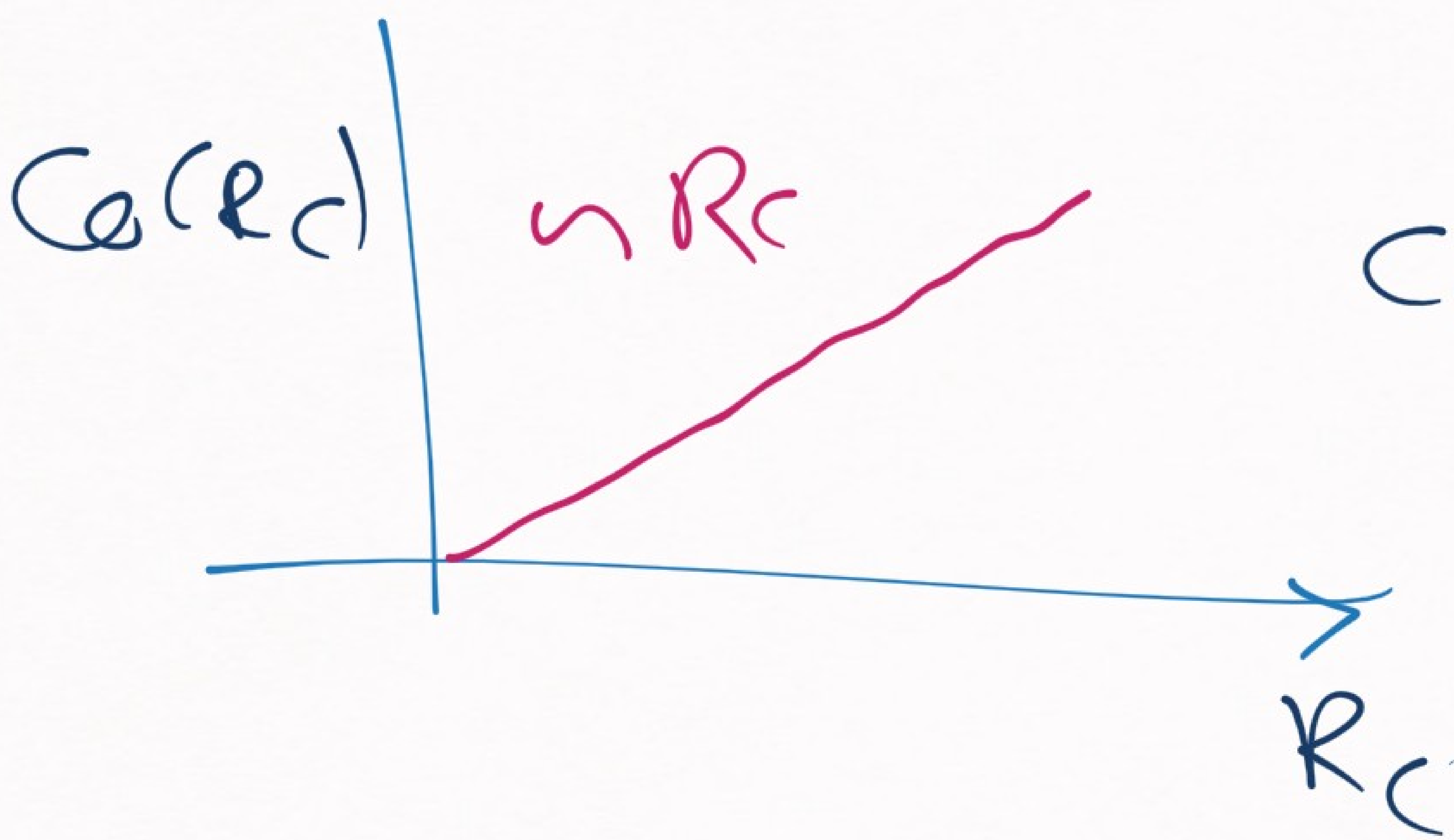
$$\lambda = \Sigma(c_0 + c_1 x + \dots)$$

