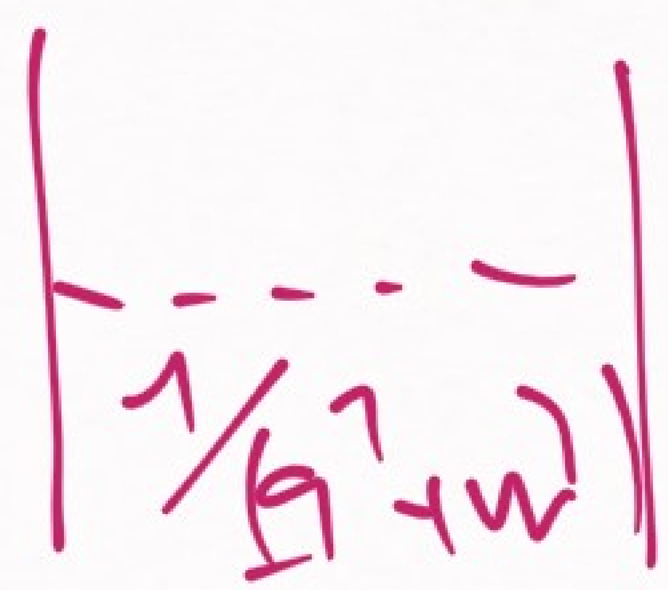
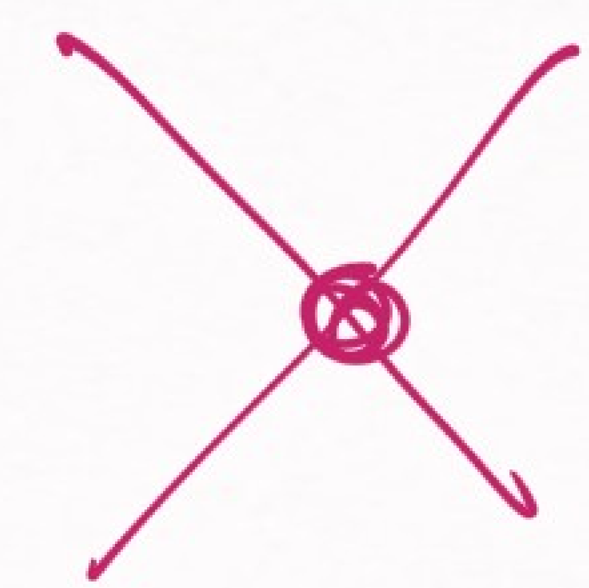
NUCLEAR PHYSICS 9

RENORMALIZATION

RECAP

(1) Renormalization: "Physics at long-distances is independent of short-distance details"

Example $\rightarrow$ 

$\longrightarrow$  $|q| \ll m$

$V(\vec{q})$ / $|q| \simeq m$

(we see the form of the potential)

$V(q) = C_0 + C_2 \vec{q}^{\,2} + C_4 \vec{q}^{\,4} + \dots$

Low-energy/long-distances

↳ We can't see the details of $V(\vec{q})$

$$V(\vec{q}) = C_0 + C_2 \vec{q}^{\,2} + C_4 (\vec{q}^{\,2})^2 + \dots$$

[Teacup & Teapot theory]

$$T - T_{ext} = (T_0 - T_{ext})\, e^{-\lambda(t-t_0)}$$

$$\lambda = S(c_0 + c_1 x + c_2 x^2 + \dots)$$

$$x = \frac{\Sigma}{\delta S}$$

$$\delta \gg 1$$

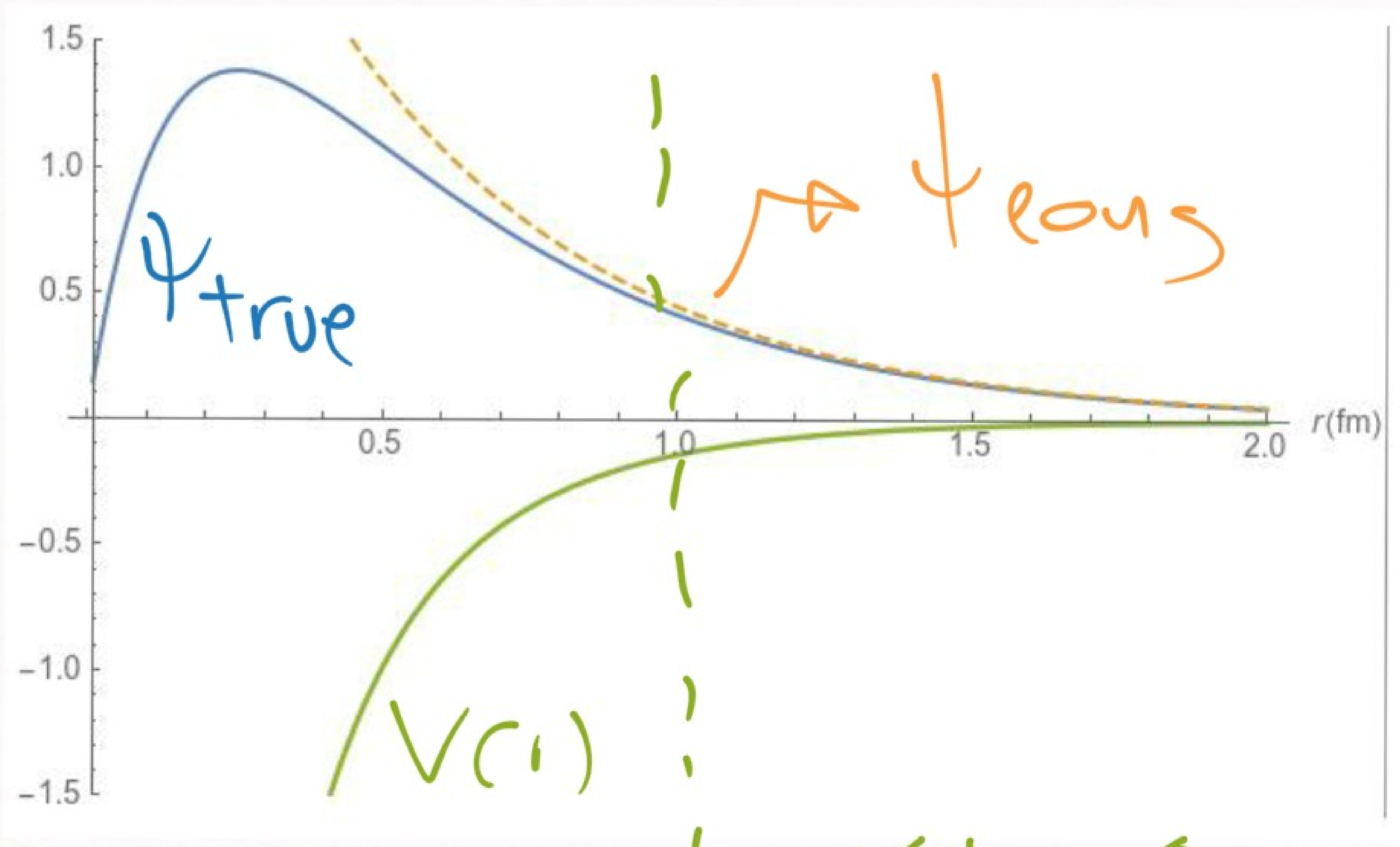
- Insensitivity to short-distance details

→ $V(\vec{q}) = C_0 + C_2|\vec{q}|^2 + \dots$

→ Wave function

$$mr \gg 1 \longrightarrow \underbrace{\left[\psi(r) \to \frac{A_S}{\sqrt{4\pi}}\frac{e^{-\gamma r}}{r}\right]}_{\text{no details of the wave function}}$$

A_S a number



$V(r) \to 0$

$V(r) \neq 0$

$\psi_{true} \to \psi_{long}$

$$\psi_{long} = \frac{A_S}{\sqrt{4\pi}} \frac{e^{-\gamma r}}{r}$$

→ We don't have to worry about short distance details

[Renormalization] $\longrightarrow$ We understand it at the conceptual level

$\neq$

Implementation problematic though

$\boxed{\text{PROBLEM}}$

$V(\vec{q}) \longrightarrow C_0 \Rightarrow V_{eff}(\vec{q}) = C_0$

$|\vec{q}| \ll m$

$V_{eff}(\vec{r}) = C_0 \, \delta^{(3)}(\vec{r})$

$$\delta^{(3)}(\vec{r}) = \begin{cases} 0 \text{ for } |\vec{r}| \neq 0 \\ \infty \text{ for } |\vec{r}| = 0 \end{cases} \quad / \int d^3\vec{r}\, \delta^{(3)}(\vec{r}) = 1$$

(Weird potential)

→ Difficult to solve Schrödingr

→ If attractive / $\nexists$ fundamental state

(the system collapses)

