NUCLEAR PHYSICS (14)

EFFECTIVE RANGE EXPANSION & FORMAL SCATTERING THEORY

DOES ANYONE WANT TO PRESENT
SOME EXERCISE IN A
FUTURE CLASS?

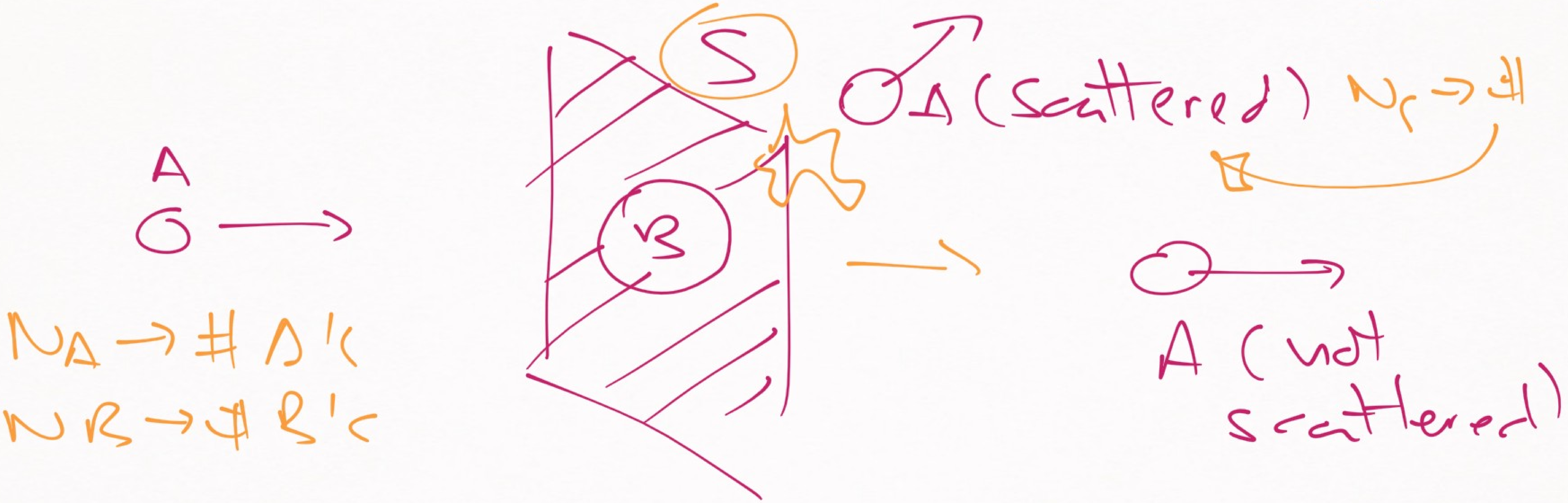
Think about it

RECAP

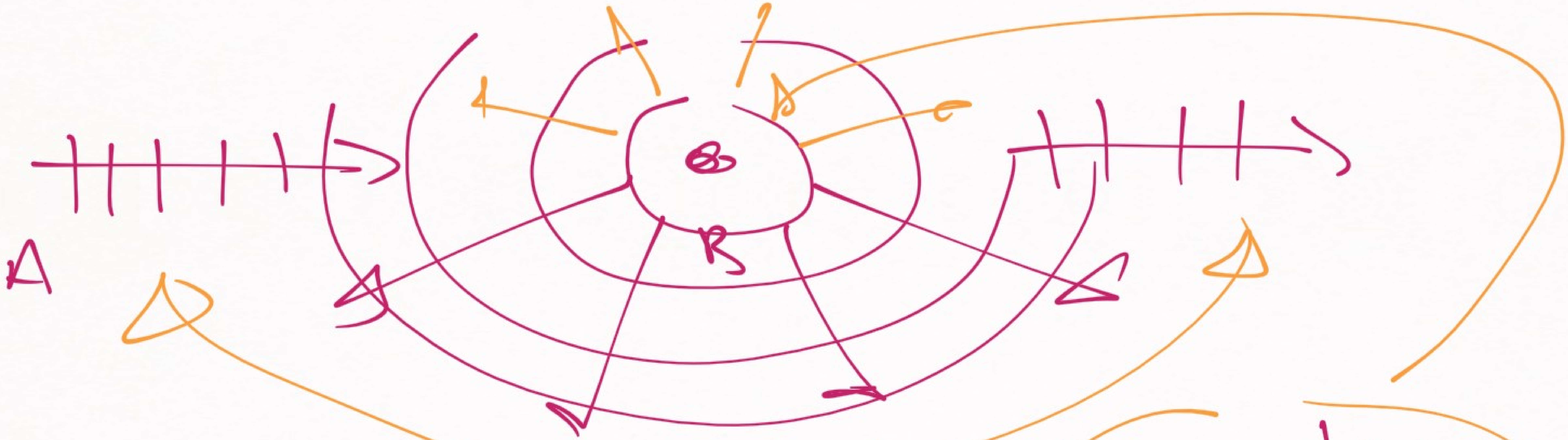
→ Scattering theory

(classical version)

$$\sigma = \frac{N_S}{N_A N_B} S$$



→ Scattering theory (QM version)



$$\psi_k(\vec r) \xrightarrow[|\vec r|\to\infty]{} e^{i\vec k\cdot\vec r} + f(\theta)\,\frac{e^{ikr}}{r}$$

$$\sigma = \frac{N_{sc}}{N_A N_B}$$

$$\psi_k(\vec{r}) \rightarrow e^{i\vec{k}\cdot\vec{r}} + f(\Omega)\,\frac{e^{ikr}}{r}$$

$$N_{sc} \rightarrow \psi_{out}(\vec{r}) \qquad \rightarrow \psi_{in}(\vec{r}) + \psi_{out}(\vec{r})$$

$$N_A N_B \rightarrow \psi_{in}(\vec{r})$$

fluxes (# of particles per unit time)

$$\psi_{in} \rightarrow |\vec{J}_{in}| = \left|\frac{k}{m}\right|$$

$$\psi_{out} \rightarrow \int \vec{J}_{out}\cdot d\vec{S} = \frac{k}{m}\int |f(\Omega)|^2 d\Omega$$

$$\sigma = \int |f(\Omega)|^2 \, d\Omega$$ → QM cross section

$$\frac{d\sigma}{d\Omega} = \underbrace{|f(\Omega)|^2}$$ → differential cross section

Partial wave expansion is possible → separate different ℓ contributions

$$f(\theta) = \sum_{l} (2l+1)\, \underbrace{f_l(k)} \, P_l(\cos\theta)$$

$$f_l(k) = \frac{e^{i\delta_l} \sin \delta_l}{k} = \frac{1}{k \cot \delta_l - ik}$$

phase shift (≃ wave function at long distances)

$$\sigma = \int |f(\Omega)|^2 d\Omega = \frac{4\pi}{k^2} \sum_l \sin^2 \delta_l$$

$$\delta_l(k) \xrightarrow[k \to 0]{} -a_l k^{2l+1} + \dots$$

SIMPLES
ANALYSIS
POSSIBLE

$$\sigma \xrightarrow[k \to 0]{} 4\pi |a_0|^2$$

→ add spin

→ add tensor forces

→ inelasticities

A few comments about σ, δ_ℓ :

1) Experimentally we measure $\underbrace{\sigma, \frac{d\sigma}{d\Omega}, \text{etc.}}$

experimental input

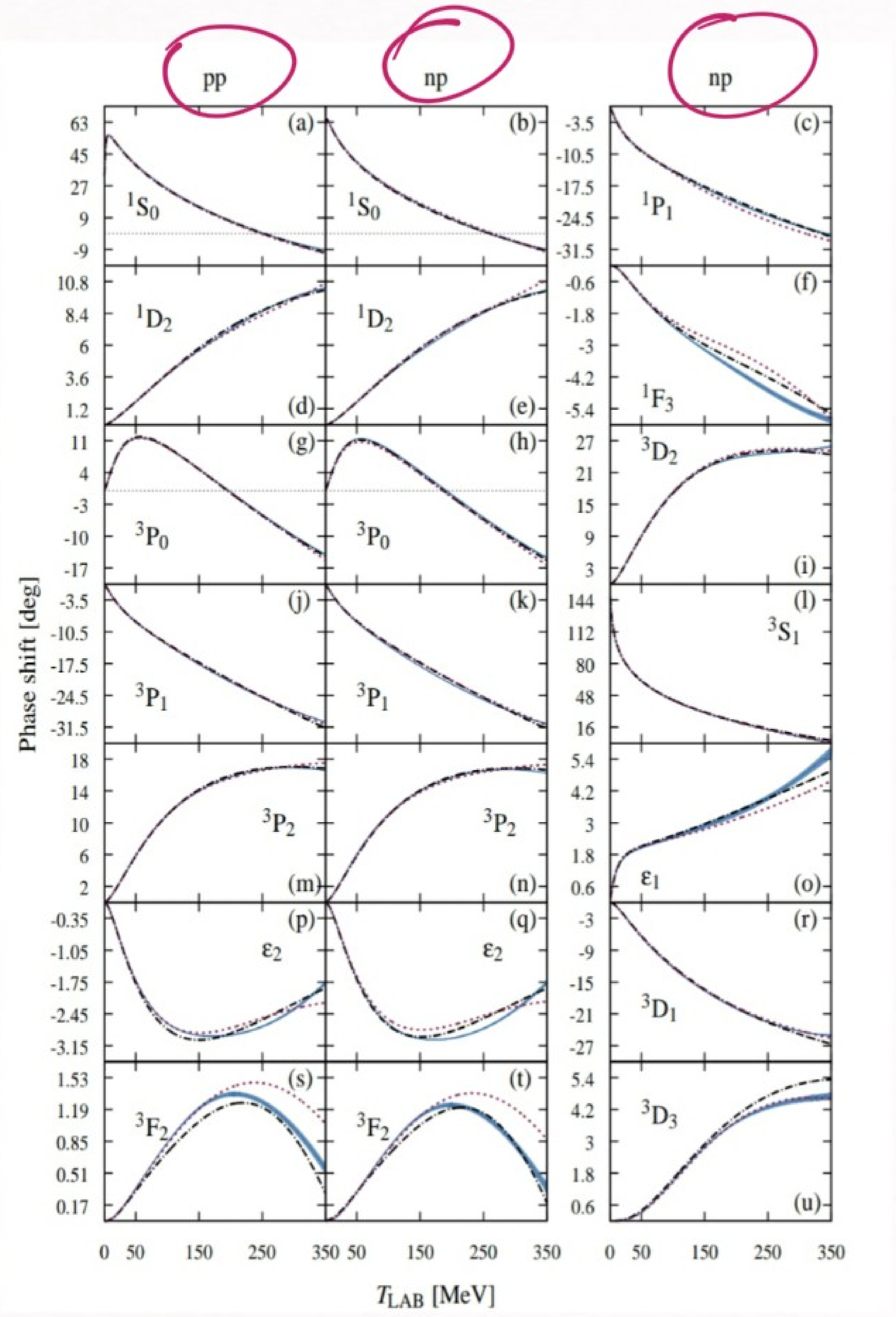
2) Phase shifts are extracted by means of theoretical models

$\rightarrow$ $\boxed{\text{PWA}}$ (Partial wave analysis)

$\boxed{\text{PWA}}$ → sophisticated analysis of σ, $\frac{d\sigma}{d\Omega}$ etc

→ this is not completely model-independent

($\exists$ small ambiguity in the definition of phase shifts)



