NUCLEAR PHYSICS 3

→ SCALES : HOW TO IDENTIFY THEM

→ UNNATURAL (FINE-TUNED) SYSTEMS

HOW TO UNDERSTAND FINE-TUNING

RECAP

CLASSIFICATION → ⊛

1) PHYSICAL SYSTEMS → CHARACTERISTIC SCALE

→ 2) NATURAL SYSTEMS :

2.a) $\exists$ a characteristic scale / describes the system $Q \sim \frac{1}{R}$

if we use units of $[E] \to Q$ (momentum/energy scale)

if we use units of $[L] \to R$ (length scale)

2.b) $\forall$ observable quantity will be $\mathcal{O}(1)$ in terms of the proper units of Q/R

$$\langle p \rangle \sim Q \sim \frac{1}{R} \quad \text{(dimensions of } [E])$$

$$\langle r \rangle \sim \frac{1}{Q} \sim R \quad \text{(dimensions of } [L])$$

etc.

2.c) In general, a natural system is easy to deal with

3) MOST SYSTEMS $\longrightarrow$ MANY SCALES

$\longrightarrow$ [MULTISCALE SYSTEMS]

$$Q \text{ or } R \longrightarrow \left.\begin{array}{l} \{Q_1, Q_2, \ldots, Q_n\} \\ \{R_1, R_2, \ldots, R_n\} \end{array}\right]$$

many scales involved

Example → H atom : $\boxed{a_B}$, $\frac{1}{m_e}$, r_p

(Bohr radius)

MOST IMPORTANT

Hierarchy

$a_B \gg \frac{1}{m_e} \gg r_p$

FINE STRUCTURE

HYPERFINE STRUCTURE

3.a) Usually, $\exists$ hierarchy ($\exists$ separation of scales)

$\{Q_1, Q_2, \ldots, Q_n\}$

$$\underbrace{Q_1}_{\text{general description}} \ll \underbrace{Q_2 \ll Q_3 \ll \cdots}_{\text{smaller \& smaller corrections}}$$

Example: H atom

general description

smaller & smaller corrections

3.b) Sometimes, no clear hierarchy $\exists$

$$\boxed{Q_1 \sim Q_2 \ll Q_3 \ll \ldots}$$

New effects possible $\rightarrow$ there could be $\boxed{\text{FINE-TUNING}}$

(explain it later)

$\boxed{\text{NATURAL SYSTEMS}}$

H atom

$a_B \sim 53000\,\text{fm}$

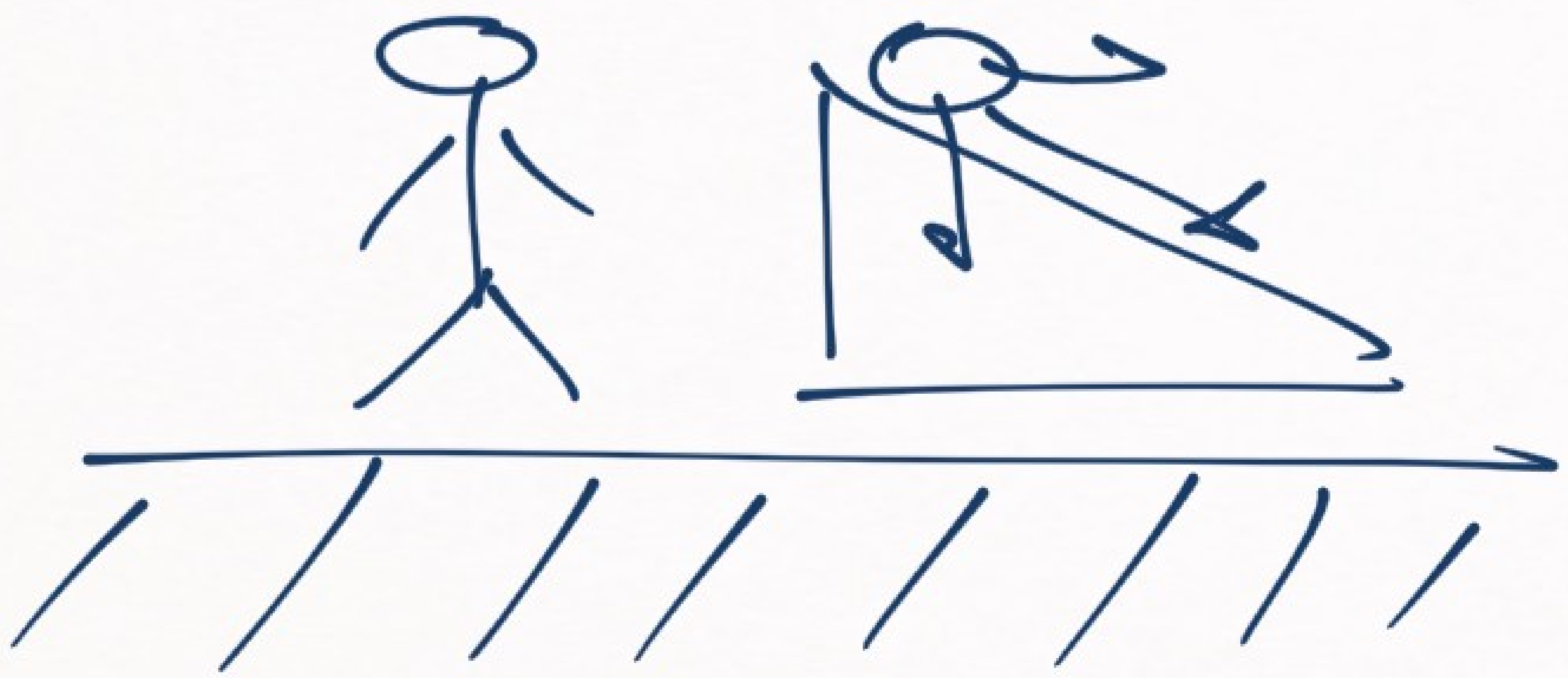
$1/m_e \sim 400\,\text{fm}$

$r_p \sim 1\,\text{fm}$

Excellent scale separation $\Rightarrow$

GALILEAN & NEWTONIAN GRAVITY IN EVERYDAY LIFE

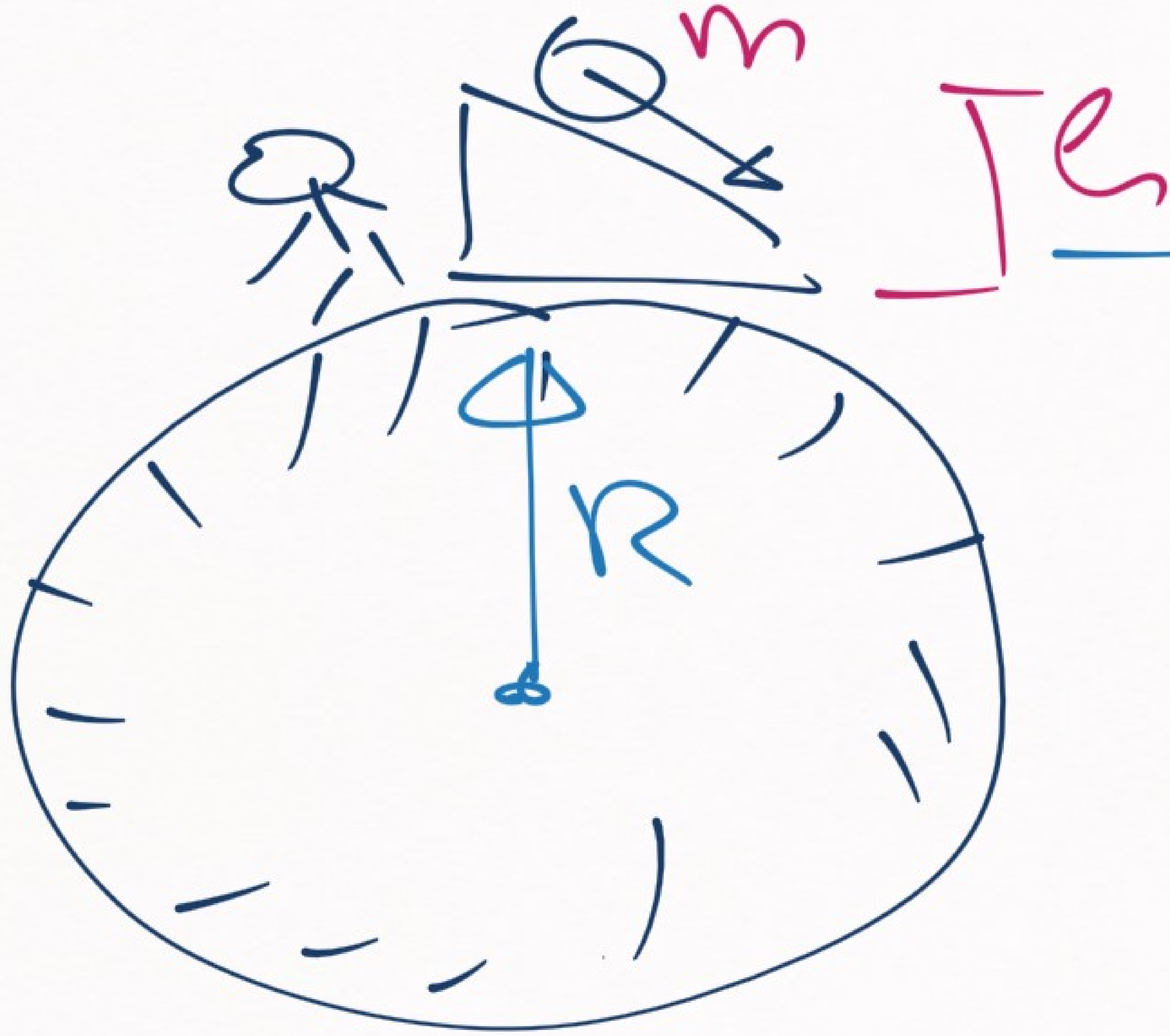
Galilean gravity → $\underbrace{\Delta V_G}_{\text{potential energy difference}} = \underbrace{m}_{\text{mass}}\, \underbrace{g}_{9.8\, m/s^2}\, \underbrace{h}_{\text{height}}$



$\rightarrow \Delta V_G = mgh$

an excellent approximation

$\boxed{\text{BUT HERE, WE ARE IGNORING THE SIZE OF THE EARTH}}$



$$\boxed{\Delta V_N = \frac{GMm}{R} - \frac{GMm}{R+h}}$$

$G \to$ NEWTON'S CONSTANT

$M \to$ EARTH'S MASS

$R \to$ EARTH'S RADIUS

$2R = 12756\ \mathrm{km}$

$R \simeq 6400\ \mathrm{Km}$

→ EXPANSION IN h/R → SCALE SEPARATION

$R \sim 6400$ km

$h \sim 10^{-3}$ m to 10 km

h/R small

$$\Delta V_N = \frac{GM}{R}\, m \left(\frac{h}{R} - \left(\frac{h}{R}\right)^2 + \left(\frac{h}{R}\right)^3 - \dots \right)$$

$$g = \frac{GM}{R^2}$$

$$= mgh \left[1 - \frac{h}{R} + \left(\frac{h}{R}\right)^2 - \dots \right]$$

$$= \Delta V_G \left(1 + O(h/R) \right)$$

$$\text{Einstein's gravity} \xrightarrow{v \ll c} \text{Newtonian gravity} \xrightarrow{h \ll R} \text{Galilean gravity}$$

$$\boxed{\text{Problem of scale hierarchy}}$$

$\boxed{\text{STANDARD MODEL}}$ $\leftarrow$ BEYOND SM PHYSICS

(supersymmetry, extra dimensions, etc.)

