NUCLEAR PHYSICS 13

The two-nucleon system

→ Review of the two-body system

→ Review of scattering theory

Nuclear physics → Nuclei

↓

[Bound states of A nucleons]

$A = 2$ → Deuteron → easy to do

$A = 3$ → Triton, ^3He → doable

$A = 4$ → Alpha particle (^4He) → a bit more difficult

Nuclei $\rightarrow$ $A \uparrow$ $\Rightarrow$ they are more difficult to describe in terms of individual nucleons & their interactions

$\Rightarrow$

[$\exists$ nuclear models]

For understanding light nuclei

→ Ideally understand the A-body problem

$A = 2, 3$ → easy

→ today

$\boxed{A=2} \longrightarrow$ [review of the two-body problem]

1) Schrödinger equation:

$$\left[\frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_1} + V(\vec{r}_1 - \vec{r}_2)\right] \Psi(\vec{r}_1, \vec{r}_2)$$
$$= E_T \Psi(\bar{r}_1, \bar{r}_2)$$

$\longrightarrow$ Go to the center-of-mass coordinates

$$\vec{P} = \vec{p}_1 + \vec{p}_2 \qquad \text{(total momentum)}$$

$$\vec{p} = \frac{m_1 \vec{p}_2 - m_2 \vec{p}_1}{m_1 + m_2} \qquad \text{(relative momentum)}$$

$$\vec{R} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2} \qquad \text{($\vec{r}$ of the c.m.)}$$

$$\vec{r} = \vec{r}_2 - \vec{r}_1 \qquad \text{(relative $\vec{r}$)}$$

$$\rightarrow \left[\frac{\vec{p}^2}{2\mu} + \frac{\vec{P}^2}{2M} + V(\vec{r}) \right] \psi(\vec{r}, \vec{R}) = E_T \, \psi(\vec{r}, \vec{R})$$

$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2} \longrightarrow \text{reduced mass}$$

$$M = m_1 + m_2 \longrightarrow \text{total mass}$$

Remove the CM movement:

$$\Psi(\vec{r}, \vec{R}) = \psi(\vec{r})\,\psi_{CM}(\vec{R})$$

$$\psi_{CM}(\vec{R}) = e^{i\vec{K}\cdot\vec{R}}$$

$$E_T = E_{cm} + \frac{\vec{K}^2}{2M}$$

not particularly interesting

(~ free particle)

$\rightarrow \left[\frac{p^2}{2\mu} + V(\vec{r})\right]\psi(\vec{r}) = E_{cm}\psi(\vec{r}) \longrightarrow \circledast$

$\rightarrow$ What we usually solve when we are talking about two-body problems

—— $\circledast$ ——

$\circledast \rightarrow$ 3-dimensional diff. eq. (difficult)

Next trick → [Partial wave expansion]

Simplify/Separate the angular momentum of the system

good quantum number (if rotational symm)

$$\frac{\vec{p}^2}{2\mu} = -\frac{\vec{\nabla}^2}{2\mu} = -\frac{1}{2\mu}\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right) + \frac{\vec{L}^2}{2\mu}$$

$$\psi(\vec{r}) = \frac{u_\ell(r)}{r} Y_{\ell m}(\hat{r})$$

reduced wave function (u_ℓ)

spherical harmonics

$$\boxed{\text{RESULT}}$$

1-dim diff. eq. (easy)

$$\left[-u_\ell''(r) + \left[2\mu V(r) + \frac{\ell(\ell+1)}{r^2} \right] u_\ell(r) = \epsilon\, u_\ell(r) \right.$$

$$\left. \epsilon = 2\mu E_{cm} \right]$$

$$\epsilon = 2\mu E_{cm} \quad \begin{cases} E_{cm} > 0 \rightarrow E_{cm} = \dfrac{k^2}{2\mu} \\ E_{cm} < 0 \rightarrow E_{cm} = -\dfrac{\gamma^2}{2\mu} \end{cases}$$

1) $E_{cm} > 0$ $\quad -u_\ell'' + \left[2\mu V + \dfrac{\ell(\ell+1)}{r^2}\right] u_\ell = k^2 u_\ell$

2) $E_{cm} < 0$ $\quad -u_\ell'' + \left[2\mu V + \dfrac{\ell(\ell+1)}{r^2}\right] u_\ell = -\gamma^2 u_\ell$

(SIMPLIFY THE ORIGINAL PROBLEM)

Next → Review the solutions

I) Asymptotic solutions ($r \to \infty$)

One condition: finite-ranged potential

$$\lim_{r \to \infty} r^n V(r) \to 0$$

$(\forall\, n \geq 0)$

Example:

$$V(r) \sim \frac{e^{-mr}}{r^{\alpha}}$$

If $\lim_{r \to \infty} r^2 V(r) \to 0$, then:

1) $E < 0$, $r \to \infty \Rightarrow \left[-u_\ell'' + \frac{\ell(\ell+1)}{r^2} \right] u_\ell \simeq -\gamma^2 u_\ell$

$\longrightarrow \quad u_\ell(r) \to A_\ell \, e^{-\gamma r} \left[1 + \mathcal{O}\left(\frac{1}{\gamma r}, \frac{1}{mr} \right) \right]$

normalization factor

expected exponential decay

$\ell = 0 \to \boxed{u_s(r) \underset{r\to\infty}{\longrightarrow} A_s e^{-\gamma r}}$ (simple)

2) $E > 0$ & $r \to \infty$

$u_\ell(r) \to \underbrace{a_\ell(k)\,\hat{j}_\ell(kr) + b_\ell(k)\,\hat{y}_\ell(kr)}$ $\to$ coefficients

$$\left.\begin{aligned}\hat{j}_\ell(x) &= x\, j_\ell(x) \\ \hat{y}_\ell(x) &= x\, y_\ell(x)\end{aligned}\right\} \to j_\ell(x), y_\ell(x) \text{ spherical Bessel functions}$$

$$j_\ell(x) \xrightarrow[x\to\infty]{} \frac{1}{x}\sin\left(x - \ell\frac{\pi}{2}\right)$$

$$y_\ell(x) \xrightarrow[x\to\infty]{} -\frac{1}{x}\cos\left(x - \ell\frac{\pi}{2}\right)$$

$$\Rightarrow \quad u_\ell(r) \longrightarrow \underbrace{N_\ell}_{\text{normalization}} \sin\left(kr - \ell\frac{\pi}{2} + \delta_\ell(k)\right)$$

