NUCLEAR PHYSICS 3

- IDENTIFYING SCALES
- UNNATURAL SYSTEMS & FINE TUNING

$\boxed{\text{RECAP}}$

1) Physical systems $\rightarrow$ $\boxed{\text{scales}}$

2) Natural system:

2.a) $\exists$ a characteristic scale $\begin{cases} Q \\ R \end{cases}$ $\quad [E] \uparrow Q, R \downarrow [L]$

2.b) $\forall$ is $O(1)$ in Q / R

$\langle p \rangle \sim Q \quad , \langle r \rangle \sim R$

Natural systems $\rightarrow$ easy problems

3) Most systems $\rightarrow$ multiple scales

$Q_1, Q_2, Q_3, \dots$

Usually, separation of scales:

for corrections

$\boxed{Q_1} \ll Q_2 \ll Q_3 \ll \dots$ $\rightarrow$ dominates over the other

$\hookrightarrow$ lowest energy scale

Example: Hydrogen atom

1) Relevant/characteristic scale is a_B [L]

$Q_B \sim 1/a_B$, a_B $\quad (Q_B = m_e \alpha)$

2) Shorter-range (higher-energy) scales

2.a) Electron structure → fine structure

2.b) Proton structure → hyperfine structure

$a_B \sim 54000 \text{ fm}$

$1/m_e \sim 400 \text{ fm}$

$r_p \sim 1 \text{ fm}$

Natural

+

Separation of scale

$\downarrow$

Easy to understand

High school:

$$\boxed{V = mgh}\left(1 + \overbrace{O\left(\frac{h}{R_{earth}}\right)}^{\text{small}}\right)$$

mass

$9.8\,m/s^2$

height

$h \sim 1\,m$

$R_{earth} \sim 7 \cdot 10^6\,m$

Standard model $\rightarrow$ beyond SM physics

M_{bsm}

$\downarrow$

$$\mathcal{O}\left(\frac{Q}{M_{bsm}}\right) \rightarrow \text{small}$$

tiny effects on observables that experimentalists measure

Problem w/ nuclear physics: 1+2) → difficult

1) not natural → deuteron (2.2 MeV)

+

2) poor scale separation

↳ pion exchange ($\frac{1}{m_\pi} \simeq 1{,}4\,\mathrm{fm}$)

vs nucleon size ($0{,}5 - 1{,}0\,\mathrm{fm}$)

SCALES IN A TWO-BODY SYSTEM

→ Identify scales

→ Systems w/ two scales

  → Natural
  → Unnatural

} SQUARE WELL (good example)

Two-body Schrödinger equation:

$$\left[-\frac{1}{2\mu}\nabla^2 + V(\vec{r})\right]\psi(\vec{r}) = E\,\psi(\vec{r})$$

potential

$[E]$

wave function

energy

$\propto -\frac{d^2}{dr^2} + \dots$

Reduced mass:

$m_1 \xrightarrow{\vec{r}} m_2$

$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}$$

Dimensions:

$$\mu \rightarrow [E]$$

$$\nabla^2 \rightarrow [L]^{-2} = [E]^2$$

$$V \rightarrow [E]$$

$$\psi(\vec{r}) \rightarrow [E]^{3/2} \rightarrow \int \underbrace{d^3\vec{r}}_{[L]^3} \underbrace{|\psi(\vec{r})|^2}_{[L]^{-3}} = 1$$

Schrödinger $\rightarrow$ reduced Schrödinger

$$\left[-\frac{1}{2\mu}\vec{\nabla}^2 + V(\vec{r})\right]\psi(\vec{r}) = E\,\psi(\vec{r})$$ $\rightarrow$ Remove μ

(non-relativistic $\rightarrow$ μ not really important)

$$\left[-\vec{\nabla}^2 + \underbrace{2\mu V(\vec{r})}_{U(\vec{r})}\right]\psi(\vec{r}) = \underbrace{2\mu E}_{\substack{k^2\ (E>0)\\ -\gamma^2\ (E<0)}}\,\psi(\vec{r})$$

$$\rightarrow \left[-\nabla^2 + \underbrace{U(\vec{r})}_{\text{reduced potential}}\right]\psi(\vec{r}) = -\gamma^2 \psi(\vec{r})$$

$$\psi(\vec{r}) = \frac{u(r)}{r} Y_{\ell m}(\hat{r})$$

$$[U] = [L]^{-2} \quad \& \quad [E]?$$

↳ helps us w/ defining scales

EXAMPLE 1: HYDROGEN ATOM

$$V(\vec{r}) = -\frac{\alpha}{r} \quad [L]^{-1}$$

$$U(\vec{r}) = -2\mu\frac{\alpha}{r} \quad [L]^{-2}$$

$$= -\frac{2}{a_B r}$$

convention

Bohr radius

$$\psi(\vec{r}) \propto \frac{1}{a_B^{3/2}} e^{-(r/a_B)}$$

EXAMPLE 2: van der Waals

$$V(\vec{r}) = -\frac{C_6}{r^6} \quad [L]^{-1}$$

$$U(\vec{r}) = -2\mu \frac{C_6}{r^6} = -\frac{R_{vdW}^4}{r^6} \quad [L]^{-2}$$

$$R_{vdW} = (2\mu C_6)^{1/4}$$

EXAMPLE 3:
Inverse square-law potential

(special system)

$$V(\vec{r}) = -\frac{C_2}{r^2}$$

$$U(\vec{r}) = -\frac{2\mu C_2}{r^2}$$

$$= -\frac{g}{r^2} \qquad [L]^{-2}$$

pure number

$\rightarrow$ [there is no scale here]

What is special about $1/r^2$?

$$\left[-\nabla^2 - \frac{g}{r^2}\right]\psi(\vec{r}) = -\gamma^2\psi(\vec{r})$$

↳ why no scale? Try to zoom in!