$\boxed{M_{bSM}}$ $\longrightarrow$

$\boxed{\mathcal{O}\left(\frac{a}{M_{bSM}}\right)}$ $\longrightarrow$ $\oplus$

→ THIS IS WHY SO MUCH EFFORT IS PUT ON MEASURING VERY SMALL DISCREPANCIES WRT STANDARD MODEL PREDICTIONS

$\mathcal{O}\left(\frac{Q}{M_{BSM}}\right) \to$ deduce the $\exists$ of beyond SM effects

$\boxed{\text{NUCLEAR PHYSICS}}$ → not like the previous examples

1) Not natural → DEUTERON (very large, $\sim 5\,\mathrm{fm}$)

needs an explanation

2) Poor separation of scales

pion exchanges $\frac{1}{m_\pi} \sim 1.4\,\mathrm{fm} \simeq$ nucleon size $r_N \sim (0.5 - 1.0)\,\mathrm{fm}$

TWO BODY SYSTEMS → IDENTIFYING THEIR SCALES

1

2 → WHAT HAPPENS WHEN $\exists$ TWO (OR MORE) SCALES?

Two-body system $\longrightarrow$ Schrödinger equation

$$\Big[\underbrace{-\frac{1}{2\mu}}_{\text{reduced mass}}\vec{\nabla}^2 + \underbrace{V(\vec{r})}_{\text{potential}}\Big]\underbrace{\psi(\vec{r})}_{\text{wave function}} = \underbrace{E}_{\text{energy}}\psi(\vec{r})$$

INPUT OR OUTPUT

reduced mass   potential

INPUT (things that we know)

wave function

OUTPUT (we want to calculate it)

What are the dimensions?

$[\mu] = [E]$ or $[L^{-1}]$

$[\vec{\nabla}^2] = [E^2]$ or $[L^{-2}]$

$[V] = [E]$ $\qquad [\psi] = [E^{3/2}]$ (for bound states)

$$\underbrace{\int d^3\vec{r}}_{[L^3]}\, \underbrace{|\psi(\vec{r})|^2}_{[L^{-3}]} = 1 \qquad [\psi] = [L^{-3/2}]$$

Schrödinger $\longrightarrow$ reduced Schrödinger

Why? $\longrightarrow$ non-relativistic systems:

[we can factor out the mass]

$$\left[-\frac{1}{2\mu}\vec{\nabla}^2 + V(\vec{r})\right]\psi(\vec{r}) = E\,\psi(\vec{r})$$

[Remove μ]

$$\left[-\vec{\nabla}^2 + 2\mu V(\vec{r})\right]\psi(\vec{r}) = 2\mu E\,\psi(\vec{r})$$

$$\left[-\nabla^2 + \underbrace{2\mu V(\vec{r})}_{U(\vec{r})}\right]\psi(\vec{r}) = \underbrace{2\mu E}_{\substack{k^2 \ (\text{if } E>0) \\ -\gamma^2 \ (\text{if } E<0)}}\psi(\vec{r})$$

$U(\vec{r})$: REDUCED POTENTIAL

$$[U(\vec{r})] = [L^{-2}]$$

or $[E^2]$

USING MOMENTA MORE CONVENIENT

$$\left[-\vec{\nabla}^2 + U(\vec{r})\right]\psi(\vec{r}) = -\underbrace{\gamma^2}_{\text{bound states}}\psi(\vec{r})$$

bound states $\rightarrow$ they are easier

If S-wave bound states:

$$\psi(\vec{r}) = \frac{1}{\sqrt{4\pi}}\frac{u(r)}{r}$$

$\rightarrow$ will be useful later

$[U] = [L^{-2}] \rightarrow$ help w/ scales

EXAMPLE I → H ATOM

$V(\vec{r}) = -\frac{\alpha}{r}$ $[L^{-1}]$ (Coulomb)

How did we obtain a_B?

convention

$2\mu V(\vec{r}) = U(\vec{r}) = -2\mu\frac{\alpha}{r} = -\frac{2}{a_B r}$

$[L^{-2}]$ $[L^{-2}]$

$$u(\vec{r}) = -\frac{2}{a_B r} \qquad [L^{-2}]$$

Origin of this convention:

$$\psi(\vec{r}) = \frac{1}{\sqrt{4\pi}} \frac{2}{a_B^{3/2}} e^{-r/a_B}$$

$\rightarrow \exists$ arbitrary numerical factors in defining scales $\rightarrow$ a bit of an art

$\boxed{\text{EXAMPLE 2}}$ $\longrightarrow$ van der Waals potential
(Cesium atoms)

$$V(\vec{r}) = -\frac{C_6}{r^6} \quad [L^{-1}]$$

$$2\mu V(\vec{r}) = U(\vec{r}) = -2\mu \frac{C_6}{r^6} = -\frac{R_6^{4}}{r^{6}} \quad [L^{-2}]$$

$$R_6 = (2\mu C_6)^{1/4} \simeq 203\ a_B$$
(Cs-Cs)

$$4 - 6 = -2$$

EXAMPLE 3 $\longrightarrow$ weird

Inverse square-law potential

$$V(\vec{r}) = -\frac{c_2}{r^2} \quad [L^{-1}]$$

pure number

$$2\mu V(\vec{r}) = U(\vec{r}) = -2\mu\frac{c_2}{r^2} = -\frac{g}{r^2}[L^{-2}]$$

$\longrightarrow$ [$\exists$ no scale]

[What is so special about not having a scale?]

$$\left[-\vec{\nabla}^2 - \frac{g}{r^2}\right]\psi(\vec{r}) = -\gamma^2\psi(\vec{r})$$

↳ what happens when there is no scale?

↳ symmetry

$$\underbrace{\left[-\vec{\nabla}^2 - \frac{g}{r^2}\right]}_{1}\psi(\vec{r}) = -\underbrace{\gamma}_{2}{}^2\,\psi(\vec{r})$$

1

$$\left.\begin{array}{l} \vec{r} \to \lambda \vec{r} \\ \vec{\nabla} \to \dfrac{1}{\lambda}\vec{\nabla} \end{array}\right] \to \left(-\vec{\nabla}^2 - \frac{g}{r^2}\right) \to \overbrace{\frac{1}{\lambda^2}\left(-\vec{\nabla}^2 - \frac{g}{r^2}\right)}^{1}$$

$$\left.\gamma \to \frac{1}{\lambda}\gamma\right]$$

$$\overbrace{\gamma^2 \to \frac{1}{\lambda^2}\gamma^2}^{2}$$

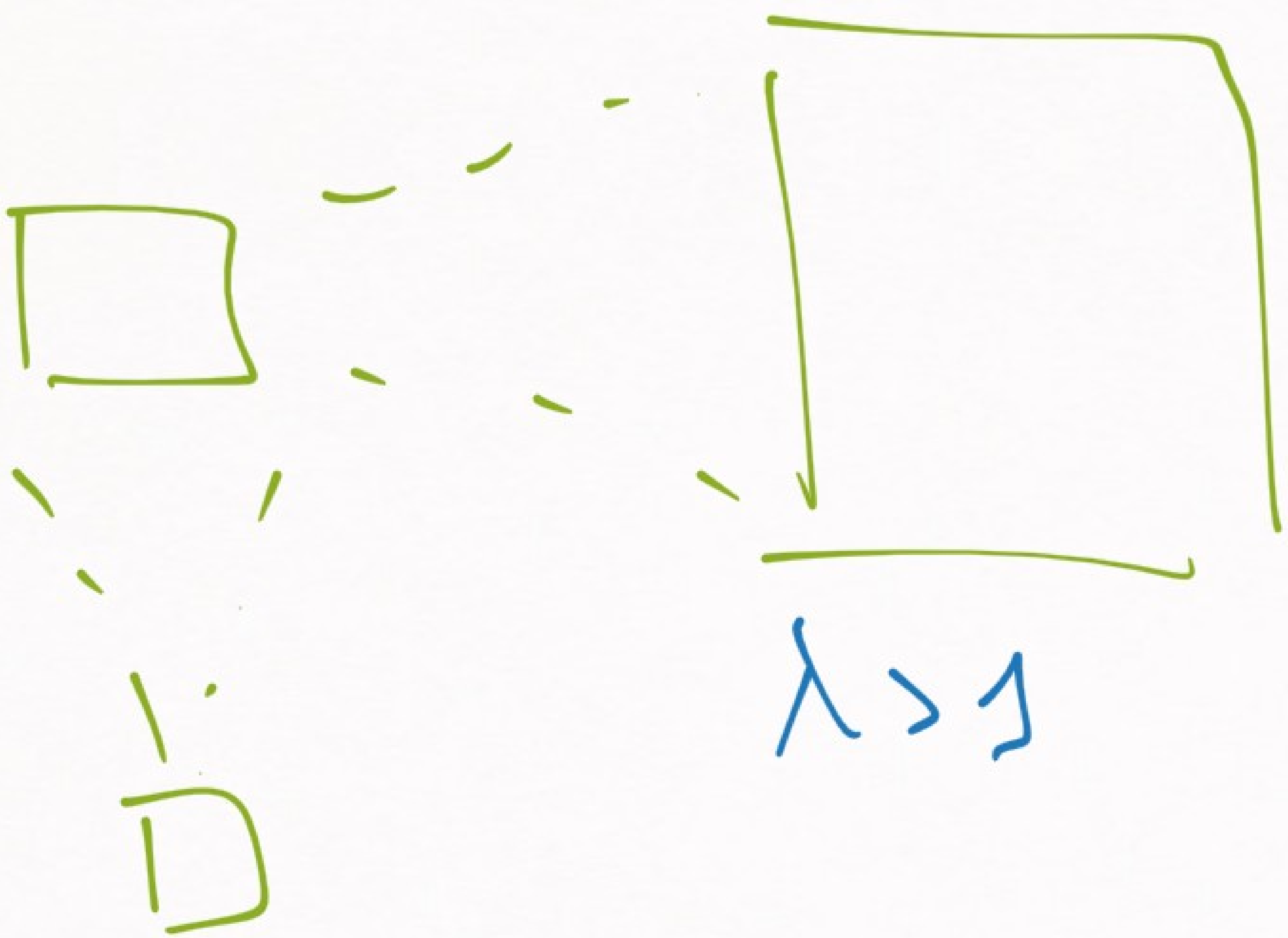
$$\left.\begin{aligned}\vec{r} &\to \lambda\vec{r}\\ \vec{p} &\to \frac{1}{\lambda}\vec{p}\end{aligned}\right\} \to \text{there is no change in the Schrödinger eq.}$$

