# NUCLEAR PHYSICS 15

## T-MATRIX & A FEW SOLUTIONS OF IT

$\boxed{\text{RECAP}}$ → You can consider QM from different viewpoints

View 1) Wave functions + differential equations

$$\psi(\vec{r}) \longrightarrow e^{i\vec{k}\cdot\vec{r}} + f(\Omega)\frac{e^{ikr}}{r}$$

$$\frac{d\sigma}{d\Omega} = |f(\Omega)|^2 \qquad \left(-\frac{1}{2\mu}\vec{\nabla}^2\psi + V\psi\right) = E\psi(\vec{r})$$

View 2) Operators & vectors on a Hilbert space

$$|\psi\rangle = |\vec{k}\rangle + G_0 T |\vec{k}\rangle$$

$$\frac{d\sigma}{d\Omega} = |f(\Omega)|^2 , \quad f(\Omega) = -\frac{\mu}{2\pi}\langle \vec{k}' | \underbrace{T}_{\text{T-matrix}} | \vec{k}\rangle$$

$$(H+V)|\psi\rangle = E|\psi\rangle$$

View 2) $\rightarrow$ RECAST THE SCATTERING PROCESS IN TERMS OF THE T-MATRIX

How to derive what the T-matrix is?

Which equation does it follow?

## Derivation

1) $H|\phi\rangle = E|\phi\rangle$

2) $H = H_0 + V$ ($H_0 \to$ kinetic term)

3) $(E - H_0)|\phi\rangle = V|\phi\rangle$ (1+2)

4) $|\phi\rangle = |\vec{k}\rangle + G_0(E)V|\phi\rangle$

$$G_0(E) = \frac{1}{E - H_0}$$

Green's function method

Just a method for solving differential equations

4') $\phi(\vec r) = e^{i\vec k\cdot\vec r} + \int d^3\vec r\,'\, G(\vec r-\vec r\,')\, V(\vec r\,')\, \phi(\vec r\,')$

$$\left(-\frac{\nabla^2}{2\mu} - E\right) G_0(\vec r) = \delta^{(3)}(\vec r)$$ ← check!

5) $G_0(\vec r; E \pm i\epsilon) = -\frac{\mu}{2\pi}\frac{e^{\pm ikr}}{r}$

→ integration contour

(≃ boundary condition)

$$G_0(\vec p) = \frac{1}{E - \frac{p^2}{2\mu}}$$

6) $\phi^{\pm}(\vec{r}) = e^{i\vec{k}\cdot\vec{r}} + \int d^3\vec{r}\,' \, G_0(\vec{r}\,' - \vec{r})\, V(\vec{r}\,')\, \phi^{\pm}(\vec{r}\,')$

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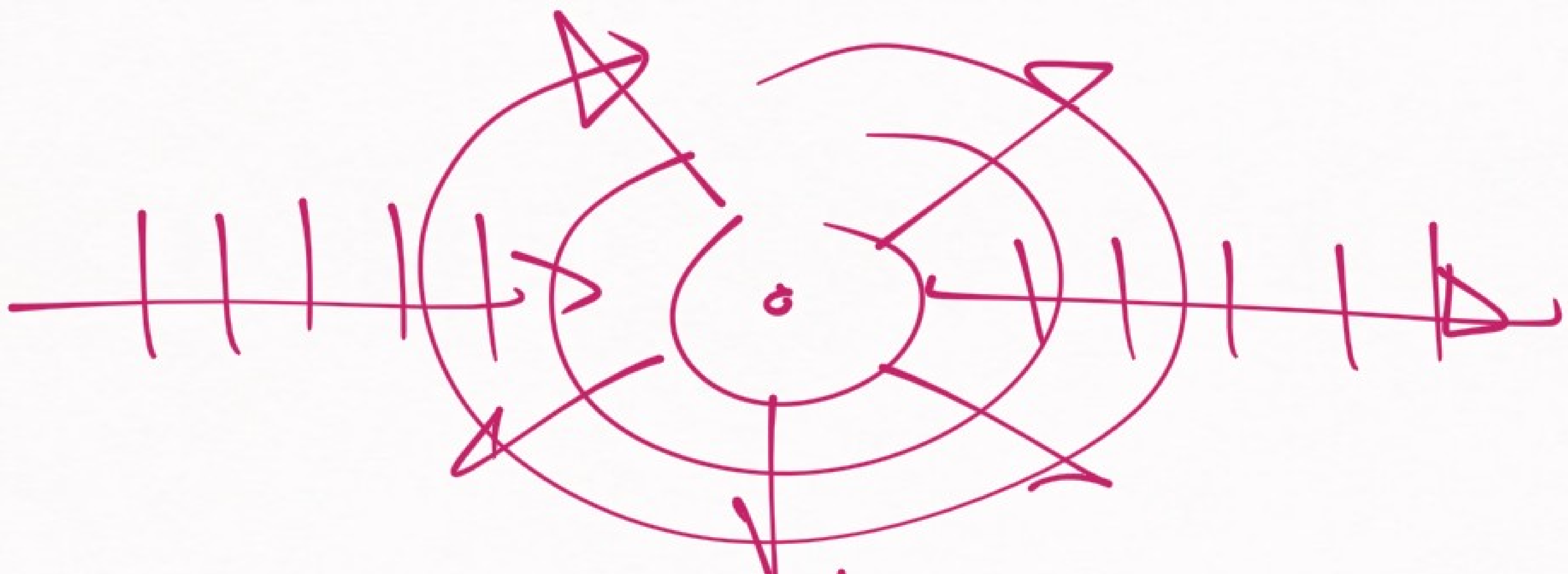

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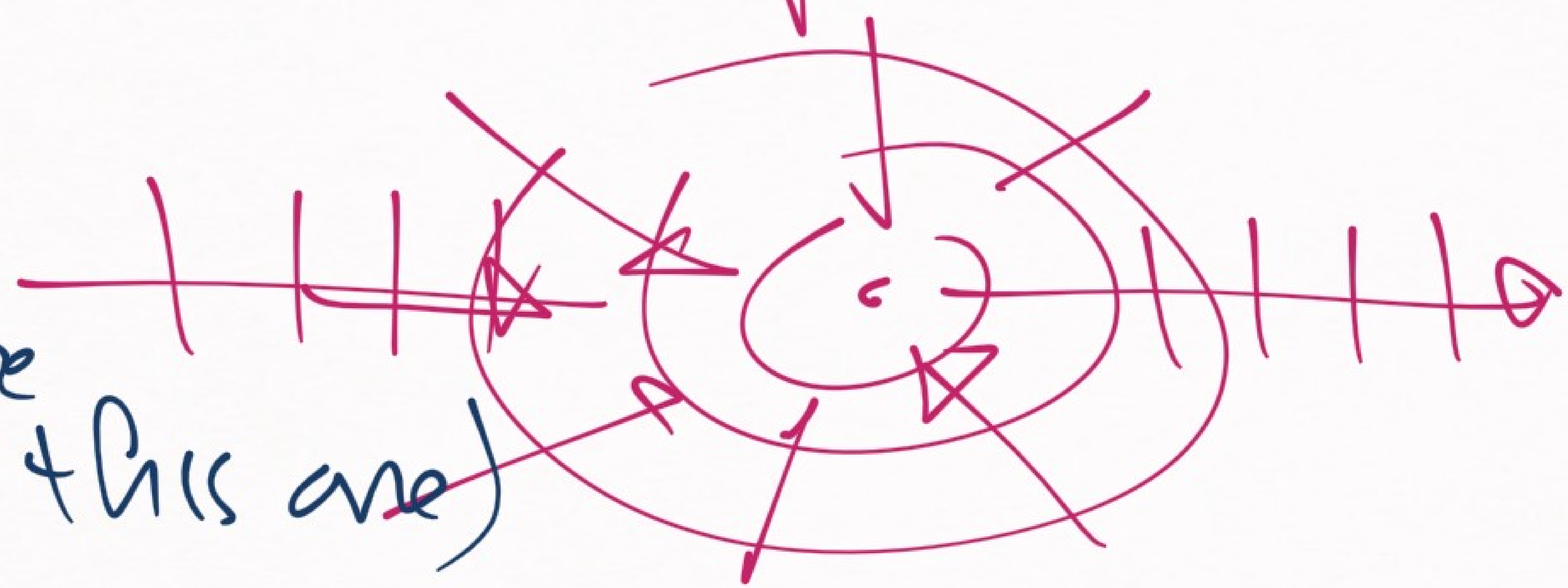
(ignore this one)



non-physical "time-inverted" solution

7) $\psi^+(\vec{r}) = e^{i\vec{k}\cdot\vec{r}} + \int d^3\vec{r}\,' G(\vec{r}-\vec{r}\,') V(\vec{r}\,') \psi(\vec{r}\,')$

?

$$\psi(\vec{r}) \longrightarrow e^{i\vec{k}\cdot\vec{r}} + f(\Omega)\frac{e^{ikr}}{r}$$

$|\vec{r}| \to \infty$

?