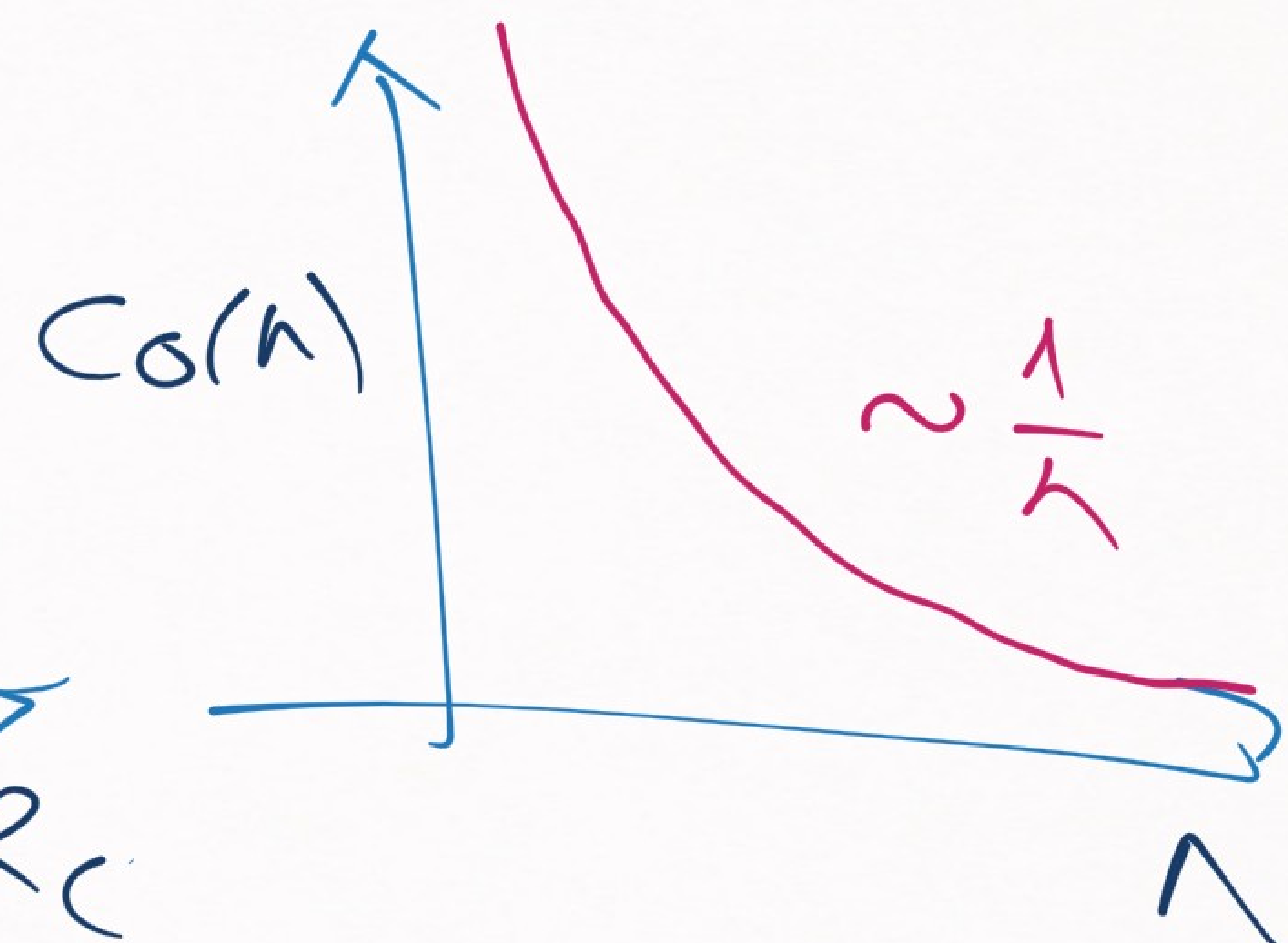
"Pot" power counting