$\to$ conformal transformation $\} \to$ dilation

$1/r^2$ potential $\longrightarrow$ system that is invariant under dilations

$\downarrow$

looks the same if we <u>zoom</u> in/out



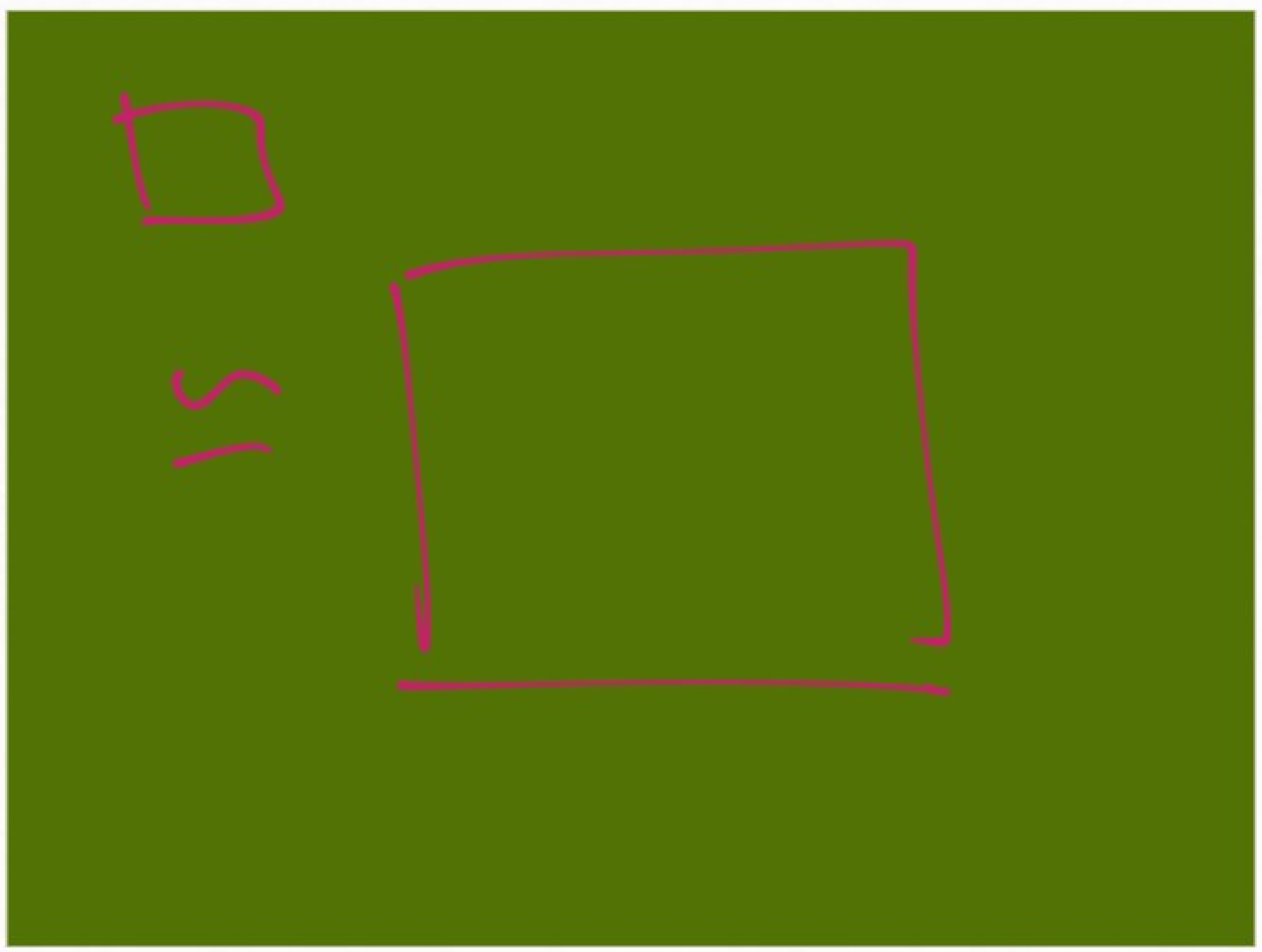
$\lambda > 1$

$\lambda < 1$

What type of systems have this symmetry?

(structure)

a)



(no structure)

b)

repeats itself

Scale invariance $\begin{cases} \text{(a) continuous} \\ \text{(b) discrete} \end{cases}$

(a)

(b)

$\vec{r} \to \lambda \vec{r}, \ \forall \lambda$

$\vec{r} \to \lambda_0 \vec{r}$ (λ_0 specific)

→ type of system described by $1/r^2$ pot.

$$\left[-\nabla^2 - \frac{g}{r^2}\right]\psi(\vec{r}) = -\gamma^2\psi(\vec{r})$$

In principle, this should belong

to type (a), but...

$$U(\vec{r}) = -\frac{g^2}{r^2}$$

$g^2 < 1/4 \Rightarrow$ (a) continuous

$g^2 > 1/4 \Rightarrow$ $\boxed{\text{(b) discrete}}$ how come?

$\boxed{g^2 > 1/4}$ → EASIEST EXAMPLE THAT WE HAVE OF AN ANOMALY

$\boxed{\text{ANOMALY}}$ → a classical symmetry that is broken by the quantization process

→ most anomalies happen in $\boxed{\text{QFT}}$

→ ($1/r^2$ anomaly → QM

(Quantum mechanics)