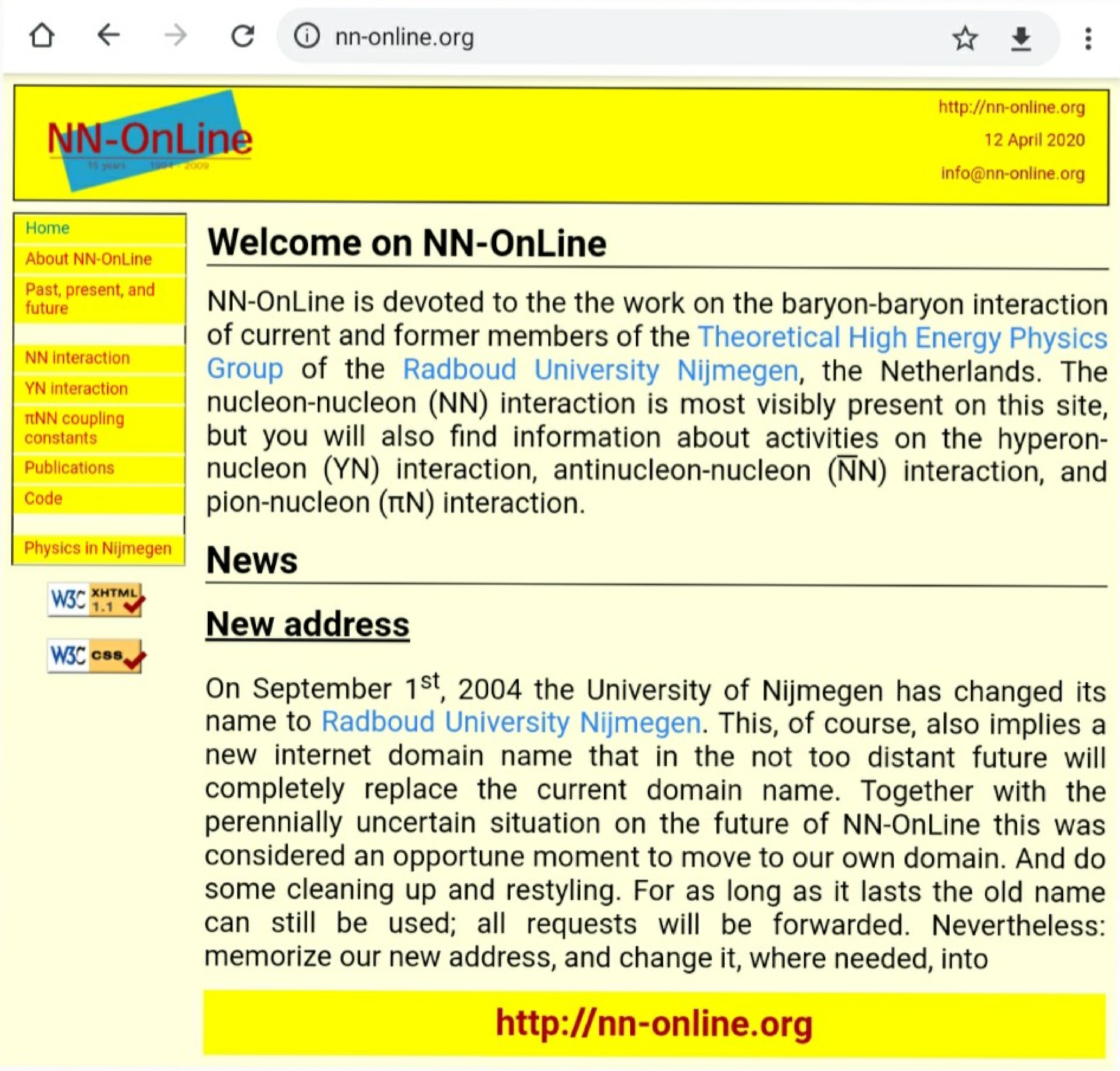
Example of NN phase shifts

(arxiv: 1304.0895, navarro, amaro, arriola)

→ pp, np (nn usually absent because we can't accelerate neutral particles)

→ $^{2S+1}L_J$ (S, L, J → angular momenta)

$^1S_0$ → S=0, L=0, J=0



NN-OnLine

http://nn-online.org

12 April 2020

info@nn-online.org

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- About NN-OnLine
- Past, present, and future
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Welcome on NN-OnLine

NN-OnLine is devoted to the the work on the baryon-baryon interaction of current and former members of the Theoretical High Energy Physics Group of the Radboud University Nijmegen, the Netherlands. The nucleon-nucleon (NN) interaction is most visibly present on this site, but you will also find information about activities on the hyperon-nucleon (YN) interaction, antinucleon-nucleon ($\overline{N}N$) interaction, and pion-nucleon (πN) interaction.

News

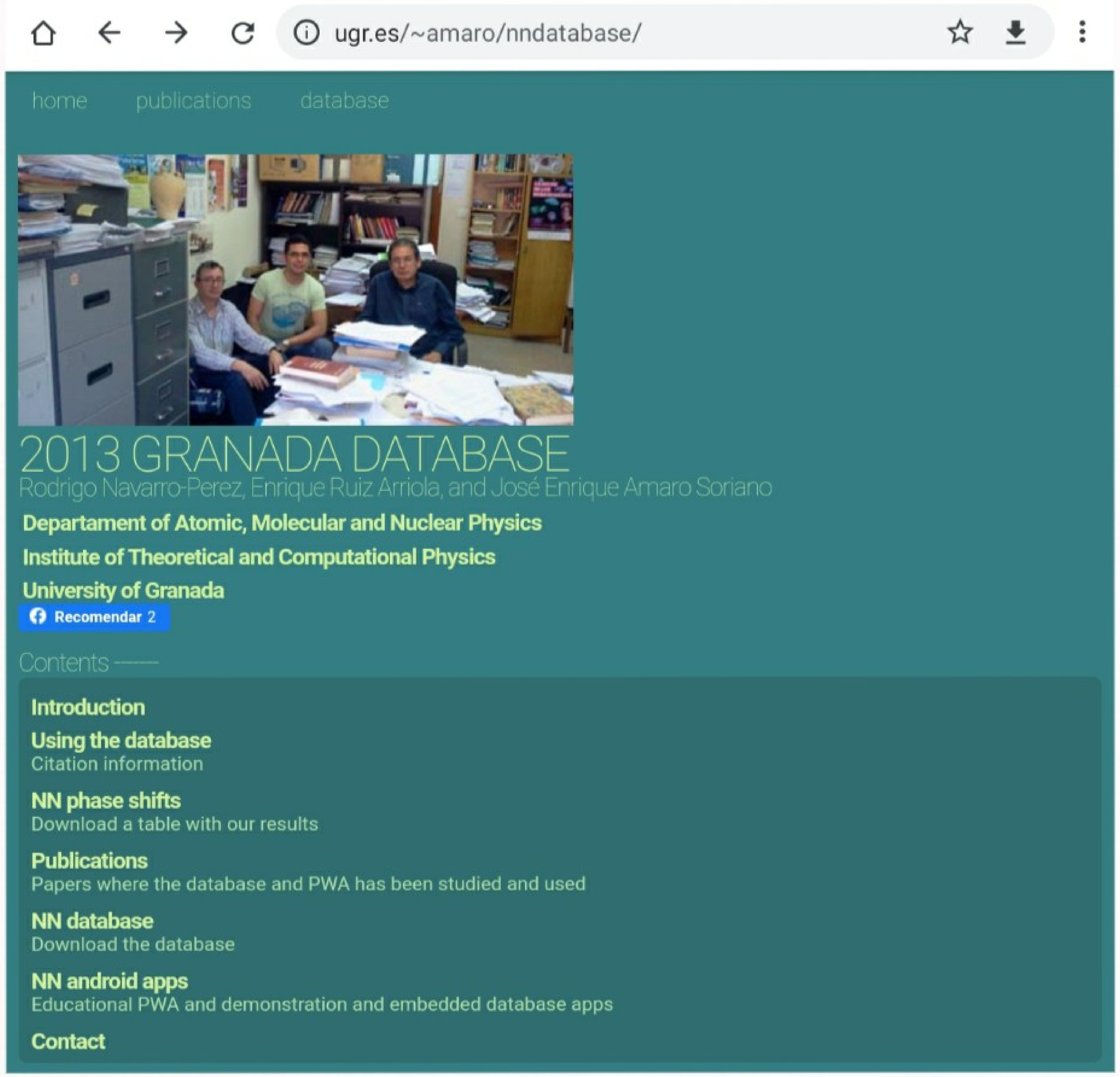
New address

On September 1^{st}, 2004 the University of Nijmegen has changed its name to Radboud University Nijmegen. This, of course, also implies a new internet domain name that in the not too distant future will completely replace the current domain name. Together with the perennially uncertain situation on the future of NN-OnLine this was considered an opportune moment to move to our own domain. And do some cleaning up and restyling. For as long as it lasts the old name can still be used; all requests will be forwarded. Nevertheless: memorize our new address, and change it, where needed, into

http://nn-online.org

→ Nijmegen group

→ check the phase shifts, different potentials, papers from Nijmegen people, etc



2013 GRANADA DATABASE

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Institute of Theoretical and Computational Physics
University of Granada

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Download a table with our results

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NN database
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NN android apps
Educational PWA and demonstration and embedded database apps

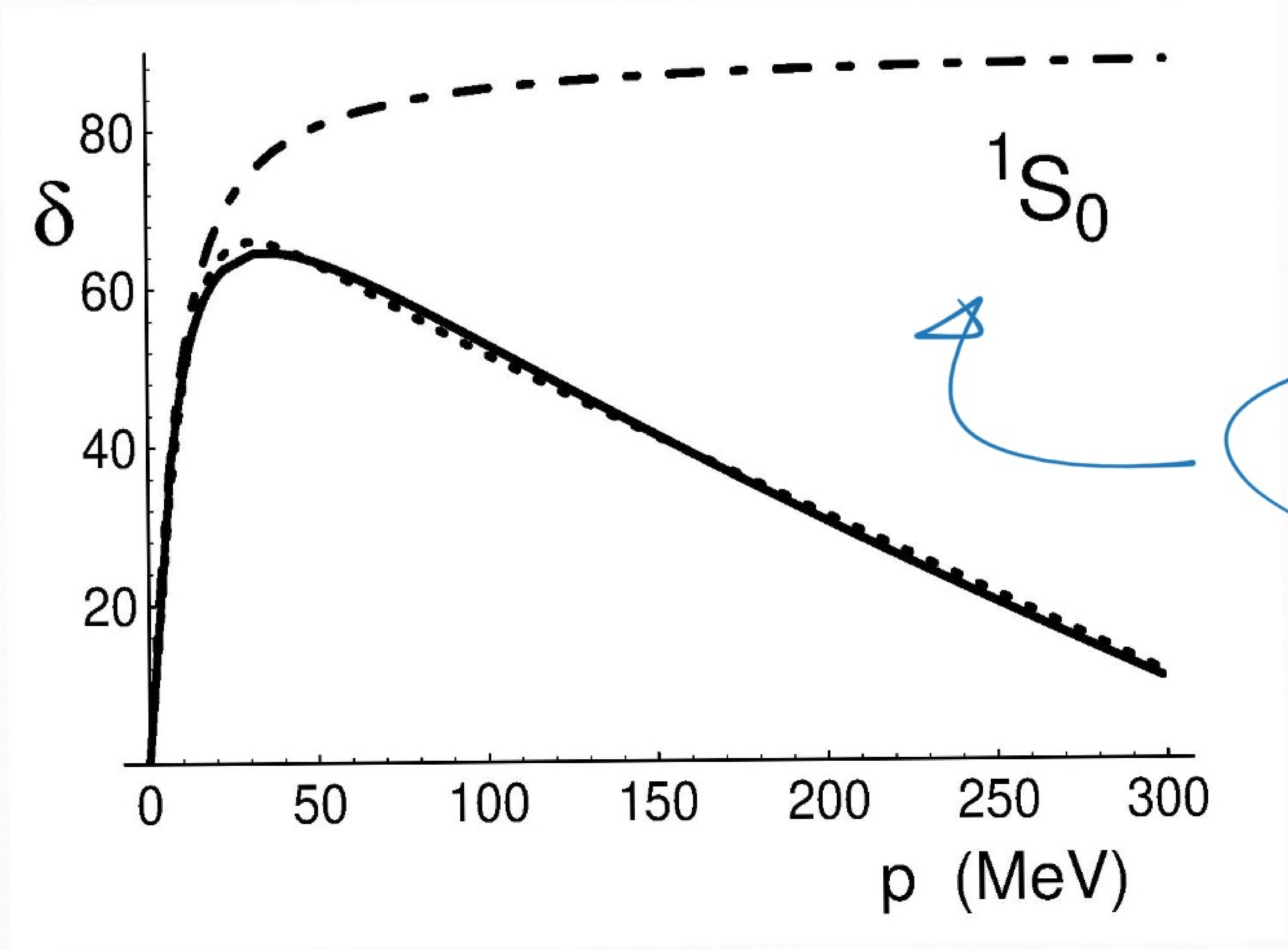
Contact

→ From the Granada group (more recent PWA)

It contains TPE effects

→ Before they had an android app w/ potentials, phase shifts, light nuclei

S-waves are always important :