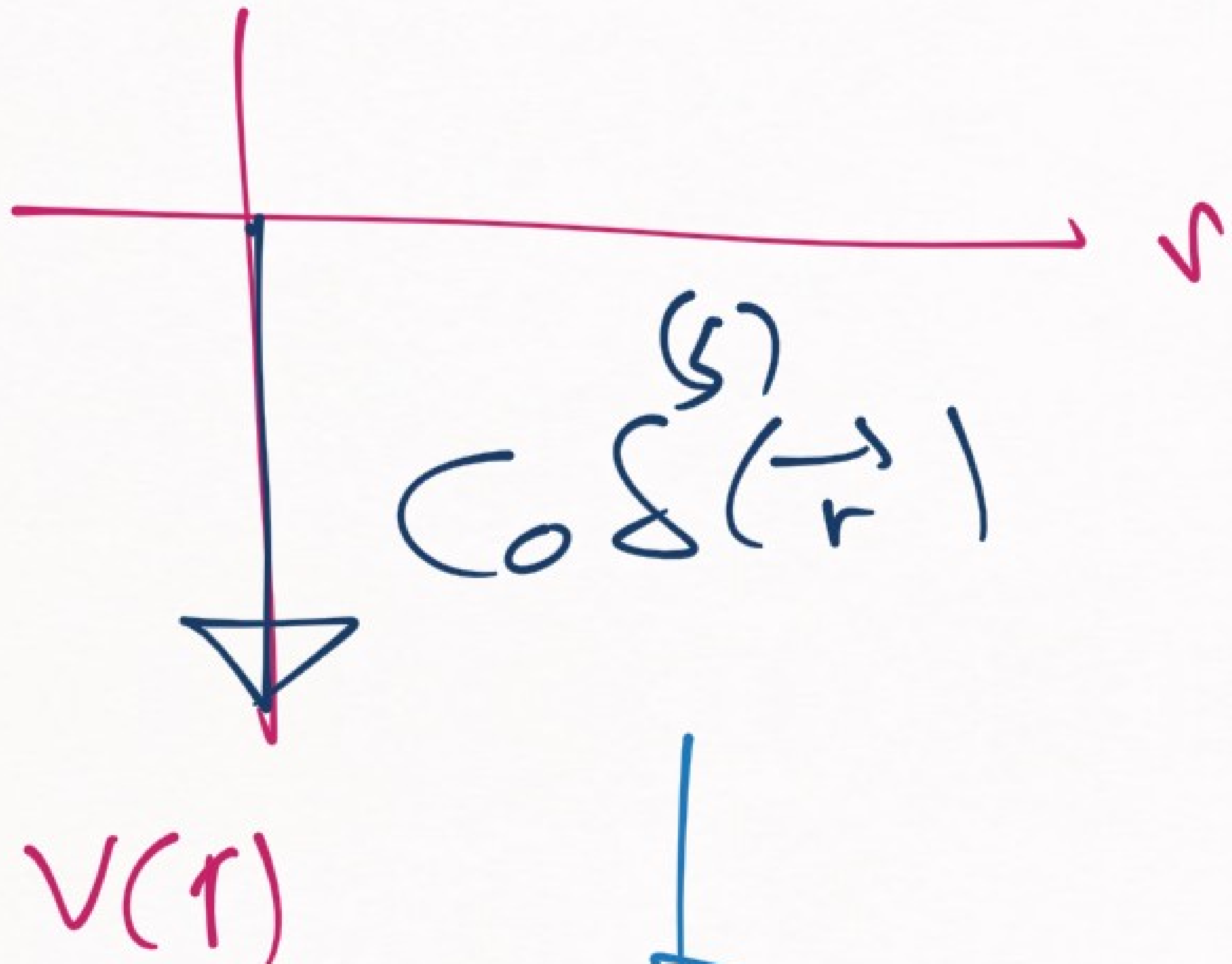
$\boxed{\text{Collapse}}$ → Variational principle

$$\left.\begin{aligned} V_{eff}(\vec{r}) &= C_0\, \delta^{(3)}(\vec{r}) \\ C_0 &< 0 \end{aligned}\right\} \Rightarrow \quad E_{ground} \leq \langle \psi_{trial} | H | \psi_{trial} \rangle$$

(ψ_{trial} → an arbitrary wave function)

$$\langle \psi_{trial} | V | \psi_{trial} \rangle = C_0 |\psi_{trial}(0)|^2$$

$\Rightarrow$ grow arbitrarily



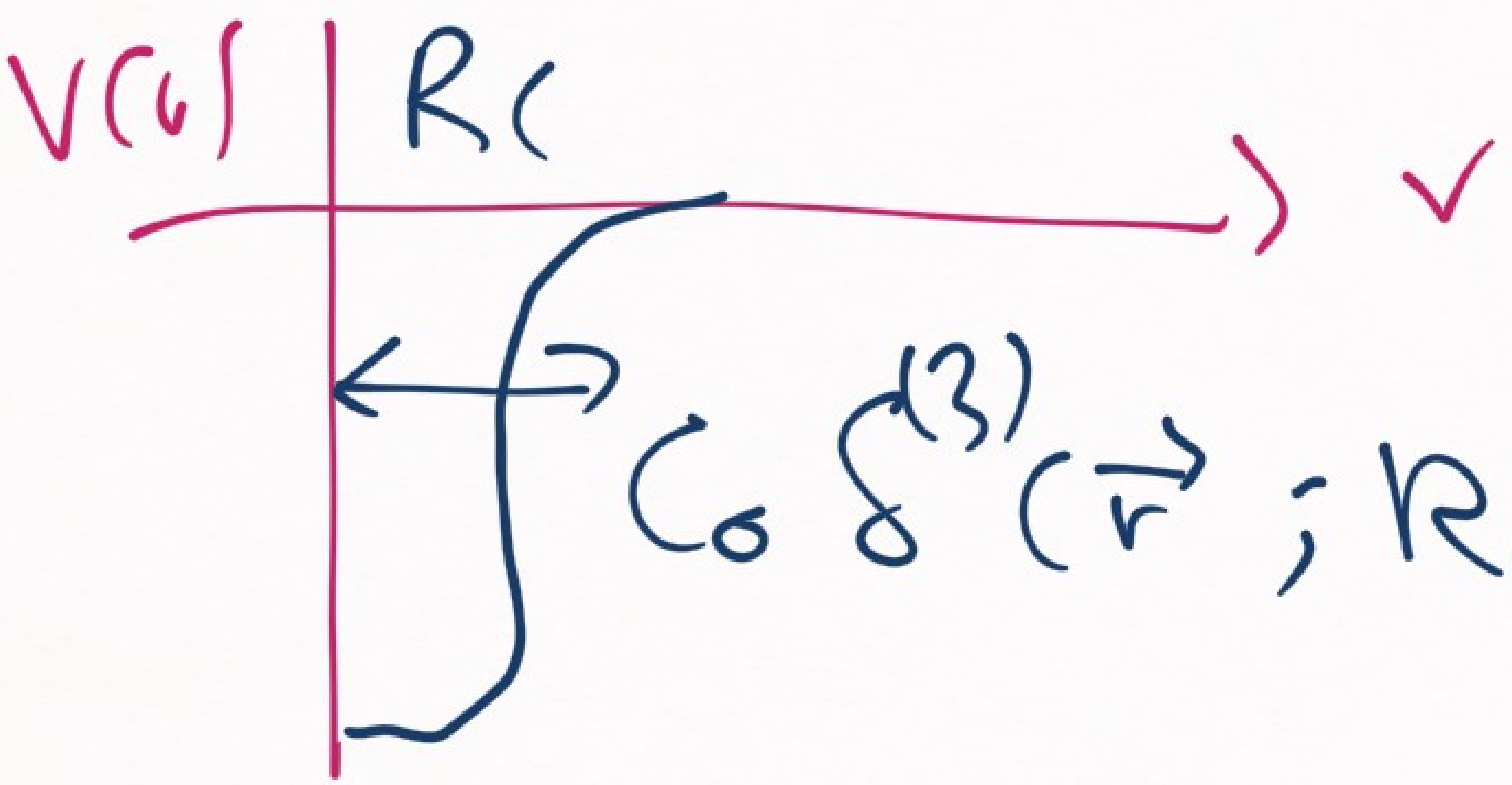
PROBLEM → this potential has some pathologies

↓

SOLUTION

↳ remove them

[REGULARIZATION]



$C_0\delta^{(3)}(\vec{r}\,; R_c)$ → soften the δ

smearing the δ

REGULARIZATION

1) Include a cutoff $\left[\begin{array}{ll} r\text{-space} & R_c \\ p\text{-space} & \Lambda \end{array}\right.$

2) Introduce a regulator $\left[\begin{array}{l} f(r/R_c) \\ f(|\vec{p}|/\Lambda) \end{array}\right.$

3) When $\Lambda \to \infty$ ($R_c \to 0$) we recover the original unregularized result

EXAMPLES OF REGULARIZATION:

1) $V(\vec{q}) = C_0 \longrightarrow V(\vec{q};\Lambda) = C_0\, f\left(\frac{|\vec{q}|}{\Lambda}\right)$

$$= C_0\, e^{-(|\vec{q}|/\Lambda)^2}$$

(Gaussian)

FT

$$V(\vec{r};\Lambda) = C_0 \frac{\Lambda^3}{8\pi^{3/2}} e^{-\frac{1}{4}(\Lambda r)^2} = C_0 \frac{e^{-(r/R_c)^2}}{R_c^3\, \pi^{3/2}}$$

$(\Lambda R_c = 2)$

2) $V(\vec{q};\Lambda) = C_0 \dfrac{\Lambda^2}{\Lambda^2 + \vec{q}^{\,2}}$ (Lorentzian regulator)

3) $V(\vec{q};R_c) = C_0 \dfrac{\delta(r-R_c)}{4\pi R_c^2}$ (delta-shell)

$\longrightarrow$ [Infinite possibilities to REGULARIZE]