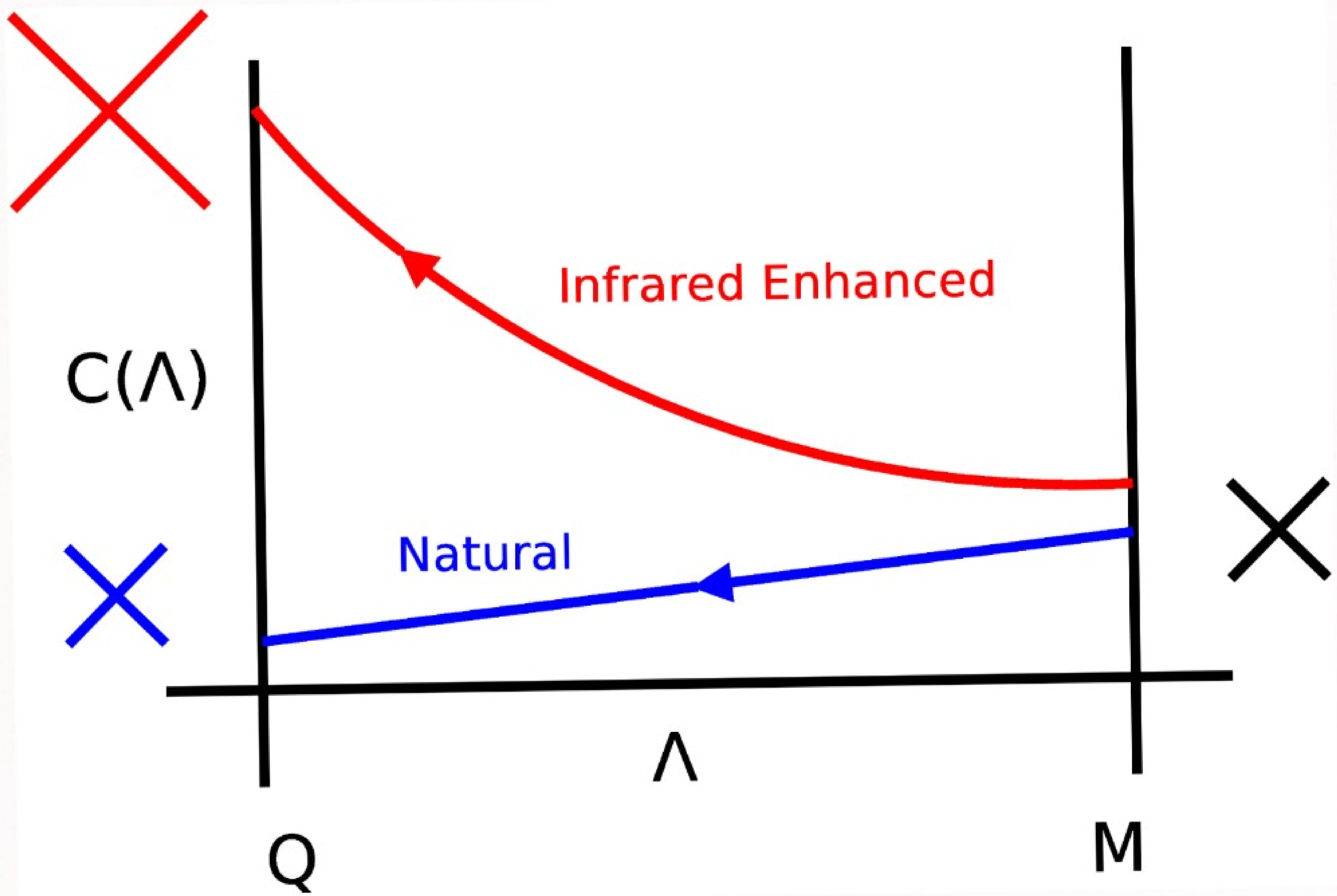
$$\lambda = \Sigma\left(d_0 + \frac{d_1}{x} + \dots\right)$$