Discrete scale invariance $\to$ anomaly

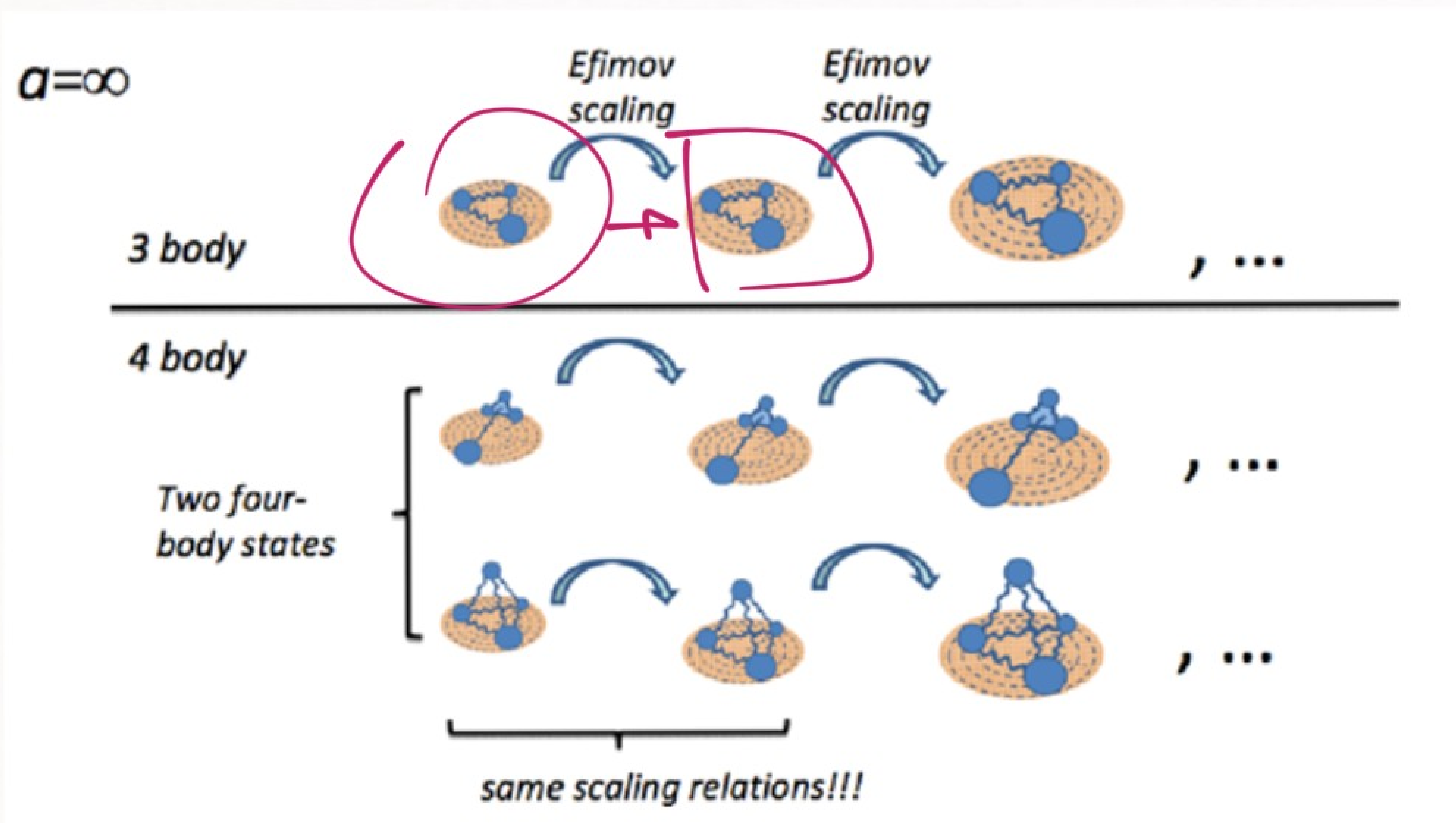
a) two-body $1/r^2$ potential

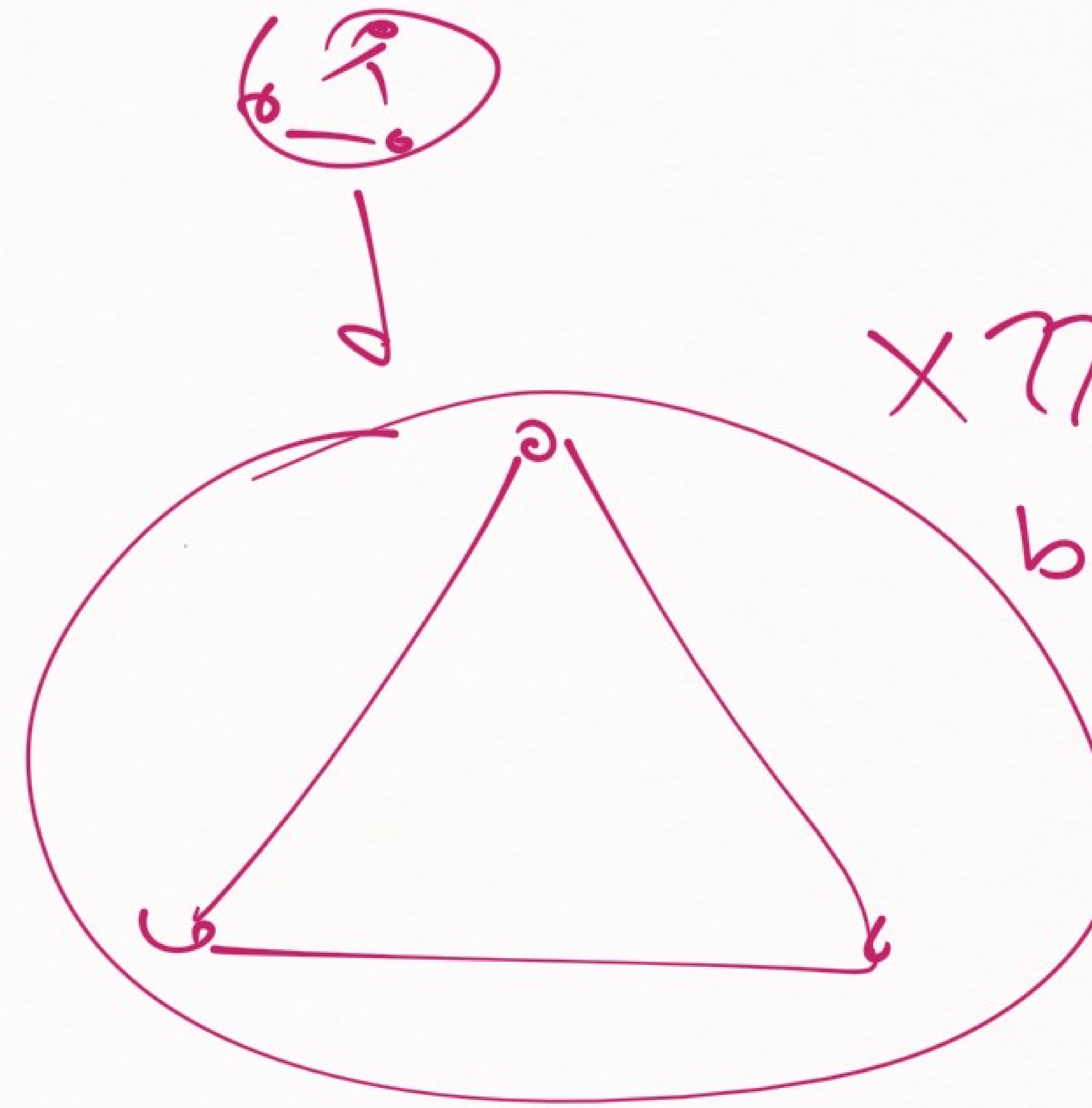
b) three-body system / two-body system has a zero-energy bound state

experimentally checked $\to$ $\boxed{\text{EFIMOV EFFECT}}$

$\boxed{\text{Efimov effect}}$ → Anomaly & discrete scale invariance

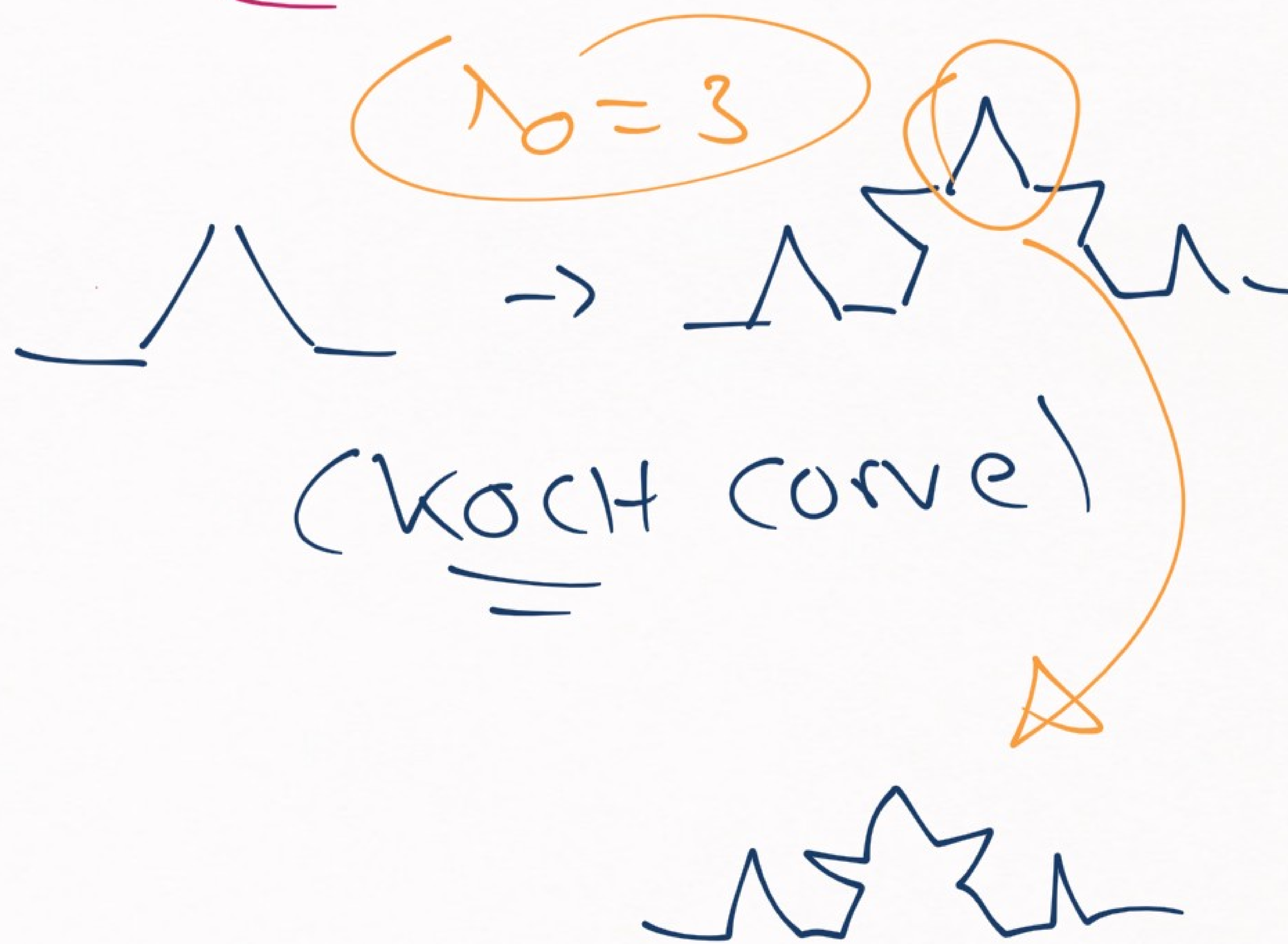
$$\vec{r} \to \lambda_0 \vec{r}\,, \quad \lambda_0 \simeq 22.7$$





×22.7 bigger

Other example → fractals



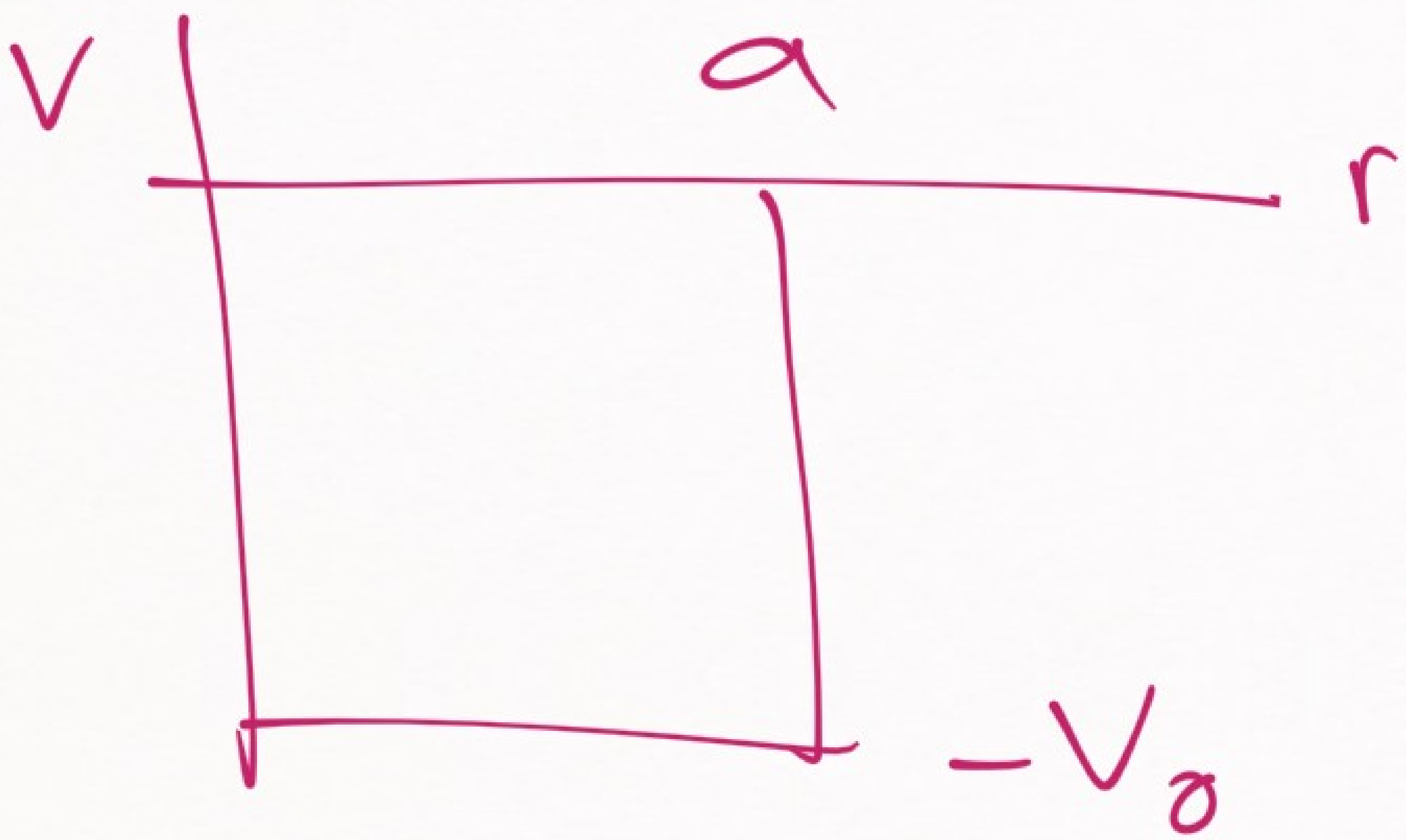
(KOCH curve)