$\boxed{\text{PROBLEM}}$ → ⊛ Cutoff is not a physical quantity

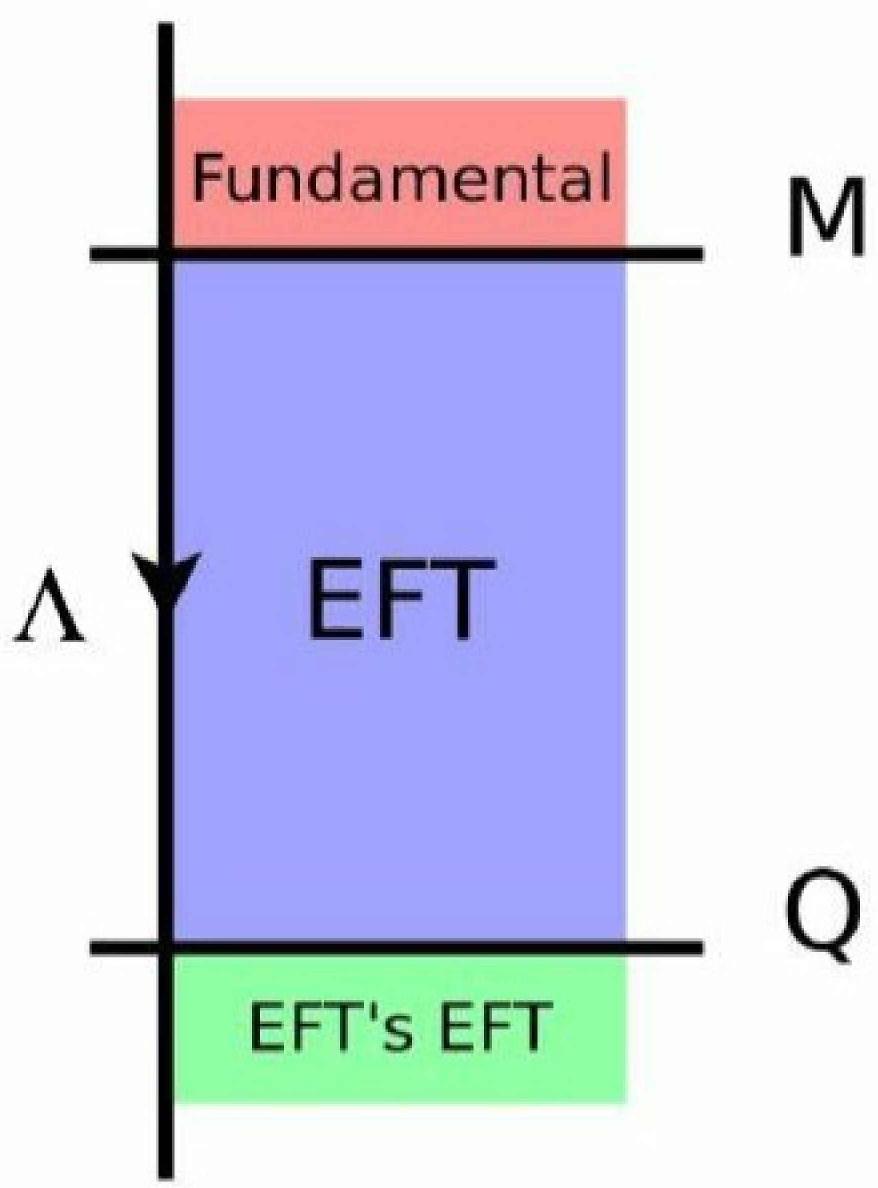
→ Cutoff is a property of our theories

(Cutoff is psycology)

$\boxed{\text{SOLUTION}}$ → ⊛ Physics must be independent of the cutoff

SOLUTION → Cutoff independence

(aka RENORMALIZATION)



Physics is unique, but choice of theory depends on resolution Λ:

- $\Lambda \geq M$: Fundamental
- $M \geq \Lambda \geq Q$: EFT

For equivalent descriptions:

$$\boxed{\frac{d}{d\Lambda}\langle \Psi | \mathcal{O} | \Psi \rangle = 0}$$

Renormalization group invariance

$\boxed{\text{Observable quantities should be cutoff independent}}$

[HOW TO RENORMALIZE]

$$\underbrace{V_c(\vec{q}) = C_0}_{\text{singular}} \xrightarrow{\text{Regularize}} \underbrace{V_c(\vec{q}) = C_0 f\left(|\vec{q}|/\Lambda\right)}_{\Lambda\text{-dependent}}$$

Renormalize

$$V(q) = \boxed{C_0(\Lambda)} f\left(\frac{|\vec{q}|}{\Lambda}\right)$$

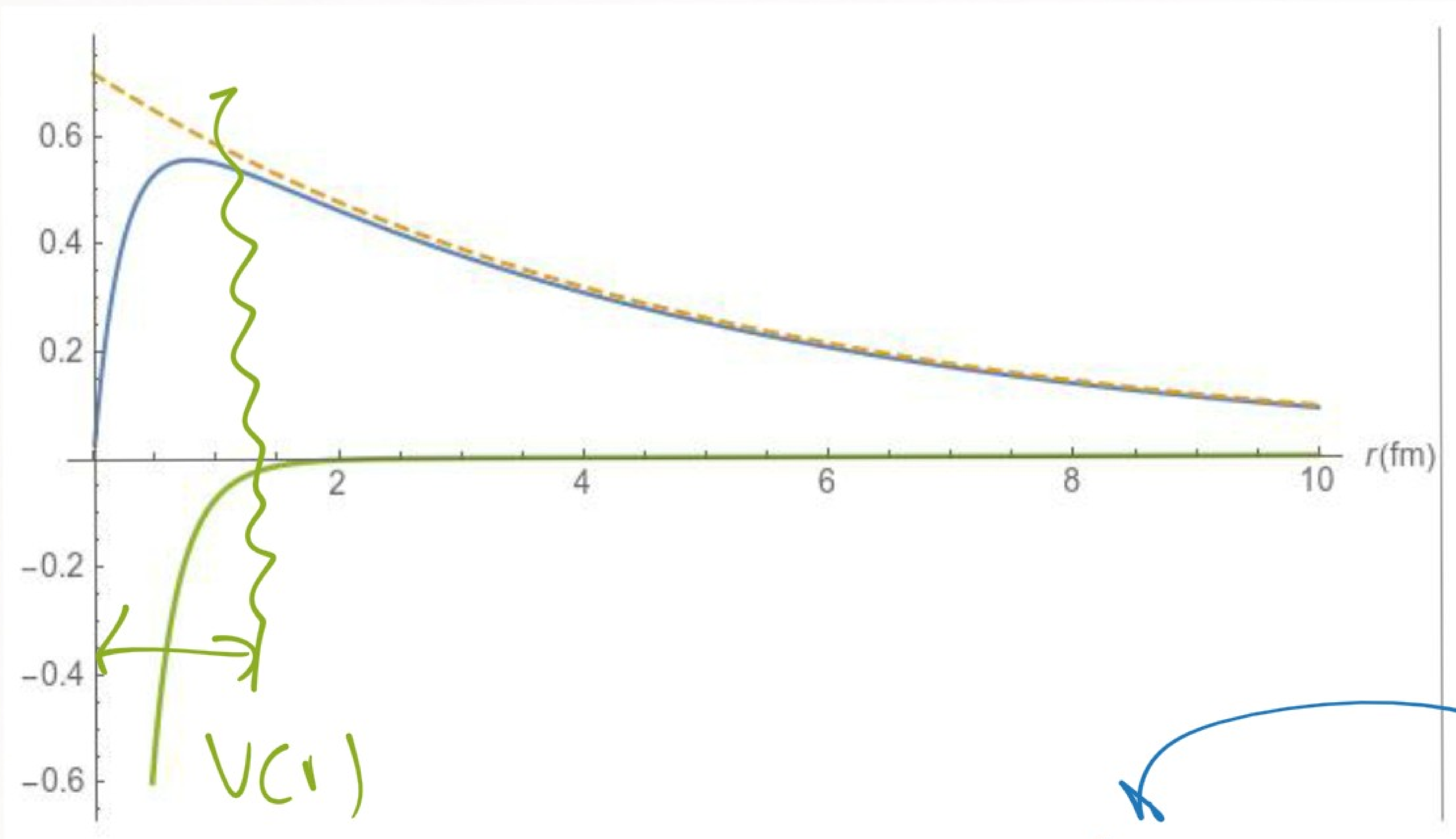
/ Observables independent of Λ

[CLARIFICATION] → Singular potential does not have a unique solution

$$\underbrace{V(\vec{r}) = C_0\, \delta^{(3)}(\vec{r})}$$

→ We can only find solutions by regularizations

RENORMALIZATION (Example)



$$V_Y(r) = -\frac{g^2}{4\pi}\frac{e^{-mr}}{r}$$

$$m \simeq 500\,\text{MeV}$$

$$\frac{g^2}{4\pi}/\gamma \simeq 0.075\,m$$

$(\gamma \ll m)$

wave function larger than $V(r)$

Previous wave function → [Contact interaction
ok ✓]

$$V_c(\vec{r}) = C_0 \underbrace{\delta^{(3)}(\vec{r})}_{\text{contact interaction}}$$

($V_c \neq 0$ for $r \neq 0$ ⇒ particles have to have direct contact w/each other)

$$V_c(\vec{q}) = C_0 \longrightarrow V_c(\vec{q};\Lambda) = C_0(\Lambda)\, e^{-\left(\frac{q^2}{\Lambda^2}\right)}$$