$\Rightarrow$ How $f(\Omega)$ and $\int d^3\vec{r}\,'(\ldots)$ relate?

7.a) $G_0(\vec{r}; E+i\epsilon) = -\frac{\mu}{2\pi}\frac{e^{+ikr}}{r}$

$$\phi^+(\vec{r}) = e^{i\vec{k}\cdot\vec{r}} - \frac{\mu}{2\pi}\int d^3\vec{r}\,' \frac{e^{ik|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|} V(\vec{r}\,')\phi^+(\vec{r}\,')$$

$$\frac{e^{ik|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|} \xrightarrow[|\vec{r}|\to\infty]{} \frac{e^{ikr\sqrt{1+\frac{r'^2}{r^2}-\frac{2\vec{r}\cdot\vec{r}\,'}{r^2}}}}{r\sqrt{\ldots\ldots}}$$

$\frac{r'^2}{r^2} \to \frac{1}{r^2}$

$\frac{2\vec{r}\cdot\vec{r}\,'}{r^2} \to \frac{1}{r}$

$$\sqrt{1+x} \simeq 1 + \frac{x}{2} \qquad k\frac{\vec{r}\cdot\vec{r}'}{r} = \vec{k}'\cdot\vec{r}'$$

$$\psi^+(\vec{r}) \to e^{i\vec{k}\cdot\vec{r}} - \frac{\mu}{2\pi}\frac{e^{ikr}}{r}\int d^3\vec{r}'\, e^{-i\vec{k}'\cdot\vec{r}'}\, V(\vec{r}')\psi^+(\vec{r}')$$

$$\psi(\vec{r}) \to e^{i\vec{k}\cdot\vec{r}} + f(\Omega)\frac{e^{ikr}}{r}$$

Comparison $\to$ $$f(\Omega) = -\frac{\mu}{2\pi}\int d^3\vec{r}'\, e^{-i\vec{k}'\cdot\vec{r}'}\, V(\vec{r}')\psi^+(\vec{r}')$$

$$= -\frac{\mu}{2\pi}\langle \vec{k}' | V | \psi^+ \rangle$$

$$\boxed{f(\Omega) = -\frac{\mu}{2\pi}\langle \vec{k}' | V | \psi^+ \rangle} \rightarrow \text{closer to the T-matrix}$$

8) $$V|\psi^+\rangle = T|\vec{k}\rangle \rightarrow \text{a possible definition}$$

More elaborate derivation:

$$|\psi^+\rangle = |\vec{k}\rangle + G_0(E+i\epsilon)\, V\, |\psi^+\rangle$$

$$= |\vec{k}\rangle + G_0 V|\vec{k}\rangle + G_0 V G_0 V |\psi^+\rangle$$

$$= |\vec{k}\rangle + G_0 V|\vec{k}\rangle + G_0 V G_0 V|\vec{k}\rangle + \ldots$$

Iterative equation:

$$\left[\,|\phi^{(1)}\rangle = (1 + G_0V + (G_0V)^2 + (G_0V)^3 + \dots)\,|\vec{k}\rangle\,\right]$$

Alternatively: (full integral form → more difficult)

$$\phi^+(\vec{r}) = e^{i\vec{k}\cdot\vec{r}} + \int d^3\vec{r}'\, G_0(\vec{r}-\vec{r}')V(\vec{r}')\, e^{i\vec{k}\cdot\vec{r}'}$$

$$+ \int d^3\vec{r}'\, d^3\vec{r}''\, G_0(\vec{r}-\vec{r}')V(\vec{r}')G_0(\vec{r}'-\vec{r}'')V(\vec{r}'')\, e^{i\vec{k}'\cdot\vec{r}'} + \dots$$

$$f(\Omega) = -\frac{\mu}{2\pi}\langle \vec{k}'|V|\phi^{+}\rangle = -\frac{\mu}{2\pi}\Big(\langle \vec{k}'|V|\vec{k}\rangle + \langle \vec{k}'|VG_0V|\phi^{+}\rangle\Big)$$

$$= -\frac{\mu}{2\pi}\Big(\langle \vec{k}'|\big(V + VG_0V + VG_0VG_0V + \ldots\big)|\vec{k}\rangle\Big)$$

$$= -\frac{\mu}{2\pi}\langle \vec{k}'|T|\vec{k}\rangle$$

$$T = V + V G_0 V + V G_0 V G_0 V + \dots$$

$$= V + V G_0 \left( V + V G_0 V + V G_0 V G_0 V + \dots \right)$$

$$= V + V G_0 T$$

$$\boxed{T = V + V G_0 T}$$

LIPPMANN-
SCHWINGER
EQUATION

If you solve the LSE $\Rightarrow$ you get $f(\omega)$

$$\boxed{T = V + V G_0 T}$$

$$\frac{d\sigma}{d\Omega} = |f(\Omega)|^2$$

Somewhat abstract ...

$\hookrightarrow$ by taking matrix elements
(by sandwiching it)

$$T = V + V G_0 T$$

$$\langle \vec{k}' | \qquad | \vec{k} \rangle$$

$$\langle \vec{k}' | T | \vec{k} \rangle = \langle \vec{k}' | V | \vec{k} \rangle + \langle \vec{k}' | V G_0 T | \vec{k} \rangle$$

$$\mathbb{1} = \int \frac{d^3\vec{e}}{(2\pi)^3} |\vec{e}\rangle\langle\vec{e}|$$

$$\boxed{\langle \vec{k}' | T | \vec{k} \rangle = \langle \vec{k}' | V | \vec{k} \rangle + \int \frac{d^3\vec{e}}{(2\pi)^3} \frac{\langle \vec{k}' | V | \vec{e} \rangle \langle \vec{e} | T | \vec{k} \rangle}{E - \frac{e^2}{2M}}}$$

→ Usual form of LSE

Two comments:

1) $$G_0(E)|\vec{p}\rangle = \frac{1}{E-H_0}|\vec{p}\rangle = \left\{ H_0|\vec{p}\rangle = \frac{\vec{p}}{2m}|\vec{p}\rangle \right\}$$

$$= \frac{1}{E-\frac{p}{2\mu}}$$

2) $$\langle \vec{k}'|\vec{k}\rangle = (2\pi)^3\,\delta^{(3)}(\vec{k}'-\vec{k})$$

$$\longrightarrow \quad \mathbb{1} = \int \frac{d^3\vec{k}}{(2\pi)^3}|\vec{k}\rangle\langle\vec{k}|$$

A few more comments:

1) Connections between the T-matrix and

1.a) Perturbative expansion

1.b) Feynman diagrams

2) Why to use the LSE?