$\Rightarrow$ $\boxed{\text{TWO BODY SYSTEM IS THE SAME}}$

If $\exists$ a bound state $\Rightarrow C_0(R_c)$







RGEs have several solutions

$\Rightarrow$

Depend on our assumptions

$\boxed{\text{Two-body system}}$

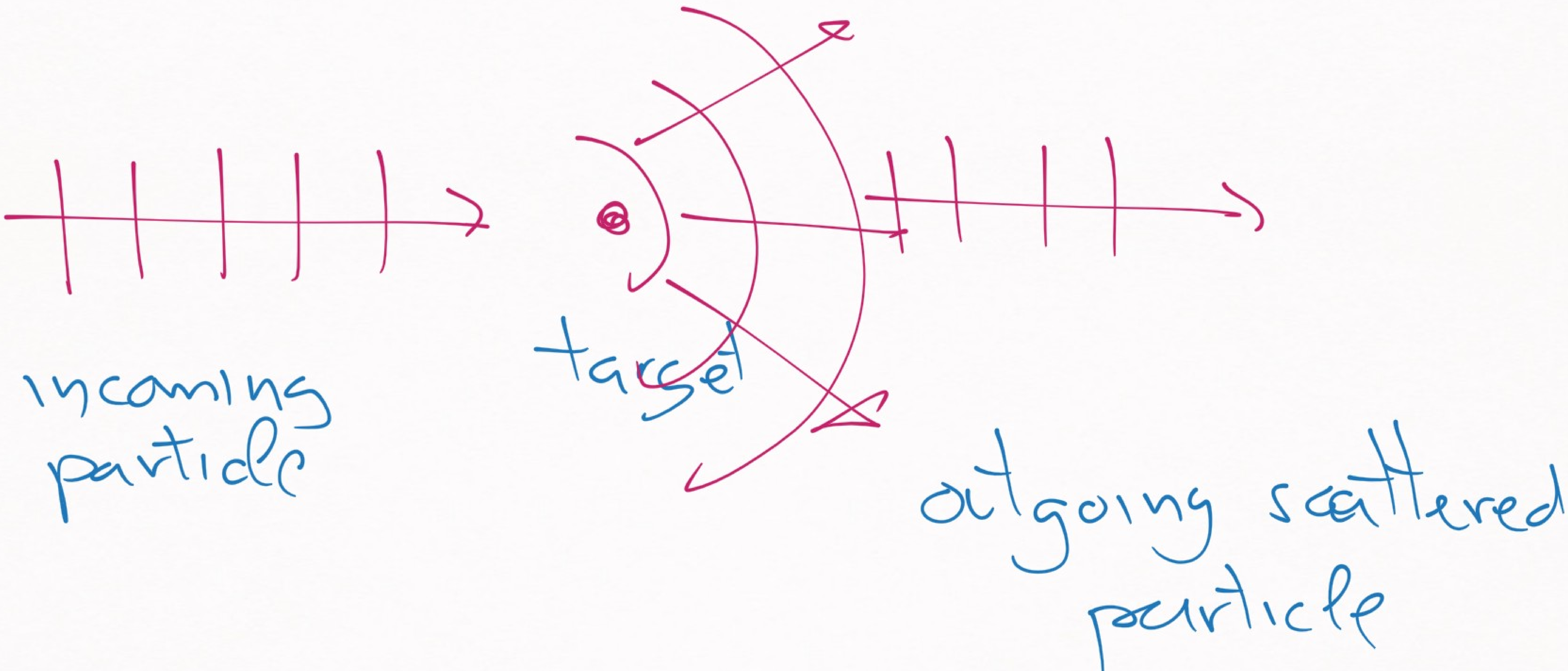
1) $\exists$ a shallow bound state $(\gamma \ll m)$

how to do it?