Regolarize
+
Renormalize

w/ $C_0(\Lambda)$ such that

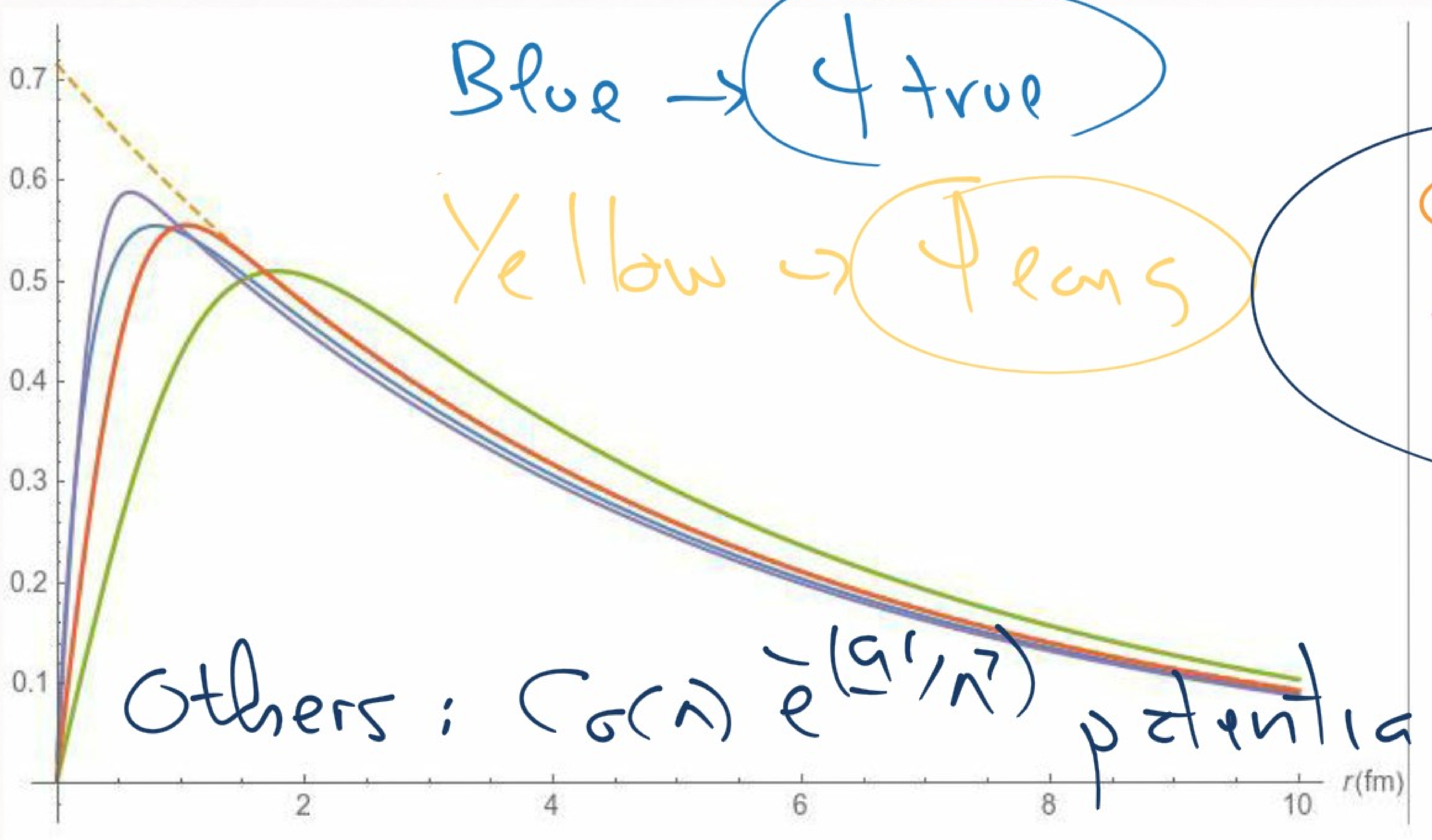
$$\gamma = 0.675\, m$$

[1) choose ρ, 2) choose Λ, 3) determine $C_0(\Lambda)$]

RESULTS OF RENORMALIZING THIS SYSTEM

Blue → ψ true

Yellow → ψ lens



Others : $C_\sigma(r)\, e^{-(g'/r^2)}$ potential

$\simeq$ not that good

Green : $\Lambda = m/2$

Orange : $\Lambda = m$

Violet : $\Lambda \simeq 2m$

better

$$G_0(\Lambda)\, e^{-(\vec{s}^2/\Lambda^2)} \longleftrightarrow -\frac{g^2}{4\pi}\frac{e^{-mr}}{r}$$

(Effective V) (true V)

Similar results (especially if $\Lambda \gtrsim m$)

→ It doesn't matter if we use V_{eff} or V_{true} (provided $\Lambda \gtrsim m$)

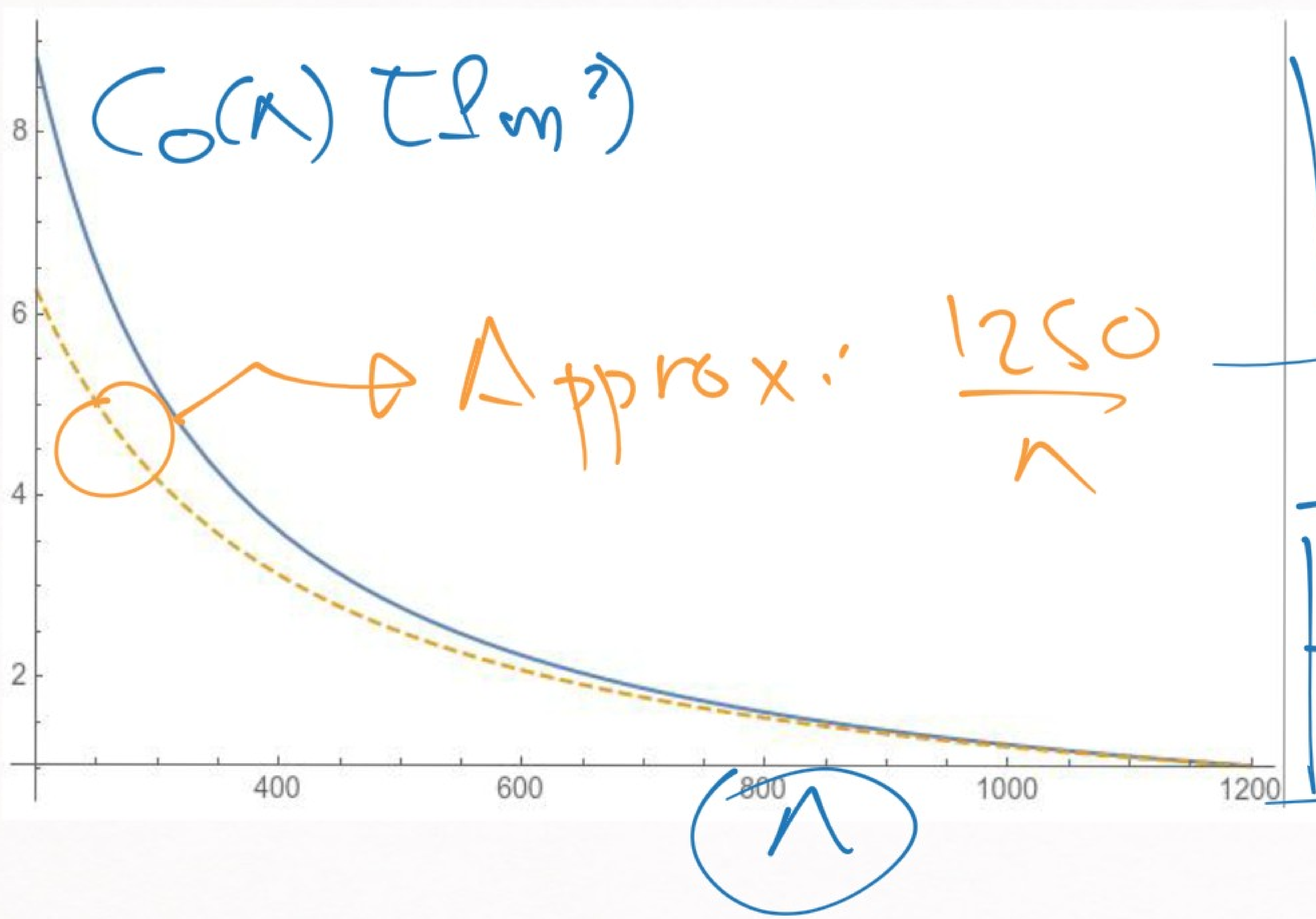
→ "Physics at long-distances does not depend on short-distance details"

If $r \ll m \Rightarrow$ it does not matter which is the true potential at $mr \sim 1$

$$\boxed{\text{RENORMALIZATION}}$$

$C_0 = C_0(\Lambda) \longrightarrow$ by reproducing γ (E_B)

$(E_B = -\gamma^2/2\mu)$



$$\boxed{C_0(\Lambda) \propto \frac{1}{\Lambda}}$$

$\Downarrow$

$$\boxed{\frac{d}{d\Lambda}\left[\Lambda C_0(\Lambda)\right] \simeq 0}$$

$$\frac{d}{d\Lambda}\left[\Lambda C_0(\Lambda)\right] \simeq 0$$ → Example of a renormalization group equation (RGE)

Approx.

$$\boxed{\frac{d}{d\Lambda}\langle\psi|H|\psi\rangle = 0}$$ → Binding energy is independent of the cutoff

RECAP $\longrightarrow$ The EFT / Renormalization Algorithm

1) Consider some system

1.a) "True potential" not necessary

1.b) Only need to know the scales
(size of bound state, range of interaction, etc)

2) Consider an effective interaction

$$V_c(\vec{r}) = C_0 \delta^{(3)}(\vec{r}) \quad , \quad V_c(\vec{q}) = C_0$$

3) Regularize & Renormalize

3.a) Regularize $V_c(\vec{q}) \Rightarrow V_c(\vec{q};\Lambda) = C_0(\Lambda)\, e^{-q^2/\Lambda^2}$

(or any other regulator)

3.b) Renormalize for each Λ, you determine $C_0(\Lambda)$ from some observable quantity

4) Check that the description is correct & makes sense (if good scale separation, this should happen)

→ this is the EFT algorithm

EFT with NDA: the algorithm

1. Identify the relevant degrees of freedom
2. Identify high- and low-energy scales→expansion parameters x
3. Identify symmetries of low-energy theory
4. Choose the accuracy required. This, together with the size of x, tells you the order, n, to which you must calculate.
5. Write down all possible local operators, that have naive dimensions up that order, and are consistent with symmetries
   "NDA"
6. Derive the behaviour of loops, and calculate them.