It's obviously more difficult than Schrödinger

CONNECTION a $\rightarrow$ Perturbative expansion

$$T = V + V G_0 T$$

$$= \underbrace{V}_{\text{tree level (QFT), first order perturbation theory (QM)}} + \underbrace{V G_0 V}_{\text{one loop / 2nd order}} + \underbrace{V G_0 V G_0 V}_{\text{two loops, 3rd order}} + \cdots$$

[BORN APPROXIMATION] $\rightarrow$ check your QM undergraduate course

$$T = V + \mathcal{O}(V^2)$$

$$f(\Omega) = -\frac{\mu}{2\pi}\langle \vec{k}' | V | k\rangle + \mathcal{O}(V^2)$$

$$= \left[-\frac{\mu}{2\pi}\int d^3\vec{r}\; e^{i(\vec{k}' - \vec{k})\cdot\vec{r}}\; V(\vec{r})\right] + \mathcal{O}(V^2)$$

Application → REPRODUCE ~~RUTHERFORD~~ THOMSON SCATTERING

(two charged particles)

$$T = V + O(V^2) \longrightarrow V = V_C \quad (\text{Coulomb})$$

$$\langle \vec{k}' | V_C | \vec{k} \rangle = V_C(\vec{k}' - \vec{k})$$

$$V_C(\vec{q}) = 4\pi \frac{\alpha}{|\vec{q}|} \longrightarrow$$

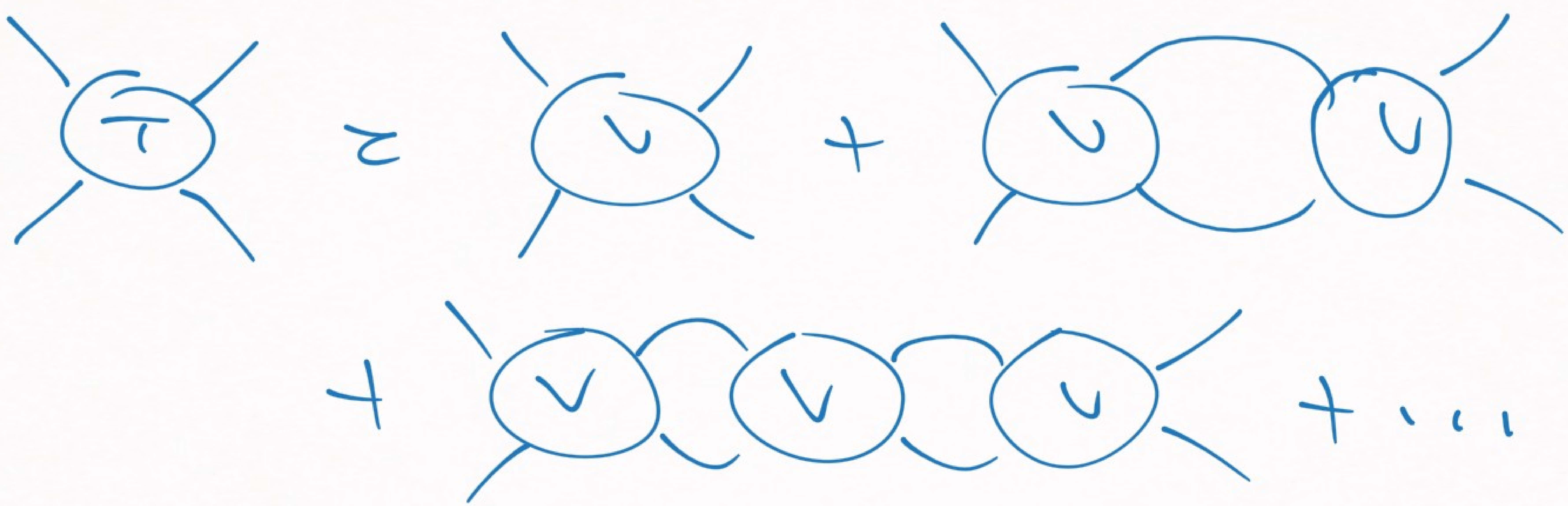
$$\rightarrow \quad f(\Omega) = -2\mu \frac{\alpha}{|\vec{k}-\vec{k}'|^2} + \mathcal{O}(\alpha^2)$$

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{4\mu^2\alpha^2}{|\vec{k}'-\vec{k}|^2} + \mathcal{O}(\alpha^2)}$$

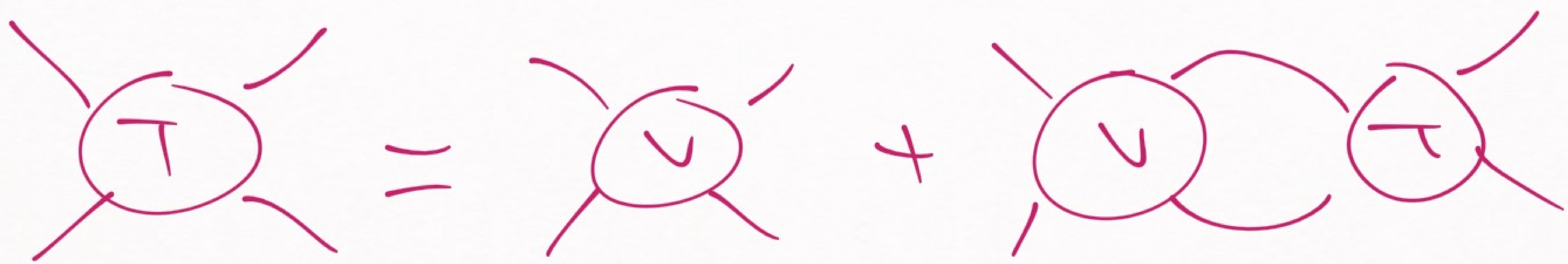
Thomson scattering

# CONNECTION TO → FEYNMAN DIAGRAMS

$$T = V + V G_0 V + V G_0 V G_0 V + \dots$$



DIAGRAMMATIC REPRESENTATION OF LSE:



WHY ON EARTH WOULD WE WANT TO USE THE LSE ?

LSE $\longrightarrow$ Advantage $\longrightarrow$ More general (than standard Schrödinger)

Note:
LSE is just Schrödinger equation

non-local potentials:

$$\langle \vec{k}' | V | \vec{k} \rangle \neq V(\vec{k}' - \vec{k})$$

1) $H|\psi\rangle = E|\psi\rangle \rightarrow$ really difficult to solve w/ this equation

2) $T = V + V G_0 T \rightarrow$ equally difficult for local & non-local potentials

# A FEW SOLUTIONS OF THE LSE

$$\langle \vec{p}\,' | T(E) | \vec{p} \rangle = \langle \vec{p}\,' | V | \vec{p} \rangle + \int \frac{d^3 \vec{\ell}}{(2\pi)^3} \frac{\langle \vec{p}\,' | V | \vec{\ell} \rangle \langle \vec{\ell} | T | \vec{p} \rangle}{E - \frac{\vec{\ell}^{\,2}}{2\mu}}$$

Easy to solve potential:

$$V_C(\vec{r}) = C_0\, \delta^{(3)}(\vec{r})$$ contact-range potential

$$\Big\downarrow \mathcal{F}$$

$$\boxed{V_C(\vec{q}) = C_0}$$

REMEMBER $\rightarrow$ Dirac-delta has to be regularized

$$V_c(\vec{p}\,' - \vec{p}) = C_0 \Rightarrow \langle \vec{p}\,' | V_c | \vec{p} \rangle = C_0 \underline{\underline{f\left(\frac{p'}{\Lambda}\right)}} f\left(\frac{p}{\Lambda}\right)$$

$$f(x) \xrightarrow[x \to 0]{} 1 \qquad \leftarrow \text{Regulator}$$

$$f(x) \xrightarrow[x \to \infty]{} 0$$

$$\langle \vec{p}' | T(E) | \vec{p} \rangle = \boxed{C_0(\Lambda)\, f\left(\frac{\vec{p}'}{\Lambda}\right) f\left(\frac{\vec{p}}{\Lambda}\right)}$$

$$+ C_0 f\left(\frac{\vec{p}'}{\Lambda}\right) \int \frac{d^3\vec{\ell}}{(2\pi)^3} f\left(\frac{\ell}{\Lambda}\right) \frac{\langle \vec{\ell} | T | \vec{p} \rangle}{E - \frac{\ell^2}{2m}}$$

$\rightarrow$ you could try to iterate ...