$\boxed{\text{phase shift}}$

very important for scattering

$$\ell = 0 \rightarrow \boxed{u_0(r) \rightarrow \sin(kr + \delta_0(k)) \, N_0}$$

3) $E \approx 0$ $\quad \delta_\ell(k) \rightarrow -a_\ell k^{2\ell+1} + O(k^{2\ell+3})$

$$\ell = 0 \rightarrow \delta_0(k) \rightarrow -a_0 k + O(k^3)$$

[Scattering length] → check previous lessons

$\rightarrow$ σ (cross-section) $\rightarrow$ $\boxed{\begin{array}{l}\sigma \longrightarrow 4\pi |a_0|^2 \\ k \to 0\end{array}}$

Wave function for

$E = 0, \ell = 0$;

$$\boxed{u_0(r) \longrightarrow 1 - \frac{r}{a_0}} \longrightarrow \text{Important formula}$$

Next thing → what happens w/ the wave functions near the origin?
($r \to 0$)

2) Behavior of wave functions for $r \to 0$

Regularity condition → $\boxed{u_\ell(0) = 0}$

→ $\langle \psi | \psi \rangle = \int dr \, |u_\ell(r)|^2 = 1$ (finite)
for $E < 0$

Regularity condition → depends on potential

Case a) Regular potential

$$\Downarrow$$

$$\boxed{\lim_{r \to 0} r^2 V(r) = 0} \to \text{good features}$$

why → $|V(r)| \ll \frac{\ell(\ell+1)}{r^2}$ (centrifugal barrier)

Reminder $\rightarrow$ Schrödinger equation is a second order differential equation

Two-linearly independent solutions

$u_\ell(r) \sim r^{\ell+1}$ ✓ , $u_\ell(r) \sim \frac{1}{r^\ell}$ ✗

Regularity condition $u_\ell(0) = 0 \Rightarrow \boxed{u_\ell(r) \sim r^{\ell+1}}$

Case b) Singular potential (most general type of potential)

$$\boxed{\lim_{r \to 0} r^2 V(r) \neq 0}$$

First we consider $\lim_{r \to 0} r^2 V(r) \to \infty$

Two-body problem → potential is well-known at long distances

$$V(r) = \sum_{n=n_0}^{\infty} a_n \frac{e^{-mr}}{r^n}$$

($n \geq 2$ → singular potential)

Case b.i) Repulsive singular potential

$$2\mu V(r) \xrightarrow[r \to 0]{} + \frac{a^{n-2}}{r^n} \quad (n > 2)$$

length scale

$$2\mu V(r) \gg \frac{l^2}{r^2} \quad (r \to 0)$$

General solution:

$$
\begin{aligned}
w(r) = {} & c_+ \left(\frac{r}{a}\right)^{n/4} \exp\left[+\frac{2}{n-1}\left(\frac{a}{r}\right)^{\frac{n-1}{2}}\right] \quad \text{①} \\
& + c_- \left(r/a\right)^{n/4} \exp\left[-\frac{2}{n-1}\left(\frac{a}{r}\right)^{\frac{n-1}{2}}\right] \quad \text{②}
\end{aligned}
$$

① $\to$ irregular at $r \to 0$ ($\to \infty$)

② $\to$ regular at $r \to 0$ ($\to 0$)

$\boxed{u_\ell(0) = 0} \Rightarrow \boxed{C_+ = 0}$

Repulsive singular potentials

→ similar to regular potentials

(in terms of solutions)

Case b.ii) Attractive singular potential

$$2\mu V(r) \to -\frac{a^{n-2}}{r^n}$$

$$u_\ell(r) = c_1 \left(\frac{r}{a}\right)^{n/4} \sin\left(\frac{2}{n-2}\left(\frac{a}{r}\right)^{\frac{n-2}{2}}\right) + c_2 \left(\frac{r}{a}\right)^{n/4} \cos\left(\frac{2}{n-2}\left(\frac{a}{r}\right)^{\frac{n-2}{2}}\right)$$

$\to$ $\exists$ a problem w/ this wf

$\boxed{u_\ell(0) = 0}$ $\longrightarrow$ the solution here is always regular

$\Rightarrow$

[UNABLE TO DETERMINE THE PHYSICAL SOLUTION]

→ Attractive singular interactions always give you a regular wave function

Physical meaning: these potential are incomplete ($\exists$ only 1 physical solution)

→ we need to supplement them w/ short-range physics

→ Singular potentials appear everywhere in EFT treatments

why? because they are long-range expansions

RECOMMENDED READING:

1. (108) **Singular potentials and limit cycles**
S.R. Beane, Paulo F. Bedaque, L. Childress, A. Kryjevski, J. McGuire (Washington U., Seattle), U. van Kolck (Arizona U. & RIKEN BNL & Caltech, Kellogg Lab). Oct 2000. 8 pp.
Published in **Phys.Rev. A64 (2001) 042103**
NT-UW-00-023, DOE-ER-41132-102-INT-00, RBRC-140, KRL-MAP-271
DOI: 10.1103/PhysRevA.64.042103
e-Print: **quant-ph/0010073** | **PDF**
References | BibTeX | LaTeX(US) | LaTeX(EU) | Harvmac | EndNote
ADS Abstract Service; OSTI.gov Server
Registro completo - Citado por 108 registros 100+

↳ you can make practical use of what we have learned in the previous slides

Case b.iii) Inverse square potential

$$\left[2\mu V(r) = \frac{g}{r^2}\right] \rightarrow \text{borderline case}$$

$$\left.\begin{aligned} \ell &= 0 \\ k &= 0 \end{aligned}\right] \rightarrow \boxed{-u_0'' + \frac{g}{r^2} u_0(r) = 0}$$

✓

easy to solve analytically

Invariant under $r \Rightarrow r' = \lambda r$ (dilations)