All operators needed for renormalization at this order should be present →Model independence

(NDA: naive dimensional analysis)

→ Talk from D.R. Phillips

LO, NLO, N^2LO, ...

→ advanced stuff

3. **A Renormalization group treatment of two-body scattering**
(196) Michael C. Birse, Judith A. McGovern, Keith G. Richardson (Manchester U.). Jul 1998. 4 pp.
Published in **Phys.Lett. B464 (1999) 169-176**
MC-TH-98-11
DOI: 10.1016/S0370-2693(99)00991-0
e-Print: **hep-ph/9807302** | **PDF**

References | BibTeX | LaTeX(US) | LaTeX(EU) | Harvmac | EndNote
ADS Abstract Service

Detailed record - Cited by 196 records 100+

Abstract, but worth giving it a try

A $\boxed{\text{RGE}}$

1. **Building light nuclei from neutrons, protons, and pions**
(45) Daniel R. Phillips (Ohio U.). Mar 2002. 54 pp.
Published in **Czech.J.Phys. 52 (2002) B49**
DOI: 10.1007/s10582-002-0079-z
To appear in the proceedings of Conference: C01-07-09.13 Proceedings
e-Print: **nucl-th/0203040** | **PDF**

References | BibTeX | LaTeX(US) | LaTeX(EU) | Harvmac | EndNote
ADS Abstract Service

Detailed record - Cited by 45 records

a lot of EFT example

We will try to understand RENORMALIZATION further ...

Feynman 2

- Therefore psychologically we must keep all the theories in our heads, and every theoretical physicist who is any good knows six or seven different theoretical representations for exactly the same physics.
  - chapter 7, “Seeking New Laws,” p. 168

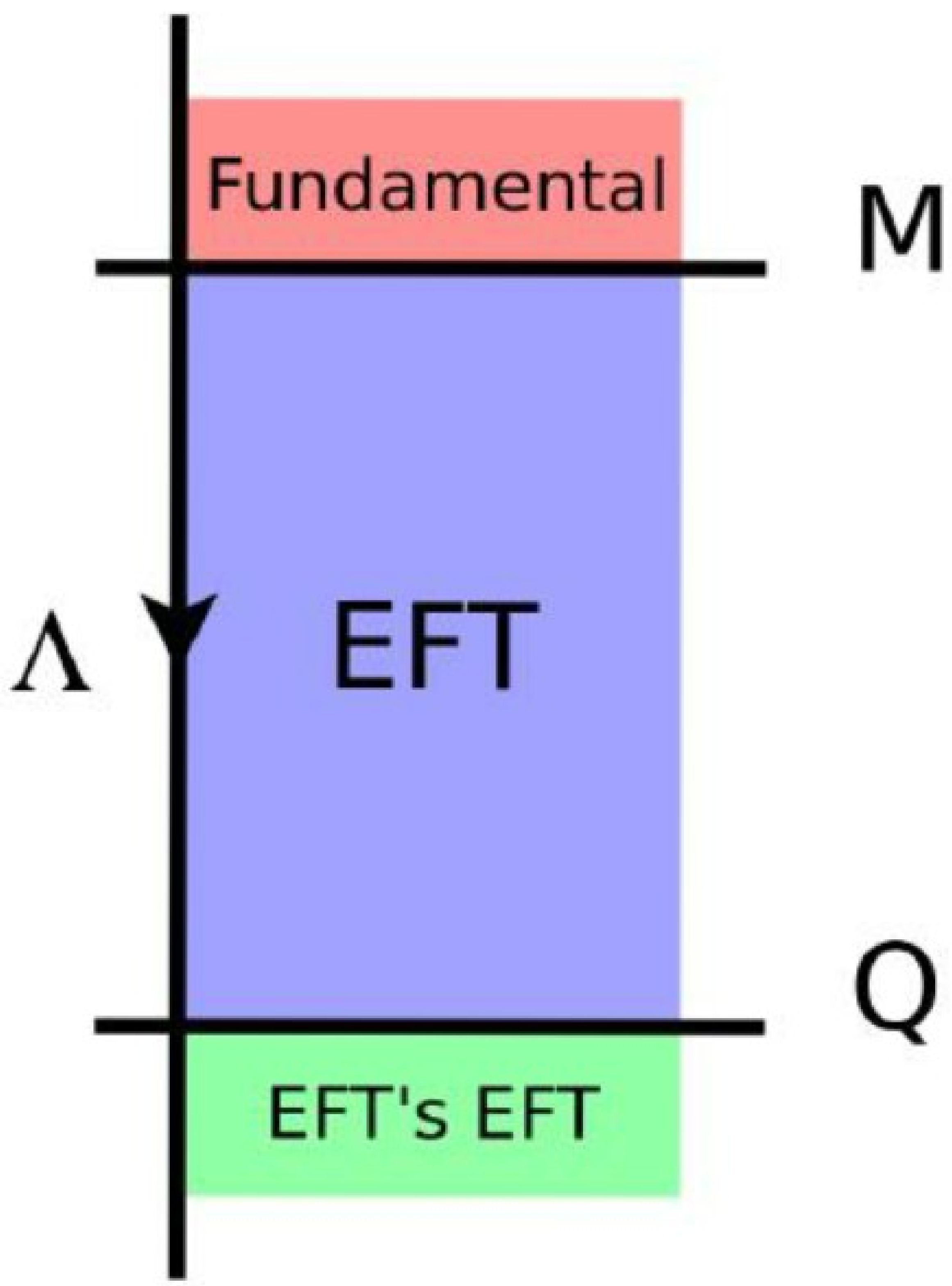
→ To understand important physical results, we must understand them from many angles

Try to have several equivalent mental representations of RENORMALIZATION

VIEW 1) $V_c(\vec{q}) = C_0(\Lambda)\, e^{-(q^2/\Lambda^2)}$

use it to explain shallow bound states $\boxed{E_B \to 0}$

VIEW 2) To try RGA (Renormalization group analysis)



Physics is unique, but choice of theory depends on resolution Λ:

- $\Lambda \geq M$: Fundamental
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For equivalent descriptions:

$$\boxed{\frac{d}{d\Lambda}\langle\Psi|\mathcal{O}|\Psi\rangle = 0}$$

Renormalization group invariance

Try to use explicitly this equation

How to do this?

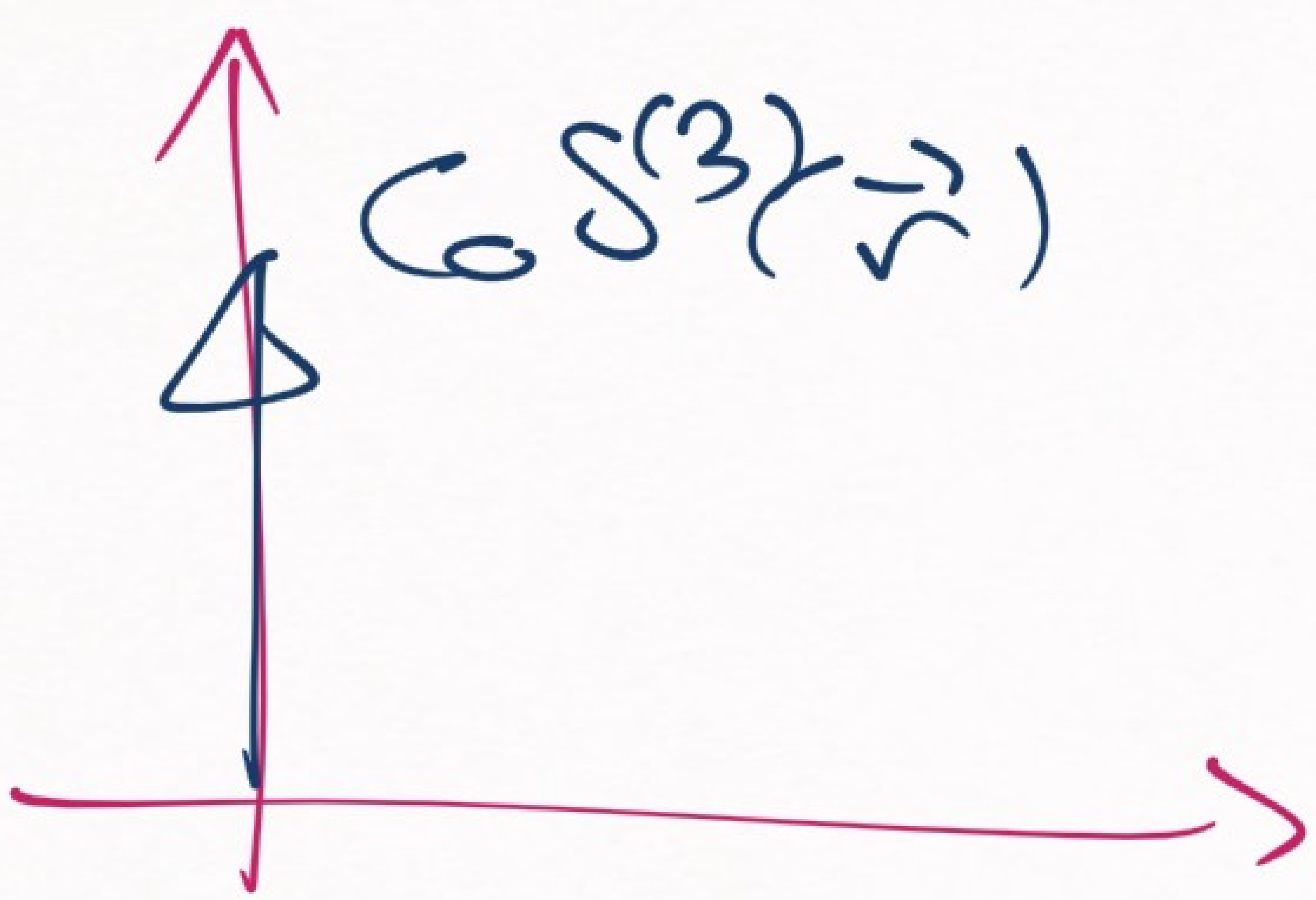
→ Starting point: $\boxed{\frac{d}{d\lambda}\langle\psi|\hat{O}|\psi\rangle = 0}$