$\exists$ ANSATZ $\longrightarrow$ $\langle \vec{p}' | T(E) | \vec{p} \rangle = \boxed{\tau(E)\, f\left(\frac{\vec{p}'}{\Lambda}\right) f\left(\frac{\vec{p}}{\Lambda}\right)}$

(hypothesis for a solution)

$$\Rightarrow \quad T(E) = C_0 + C_0\, T(E)\, I(E;\Lambda)$$

$$\text{w/} \quad I(E;\Lambda) = \int \frac{d^3\vec{\ell}}{(2\pi)^3}\, \frac{g^2(\ell/\Lambda)}{E - \frac{\ell^2}{2\mu}}$$

The loop function $\rightarrow$ you can solve it in many ways

(contour integration)

$$\boxed{T(E) = \frac{1}{\frac{1}{C_0(\Lambda)} - J(E,\Lambda)}}$$

→ very compact & convenient expression

Separable potentials

$$\langle \bar{p}' | V | \bar{p} \rangle = \lambda \, g(p') g(p)$$

$$\Rightarrow \boxed{T(E) = \frac{1}{1/C_0(\Lambda) - I(E;\Lambda)}} \rightarrow$$ Solve it analytically for a few choices of $f(x)$

EXAMPLE

$$f(x) = \Theta(1-x)$$

$$I(E;\Lambda) = \int \frac{d^3\vec{\ell}}{(2\pi)^3} \frac{\Theta(\Lambda-|\vec{\ell}|)}{E-\vec{\ell}^2/2\mu} \longrightarrow$$

$\rightarrow$ $$I(E \pm i\epsilon; \Lambda) = \frac{\mu}{\pi^2}\left[\mp i\frac{\pi}{2}k - \Lambda + \frac{k}{2}\log\left|\frac{\Lambda+k}{\Lambda-k}\right|\right]$$

(remember $G_0(E \pm i\epsilon)$)

$$k = \sqrt{2\mu E}$$

$\rightarrow$ you can combine it w/ the ansatz

$$\tau(E) = \frac{1}{1/C_0(E) - I(E,\Lambda)}$$

Let's make some additional connections:

1) $\sigma(E) \xrightarrow[E \to 0]{} 4\pi |a_0|^2$ ($a_0 \to$ scattering length)

2) $\sigma = \int |f(\Omega)|^2 \, d\Omega$

3) $f(\Omega) = -\frac{\mu}{2\pi} \langle \vec{p}' | T(E + i\epsilon) | \vec{p} \rangle$

$$\rightarrow \quad \sigma = 4\pi \left| \frac{\mu}{2\pi} T(E+i\epsilon) \right|^2 \xrightarrow[E\to 0]{} 4\pi |a_0|^2$$

$$\Downarrow$$

$$\frac{\mu}{2\pi} T(E+i\epsilon) \xrightarrow[E\to 0]{} a_0 \,, \quad \boxed{T(E+i\epsilon) \xrightarrow[E\to 0]{} \frac{2\pi}{\mu} a_0}$$

We can renormalize the Dirac-delta:

$$\langle \vec{p}' | V_c | \vec{p} \rangle = C_0(\Lambda) f\left(\frac{p'}{\Lambda}\right) f\left(\frac{p}{\Lambda}\right)$$

1) $$t(E) = \frac{1}{1/C_0(\Lambda) - I(E;\Lambda)}, \quad E = 0$$

2) $$I(E=0,\Lambda) = -\frac{\mu}{\pi^2}\Lambda$$

3) $$t(0) = \frac{2\pi}{\mu} a_0 = \frac{1}{\frac{1}{C_0(\Lambda)} + \frac{\mu}{\pi^2}\Lambda}$$

4) $\dfrac{d\tau(0)}{d\Lambda} = 0$

$$\Rightarrow \boxed{\frac{1}{G_0(\Lambda)} = \frac{\mu}{2\pi}\left(\frac{1}{a_0} - \frac{2}{\pi}\Lambda\right)} \Rightarrow \tau(0) = \frac{2\pi}{\mu} a_0$$

RG-Equation of $G_0(\Lambda)$

# SUMMARY

1) $T = V + V G_0 T$ is solvable for
$$\langle \vec{p}' | V | \vec{p} \rangle = \lambda g(p') g(p)$$

2) $\langle \vec{p}' | V | \vec{p} \rangle = C_0$ is a Dirac-delta in p-space

3) $\langle \vec{p}' | V | \vec{p} \rangle = C_0(\Lambda) f\left(\frac{p'}{\Lambda}\right) f\left(\frac{p}{\Lambda}\right)$ regularized the Dirac-delta ($f$ fulfills 1))

4) $f(x) = \theta(1-x)$ $\rightarrow$ analytic solutions

5) $\frac{d}{d\Lambda} \langle \vec{p}' | T(E) | \vec{p} \rangle = 0$ for a given $E$ (usually $E=0$)

$\Rightarrow$ RGE for $C_0(\Lambda)$

[ 3) is a really good toy model for understanding the LSE and RG ]

Another connection:

$$f(\omega) = \sum_\ell (2\ell+1) f_\ell(k) P_\ell(\cos\theta)$$

$$f_\ell(k) = \frac{1}{k\cot\delta_\ell - ik} = \frac{e^{i\delta_\ell}\sin\delta_\ell}{k}$$

$$\tau(E) f(\tfrac{p'}{h}) f(\tfrac{p}{h}) \longrightarrow \text{S-wave solution}$$

$$\left[ \quad \tau(E+i\epsilon) = -\frac{2\pi}{\mu} \frac{1}{k\cot\delta_0 - ik} \quad \right]$$

Try to understand the $\Lambda \to \infty$ limit of

$$\langle \vec{p}' | V | \vec{p} \rangle = C_0(\Lambda) f\left(\frac{p'}{\Lambda}\right) f\left(\frac{p}{\Lambda}\right)$$

6

$$I(E + i\epsilon, \Lambda) \xrightarrow[\Lambda \to \infty]{} \frac{\mu}{\pi^2}\left[-i\frac{\pi}{2}k - \Lambda + O\left(\frac{k^2}{\Lambda}\right)\right]$$

$$f(x) = \theta(1 - x)$$

(sharp cutoff)

$$\frac{1}{\tau(E)} = \frac{1}{C_0(\Lambda)} - I(E;\Lambda)$$

$$\hookrightarrow \quad \tau(E+i\epsilon) = \frac{1}{\frac{1}{C_0(\Lambda)} + \frac{M}{\pi^2}\Lambda + \frac{M}{2\pi} ik + \dots}$$

$$\xrightarrow{\Lambda \to \infty}$$

$$\boxed{\tau(E+i\epsilon) = \frac{2\pi}{M} \frac{1}{1/a_0 - ik}}$$

$$\tau(E) = \frac{2\pi}{\mu}\frac{1}{1/a_0 + ik} = -\frac{2\pi}{\mu}\frac{1}{k\cot\delta_0 - ik}$$

First term of ERE:

$$\boxed{k\cot\delta_0 = -\frac{1}{a_0}} \longrightarrow$$

$r_0 = 0$

pretty logical b/c Dirac delta has zero range

# [T-MATRIX & BOUND STATES]

full propagator

free propagator

$$T = V + V G_0 T$$

$$G = G_0 + G_0 V G_0 + G_0 V G_0 V G_0 + \dots$$

$$T = V + V G V$$

$$(G = G_0 + G_0 V G)$$

$$T(E) = V + V\,G(E)\,V$$

$$G(E) = \frac{1}{E - H} \qquad G_0(E) = \frac{1}{E - H_0}$$

full Hamiltonian

$$A|\lambda_i\rangle = \lambda_i|\lambda_i\rangle, \quad i = 1, \ldots, n$$

$$\mathbb{1}_{n \times n} = \sum_i |\lambda_i\rangle\langle\lambda_i|$$

$$H \longrightarrow \mathbb{1} = \underbrace{\sum_{i=1}^{n_B} |B_i\rangle\langle B_i|}_{\text{bound states}} + \underbrace{\int \frac{d^3\vec{k}}{(2\pi)^3} |\phi_k^+\rangle\langle\phi_k^+|}_{\text{continuum}}$$

$$H_0 \longrightarrow \mathbb{1} = \int \frac{d^3\vec{k}}{(2\pi)^3} |\vec{k}\rangle\langle\vec{k}|$$

$$G(E) = \frac{1}{E - H} \times \mathbb{1}$$

$$= \sum_{i=1}^{n_B} |B_i\rangle \frac{1}{E - B_i} \langle B_i| + (\text{continuum})$$

$n_B \to$ # of bound states

$$G(E) \longrightarrow \frac{|B_i\rangle\langle B_i|}{E - B_i} + \text{corrections}$$

$E \to B_i$ $\longrightarrow \infty$

$\longrightarrow$ $G(E)$ has a pole at $E = B_i$

$\rightarrow \quad T(E) = V + V\,G(E)\,V$

$T(E)$ is also going to have a pole

for $E \rightarrow B_i$

$$\underset{E = B_i}{\mathrm{Res}}\, T(E) = V|B_i\rangle\langle B_i|V$$ $\rightarrow$ a bit later ?