TWO SCALE SYSTEM + FINE-TUNING

Example → Square-well { two-scale
we can tune the scales

↓

[unnatural system]



$$V(\vec{r}) = -V_0\,\Theta(a-r)$$

$$\boxed{V_0 = \frac{1}{2\mu}\frac{1}{R_S^2}}$$

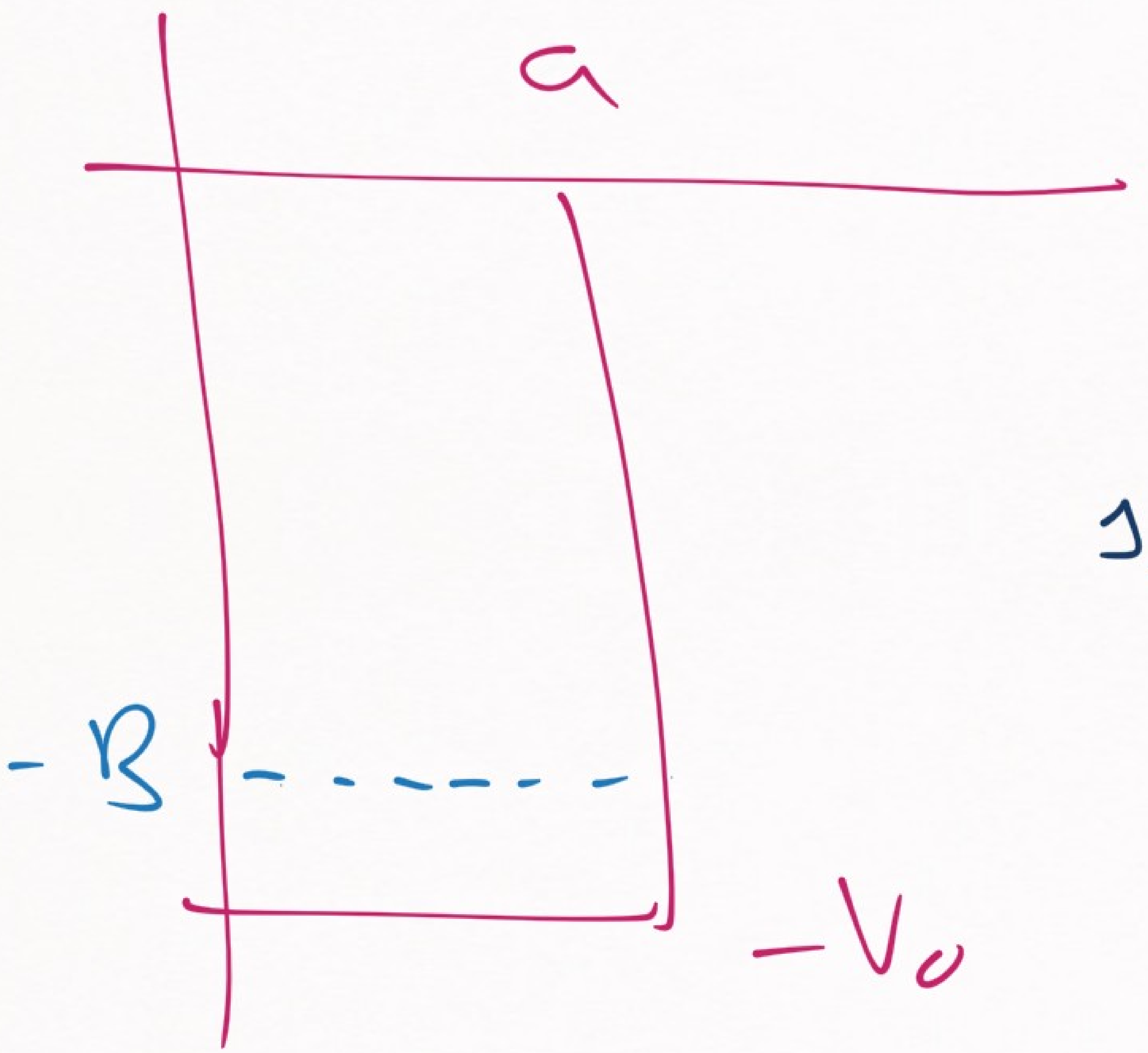
scale related to the depth of the well

$\boxed{\text{SQUARE WELL}}$

→ Scale 1 : a (the size of the well)

→ Scale 2 : $\frac{1}{R_S}$ (the depth of the well)

$\boxed{\text{What does a natural square well look like?}}$



→ Suppose we have a bound state

1) $B \leq V_0$ (dynamical constraint)

1,a) $B \sim \mathcal{O}(V_0)$ (natural)

1,b) $B << \mathcal{O}(V_0)$ (unnatural)

Imagine we don't know anything about the dynamics of the system:

$\rightsquigarrow$ $B \in [0, V_0]$ random? (actually not the case)

(mental experiment)

$B \sim 0.5 V_0 \sim \mathcal{O}(V_0)$ → most common situation

$B \sim 0.01 V_0$ → will happen rarely

$\boxed{\text{mental experiment}}$ $\Rightarrow$ Indeed, the most common thing is:

$$\boxed{B \sim O(V_0)}$$

Meanwhile, this will be weird:

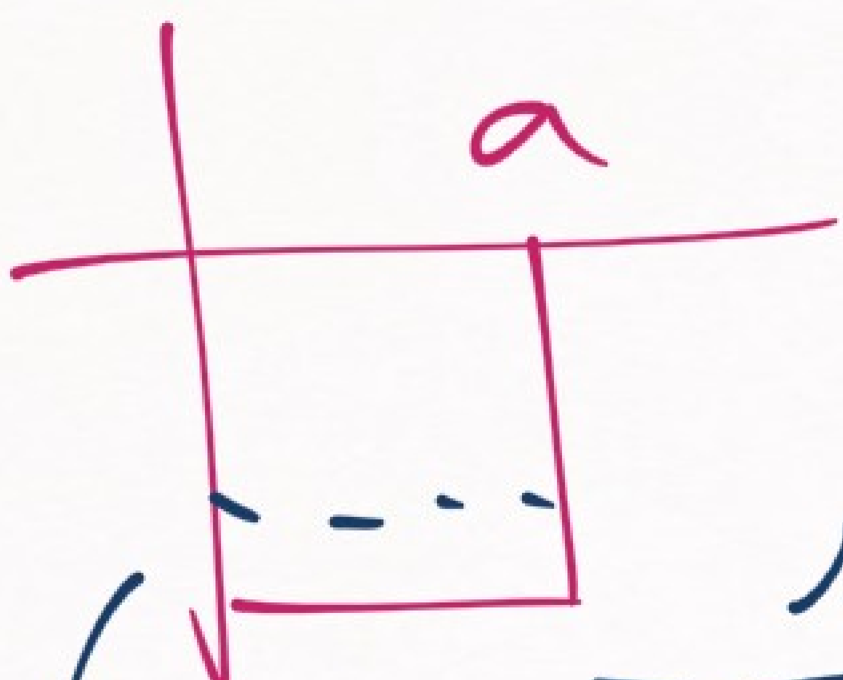
$$B \ll O(V_0)$$

will require explanation

General idea $\rightarrow$ $\boxed{\text{CONCRETE CALCULATIONS}}$

Reminder of the square well:

(check it)



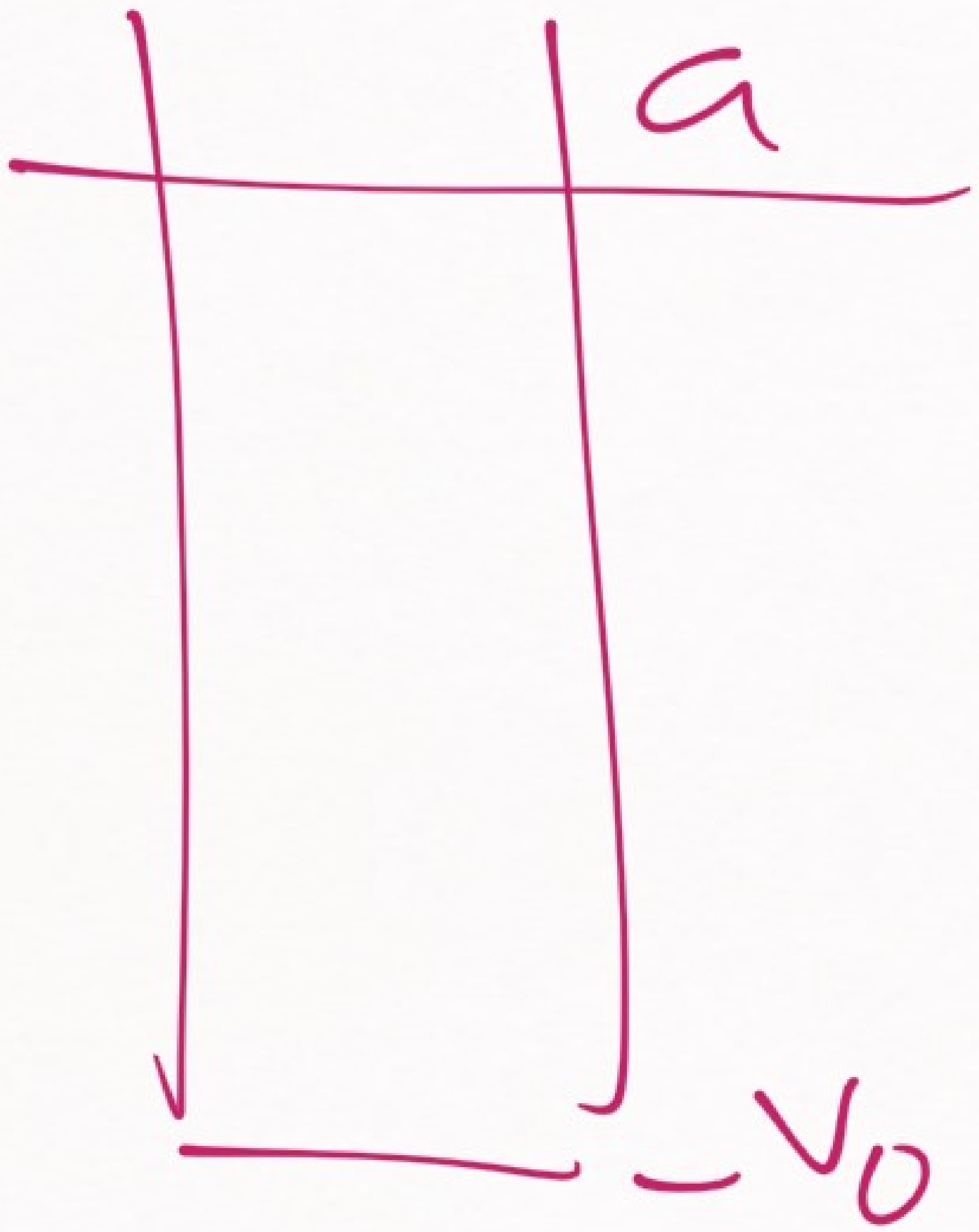
$$-\frac{1}{2\mu}\frac{1}{R_s^2} = -V_0$$