(or scale invariance)

$$-\frac{d^2}{dr^2} u_0(r) + \frac{g}{r^2} u_0(r) = 0$$

$$\Rightarrow -\frac{d^2}{dr'^2} \tilde{u}_0(r') + \frac{g}{r'^2} \tilde{u}_0(r') = 0$$

Previous lessons → scale invariance

$$\boxed{g > g_{crit}} \qquad g_{crit} = -\frac{1}{4}$$

$$\Rightarrow u_0(r) = c_+ r^{1/2+\nu} + c_- r^{1/2-\nu}$$

$$\nu = \nu(g)$$

$$\boxed{g < g_{crit}}$$

$$u_0(r) = r^{1/2} \sin(\alpha \log r + \varphi)$$

$$u_0(r) = r^{1/2}\sin(\alpha \log r + \varphi)$$

Discrete scale invariance

$$= r^{1/2}\sin\left(\alpha \log \frac{r}{R^*}\right)$$

Symmetry of the system

$$r \to \lambda r$$

$$\sin\left(\alpha \log \frac{r}{R^*} + \alpha \log \lambda\right)$$

if $\alpha \log \lambda = \pi \Rightarrow \boxed{\lambda_0 = e^{\pi/\alpha}} \to r \to \lambda_0 r_0$

for $1/r^2$ potential + $g < g_{crit}$

$\Rightarrow \exists \, \lambda_0 \ (\lambda_0 = e^{\pi/\alpha}) \, / \, r \to \lambda_0 r$ is a symmetry of the system

$\Downarrow$

$\equiv$ discrete scale invariance

Interesting two-body potential

→ $\boxed{\text{Delta-shell potential}}$

$$V(r;R_c) = \frac{g(R_c)}{4\pi R_c^2}\,\delta(r-R_c)$$

→ Extremely useful potential

Review $\boxed{\text{SCATTERING THEORY}}$

Classical setting:

Projectile

Target

Describe this

$\boxed{\text{Cross Section}}$

Effective area that a target offers to a projectile

Definition:

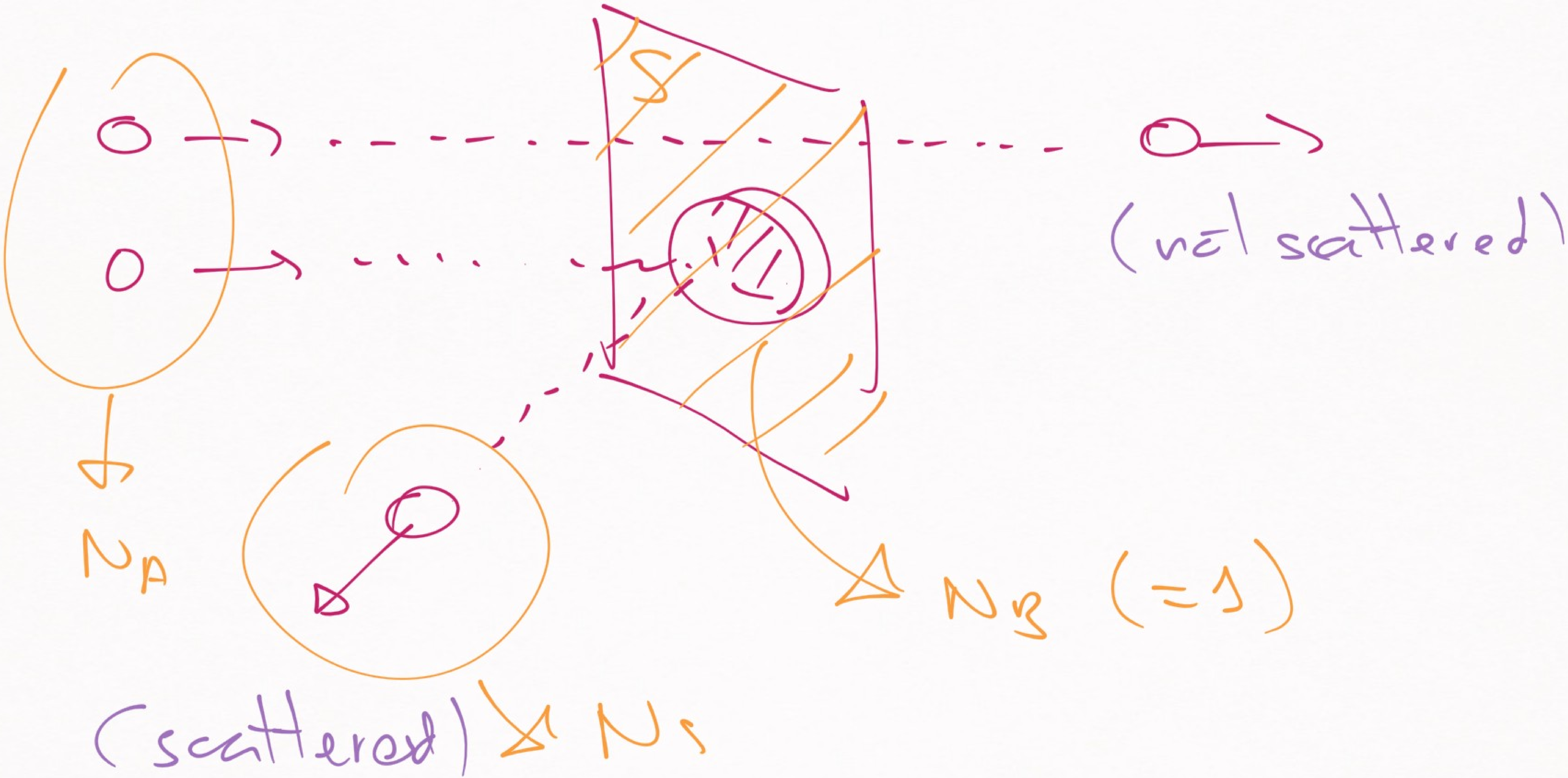
$$\sigma = \frac{N_s}{N_A\, N_B\, S}$$

scattered particles

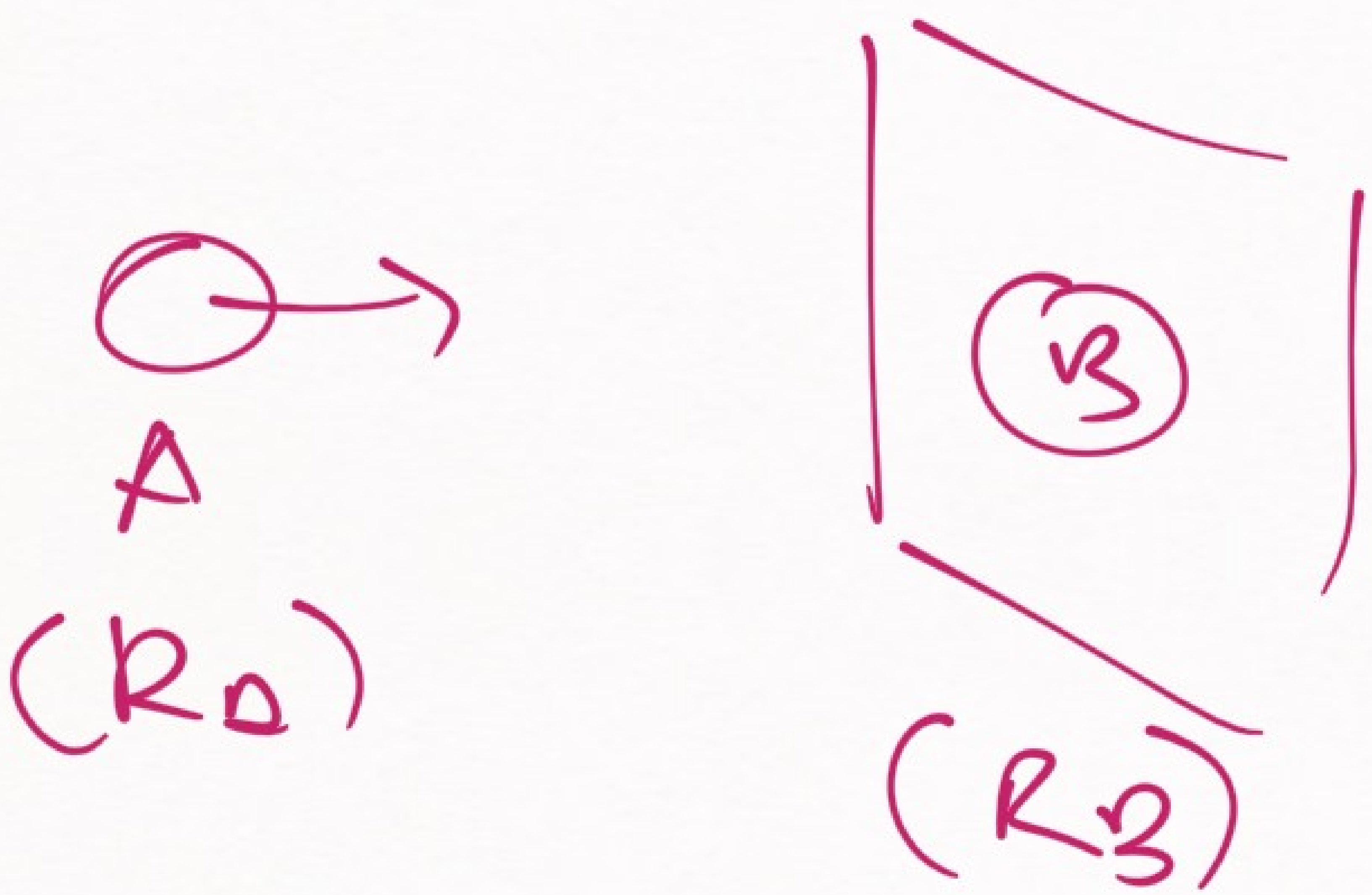
of projectiles

# of targets

area in which the projectile & the target can collide

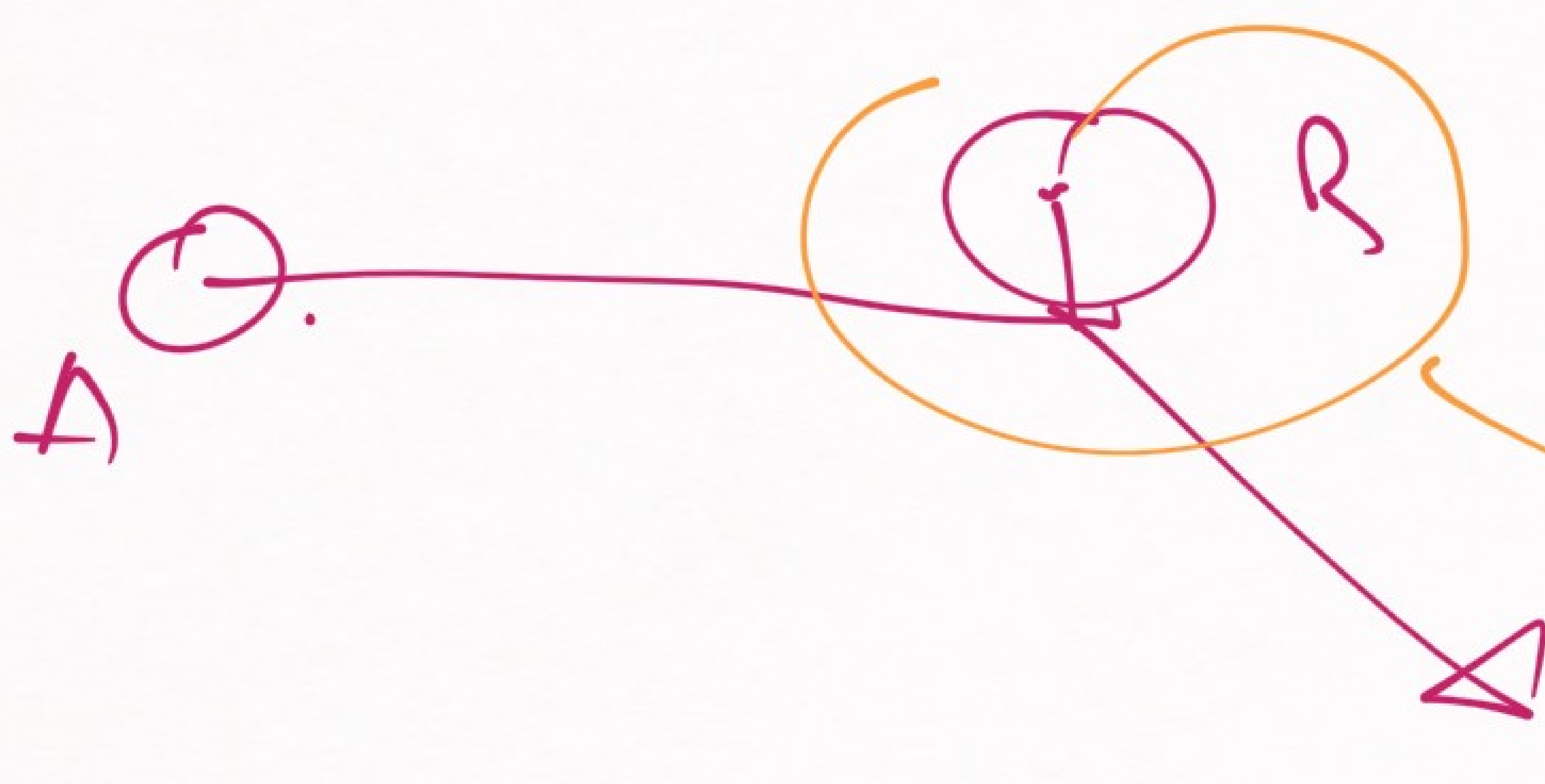

S
(not scattered)
N_A
N_B (=1)
(scattered)
N_s

EXAMPLE → HARD BALLS SCATTERING TOGETHER



n_A $(S_A = \pi R_A^2)$

n_B $(S_B = \pi R_B^2)$



$b < R_A + R_B$

$S_{eff} = \pi (R_A + R_B)^2$

$$N_s = N_A \times \left(N_B \frac{S_B^{eff}}{S} \right) \qquad \sigma = \frac{N_c}{N_A N_B} S$$

$$= \frac{N_A N_B}{S} \pi (R_A + R_B)^2$$

$$\frac{N_s}{N_A N_B} S = \boxed{\sigma = \pi (R_A + R_B)^2}$$

Conceptually simple for classical hard balls

$$\boxed{\sigma = S_B^{eff} = \pi (R_A + R_B)^2}$$

$\rightarrow$ THE QUANTUM MECHANICAL VERSION OF THIS

BREAK → OVER

Singular potential → $r \to 0$ (ultraviolet behavior)

Infrared singularities ($r \to \infty$)

Coulomb potential $V_c(r) \to \frac{\alpha}{r}$

Infrared singular potentials

$\rightarrow u_0(r) \not\rightarrow \sin(kr + \delta_0)$

(especial definitions of phase shifts)

$\rightarrow V(r) \rightarrow \frac{1}{r^n}$ $\quad \sigma \not\rightarrow 4\pi |a_0|^2$

$n < 3 \rightarrow \boxed{a_0 \text{ is not defined}}$

RECAP → $\boxed{\text{Cross section}}$

$$\sigma = \frac{N_S}{N_A N_B} S$$

↓

Cross section

(units of area)

N_A → # of incoming particles (beam/projectile/...)