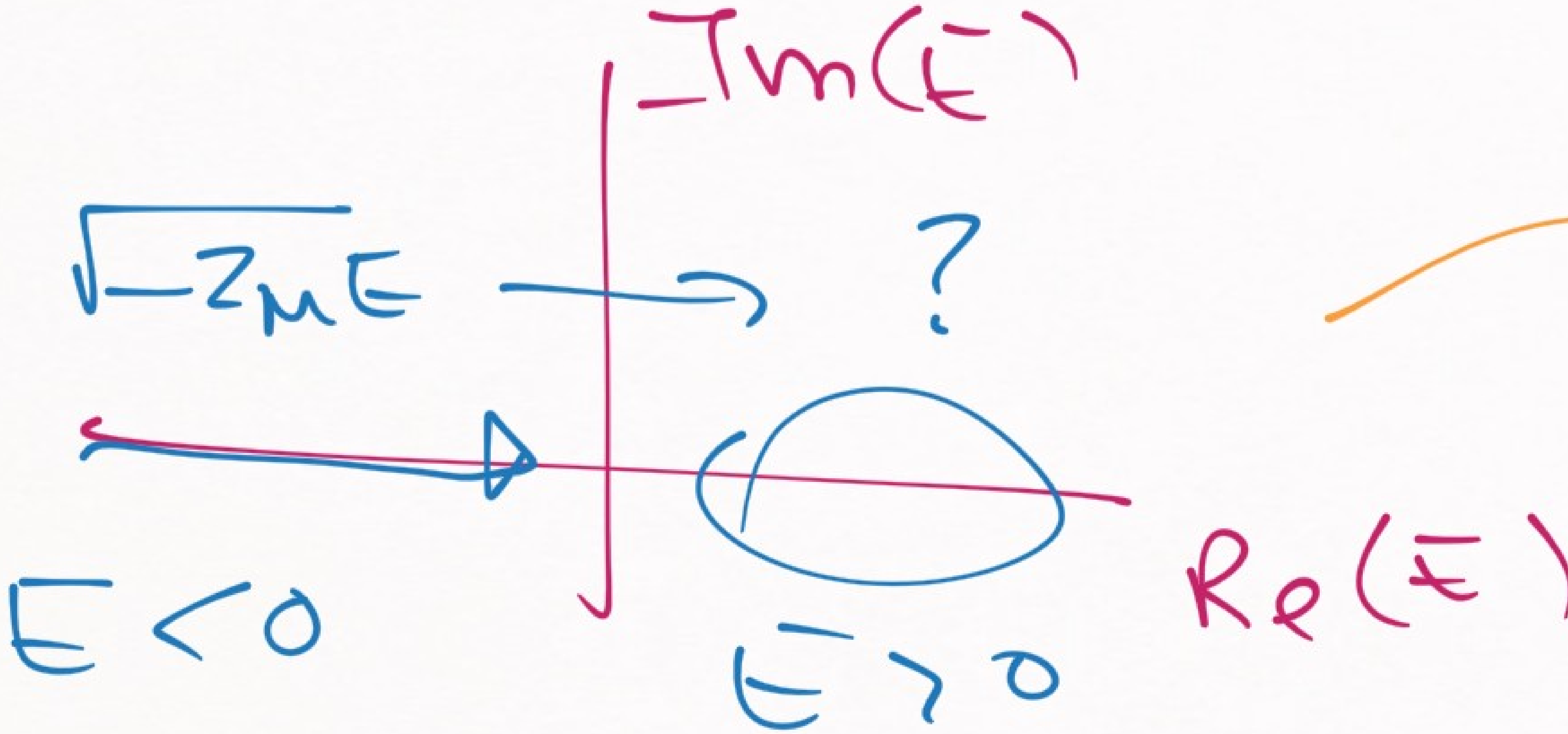
EXAMPLE

$$\underbrace{\tau(E+i\epsilon)}_{E>0} = \frac{2\pi}{\mu} \frac{1}{1/a_0 + ik} \longrightarrow \circledast$$

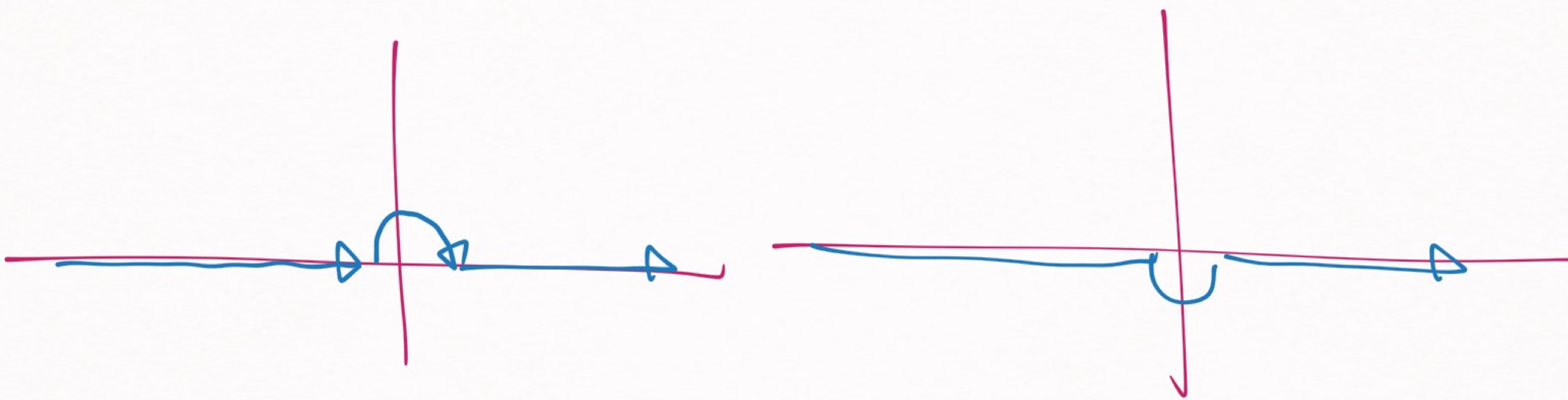
Reminder → bound states happen at $E<0$

$\circledast \longrightarrow k = \sqrt{2\mu E} \quad (E>0)$

how $\tau(E)$ looks for $E<0$ ?