→ Observation: r- or p-space are okay

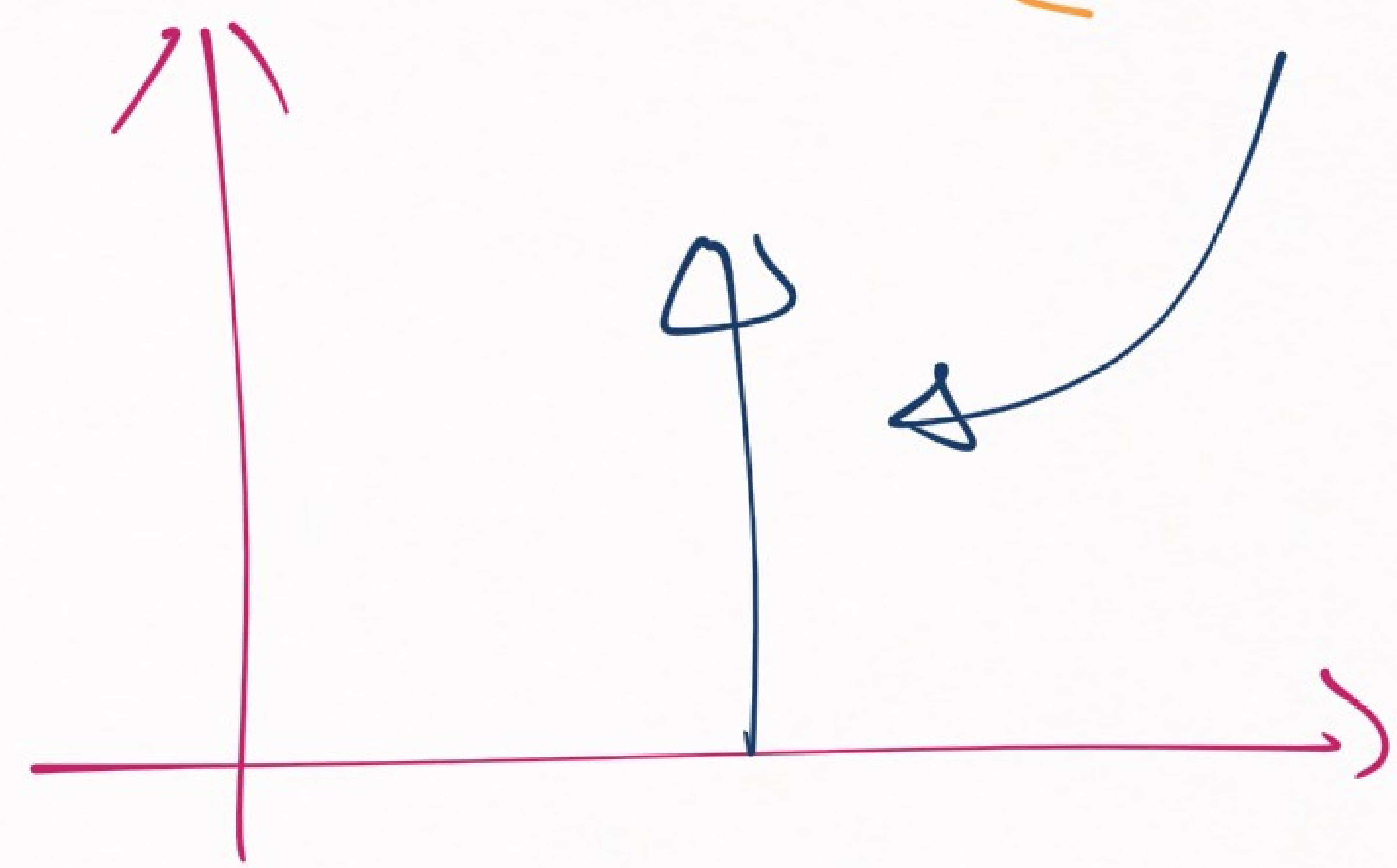
Easyness → $\boxed{\text{r-space}}$ $\frac{d}{dR_c}\langle\psi|\hat{O}|\psi\rangle = 0$

Next stop → regularize the Dirac delta in r-space

$$V(\vec{r}) = C_0\,\delta^{(3)}(\vec{r}) \longrightarrow V(\vec{r};R_c) = C_0(R_c)\,\frac{\delta(r-R_c)}{4\pi R_c^2}$$



→



Dirac-delta only singular for $d > 2$

$$\delta^{(d)}(\vec{r})$$

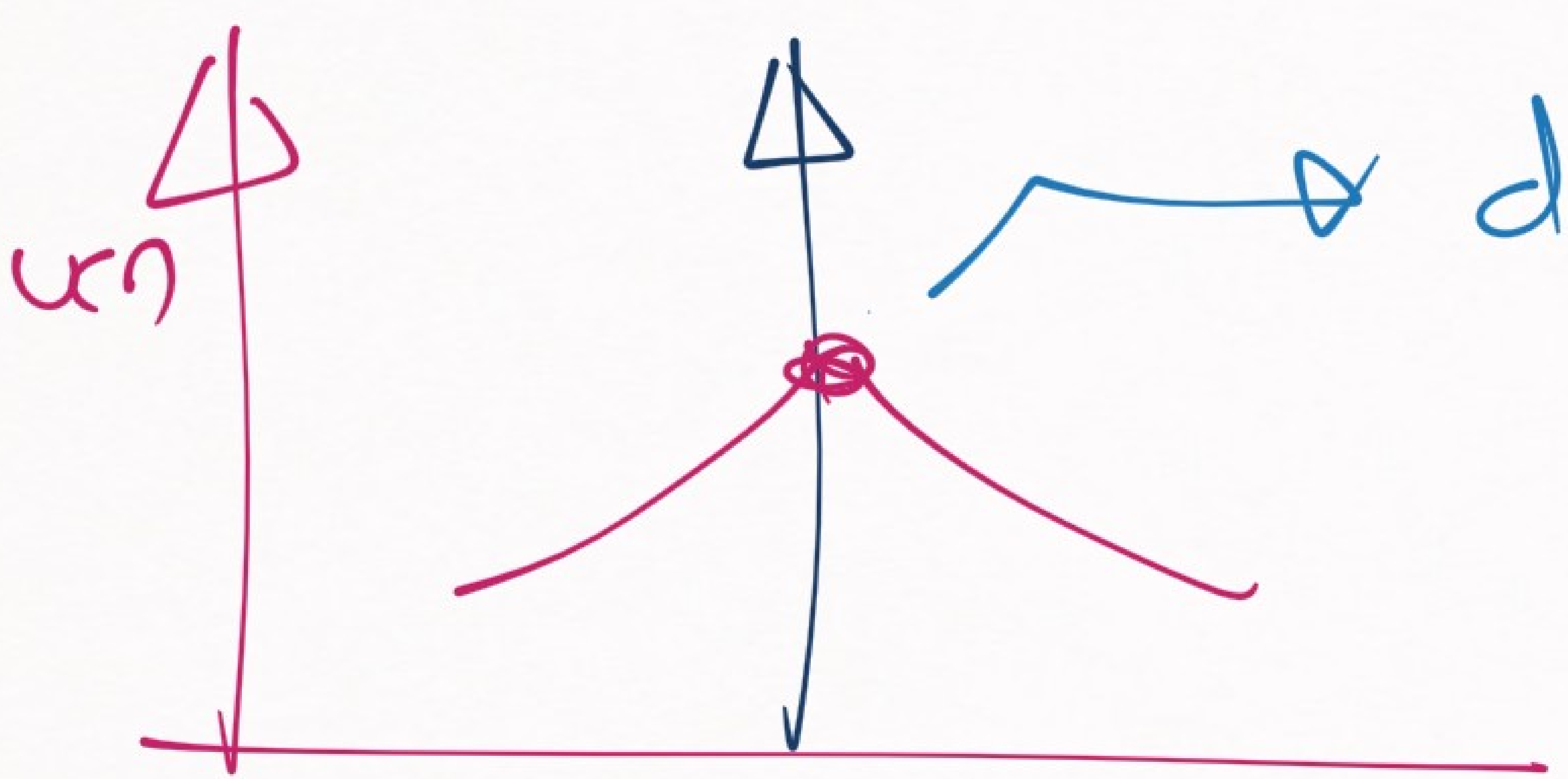
Delta-shell:

$$c_0(R_c)\,\frac{\delta(r-R_c)}{4\pi R_c^2}$$

Maybe it's a potential that you have studied before

Solution of Delta-shell:

$$\psi(\vec{r}) \propto \frac{1}{\sqrt{4\pi}} \frac{u(r)}{r}$$ (S-wave wave function)



discontinuity in the derivative of the wave function

$$\lim_{\epsilon \to 0^+} \left[\frac{u'(R_c+\epsilon)}{u(R_c+\epsilon)} - \frac{u'(R_c-\epsilon)}{u(R_c-\epsilon)} \right] = \frac{2\mu C_0(R_c)}{4\pi R_c^2}$$