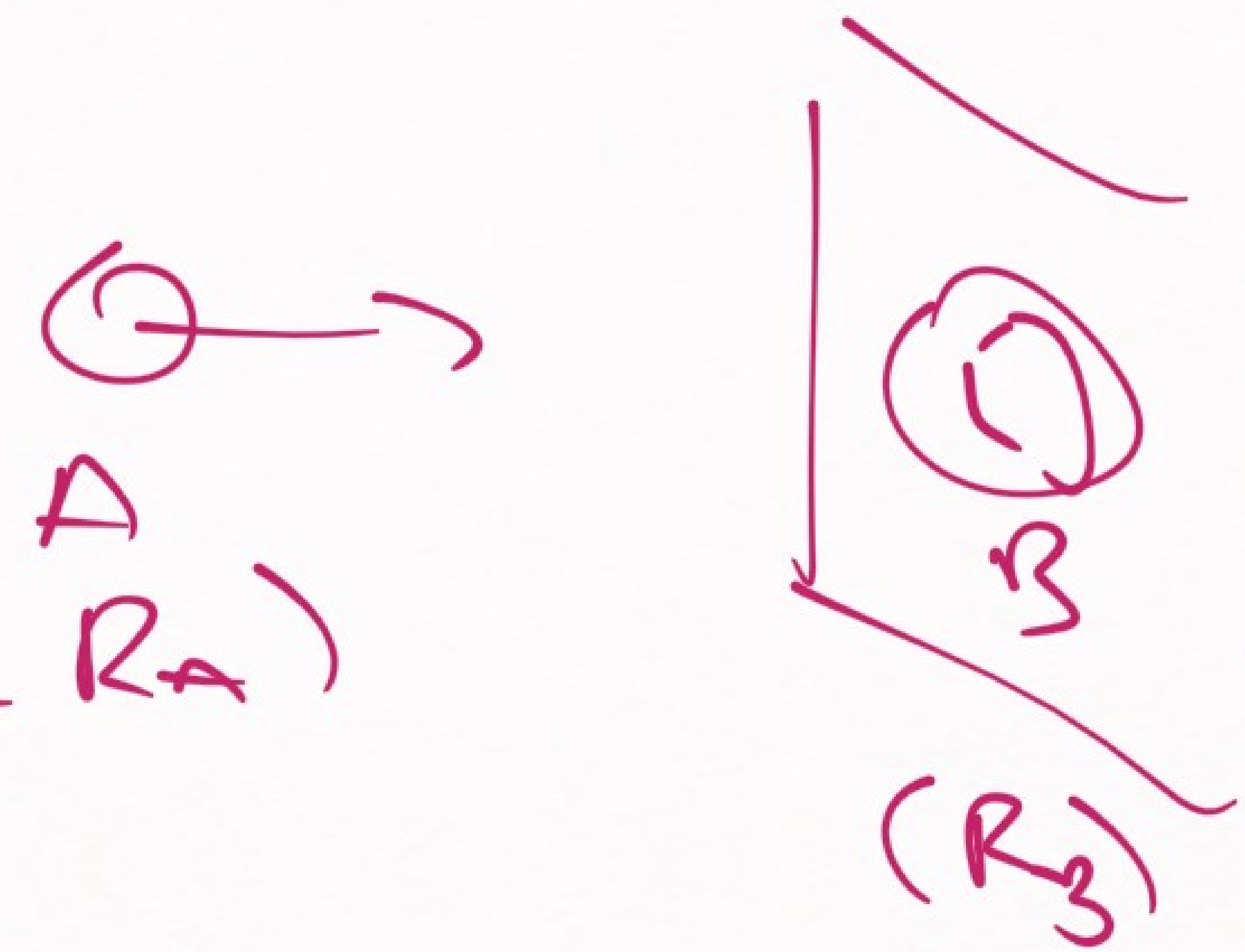
N_B → # of target particles

N_S → # of scattered particles

S → surface of beam/target

(Easiest example)