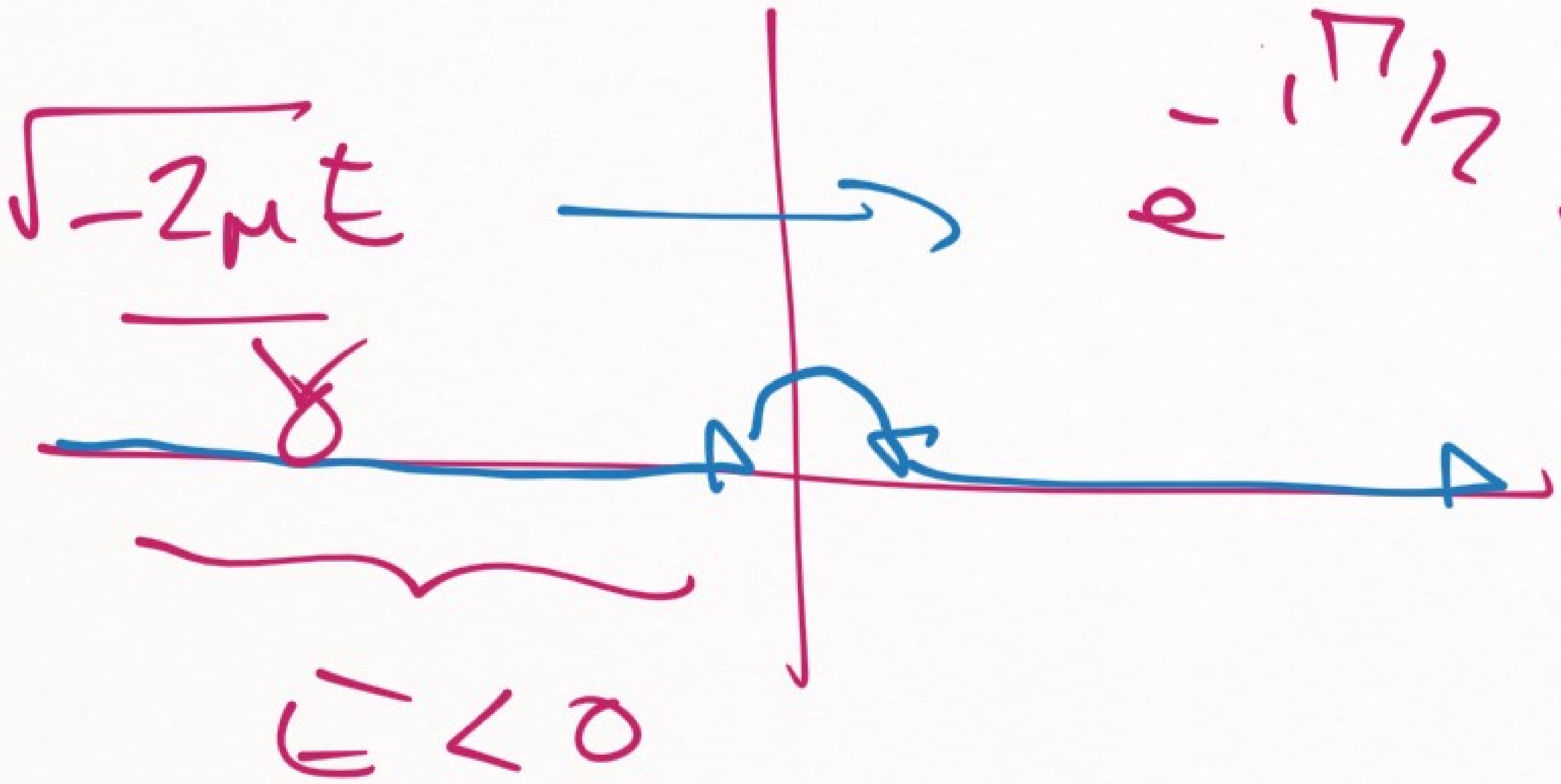



Two ways to define this

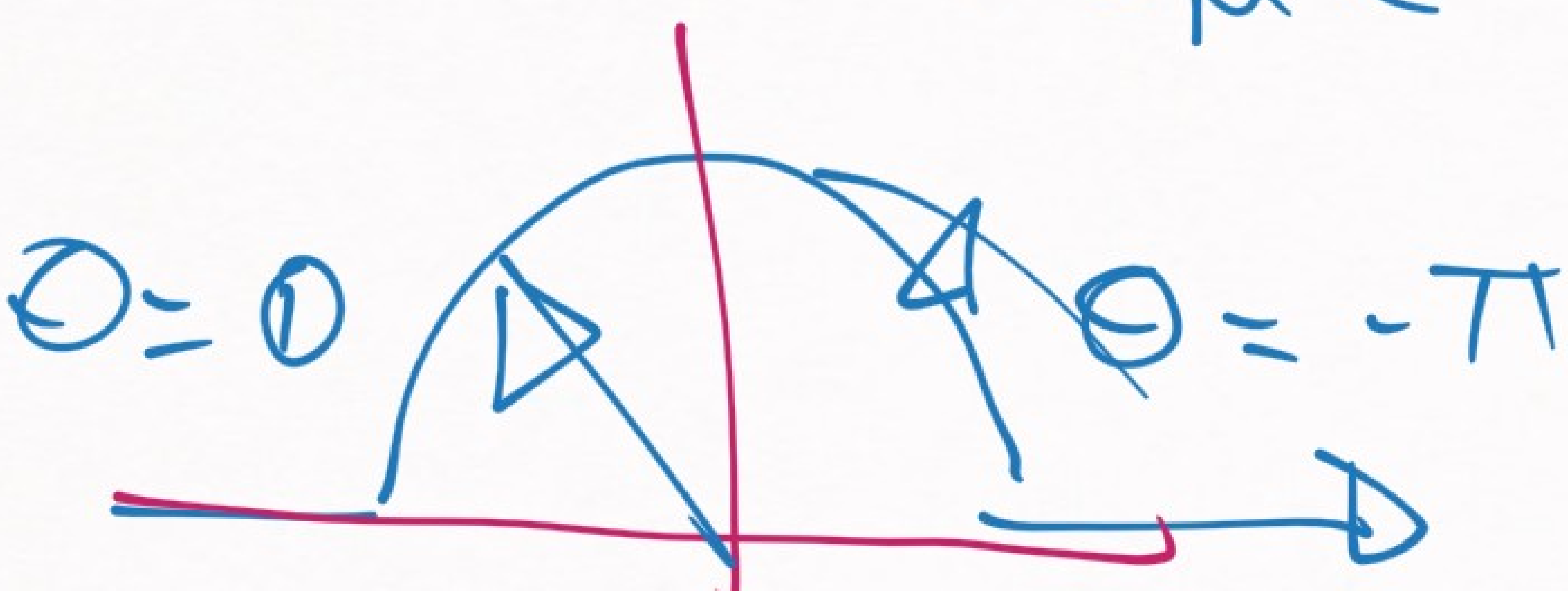


$$\sqrt{-2\mu E} \longrightarrow e^{-i\pi/2}\underbrace{\sqrt{2\mu E}}_{(E>0)} = -iK$$