NN → $S = 0, 1$

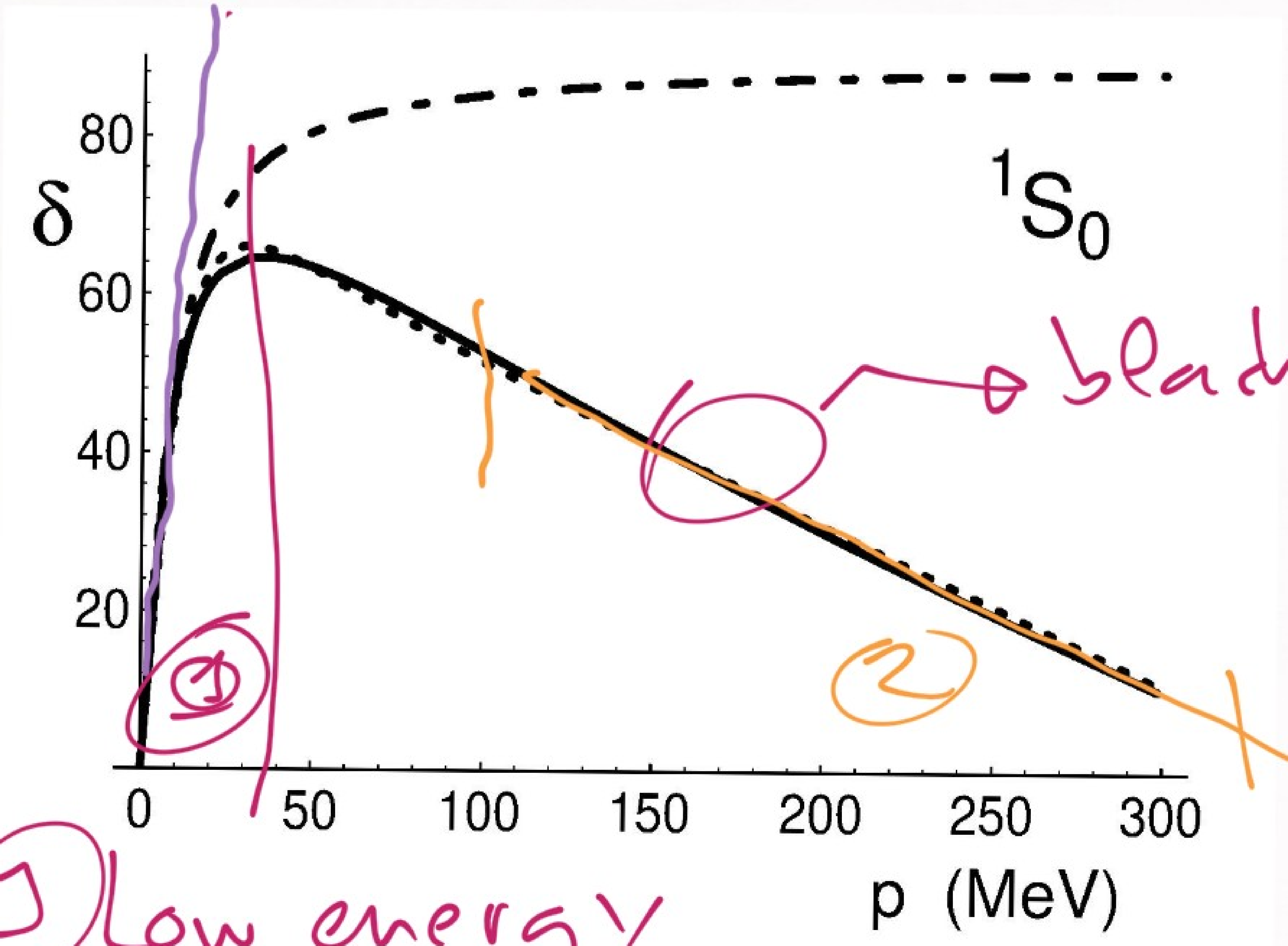
two types of S-waves :

${}^1S_0$ (singlet)

${}^3S_1$ (triplet)

↓

deuteron



dotted ones → some theory

black : PWA ($\simeq$ exp. data)

① Low energy

$$\delta_0(p) \to -a_0 p + O(p^3)$$

$$\boxed{a_0 \simeq -23.7\,\text{fm}}$$

② Higher energies

$$\delta_0 \sim -R(p-p_0)$$

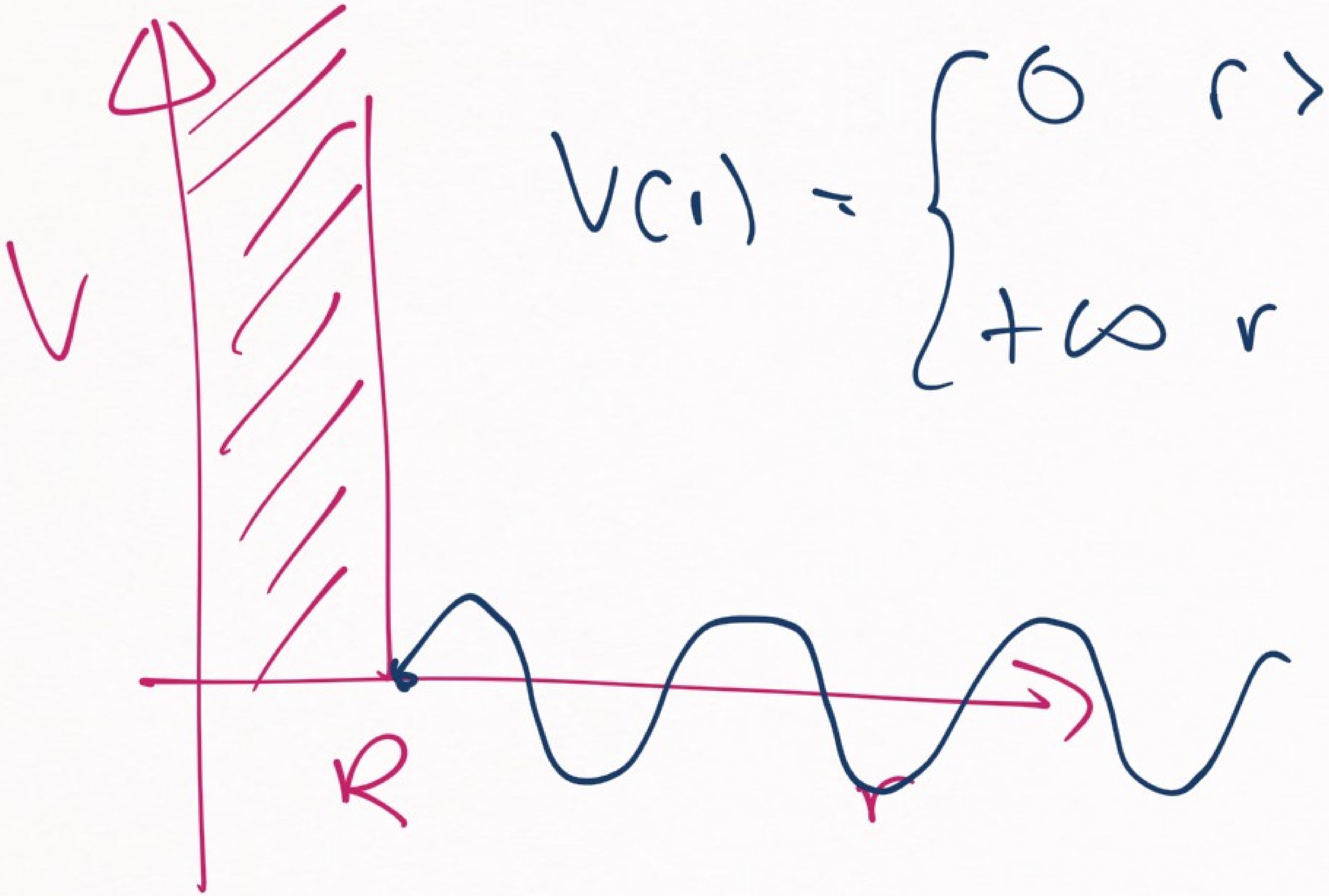
(straight line)

↓

(interesting)

WHY INTERESTING ?

→ Shows the existence of a repulsive core



$$V(r) = \begin{cases} 0 & r > R \\ +\infty & r \leq R \end{cases}$$

Solution :