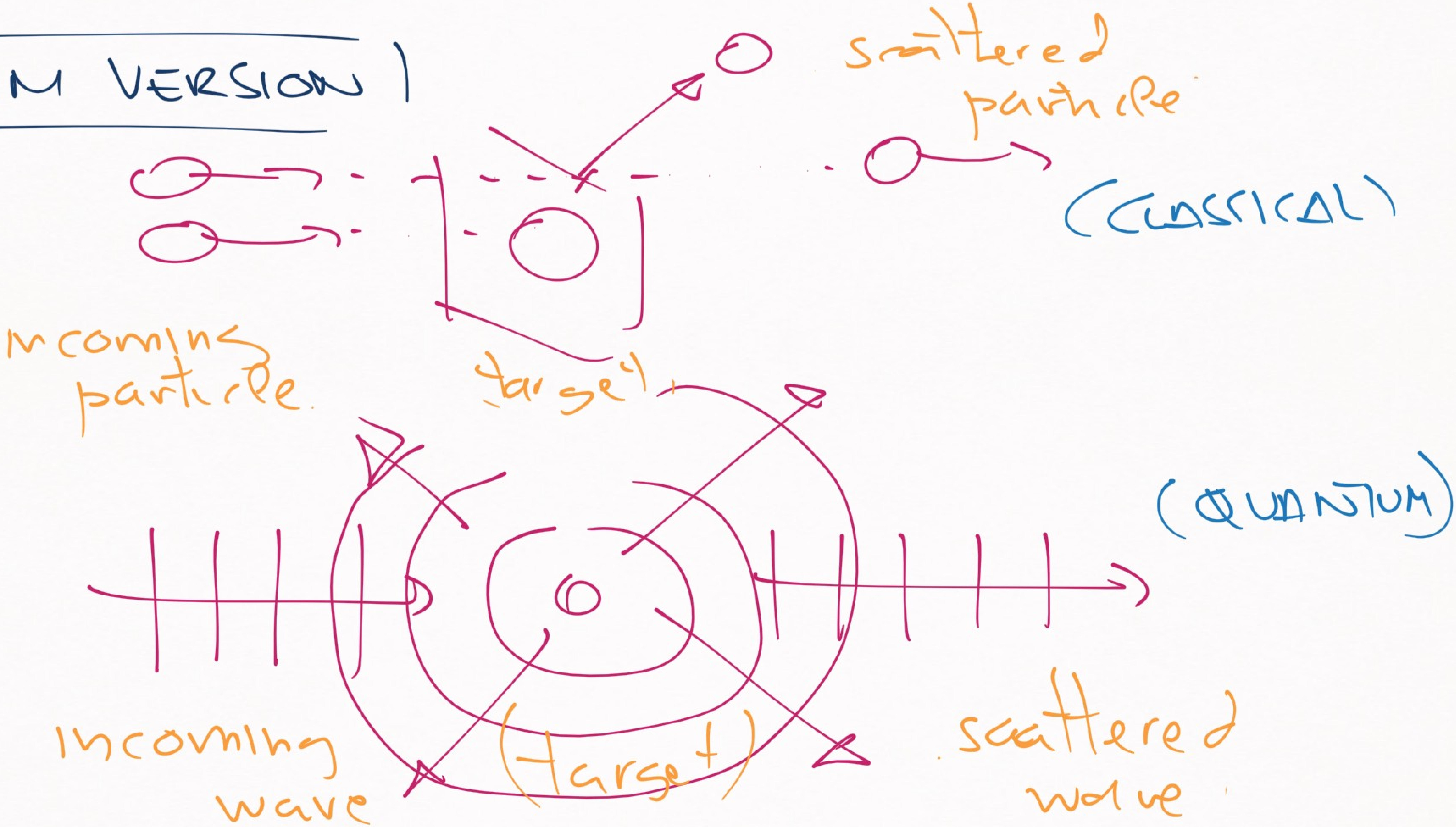
$$\boxed{\sigma = \pi (R_A + R_B)^2}$$



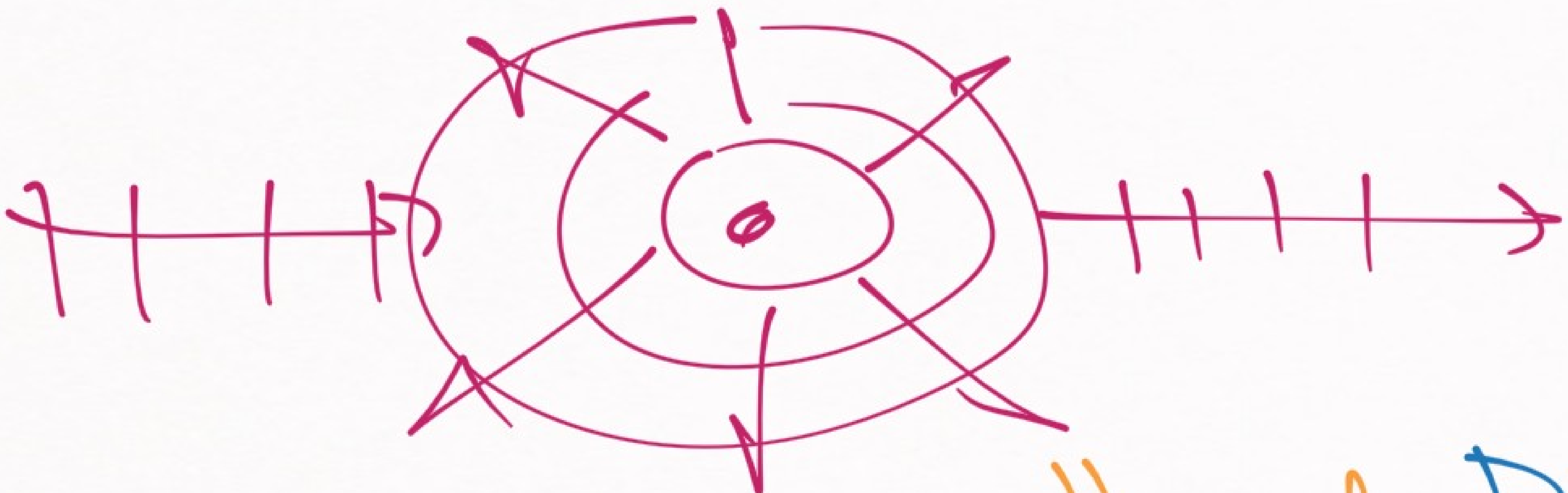
(Classical hard balls)

$\rightarrow$ What is the QM version of this?

QM VERSION



scattered particle
(CLASSICAL)
incoming particle
target
(QUANTUM)
incoming wave
(target)
scattered wave



incoming wave (plane wave)

scattered wave (spherical wave)

(QUANTUM)

↓

Described by a wave function (ψ)

$$\psi_{\vec{k}}(\vec{r}) \to \boxed{e^{i\vec{k}\cdot\vec{r}}} + \underline{f(\Omega)}\,\boxed{\frac{e^{ikr}}{r}}$$

angular dependence

Next step → traslate the classical terms into QM

$$\psi_{\vec{k}}(\vec{r}) \rightarrow e^{i\vec{k}\cdot\vec{r}} + f(\theta)\frac{e^{ikr}}{r}$$

$$\rightarrow \psi_{in}(\vec{r}) + \psi_{out}(\vec{r})$$

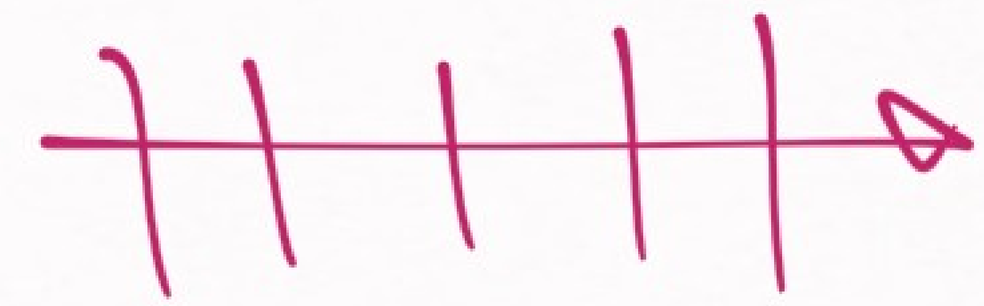
$$\boxed{\sigma = \frac{N_S}{N_A N_B} S}$$

$N_A N_B \sim \psi_{in}(\vec{r})$

$N_S S \sim \psi_{out}(\vec{r})$

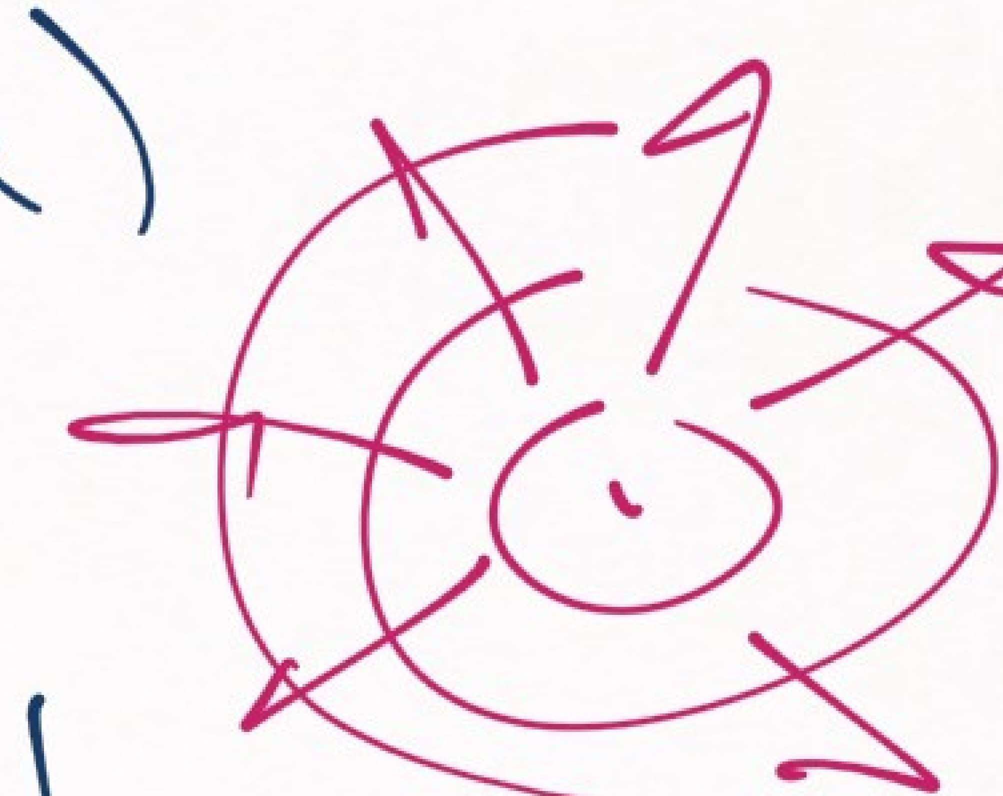
→ Same relation

Figure out the relation:



1) $N_B = 1$ (simplification)

$\rightarrow \frac{N_A}{T} \propto \Phi_{in}$ (incoming flux)



$\rightarrow S \frac{N_S}{T} \propto \int d\vec{S} \cdot \vec{\Phi}_{out}$ (outgoing flux)

(per unit time)

2) $j_{in} = |\vec{j}_{in}|$

$$\vec{j}_{in} = -\frac{i}{2m}\left[\psi^*_{in}\vec{\nabla}\psi_{in} - \psi_{in}\vec{\nabla}\psi^*_{in}\right]$$

$(\vec{\nabla}\cdot j_{in} = 0)$

$$\Rightarrow \boxed{j_{in} = \frac{k}{m}}$$

3) $$\vec{j}_{out} = -\frac{i}{2m}\left[\psi^*_{out}\,\hat{r}\cdot\vec{\nabla}\psi_{out} - \psi_{out}\,\hat{r}\cdot\vec{\nabla}\psi^*_{out}\right]$$

$$\Rightarrow \boxed{\vec{j}_{out} = \frac{k}{m}|f(\Omega)|^2\frac{\hat{r}}{R^2}}$$

4) $\lim\limits_{R\to\infty} \int d\vec{S}\cdot\vec{J}_{out} \quad \left(d\vec{S} = R^2\, d\Omega\, \hat{r} \right)$

$$\lim_{R\to\infty} \int d\vec{S}\cdot\vec{J}_{out} = \frac{k}{m}\int |f(\Omega)|^2 \underbrace{d\Omega}_{\text{solid angle}}$$

$$\left.\begin{aligned} N_A N_B &\to \frac{k}{m} \\ N_S S &\to \frac{k}{m}\int |f(\Omega)|^2 d\Omega \end{aligned}\right] \Rightarrow$$

$$\Rightarrow \boxed{\sigma = \int |f(\Omega)|^2 \, d\Omega}$$

(total cross section)

$$\psi(\vec{r}) \to e^{i\vec{k}\cdot\vec{r}} + f(\Omega)\frac{e^{ikr}}{r}$$

f

If we want angular dependence

$$\boxed{\frac{d\sigma}{d\Omega} = |f(\Omega)|^2}$$

differential cross section

WHAT DO WE HAVE?

1) TWO-BODY SYSTEM $\left[-\frac{\nabla^2}{2m}+V(\vec{r})\right]\psi_{\vec{k}}(\vec{r})=\frac{k^2}{2m}\psi_{\vec{k}}(\vec{r})$

2) SOLUTION OF THE TYPE

$$\psi_k(\vec{r}) \xrightarrow[r\to\infty]{} e^{i\vec{k}\cdot\vec{r}} + f(\Omega)\frac{e^{ikr}}{r}$$

3) (DIFFERENTIAL) CROSS SECTION

$$\boxed{\frac{d\sigma}{d\Omega}=|f(\Omega)|^2}$$

→ [PARTIAL WAVE EXPANSION]

We can connect $\frac{d\sigma}{d\Omega}$ w/ $\psi(\vec{r}) = \sum_{\ell m} \frac{u_\ell(r)}{r} Y_{\ell m}(\hat{r})$

→ good because $u_\ell(r)$ is easier to calculate

$$\sum$$

[PARTIAL WAVE EXPANSION]

→ Exploit the rotational symmetry of a problem to make calculations easier

Σ

$$\psi(\vec{r}) = \sum_{\ell m} \psi_\ell(r) Y_{\ell m}(\hat{r}) \quad , \psi_\ell(r) = \frac{u_\ell(r)}{r}$$

$\rightarrow e^{i\vec{k}\cdot\vec{r}}$ we should also take into account $\vec{k}$

$$e^{i\vec{k}\cdot\vec{r}} = 4\pi \sum_{\ell m} i^\ell j_\ell(kr) Y^*_{\ell m}(\hat{k}) Y_{\ell m}(\hat{r})$$

$$= \sum_\ell (2\ell+1) i^\ell j_\ell(kr) P_\ell(\hat{k}\cdot\hat{r})$$

Legendre polynomials

$$\int_{-1}^{+1} dx\, P_n(x) P_m(x) = \frac{\delta_{nm}}{2n+1} \quad (\text{check it})$$

$\rightarrow$ Expand $f(\Omega)$

$$\underbrace{f(\Omega)}_{\text{scattering amplitude}} = \sum_{\ell} (2\ell+1)\, f_\ell(k)\, P_\ell(\hat{k}\cdot\hat{r})$$

→ Connect all the expansions

$$\psi_{\vec{k}}(\vec{r}) = 4\pi \sum_{lm} i^l \frac{u_l(r,k)}{r} Y^*_{lm}(\hat{k}) Y_{lm}(\hat{r})$$

$$= \sum_l (2l+1) i^l \frac{u_l(r)}{r} P_l(\hat{k}\cdot\hat{r})$$

We can plug in the asymptotic $(r\to\infty)$ form

$$\frac{u_\ell(r)}{r} \to e^{i\delta_\ell}\left[\cos\delta_\ell(k)\, j_\ell(kr) - \sin\delta_\ell(k)\, y_\ell(kr)\right]$$

Check previous slide
(+ choice of normalization)

$\longrightarrow$ Some elaboration $\Rightarrow$

$$\left[f_\ell(k) = \frac{e^{i\delta_\ell}\sin\delta_\ell}{k} = \frac{1}{k\cot\delta_\ell - ik} \right]$$

$$\sigma = \int |f(\Omega)|^2 d\Omega = \frac{4\pi}{k^2}\sum_\ell \sin^2\delta_\ell$$

$$\sigma \xrightarrow[k\to 0]{} 4\pi|a_0|^2$$

$$\left(\delta_\ell \to -a_\ell k^{2\ell+1} + O(k^{2\ell+3}) \right)$$

We finish for today

(See you on next thursday)