$$\overline{\quad\gamma\quad}$$

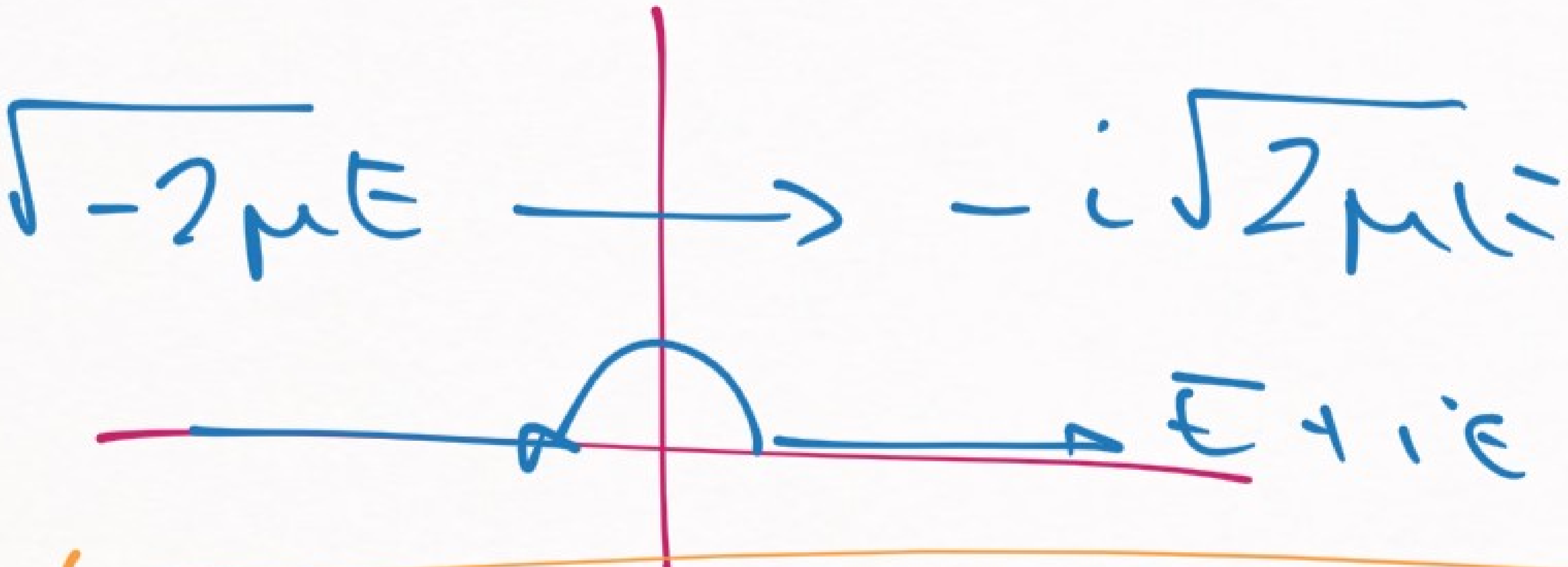


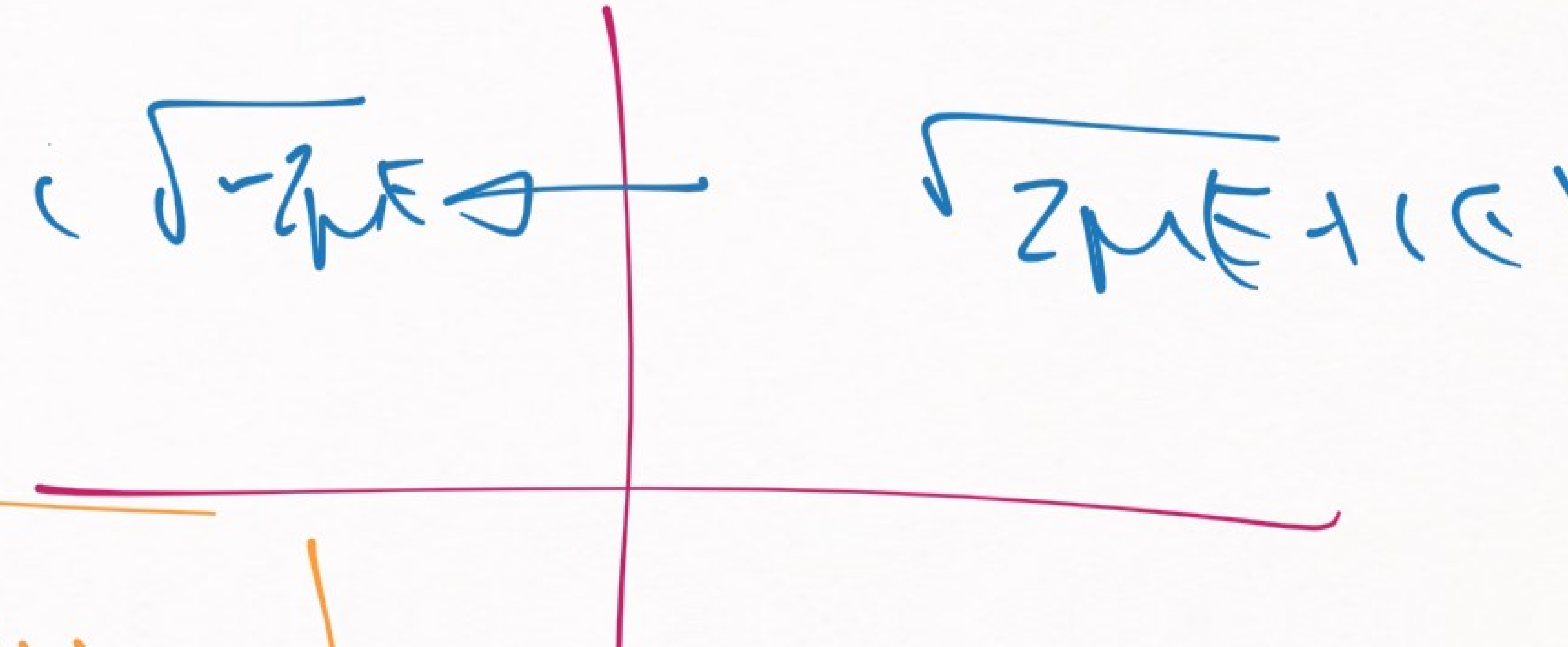

$$-2\mu E = |2\mu E|\, e^{i\Theta}$$



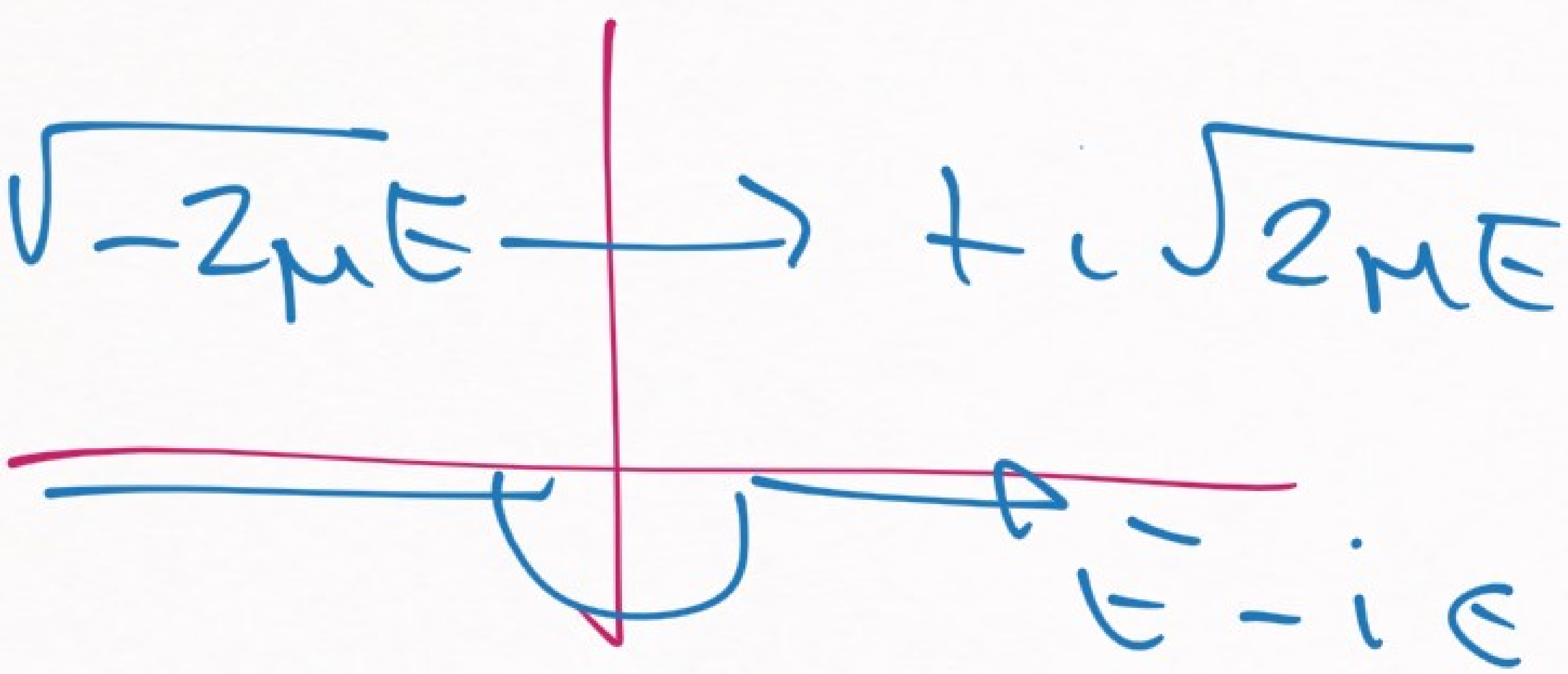

$$\sqrt{-2\mu E} = \sqrt{|2\mu E|}\, e^{i\Theta/2}$$

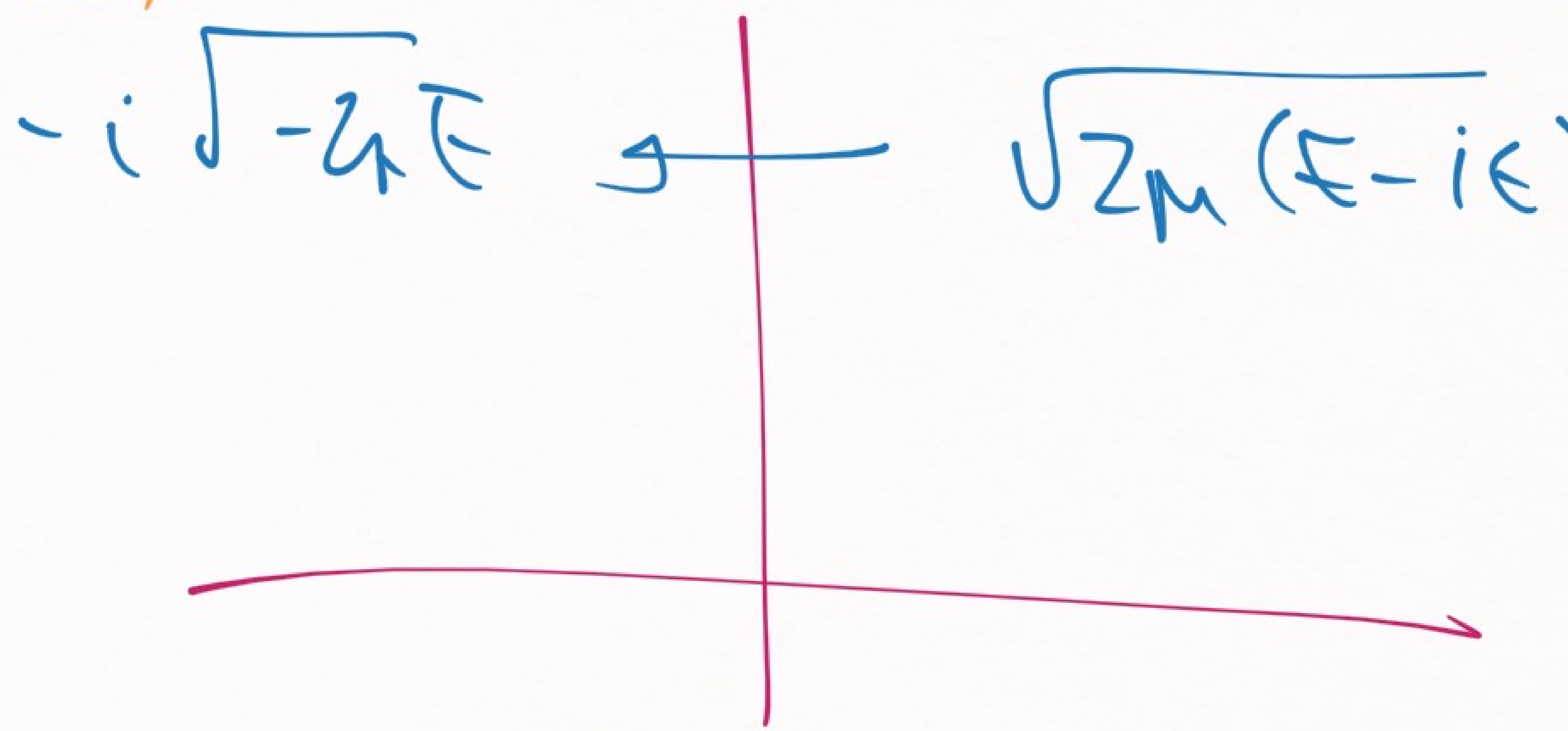
clockwise → negative




**STUDY THIS CAREFULLY**



$$\tau(E+i\epsilon) = \frac{2\pi}{\mu}\,\frac{1}{1/a_0 + ik} \longrightarrow \boxed{\tau(E<0) = \frac{2\pi}{\mu}\,\frac{1}{1/a_0 - \sqrt{-2\mu E}}}$$

T-matrix at negative energy

$$\tau(E<0) = \frac{2\pi}{\mu} \frac{1}{\frac{1}{a_0} - \sqrt{-2\mu E}}$$

If $\underline{\underline{a_0 > 0}}$ $\Rightarrow$ pole at $\sqrt{-2\mu E} = 1/a_0$

$\Rightarrow$ $E = -\frac{1}{2\mu}\left(\frac{1}{a_0}\right)^2$ pole $\Rightarrow$ bound state

REMINDER: $^1S_0 \to a_0 = -23.7$ fm

$^3S_1 \to a_0 = +5.4$ fm

bound state

"+"

$\rightarrow$ previous result is indeed consistent with what we already know

→ the complex energy plane trick makes sense ✓

→ $$\tau(E) = \frac{2\pi}{\mu}\,\frac{1}{1/a_0 + ik} = \frac{2\pi}{\mu}\,\frac{1}{1/a - \sqrt{-2\mu E}}$$

$$= \frac{2\pi}{\mu}\,\frac{1/a_0 + \sqrt{-2\mu E}}{\left(1/a_0^2 + 2\mu E\right)} \longrightarrow$$

1) $\tau(E)$ has a pole at:

$$E_B = -\frac{1}{2\mu}\left(\frac{1}{a_0}\right)^2 \quad (\text{if } a_0 > 0)$$

2) $$\underbrace{\mathop{\mathrm{Res}}_{E \to E_B} \tau(E) = \frac{\pi}{\mu^2}\frac{2}{a}}$$

We will use this now to calculate the wave function

1) $\text{Res}\,T(E) = V|B\rangle\langle B|V$

$E \to E_B$

2) $T = V + V G_0 T$

3) $E \to E_B\,, \quad T \to \dfrac{\text{Res}\,T}{E - E_B}$