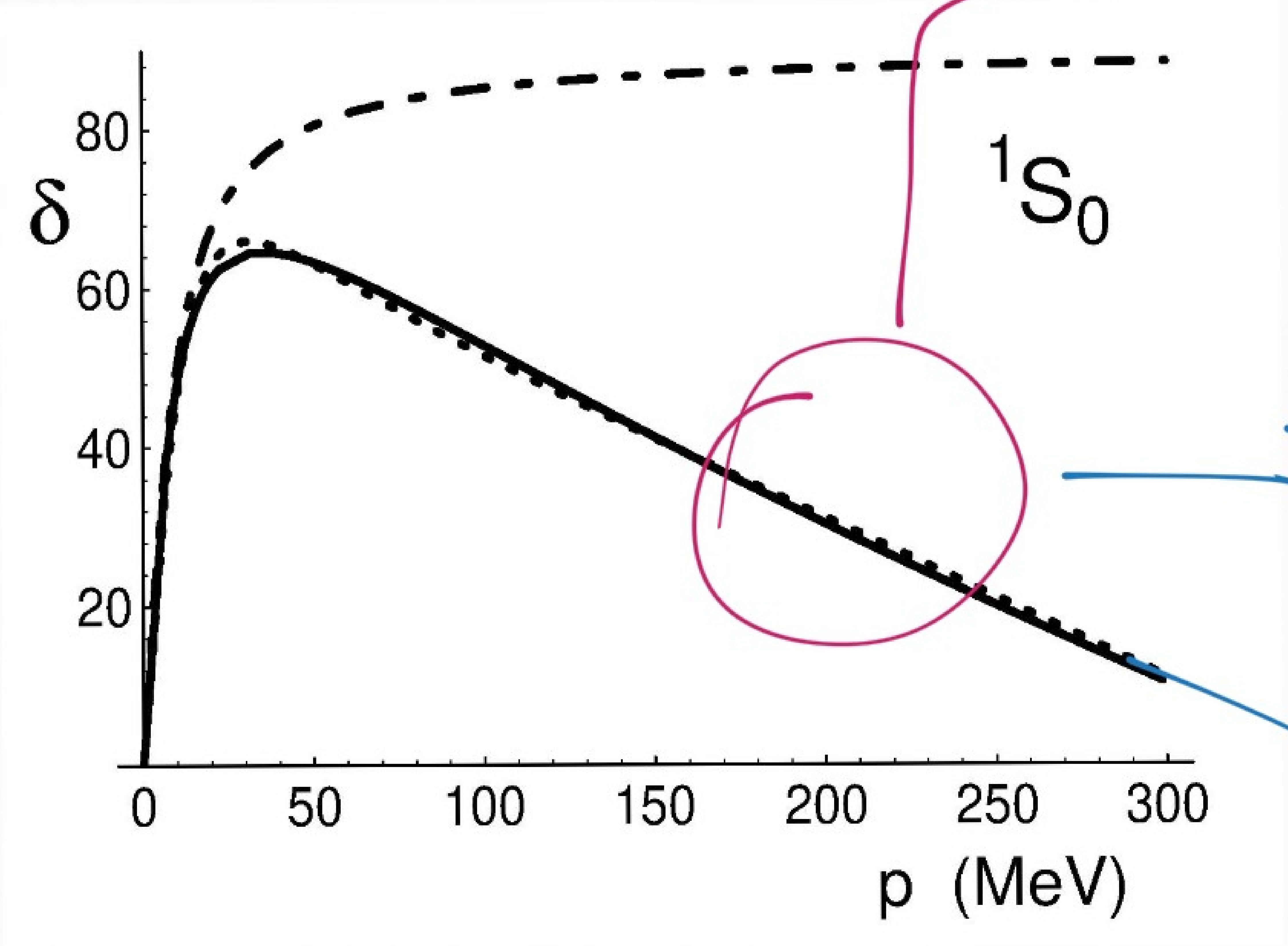
$$u(R) = 0$$

$$u(r > R) = \sin(kr + \delta)$$

$$\boxed{\delta = -kR}$$

REPULSIVE CORE

$\delta(p) = -pR$

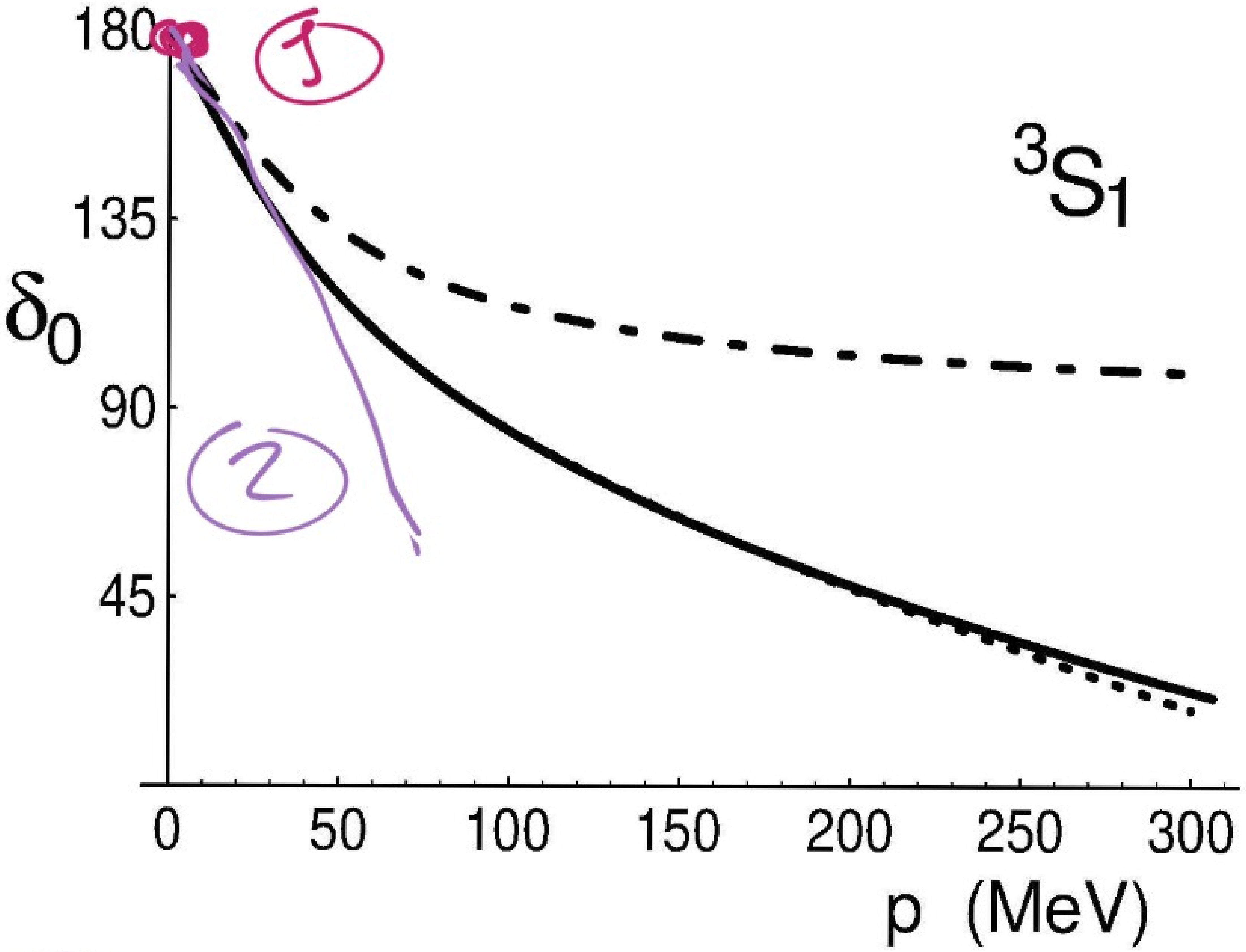


→ FORM OF $\delta(p)$

$\delta(p) \simeq -R(p-p_0)$

→ How we know about the repulsive core

(one way)



→ other important S-wave (deuteron channel)

① → $\delta(0) = 180° = \pi$

→ Levinson theorem

$\delta(p=0) - \delta(p\to\infty) = n_B \pi$

$n_B \to$ # of bound states

② $p \to 0$

$\delta(p) \to -a_0 p$

$\boxed{a_0 \simeq 5.4\,\text{fm}}$

[$^1S_0$ & $^3S_1$ COMPARISON]

① $^1S_0$ → $\boxed{a_0 < 0}$ → attraction (but not enough to bind)

② $^3S_1$ → $\boxed{a_0 > 0}$

2.a) Repulsion

2.b) Attraction + bound state
($E_b \to 0$, $a_0 \to \pm\infty$)

→ We end the comparison between

$$\boxed{{}^1S_0 \quad \& \quad {}^3S_1}$$

→ How to describe the phase shifts at low energies beyond the scattering length

$$\delta_0(k) \to -a_0 k + \underbrace{\mathcal{O}(k^3)}_{\text{which are the corrections?}}$$

$$\downarrow$$

$$k \cot \delta_0(k) = -\frac{1}{a_0} + \boxed{\mathcal{O}(k^2)}$$

(more convenient)

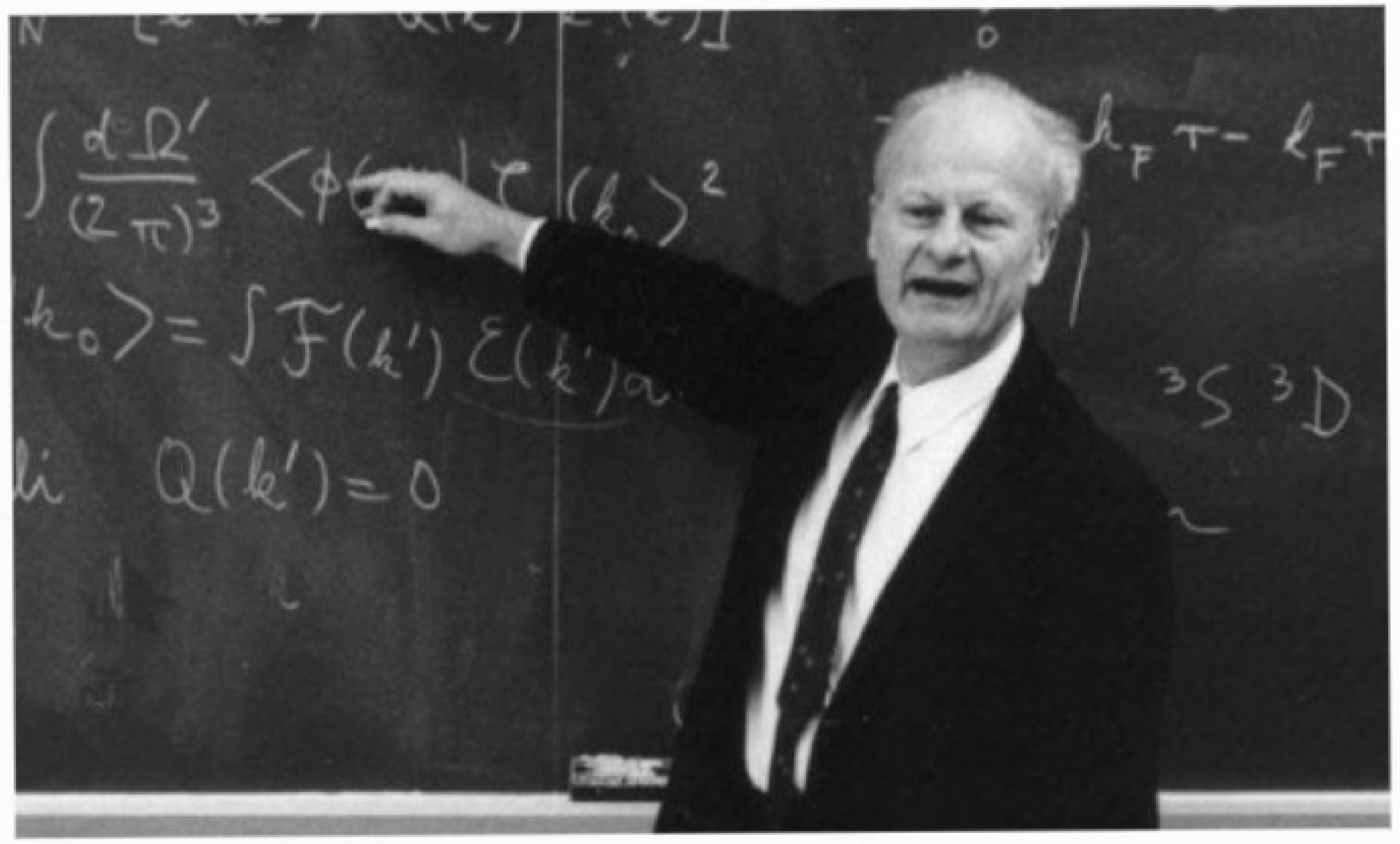
[EFFECTIVE RANGE EXPANSION]

Extension of $\delta_0(k) \to -a_0 k$ to arbitrary powers of k?

$$k\cot\delta_0 = -\frac{1}{a_0} + \frac{1}{2} r_0 k^2 + \sum_{n=2}^{\infty} v_n k^{2n} \longrightarrow \boxed{?}$$

↳ this is how the extension will look like

?



Hans Bethe

+

Schwinger /
Landau &
Smorodinsky

Strong point: translate physical problems
into ordinary differential equations
(ERE, G-matrix)

$$\left.\begin{array}{l} k\cot\delta_0 = -\frac{1}{a_0} + O(k^2) \\ (\delta_0(k) \to -a_0 k \\ \qquad + O(k^3)) \end{array}\right] \to k\cot\delta_0 = -\frac{1}{a_0} + \frac{1}{2} r_0 k^2 + \sum_n v_n k^{2n}$$

$\to$ [WRONSKIAN IDENTITY]

(Really useful trick)

Trick → compare two differential equations

Eg 1) $-u_k'' + 2\mu V(r) u_k(r) = k^2 u_k(r)$

$(u_k(r) \to \sin(kr+\delta)/\sin\delta \,,\; r \to \infty)$

Eg 2) $-u_0'' + 2\mu V(r) u_0(r) = 0$

$(u_0(r) \longrightarrow 1 - r/a_0 \,,\; r \to \infty)$

Now we build a Wronskian identity:

$$(\text{Eq. }1) \times u_0(r) - (\text{Eq. }2) \times u_k(r)$$

↓

$$-(u_k'' u_0 - u_k u_0'') = k^2 u_k u_0$$

↓

Notice: $(u_k'' u_0 - u_k u_0'') = (u_k' u_0 - u_k u_0')'$

(Exact derivative)

$$-(u_k' u_0 - u_k u_0')^{\overset{dr}{\prime}} = k^2 u_k u_0$$

$\downarrow$ (Integrate)

$$\underbrace{-(u_k' u_0 - u_k u_0')}_{\text{Wronskian}}\Big|_{r_c}^{R} = k^2 \int_{r_c}^{R} u_k u_0 \, dr$$

Second step:

Eq.3) $-u_k'' = k^2 u_k$, $u_k = \frac{\sin(kr+\delta)}{\sin\delta}$

Eq.4) $-v_0'' = 0$, $v_0 = 1 - r/a_0$

Eqs. 3, 4 → asymptotic versions of

Eqs 1, 2

→ (Eq 3) $\times v_0$ − (Eq. 4) $\times v_k$

$$-(v_k' v_0 - v_k v_0')\Big|_{r_c}^{R} = k^2 \int_{r_c}^{R} v_k v_0 \, dr$$

We combine this w/ the one for u_k, u_0 and calculate the difference

$$(u_k' u_0 - u_k u_0')\Big|_{r_c}^{R} - (v_k v_0 - v_k v_0')\Big|_{r_c}^{R} = k^2 \int_{r_c}^{R} (u_k u_0 - v_k v_0) \, dr$$

Take the $r_c \to 0$, $R \to \infty$ limits :

$$\left[\; k\cot\delta = -\frac{1}{a_0} + k^2 \int_0^\infty (v_k v_0 - u_k u_0)\, dr \;\right]$$

$\to$ u_k, v_k good k^2 expansion

$$u_k = u_0 + k^2 u_2 + k^4 u_4 + \dots$$

$$v_k = v_0 + k^2 v_2 + k^4 v_4 + \dots \quad \leftarrow \underline{\text{analytic}}$$

Σ

$$\int_0^\infty (v_k v_0 - u_k u_0)\, dr = \frac{1}{2} r_0 k^2 + v_2 k^4 + v_3 k^6 + \dots$$

$$\Rightarrow \boxed{k \cot\delta = -\frac{1}{a_0} + \frac{1}{2} r_0 k^2 + \sum_{n=2}^{\infty} v_n k^{2n}}$$