$\mu \to$ Reduced mass

$$\frac{d}{dR_c} \langle \psi | V_c | \psi \rangle = 0$$

RECAP → We are trying to solve the RGE

$$\left[\frac{d}{dR_c} \langle \psi | \hat{O} | \psi \rangle = 0 \right]$$

1) a regulator ($\frac{d}{dR_c}$ needs one) ✓
2) a wave function $|\psi\rangle$ ✗
3) an observable ✗

$\boxed{\text{NEXT STEP}}$ → find a wave function

Two options for a long-distance wave function

1) $\psi(\vec{r}) = e^{i\vec{k}\cdot\vec{r}}$ (plane wave)

2) $\psi(\vec{r}) = \frac{A_S}{\sqrt{4\pi}} \frac{e^{-\gamma r}}{r}$ (S-wave bound state)

$\hat{O} = V_c(\vec{r}; R_c)$ (observable in QM, but not in QFT)

1) $e^{i\vec{k}\cdot\vec{r}} \rightarrow \langle\psi|\hat{O}|\psi\rangle = \int d^3\vec{r}\, V_c(\vec{r})\, |\psi(\vec{r})|^2$

$$= \int d^3\vec{r}\, C_0(R_c)\, \frac{\delta(r-R_c)}{4\pi R_c^2}\, |e^{i\vec{k}\cdot\vec{r}}|^2 = C_0(R_c)$$

$$\frac{d}{dR_c}\langle\psi|V_c|\psi\rangle = \frac{d}{dR_c} C_0(R_c) = 0$$

If we have a system w/ $\psi(\vec{r}) \to e^{i\vec{k}\cdot\vec{r}}$ at long-distance, then the RGE is

$$\boxed{\frac{d}{dR_c} C_0(R_c) \simeq 0}$$

2) $\psi(\vec{r}) \sim \dfrac{e^{-\gamma r}}{r}$

$$\langle \psi | V_c | \psi \rangle = \int d^3\vec{r}\, \frac{|\Delta_s|^2}{4\pi} \frac{e^{-2\gamma r}}{r^2} C_0(R_c) \frac{\delta(r-R_c)}{4\pi R_c^2}$$

$$= |\Delta_s|^2 e^{-2\gamma R_c} \frac{C_0(R_c)}{4\pi R_c^2} \sim \frac{C_0(R_c)}{R_c^2}$$

$$\langle \psi | V_c | \psi \rangle \sim \frac{C_0(R_c)}{R_c^2}$$

If we have a system w/ $f(\bar{r}) \sim \frac{e^{-\gamma r}}{r}$

at long distances, then the RGE is

$$\boxed{\frac{d}{dR_c}\left[\frac{C(R_c)}{R_c^2}\right] \simeq 0}$$

RECAP

[RGE is not unique!]

1) $\frac{\partial}{\partial R_c}[C(R_c)] \simeq 0$

2) $\frac{\partial}{\partial R_c}\left[\frac{C(R_c)}{R_c^2}\right] \simeq 0$

not the same

What is going on here?

Remember the teacup & teapot theory:

→ "Cup" power counting

$$\lambda = \Sigma\left(c_0 + c_1 x + c_2 x^2 + \ldots\right)$$

→ "Pot" power counting

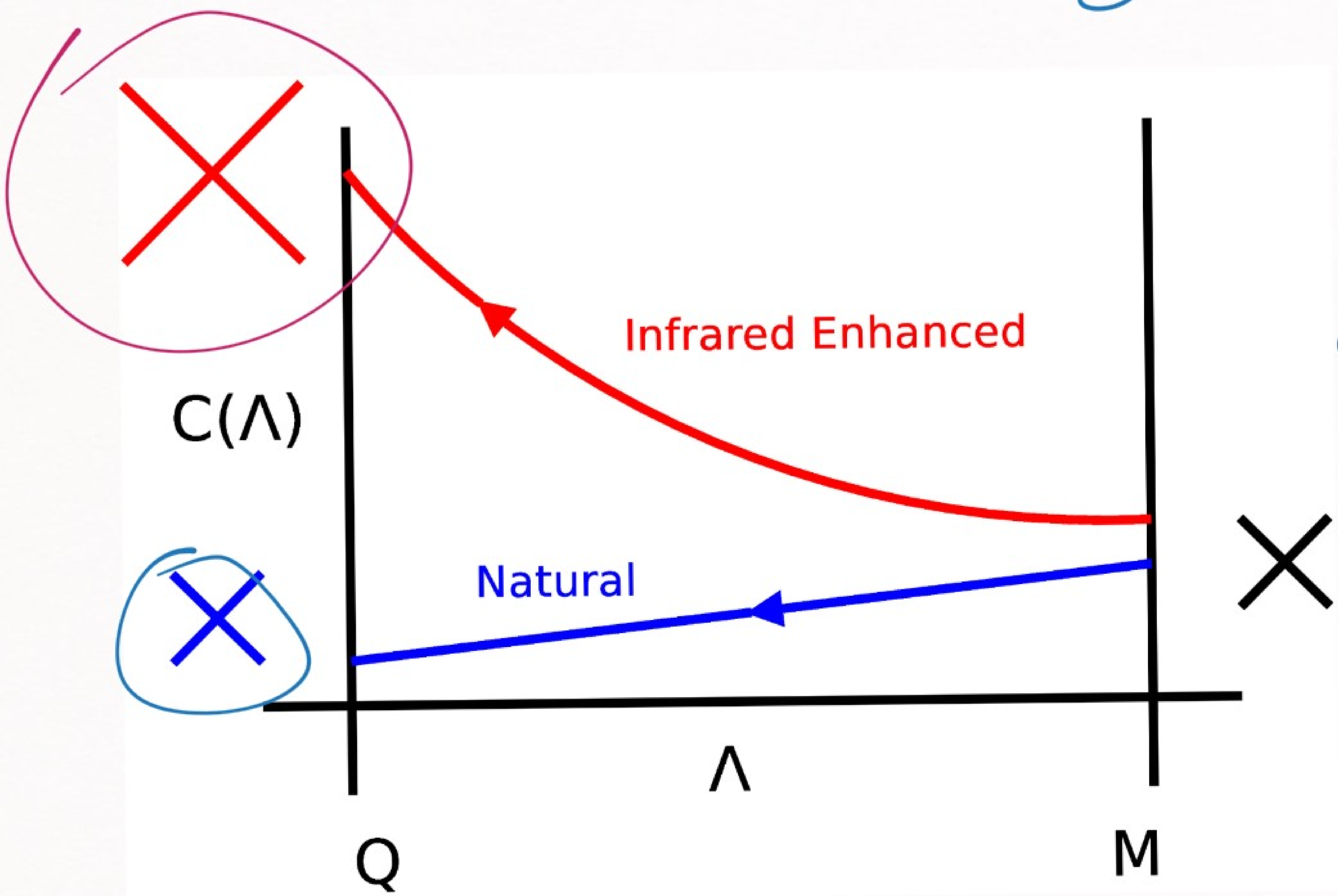
$$\lambda = \Sigma\left(d_0 + \frac{d_1}{x} + \frac{d_2}{x^2} + \ldots\right)$$

→ this two RGEs ($\frac{dC_0}{dR_c} \simeq 0$, $\frac{d}{dR_c}\left(\frac{C_0}{R_c^2}\right) \simeq 0$)

actually represent the same idea

Each one is valid for a specific physical situation

Take-home message :



$\longleftarrow$

$\Lambda \to 0$

$\longrightarrow$

$\Lambda \to \infty$

RGEs have multiple solutions (representing different physics)

1) $\dfrac{d C_0(R_c)}{d R_c} \simeq 0$ (natural system)

2) $\dfrac{d}{d R_c}\left[\dfrac{C_0}{R_c^7}\right] \simeq 0$ (unnatural system)

1) Natural system: $\frac{d}{dR_c}[C_0(R_c)] \simeq 0$

2) Unnatural system: $\frac{d}{dR_c}\left[\frac{C_0(R_c)}{R_c^2}\right] \simeq 0$