1+2+3) $\text{Res}\,T = \cancel{(E - E_B)}\,V + V G_0\,\text{Res}\,T$

$0$

$E \to E_B$

4) $\mathrm{Res}\,T = V G_0 \,\mathrm{Res}\,T$

5) 1+4) $\rightarrow$ $\cancel{V}|B\rangle\langle B\cancel{|V} = \cancel{V} G_0 (V|B\rangle\langle B\cancel{|V})$

Repetitions Repetions

6) $$\boxed{|B\rangle = G_0 V |B\rangle}$$ Bound state equation

BSE $\to$ LSE for bound states

$$\boxed{|B\rangle = G_0 V |B\rangle}$$

$\to$ apply this to <u>Rest</u>

$$\mathrm{Rest} = V|B\rangle\langle B|V$$

$$V|B\rangle = G_0^{-1}|B\rangle$$

$$= (E_B - H_0)|B\rangle = \left(E_B - \frac{p^2}{2\mu}\right)|B\rangle$$

Putting all the pieces together :
(a lot of)

$$\langle \vec{p}\,' | R_{est} | \vec{p} \rangle = \left(B - \frac{p^2}{2\mu}\right)\left(B - \frac{p'^2}{2\mu}\right) \psi_B(\vec{p})\, \psi_B^{\dagger}(\vec{p}\,')$$

$\rightarrow$ [the residue is relate to the wave function]

Contact-range theory:

1) $\mathrm{Res}\,T(E) = \frac{\pi}{\mu^2}\frac{2}{a_0}$.

$\rightarrow$ 2) $$\psi_B(\vec{p}) = \frac{\sqrt{8\pi/a}}{p^2 - 2\mu B} = \frac{\sqrt{8\pi/a}}{p^2 + 1/a^2}$$

$\gamma = \frac{1}{a_0}$

the wave number

$$\boxed{\psi_B(\vec{p}) = \frac{\sqrt{8\pi\gamma}}{p^2 + \gamma^2}}$$

The residue has gives us the $\psi$ for a contact-range theory w/ the correct normalization

$$\int \frac{d^3 \vec{p}}{(2\pi)^3} |\psi_B(\vec{p})|^2 = 1$$

$$\mathcal{F} \longrightarrow \psi_B(\vec{r}) = \frac{\sqrt{2\gamma}\, e^{-\gamma r}}{r} \frac{1}{\sqrt{4\pi}}$$

$$= \frac{u(r)}{r} Y_{\ell m}(\hat{r})$$

→ It's sort of amazing

(also somewhat difficult)

→ calculation to be repeated
at home

[END THE LESSON]