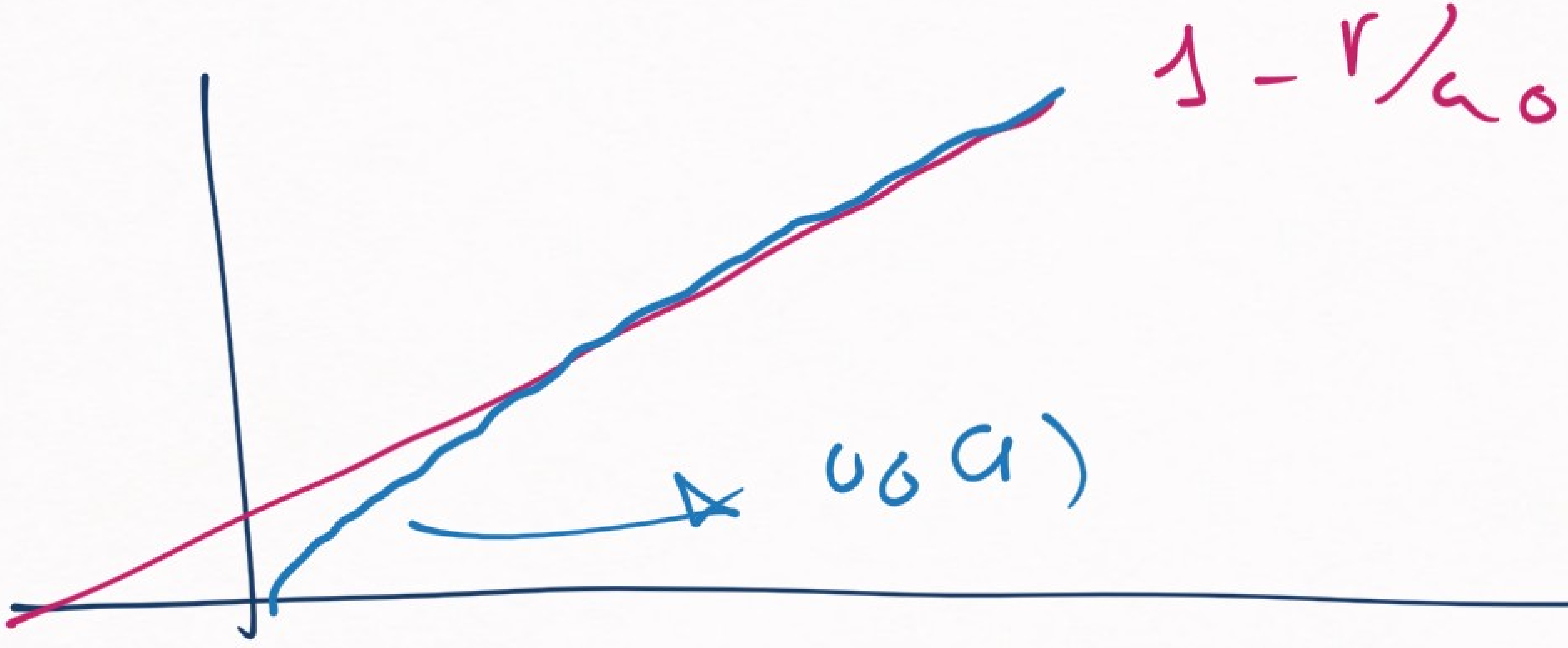
ERE (Effective range expansion)

→ why the name?

$$r_0 = 2\int_0^{\infty} \left[\left(1-\frac{r}{a_0}\right)^2 - u_0^2(r) \right] dr$$

$$u_0(r) \rightarrow \left(1 - \frac{r}{a_0}\right), \, r \rightarrow \infty$$

$$\rightarrow 0 \text{ for } r \rightarrow \infty$$



$u_0(r) \leq \left(1 - \frac{r}{a_0}\right)$

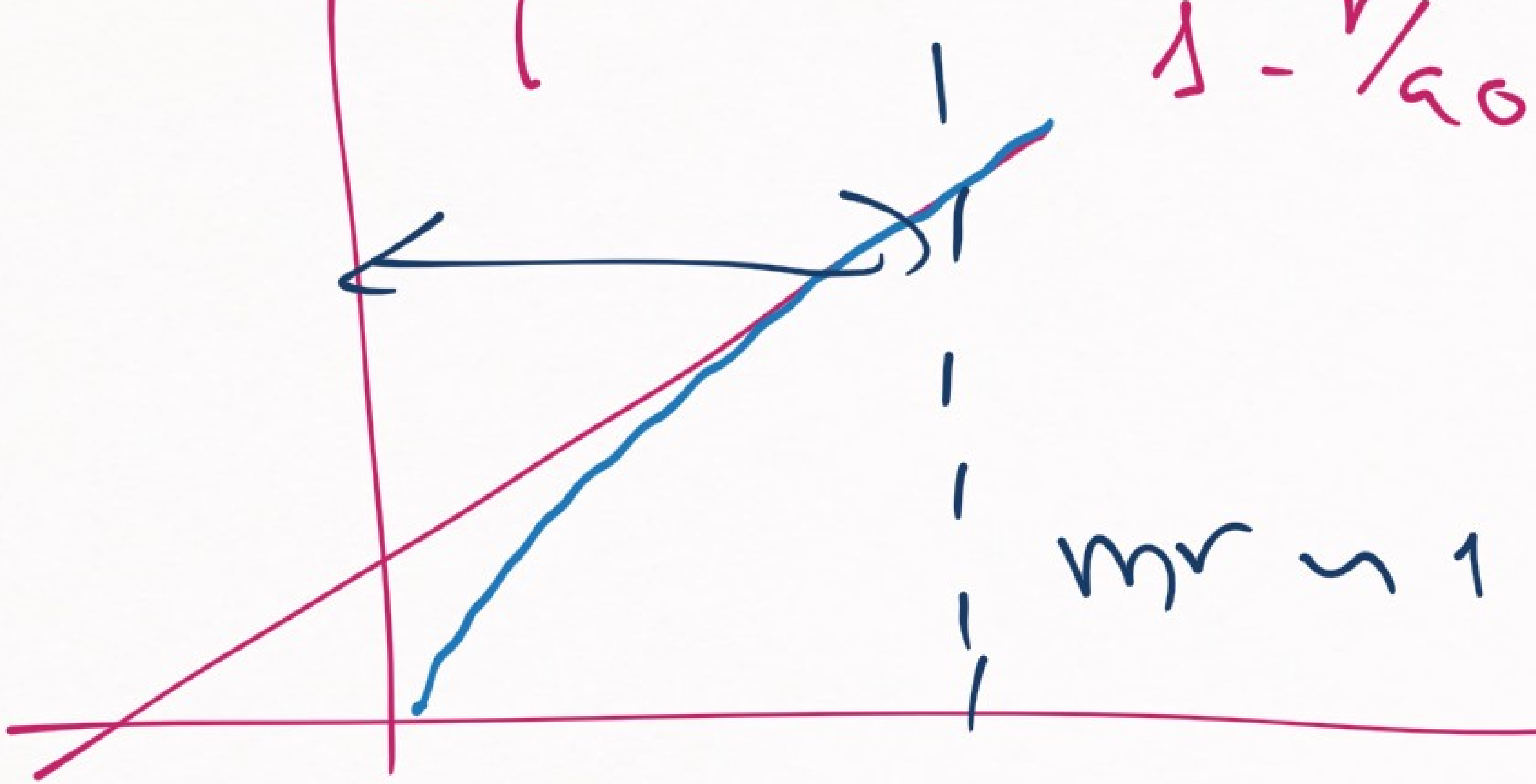
when

$V(r) \leq 0$

$$V(r) = \rho(r)\, e^{-mr} \qquad mr \gg 1 \Rightarrow V(r) \simeq 0$$

$$2\int_0^{\infty} \left[\left(1-\frac{r}{a_0}\right)^2 - u_0^2(r) \right) dr \simeq 2\int_0^{1/m} \left[\left(1 - r/a_0\right)^2 - u_0^2(r) \right] dr$$

$\uparrow r$



$$\simeq \boxed{\frac{1}{m}}$$

$r_0 \rightarrow$ Effective range because $r_0 \simeq R_0$

In the 50's people wanted to extract info about $V(r)$ from low energy phase shifts

range of the potential

e.g. $\boxed{R_0 \sim 1/m}$

ERE for NN:

$\boxed{^1S_0} \rightarrow$ $a_0 \simeq -23.7$ fm

$r_0 \simeq \boxed{2.7 \text{ fm}}$

$v_2 \simeq -0.5 \text{ fm}^3$

$\boxed{^3S_1} \rightarrow$ $a_0 \simeq 5.4$ fm

$r_0 \simeq \boxed{1.8 \text{ fm}}$

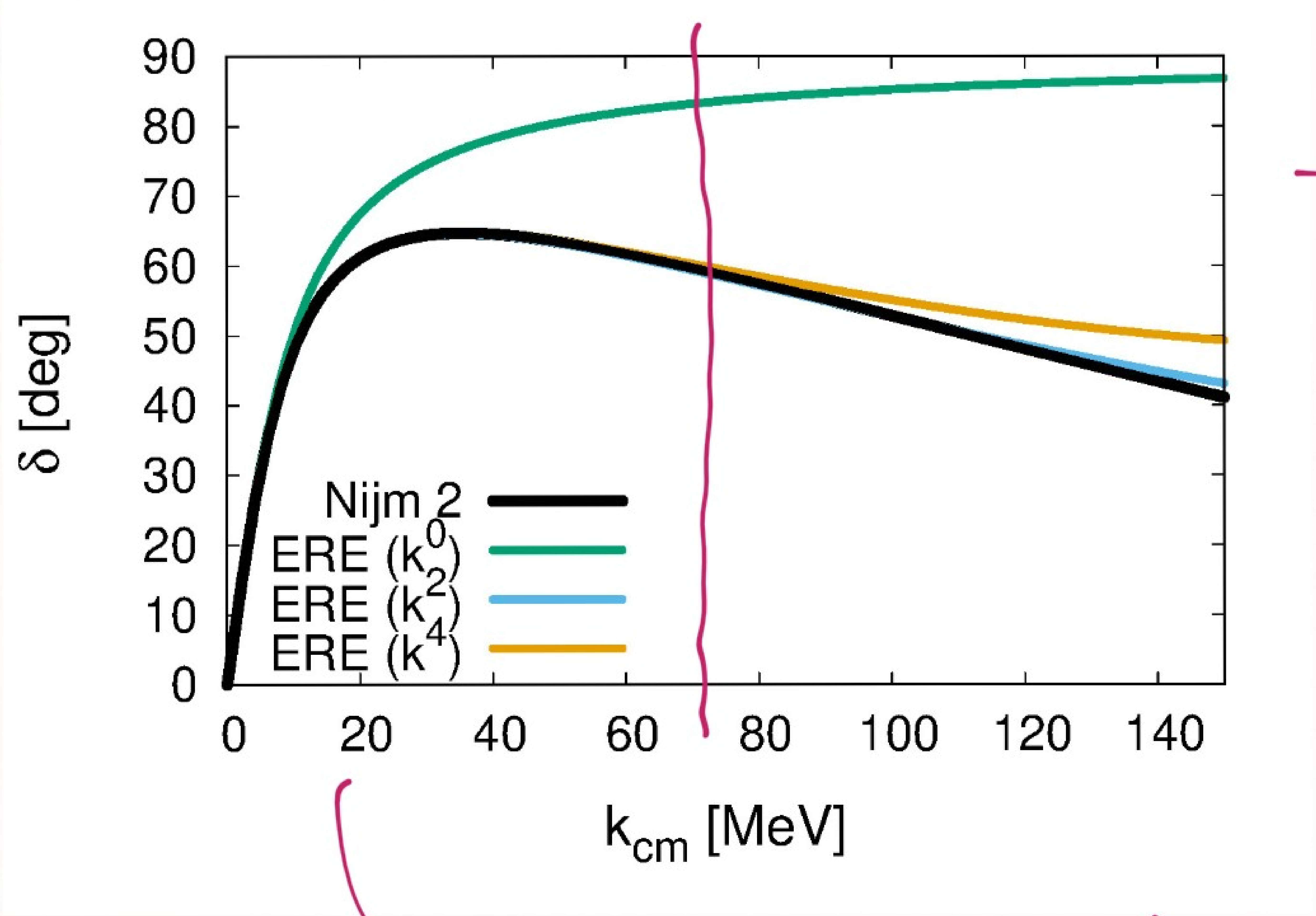
$v_2 \simeq 0.05 \text{ fm}^3$

$R_0 \sim (1.8 - 7.7)$ fm

$\sim (1.5 - 3.0)$ fm

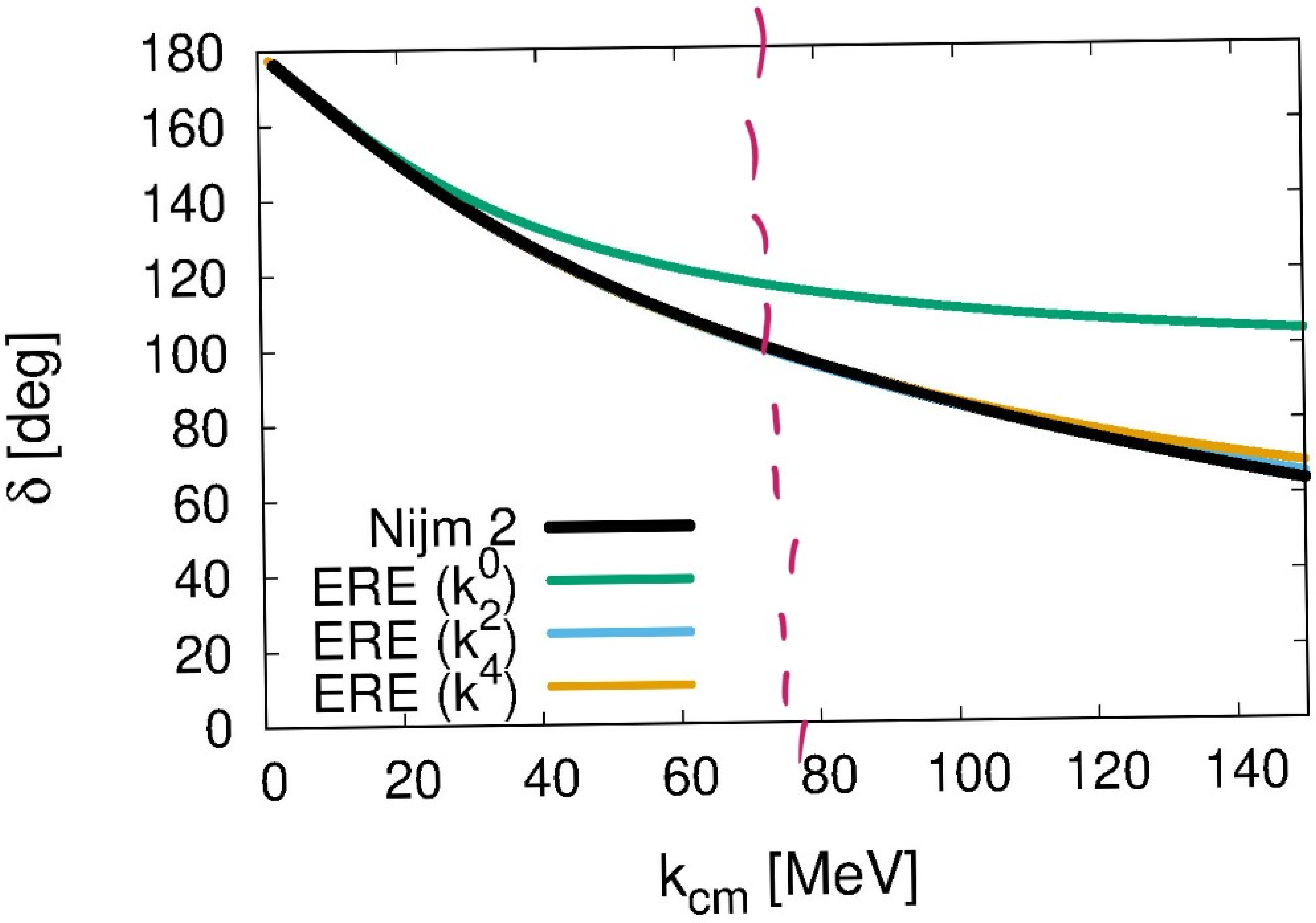
$\downarrow$

$$\boxed{\frac{1}{m_\pi} - \frac{2}{m_\pi}}$$



→ $^1S_0$

ERE (parameters in previous page)



$\rightarrow$ $^3S_1$

CAVEAT → ERE only converges at low energies

$$V(r) = f(r)\, e^{-mr} \Rightarrow k\cot\delta_0 = -\frac{1}{a_0} + \frac{1}{2} r_0 k^2 + \sum v_n k^{2n}$$

only converges for $|k| < m/2$

CHALLENGING EXERCISE TO TRY

$$\sigma = \underbrace{4\pi|a_0|^2 + O(k^2)}$$

→ add here $\boxed{r_0 \, , \, a_1}$ p-wave scattering length

→ problem: $k \to 0$ easy for theory (super difficult for experiments)

[FORMAL SCATTERING THEORY
(T-MATRIX)]

→ adding layers of abstraction

(viewing physical problem
from different perspectives)

Two views of QM :

1) Wave functions that obey differential equations

$$\left[-\frac{\nabla^2}{2m} + V(\vec{r})\right] \psi(\vec{r}) = E \psi(\vec{r})$$

2) Operators acting on vectors on a Hilbert space

$$\hat{H}|\psi\rangle = E|\psi\rangle$$

What do they imply for scattering theory?

View 1) $\psi(\vec{r}) \to e^{i\vec{k}\vec{r}} + f(\Omega)\frac{e^{ikr}}{r}$

$$\frac{d\sigma}{d\Omega} = |f(\Omega)|^2$$

View 2) $|\psi\rangle = |\vec{k}\rangle + G_0 T|\vec{k}\rangle$

$$f(\Omega) = -\frac{\mu}{2\pi}\langle k'|T|\vec{k}\rangle$$

Aim → find an operator $(\underline{\underline{T}})$ the matrix elements of which gives us the scattering amplitude $\underline{\underline{f(\Omega)}}$

$$\boxed{\text{T-matrix}}$$

Derivation (schematic):

1) $\underset{\text{Hamiltonian}}{\underline{\underline{H}}}|\phi\rangle = E\,\underset{\text{wave function}}{\underline{\underline{|\phi\rangle}}}$

2) $H = \underset{\text{kinetic energy}}{\underline{\underline{H_0}}} + \underset{\text{potential}}{\underline{\underline{V}}}$

1+2) $\rightarrow \boxed{(E - H_0)|\phi\rangle = V|\phi\rangle}$

→ Solve $(\mathcal{E}-H_0)|\phi\rangle = V|\phi\rangle \longleftarrow$

Technique → Green's function method

→ We go back to w.f. language

$|\phi\rangle \longrightarrow \phi(\vec{r}) = \langle\vec{r}|\phi\rangle$

$\rightarrow \phi(\vec{r}) = e^{i\vec{k}\cdot\vec{r}} + \int d^3\vec{r}\,'\, G_0(\vec{r}-\vec{r}\,')V(\vec{r}\,')\phi(\vec{r}\,')$

ANSATZ → a proposal of a solution

Try the ansàtz :

$(E - H_0)|\phi\rangle = V|\phi\rangle$ w/ the ansatz

(you can try yourself)

→ Previous ansatz solution if :

$$(E - H_0)\,G_0(\underline{\vec{r} - \vec{r}\,'}) = \delta^{(3)}(\vec{r} - \vec{r}\,')$$

$$H_0 = -\frac{\nabla^2}{2\mu}$$ → Green function

Step 1 : $\psi(\vec{r}) = e^{i\vec{k}\cdot\vec{r}} + \int d^3\vec{r}\,' \, G_0(\vec{r}-\vec{r}\,')V(\vec{r}\,')\psi(\vec{r}\,')$

Step 2 : $\underbrace{(E-H_0)\,G_0(\vec{r}) = \delta^{(3)}(\vec{r})}_{\text{[G_0 is a Green function of H_0]}}$

Step 3 → find a solution for G_0

↳ go to p-space

$$G_0(\vec{r}) = \int \frac{d^3\vec{\ell}}{(2\pi)^3}\, G_0(\vec{\ell})\, e^{+i\vec{\ell}\cdot\vec{r}}$$

$$(E-H_0)\,G_0(\vec{r}\,') = \delta^{(3)}(\vec{r}-\vec{r}\,')$$

$$\left[\begin{array}{l} (E-H_0)\,G_0(\vec{\ell}) = 1 \\ H_0 = \dfrac{\vec{\ell}^{\,2}}{2M} \end{array}\right] \longrightarrow \boxed{G_0(\vec{\ell}) = \frac{1}{E-\frac{\ell^2}{2M}}}$$

$$G_0(\vec{p}) = \frac{1}{E - \frac{\vec{p}^2}{2\mu}} \longrightarrow \boxed{G_0(E) = \frac{1}{E - H_0}}$$

"OPERATOR LANGUAGE"

(Resolvent operator, propagator)

Original aim : $\left| \boxed{G_0(\vec{r}-\vec{r}\,')} \right| \longrightarrow$ Fourier transform of $G_0(\vec{\ell})$

$$G_0(\vec{r}) = \int \frac{d^3\vec{\ell}}{(2\pi)^3} G_0(\vec{\ell})\, e^{i\vec{\ell}\cdot\vec{r}}$$

$$= \frac{1}{2\pi r^2}\int_0^\infty d\ell \, \frac{\ell \sin(\ell r)}{\epsilon - \frac{\ell^2}{2\mu}} = \frac{1}{4\pi r^2}\,\mathrm{Im}\left[\int_{-\infty}^{+\infty} d\ell \, \frac{\ell e^{i\ell r}}{\epsilon - \frac{\ell^2}{2\mu}}\right]$$

$$\oplus \longrightarrow \boxed{\int_{-\infty}^{+\infty} d\ell\, \ell \, \frac{e^{i\ell r}}{E-\frac{\ell^2}{2\mu}}} \longrightarrow$$ calculate it by residues in the complex plane

$$\downarrow$$

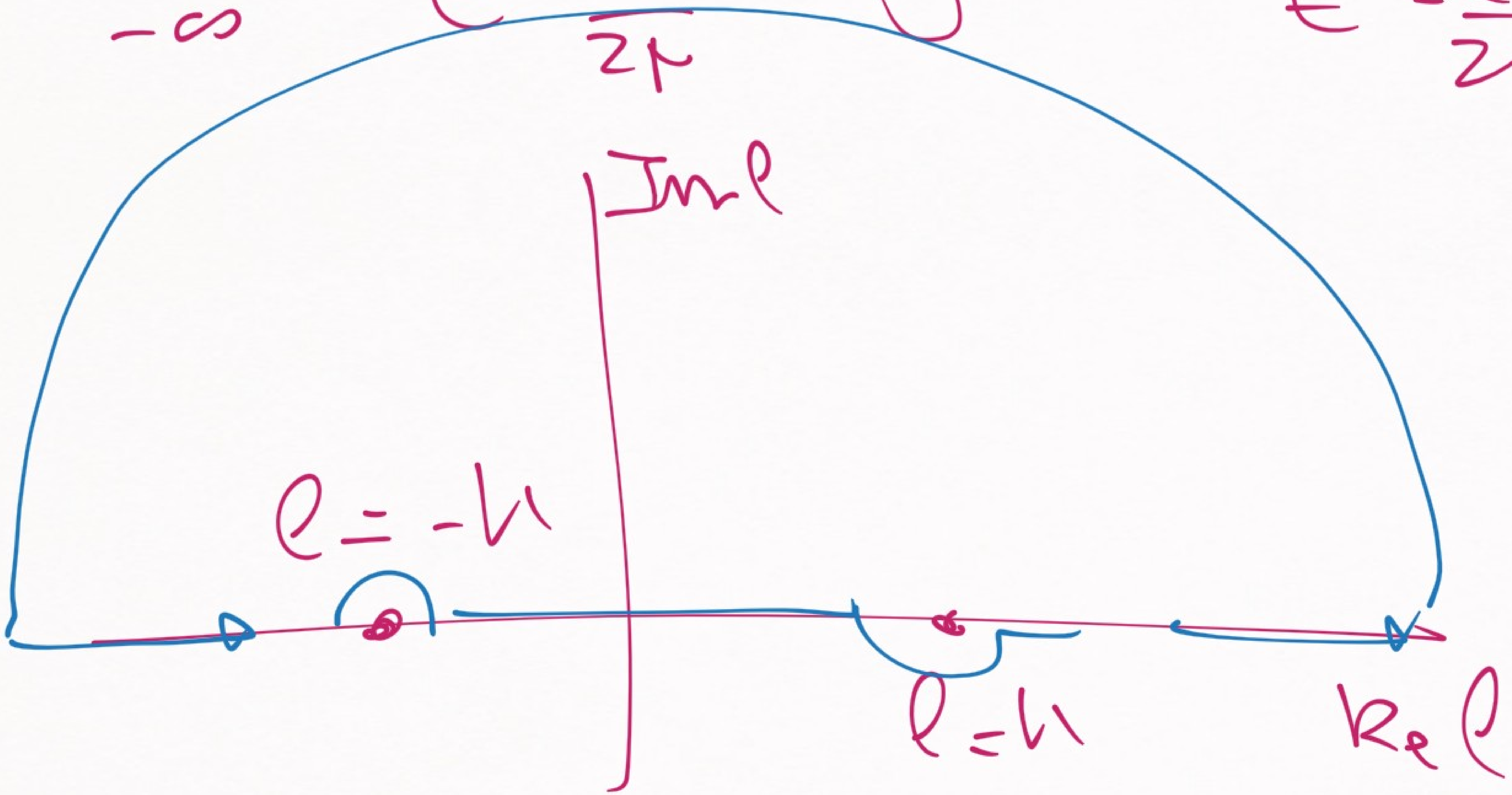
$$\ell = \pm\sqrt{2\mu E} = \pm k \,) \longrightarrow$$ poles