$\rightarrow$ eigenvalue equation:

$$k\cot(ka) = -\gamma$$

$$B = \frac{1}{2\mu}\gamma^2 = -E_s$$

$$k = \sqrt{\frac{1}{R_s^2} - \gamma^2} = \sqrt{2\mu(V_0 - B)}$$



SEPARATION OF SCALES:

$$\boxed{R_s \ll a}$$ → easiest example

$$\Downarrow$$

$$\boxed{\begin{gathered} B \sim \mathcal{O}(V_0) \\ \gamma \sim \mathcal{O}(1/R_s) \end{gathered}}$$ → coincides w/ previous assumptions of <u>naturalness</u>

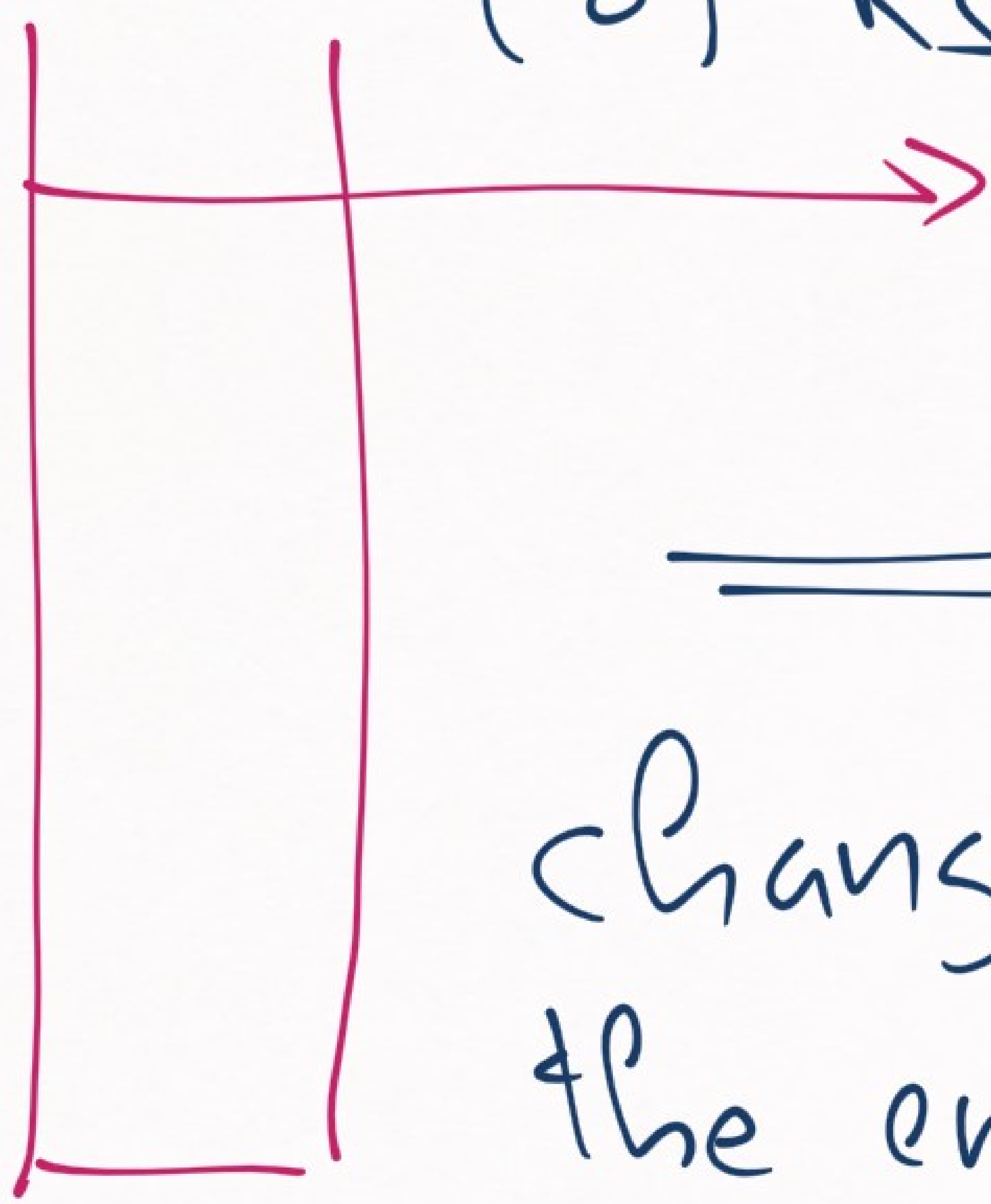
$R_s \ll a$ → what does it mean?

$\Downarrow$

[Depth of the well is more important than its range]

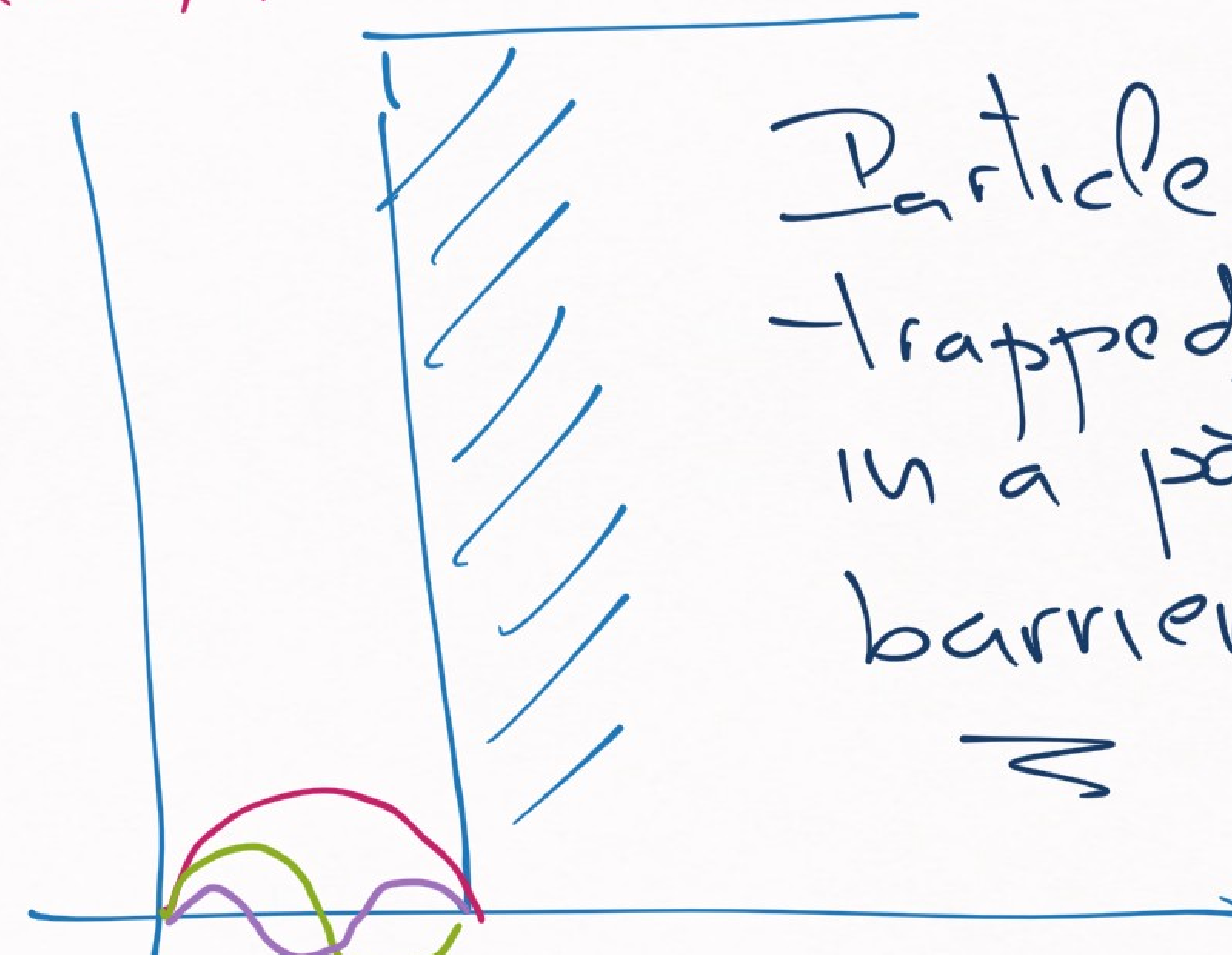
Physical interpretation:

(of $R_s \ll a$)