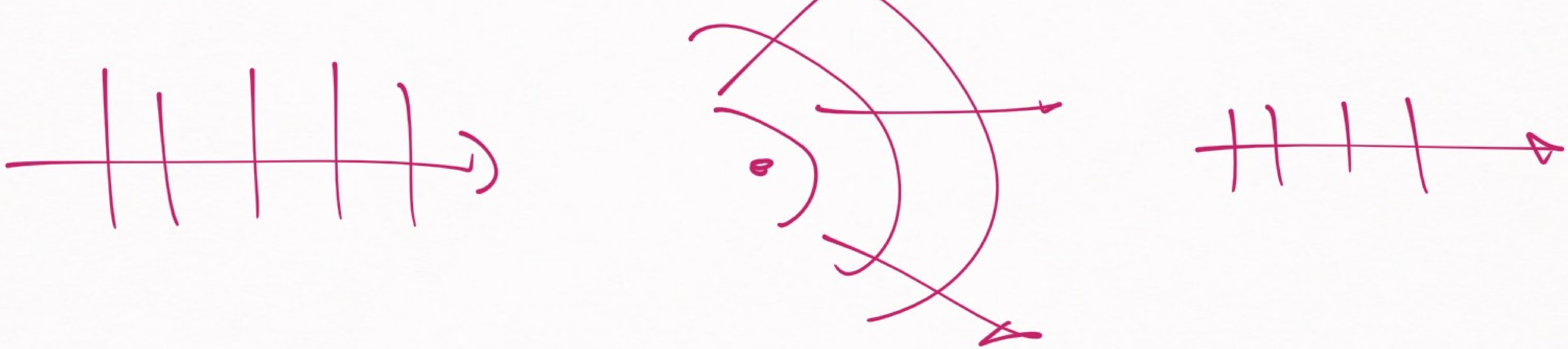
Infrared-enhanced → Why?

$\left(R_c \to \infty \longrightarrow \underline{\underline{C_0 \to \infty}} \right)$

[Let's try to find examples of natural & unnatural systems in QM]

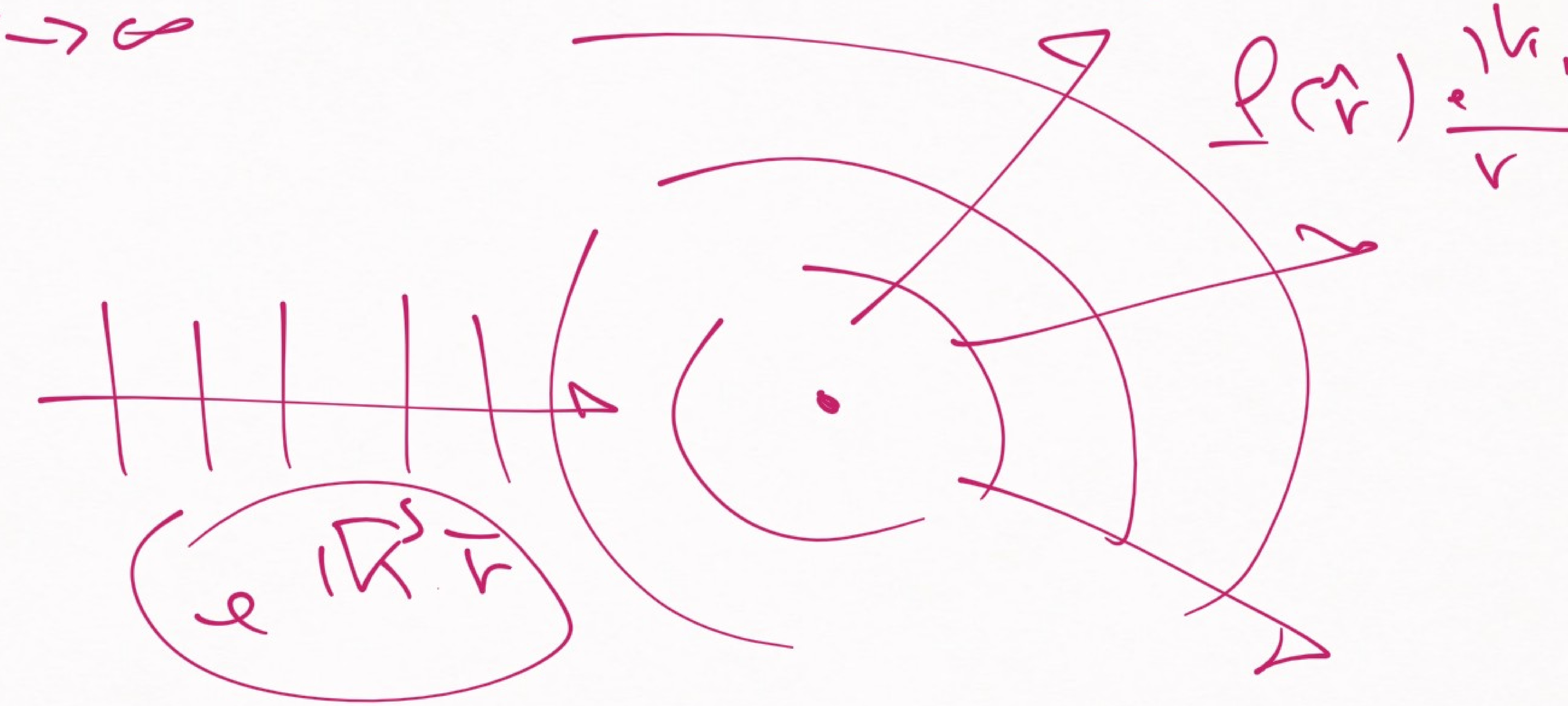
1) Natural → Key assumption $\psi(\vec{r}) \simeq e^{i\vec{k}\cdot\vec{r}}$

Scattering by a weak potential



Scattering set-up: → go back to your QM textbook

$$\psi(\vec{r}) \xrightarrow[r\to\infty]{} e^{i\vec{k}_i\cdot\vec{r}} + f(\hat{r})\,\frac{e^{ikr}}{r}$$



Scattering wave function: $\psi(\vec{r}) \to e^{i\vec{k}\cdot\vec{r}} + f(\hat{r})\frac{e^{ikr}}{r}$

QM: potential is weak → Born approximation

$$\frac{d\sigma}{d\Omega} = |f(\hat{r})|^2 \to \left[f(\hat{r}) \simeq -\frac{\mu}{2\pi}\int d^3\vec{r}\, V(\vec{r}) \times e^{-i(\vec{k}'-\vec{k})\cdot\vec{r}} \right]$$

(check any textbook)

the assumption in the Born approximation is a weak potential

$\hookrightarrow \left| e^{i\vec{k}\cdot\vec{r}} \right| \gg \left| f(\hat{r}) \frac{e^{ikr}}{r} \right|$

$\hookrightarrow$ Justifies $\psi(\vec{r}) \simeq e^{i\vec{k}\cdot\vec{r}} +$ corrections

$$\Rightarrow \boxed{\frac{d}{dR_c}\left[C_0(R_c)\right] \simeq 0} \rightarrow \text{Weak potential} \rightarrow \text{Born approx.}$$

Example → scattering by a weak Yukawa potential

Born Scattering amplitude

$$V(\vec{r}) = -\frac{g^2}{4\pi}\frac{e^{-mr}}{r} \Rightarrow f_y(\hat{r}) = \frac{M}{2\pi}\frac{g^2}{(\vec{k}-\vec{k}')^2+m^2}$$

$$f_Y(\hat{r}) = \frac{\mu}{2\pi} \frac{g^2}{(\vec{k}' - \vec{k})^2 - m^2} \simeq \frac{\mu}{2\pi} \frac{g^2}{m^2} \left[1 - \frac{(\vec{k} - \vec{k}')^2}{m^2} + \dots \right]$$

$$|\vec{k}' - \vec{k}| \ll m$$

At low energies we have:

$$\boxed{f_Y(\hat{r}) \to \frac{\mu}{2\pi} \frac{g^2}{m^2}}$$

How does this relate to EFT?

How we treat this in EFT:

1) EFT potential $\rightarrow$ $V_c(\vec{r}, R_c) = C_0(R_c)\,\dfrac{\delta(r-R_c)}{4\pi R_c^2}$

2) $f_{EFT}(\vec{r}) \simeq -\dfrac{\mu}{2\pi}\displaystyle\int d^3\vec{r}\, V(\vec{r})\, e^{-i(\vec{k}'-\vec{k})\cdot\vec{r}}$

$$\simeq -\frac{\mu\, C_0(R_c)}{2\pi}\,\frac{\sin(\underline{q}R_c)}{(qR_c)}$$

Match the theories:

$$\rho_{\gamma}(\hat{r}) \to \frac{\mu}{2\pi}\frac{g^2}{m^2}$$

$$\rho_{EFT}(\hat{r}) \to -\frac{\mu}{2\pi} C_0(R_c)$$

$$C_0(R_c) = -\frac{g^2}{m^2}$$

$$|\vec{q}| \to 0$$

Indeed $\frac{dC_0(R_c)}{dR_c} \simeq 0$

$$V_Y(\vec{q}) = -\frac{g^2}{\vec{q}^2+m^2} = -\frac{g^2}{m^2}\left(1+\dots\right)$$

$$= C_0 + C_2\vec{q}^2 + C_4\vec{q}^4 + \dots$$

$$\boxed{C_0 = -\frac{g^2}{m^2}} \longrightarrow \text{this is what we find}$$

$\rightarrow$ [All loose ends match each other]

Natural system $\rightarrow$ $\dfrac{\partial C_0}{\partial R_c} = 0$

$$C_0 = \lim_{|\bar{q}| \to 0} V_{true}(\bar{q})$$

Everything behaves as expected

NEXT LESSON

→ Unnatural systems