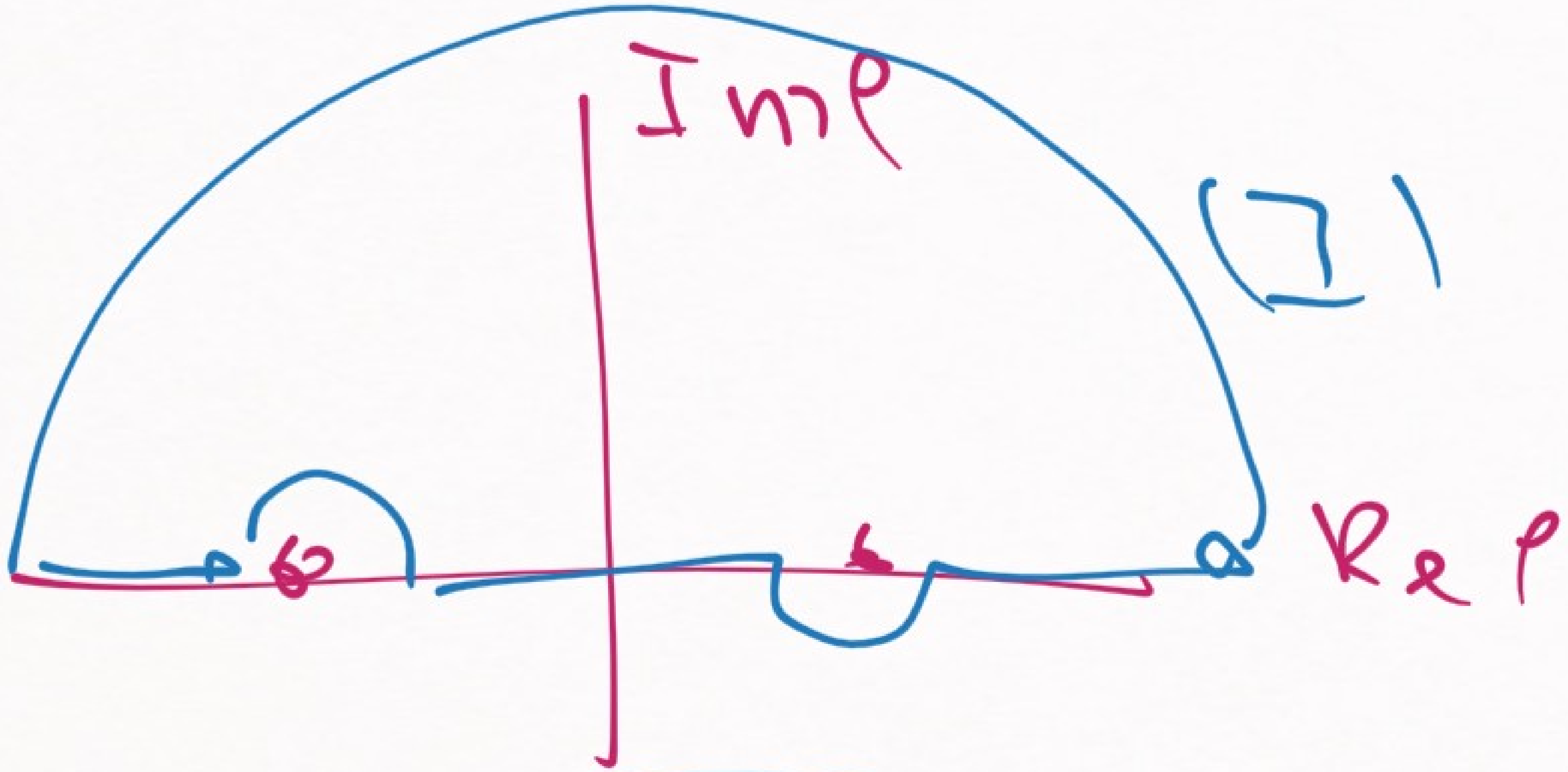
$$\left(\frac{1}{E-\frac{\ell^2}{2\mu}} \text{ diverges} \right)$$

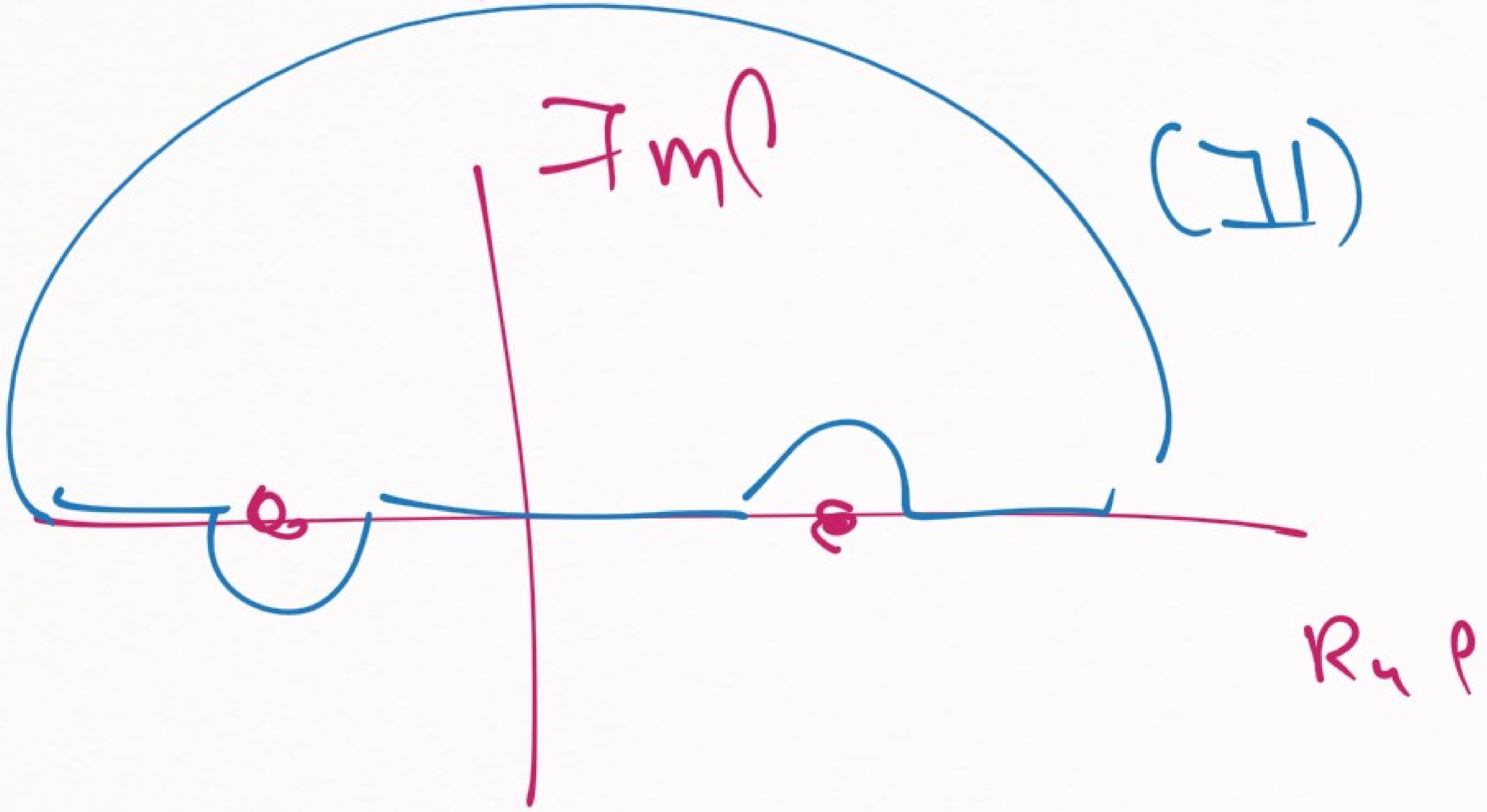
$$\int_{-\infty}^{\infty} \ell \, d\ell \, \frac{e^{i\ell r}}{E - \frac{\ell^2}{2\mu}} = \oint \ell \, d\ell \, \frac{e^{i\ell r}}{E - \frac{\ell^2}{2\mu}} \quad (= \text{sum of residues})$$

↓

depends on the contour







(I) & (II) are the two most interesting contours

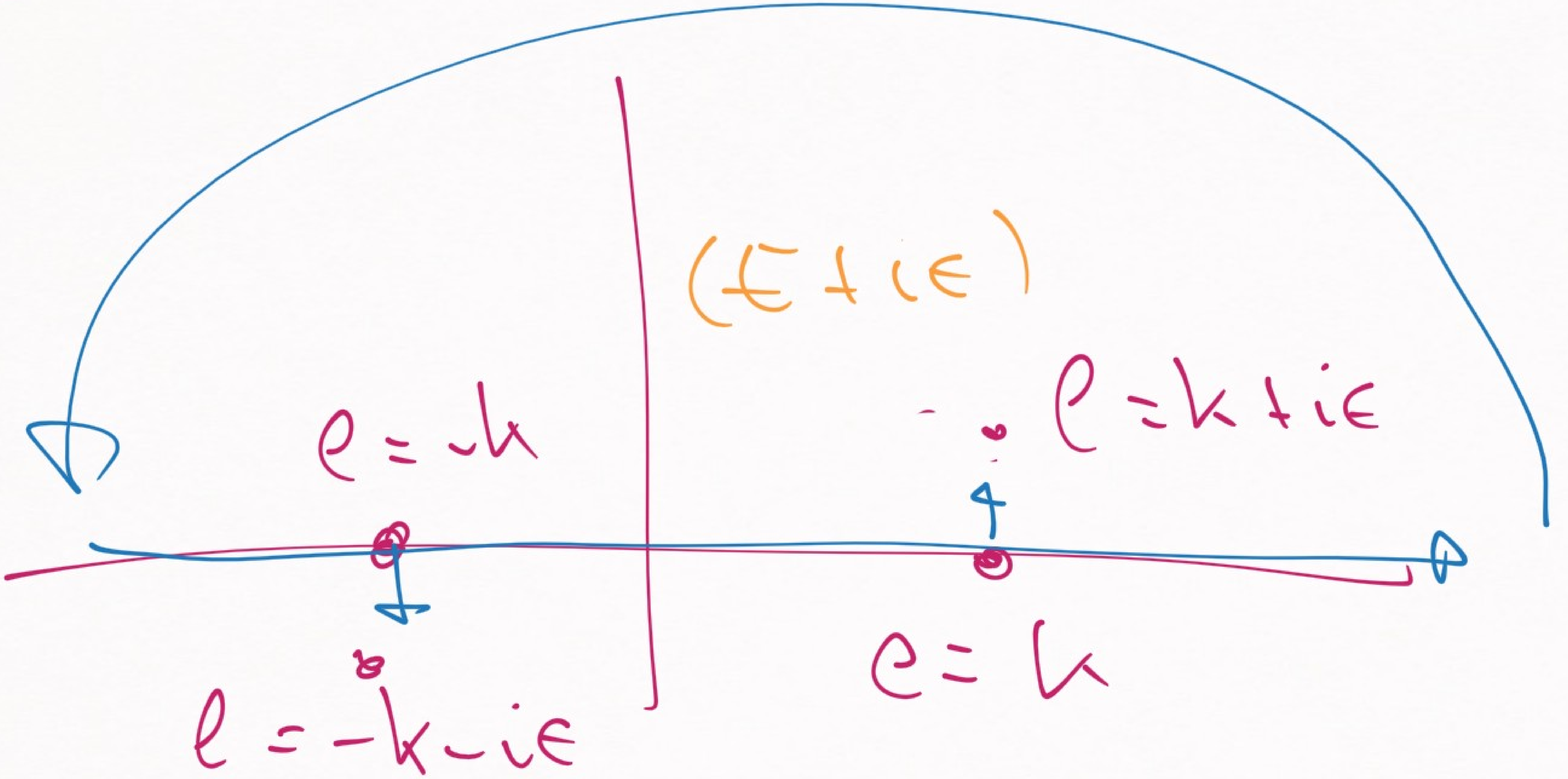
Contour I $\quad G_0^{(I)}(\vec{r}) = -\frac{\mu}{2\pi}\frac{e^{\boxed{+}ikr}}{r}$