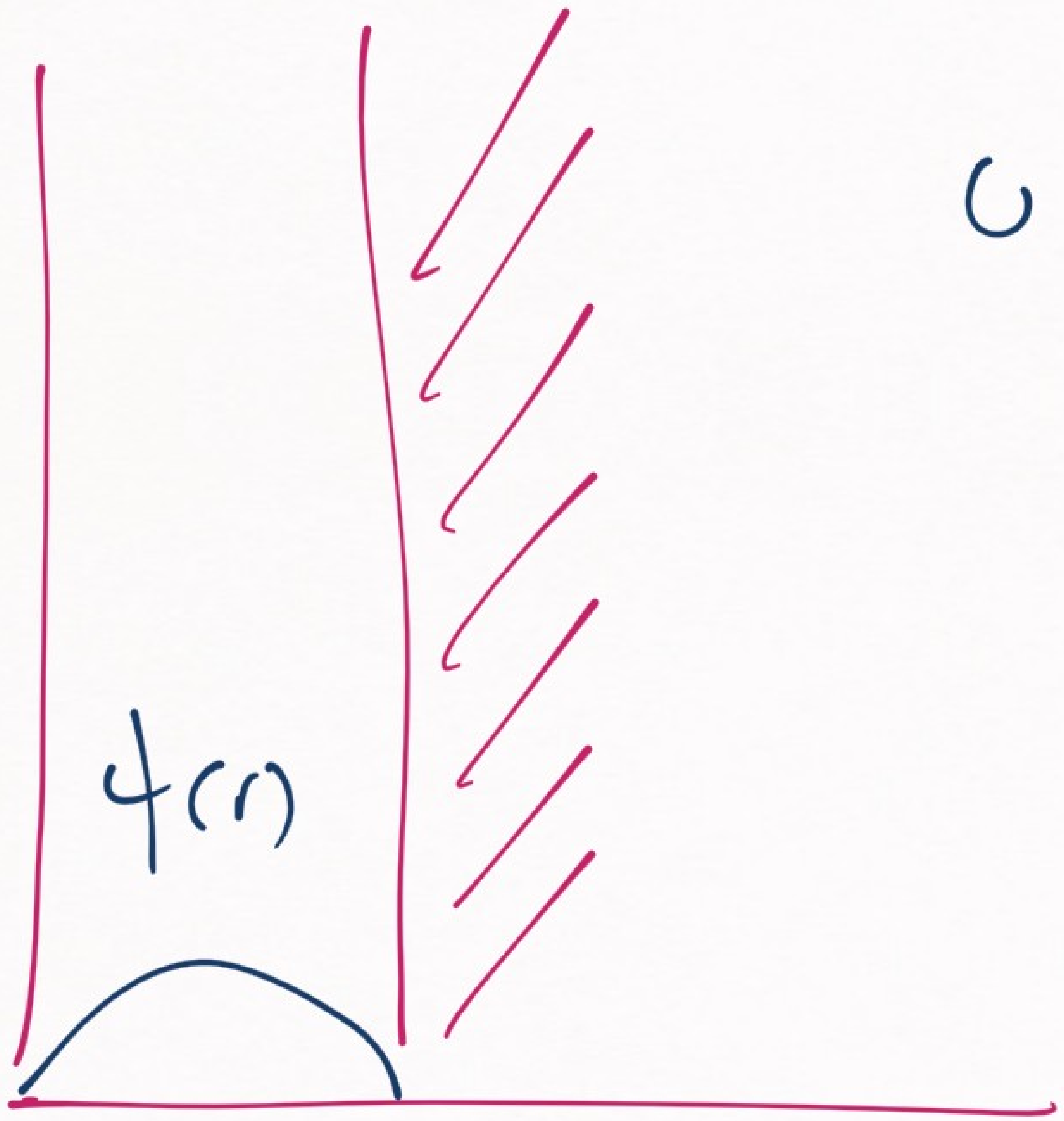
$\Longrightarrow$

change the energy origin



Particle trapped in a potential barrier

$R_s \to 0$



$$\psi(r) = \frac{u(r)}{r}$$

$\boxed{\text{ENERGY LEVELS}}$

$$u(r) = \sin(kr) + \left[\begin{array}{l} u(0) = 0 \\ u(r=a) = 0 \end{array}\right.$$

$$\to \boxed{ka = n\pi}$$

$\downarrow$

energy levels

When $R_s \ll a$, this should be a good approximation:

$$B(n) = \frac{\gamma_n^2}{2\mu} = \frac{1}{2\mu}\left(\frac{1}{R_s^2} - \underbrace{k_n^2}_{R_s \to 0 \text{ limit}}\right)\left(1 + \mathcal{O}\left(\frac{R_s}{a}\right)\right)$$

$$B(n) = \frac{1}{2\mu}\left(\frac{1}{R_s^2} - \frac{1}{a^2}(n\pi)^2\right)\left(1 + \mathcal{O}\left(\frac{R_s}{a}\right)\right) \Rightarrow$$

2D $\boxed{B(n) = V_0\left(1 - \frac{R_s^2}{a^2}(n^T l)^2 + O\left(\left(\frac{R_s}{a}\right)^3\right)\right)}$

If natural ($R_s \ll a$) $\Rightarrow$ $\boxed{B \simeq O(V_0)}$

in particular: $\underline{B = V_0} - \Delta B$, $\underline{\Delta B > 0}$

→ We can even calculate the first non-trivial R_s/a correction

$$B = V_0\left[1 - \frac{R_s^2}{a^2}(n\pi)^2 + 2\frac{R_s^3}{a^3}(n\pi)^2 + \mathcal{O}\left(\frac{R_s^4}{a^4}\right)\right]$$

[You can try to get this correction]

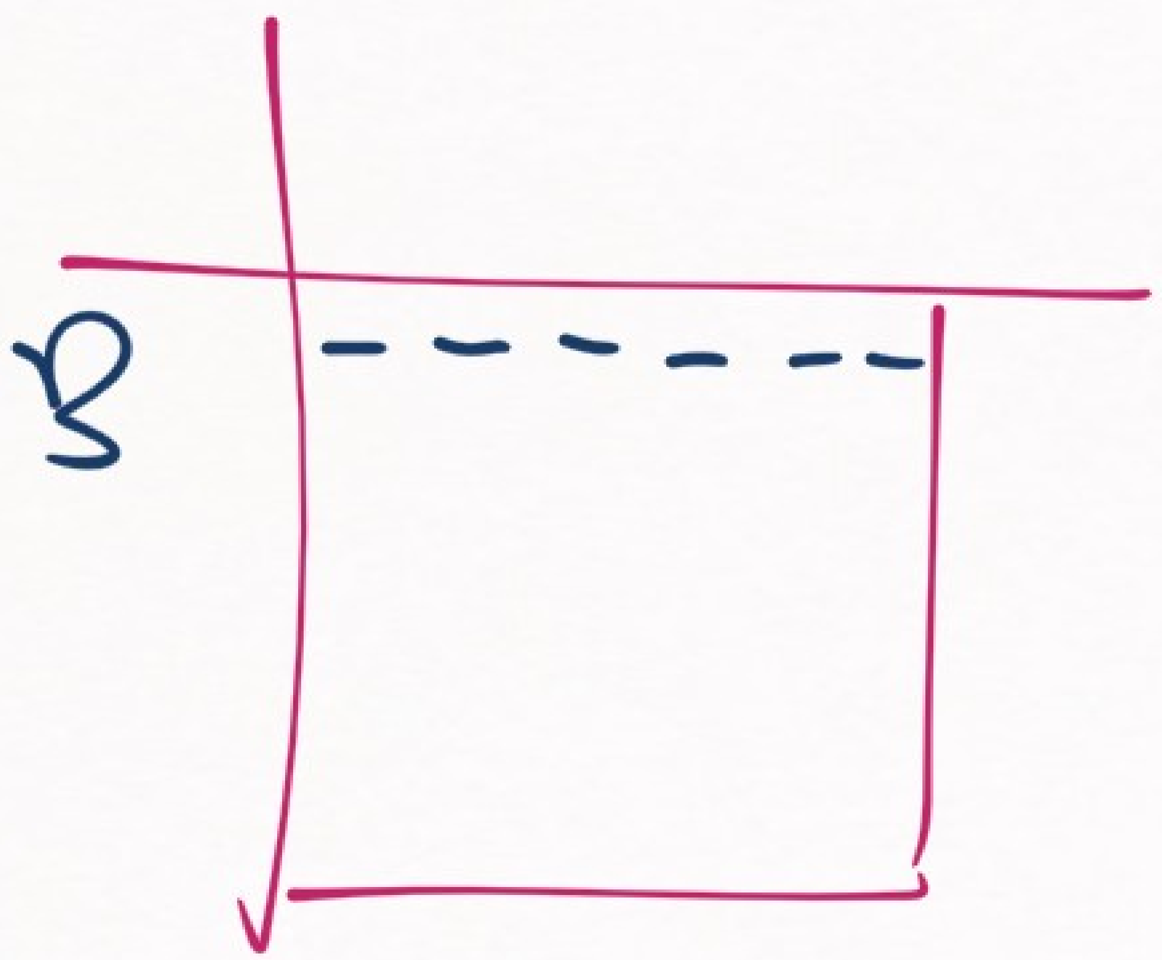
THE POINT $\rightarrow$ $\boxed{R_s \ll a}$

$\boxed{\text{Natural system}}$

$$\gamma = \frac{1}{R_s}\left(1 + \mathcal{O}\left(\frac{R_s}{a}\right)\right)$$

1) Good scale separation
2) Low energy physics dominated by R_s
3) $\exists$ corrections given by powers of $\frac{R_s}{a}$ (small number)

The problem $\rightarrow$ $\boxed{R_s \sim a}$



$\boxed{B \ll V_0}$ $\rightarrow$ unnatural (or fine-tuned system)

When does this happens?