2) $\nexists$ bound states, but we have instead a perturbative system

→ I will have to advance scattering theory (if you don't know scattering theory, ignore the following explanation until later)

2) SCATTERING BY A WEAK POTENTIAL



$$\psi(\vec{r}) \rightarrow \underbrace{e^{i\vec{k}\cdot\vec{r}}}_{\text{incoming wave}} + \underbrace{f(\hat{k})\,\frac{e^{ikr}}{r}}_{\text{scattered wave}}$$

$f(\hat{k}) \rightarrow$ scattering amplitude

$$\frac{d\sigma}{d\Omega} = |f(\hat{k})|^2$$

Scattering by a weak potential:

→ Born approximation

$$f(\hat{k}) \simeq -\frac{\mu}{2\pi}\int d^3\vec{r}\, V(\vec{r})\, e^{-i\vec{k}\cdot\vec{r}}$$

(check it in your QM textbook)

→ Born approximation $\Rightarrow f(\hat{k})$ small

$$\psi(\vec{r}) \to \underbrace{\psi(\vec{r}) \simeq e^{i\vec{k}\cdot\vec{r}}}_{\text{input for RGE}} + \text{small corrections}$$

$$\langle\psi|V_c|\psi\rangle = \int d^3\vec{r}\; C_0\,\delta^{(3)}(\vec{r};R_c)\,|\cancel{e^{i\vec{k}\cdot\vec{r}}}|^2 = C_0$$

1

$\circledast \simeq C_0(R_c) \Rightarrow \langle\Psi|V_c|\Psi\rangle \simeq C_0(R_c)$

$\boxed{RBE} \Rightarrow \underbrace{\frac{d}{dR_c}\langle\Psi|V_c|\Psi\rangle = 0}_{\Downarrow}$

$C_0(R_c) \simeq \text{constant}$

$$\boxed{\frac{d}{dR_c}[C_0(R_c)] = 0}$$

Is this true? Can we check it? YES!

$$f(\hat{k}) = -\frac{\mu}{2\pi}\int d^3\vec{r}\, V(\vec{r})\, e^{-i\vec{k}\cdot\vec{r}}$$

$$= -\frac{\mu}{2\pi} V(\vec{k}) = -\frac{\mu}{2\pi}\left[V(\vec{0}) + (\text{expansion in } \vec{k}^2)\right]$$

Low energies ($|\vec{k}|^2 \ll m^2$) $\quad \underline{\underline{f(\hat{k}) = -\frac{\mu}{2\pi} V(\vec{0})}}$

$$V_{eff}(\vec{q}) = C_0 \rightsquigarrow \rho_{eff}(\hat{k}) = -\frac{\mu}{2\pi} C_0$$

$$V_y(\vec{q}) = -\frac{g_y^2}{q^2 + m_y^2} \rightsquigarrow \rho_y(k) = -\frac{\mu}{2\pi}\left(-\frac{g_y^2}{m_y^2} + \text{corrections}\right)$$

$$O\left(\frac{\vec{q}^2/m_y^2}{}\right)$$

$$\rho_y(k) \to +\frac{\mu}{2\pi}\frac{g_y^2}{m_y^2}$$

$$\rho_C(k) \to -\frac{\mu}{2\pi} C_0$$

$$C_0 = -\frac{g_y^2}{m_y^2}$$

$$\frac{d}{dR_c}\left[c_\sigma(R_c)\right] = 0 \longrightarrow \text{weak potential}$$

$$\frac{d}{dR_c}\left[\frac{c_\sigma(R_c)}{R_c}\right] = 0 \longrightarrow \text{Strong potential w/ shallow bound state}$$

Message → DEPENDING ON THE SITUATION WE WILL HAVE DIFFERENT BEHAVIORS FOR $C_0(R_c)$

⇝ Different RGE's or EFT's

(FINISH FOR TODAY)