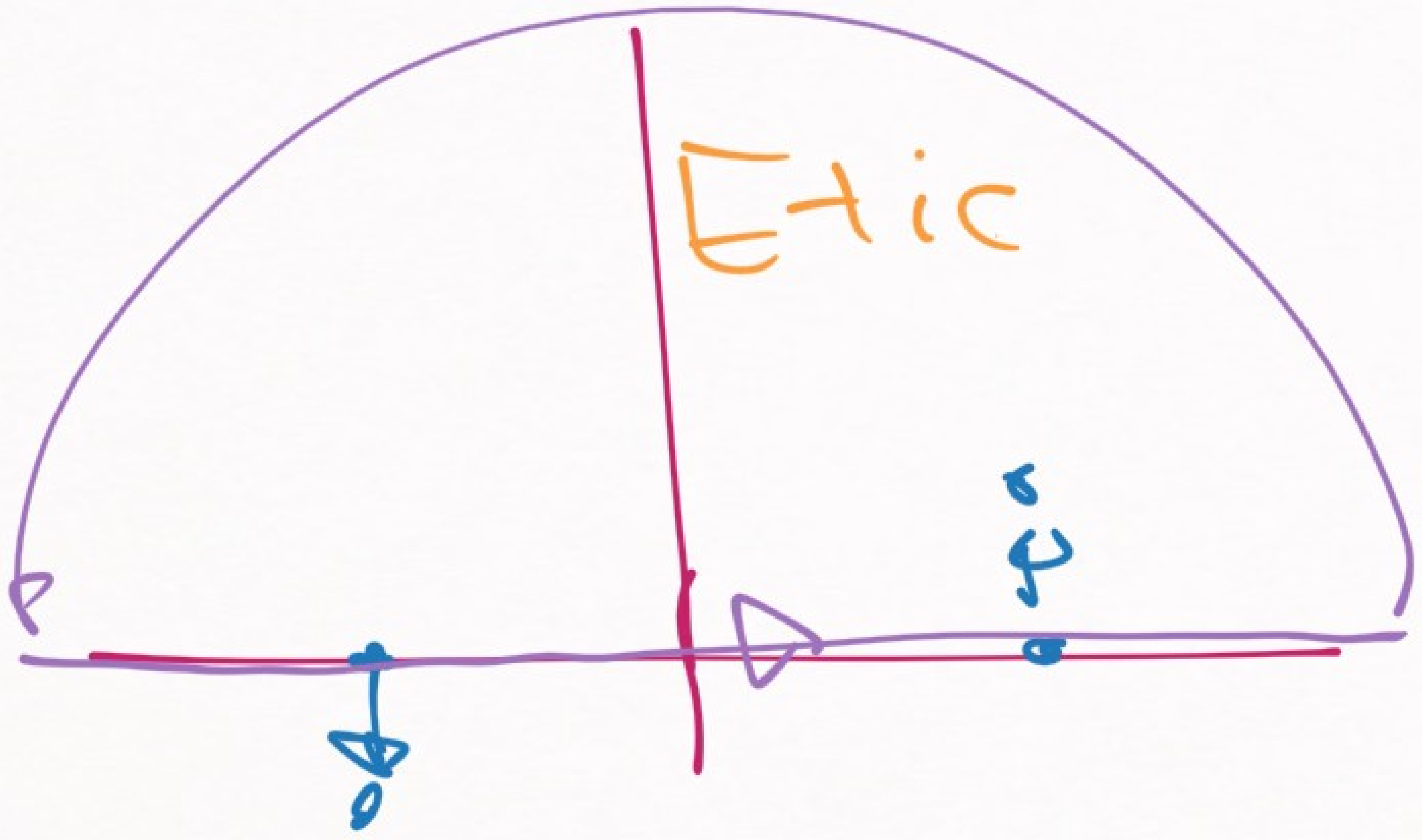
Contour II $\quad G_0^{(II)}(\vec{r}) = -\frac{\mu}{2\pi}\frac{e^{\boxed{-}ikr}}{r}$

↳ A trick here is that these two contours can be obtained simply by changing the energy at which $G_0(\hat{E})$ is evaluated

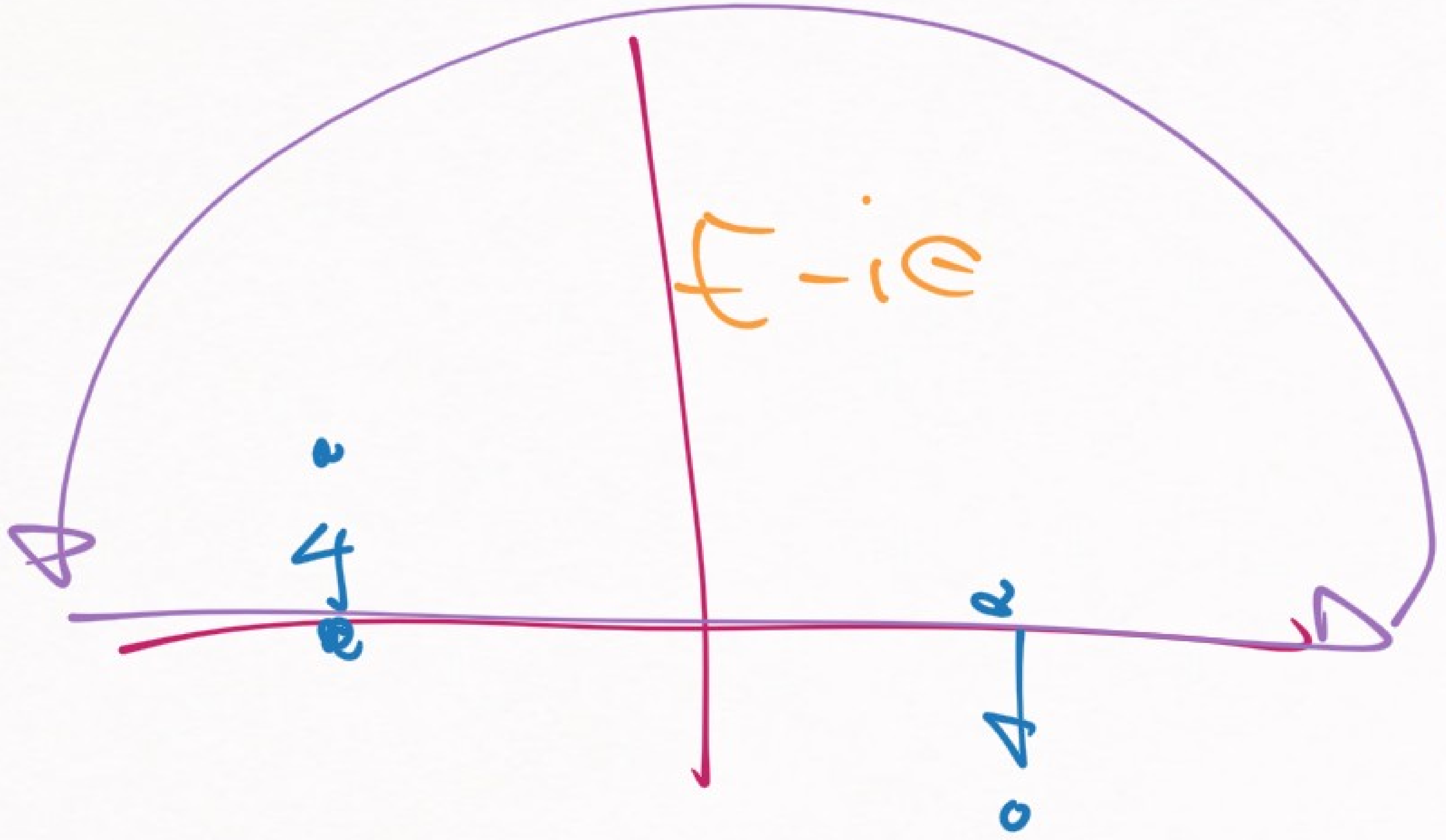
$$G_0(E) \to G_0(E \pm i\epsilon) \qquad (\epsilon \to 0^+ \; (\epsilon > 0))$$

$$\frac{1}{E - H_0} \qquad \frac{1}{E \pm i\epsilon - H_0}$$





$\underline{\sim}$ (I)



$\underline{\sim}$ (II)

RESULT OF ±iε PRESCRIPTION IS :

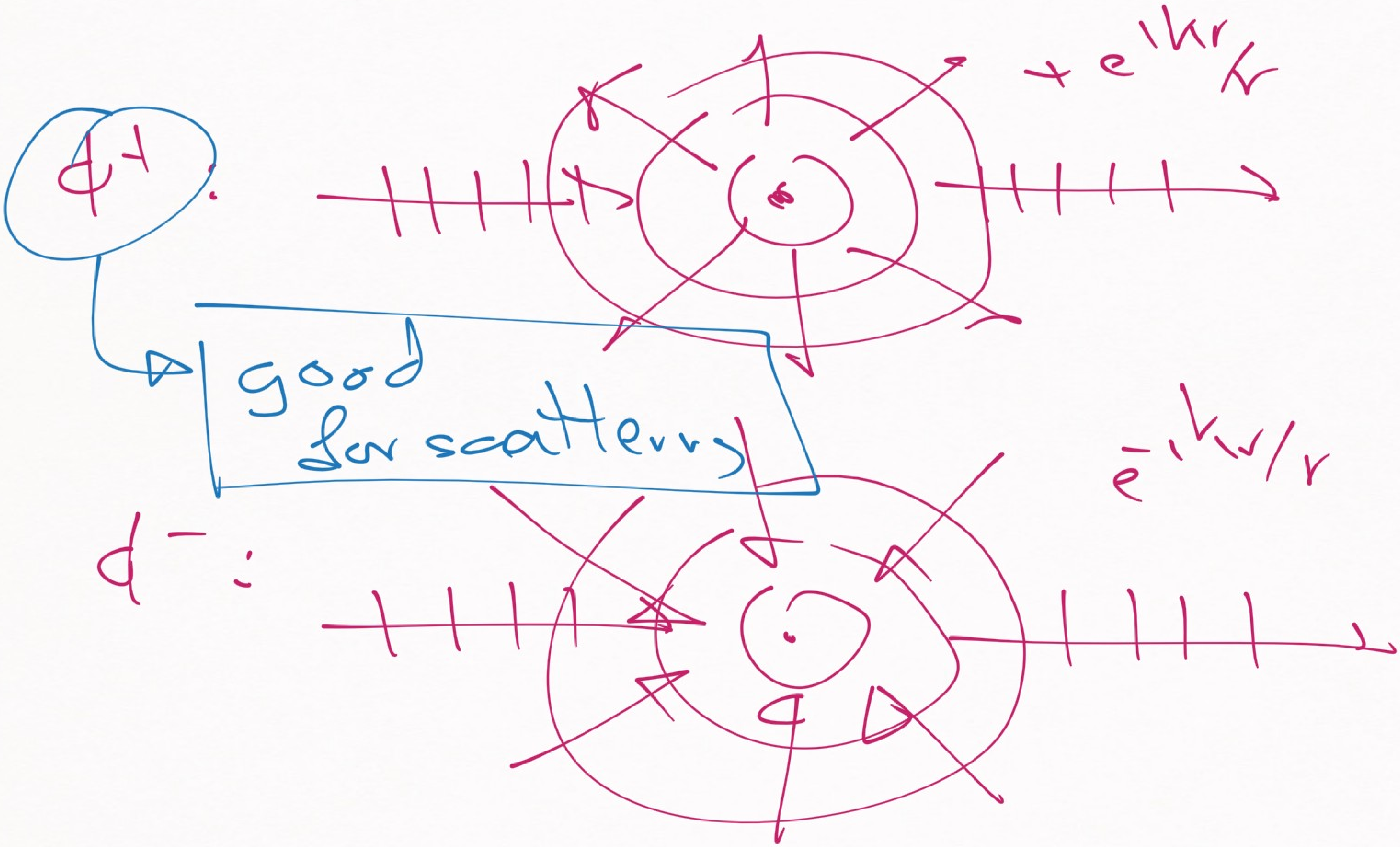
$$\boxed{G_0(\vec{r}; E \pm i\epsilon) = -\frac{\mu}{2\pi}\frac{e^{\pm ikr}}{r}}$$

(↳ A compact (use all over QM, QFT))

Physical interpretation:

$$\psi(\vec{r}) = e^{i\vec{k}\cdot\vec{r}} + \int d^3\vec{r}\, G_0(\vec{r}-\vec{r}')\, V(\vec{r}')\,\psi(\vec{r}')$$

$$\hookrightarrow \quad \psi^{\pm}(\vec{r}) \quad \left(\sim \underset{\checkmark}{e^{i\vec{k}\cdot\vec{r}}} + f(\Omega)\,\frac{e^{\pm ikr}}{r} \right)$$


d^+:
$+e^{ikr}/r$
good for scattering
d^-:
e^{-ikr}/r

CONTINUE NEXT
MONDAY