(1) $\boxed{k\cot(ka) = -\gamma}$

(eigenvalue equation)

Unnatural system:

$\boxed{B << V_0}$

$\Downarrow$

(2) $\boxed{\frac{1}{\gamma} << a, R_s}$

$\Downarrow$

$k \simeq \frac{\Lambda}{R_s}$

(1) + (2)

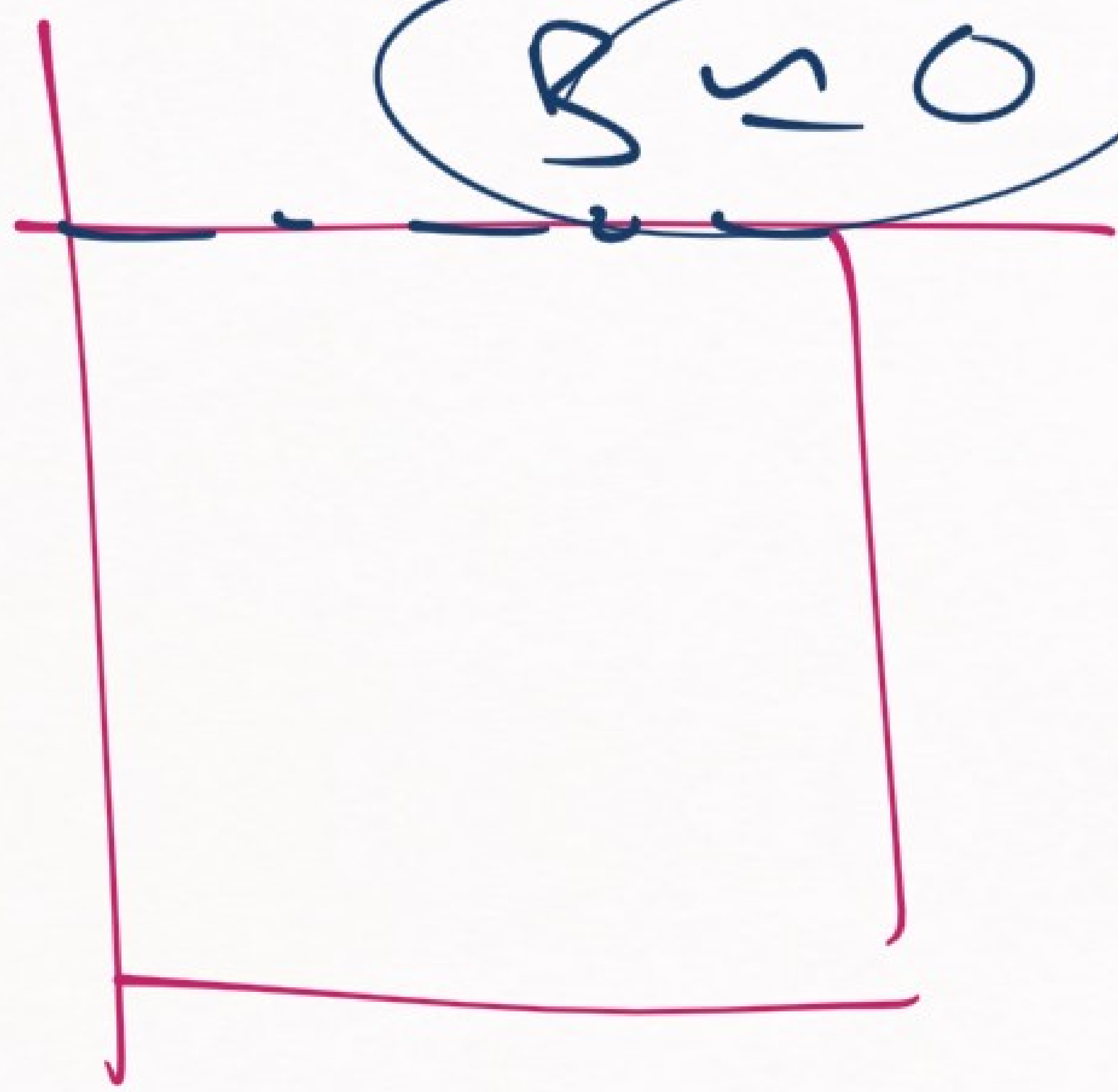
$$k\cot(ka) = \mathcal{O}(\gamma R_s)$$
$$= \mathcal{O}(\gamma a)$$

$(1) + (2) \Rightarrow \underbrace{k \cot(ka)}_{} = 0 + O(\gamma R)$

$r \sim R_s, a$

$$\boxed{ka = (2n+1)\frac{\pi}{2}} + O(\gamma R)$$

$n = 0 \longrightarrow ka = \frac{\pi}{2} \longrightarrow \gamma = 0$



$B \simeq 0 \Rightarrow \boxed{ka = \frac{\pi}{2}}$

$\Downarrow$

$\frac{a}{R_s} = \frac{\pi}{2} \rightarrow a \sim R_s$

$a = \frac{\pi}{2} R_s$

Shallow bound state ($B \ll V_0$) $\Leftarrow$ poor scale separation ($a \sim R_s$)

$B \ll V_0$ → does not only require $a \ll R_S$, but a very specific proportionality:

$$\boxed{\frac{a}{R_S} = \frac{\pi}{2}}$$

→ [IF WE VARY THIS RATIO, QUICKLY THE SYSTEM WILL BECOME NATURAL]

[EXPLICIT CALCULATION]

$$\frac{a}{R_s} = \frac{\pi}{2}(1+x)$$

x small, $x \geq 0$

(for having a bound state)

$\rightarrow$ 1) $x = 0 \Rightarrow \gamma = 0$

2) $x = 0.1 \Rightarrow \gamma \simeq \dfrac{0.16}{R_s}$

$\rightarrow$ unnatural

3) $x = 0.2 \Rightarrow \gamma \simeq \dfrac{0.37}{R_s}$ $\hat{=}$ middle

4) $x = 0.3 \Rightarrow \gamma \simeq \dfrac{0.51}{R_s}$

5) $x = 0.4 \Rightarrow \gamma \simeq \dfrac{0.73}{R_s}$

$\rightarrow$ natural

For having $\beta << V_0$ (or $r << R_s, a$)

→ we need to be extremely close to $\boxed{\frac{a}{R_s} = \frac{\pi}{2}}$

$$\boxed{\frac{a}{R_s} = \frac{\pi}{2}(1+x)}$$ → dependence on x is huge

Huge dependence on a very specific condition $\rightarrow$ $\boxed{\text{fine-tuning}}$

$\boxed{\text{UNNATURAL} \simeq \text{FINE-TUNING}}$

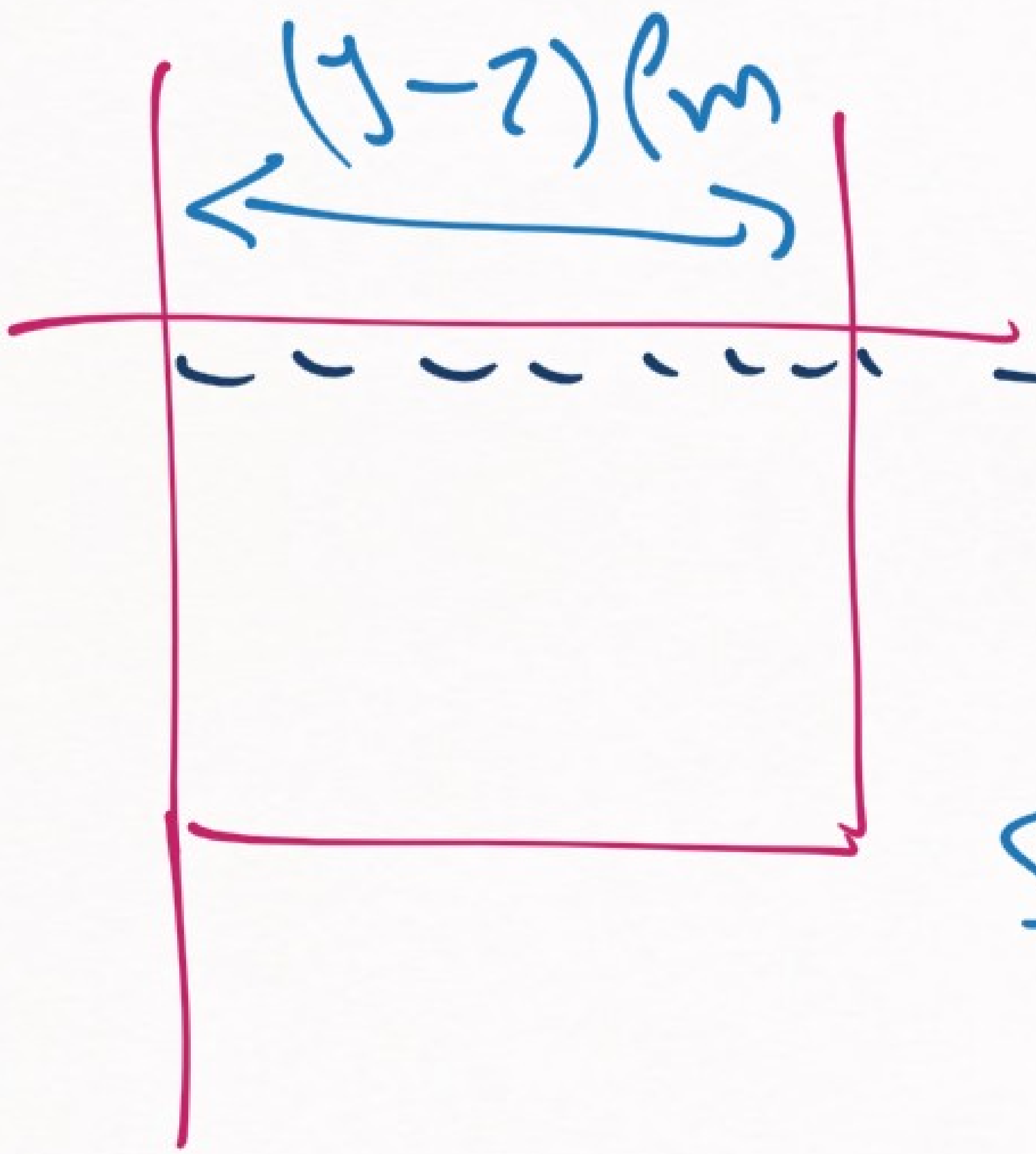
$\rightarrow$ [VERY SPECIFIC CANCELLATIONS]

[SQUARE WELL EXAMPLE]

→ $\boxed{\text{EXPLAINS FINE-TUNING IN THE DEUTERON}}$

how is this?

DEUTERON → np bound state



$B_d \simeq 2.2 \text{ MeV} \simeq 2 \text{ MeV}$

50 MeV depth } V_0

$\boxed{B \ll V}$

fine-tuning of $\frac{2}{50}$

or $\frac{1}{25}$

$$-B = \langle T \rangle + \langle V \rangle$$
$$-2 \simeq 48 - 50$$

BINDING ENERGY OF THE DEUTERON

⇒ A FINE-TUNING OF MAYBE $\underline{\underline{1/25}}$

→ this is not the most extreme example in nuclear physics

[VIRTUAL STATE IN np SCATTERING]

$\boxed{\text{VIRTUAL STATE}}$ $\rightarrow$ A system that is almost bound (but not quite)

bound : $\psi(\vec{r}) \sim \dfrac{e^{-\gamma r}}{r}$

virtual : $\psi(\vec{r}) \sim \dfrac{e^{+\gamma r}}{r} \Rightarrow B_v = -\dfrac{\gamma_v^2}{2\mu}$

Virtual state is "visible" in reactions

energy close to "virtual state" energy

$\Rightarrow$ very large "cross section"

$\boxed{B_v \sim 0.07\ \text{MeV}} \to np \quad S=0$

($\text{deuteron } np \quad S=1$)

$\downarrow$

repeat the arguments of fine tuning & square well

$\to \dfrac{1}{1600} \to$ a lot of finetuning

NUCLEAR PHYSICS

$\Downarrow$

LOTS OF FINE-TUNING

SEE YOU ON FRIDAY