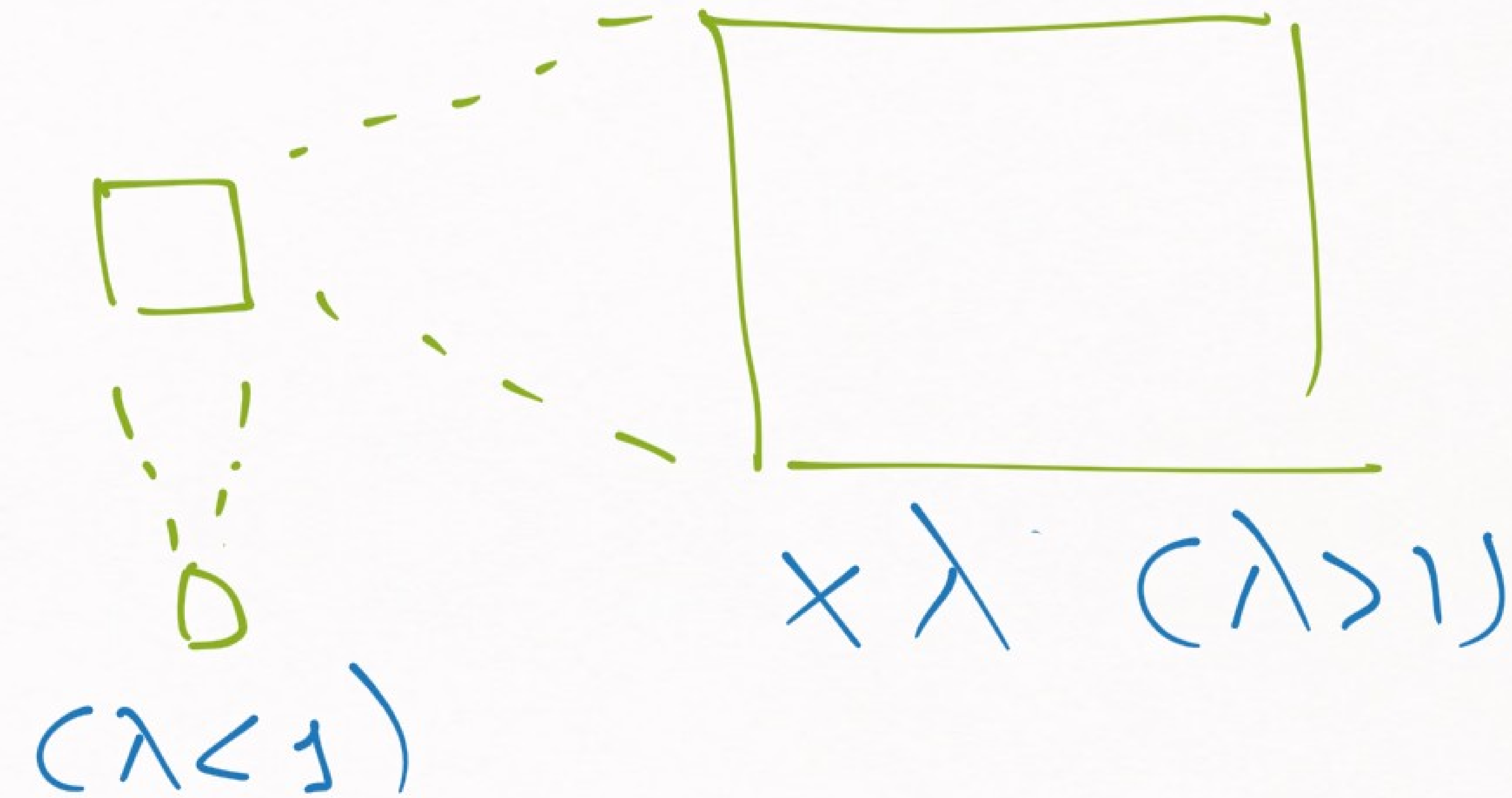
$$\vec{r} \to \lambda\vec{r} \qquad \gamma \to \frac{1}{\lambda}\gamma$$
$$\vec{\nabla} \to \frac{1}{\lambda}\vec{\nabla}$$

what is this?

$\vec{r} \to \lambda \vec{r}$ (dilation)

$\vec{\nabla} \to \frac{1}{\lambda} \vec{\nabla}$

$\gamma \to \frac{1}{\lambda} \gamma$



$$\left[-\vec{\nabla}^2 - \frac{g}{r^2}\right] \psi(\vec{r}) = -\gamma^2 \psi(\vec{r})$$

$\to$ invariant under dilations

$\frac{1}{r^2}$ describes a system that looks identical when zooming in

a)

b)

→ Scale invariance $\begin{cases} \text{a) continuous } (\forall \lambda) \\ \text{b) discrete } (\lambda = \lambda_0) \end{cases}$

$\vec{r} \to \lambda \vec{r}\ ,\ \forall \lambda$

$\vec{r} \to \lambda_0 \vec{r}$ (λ_0 specific)

$1/r^2$ potential $\rightarrow \left[-\nabla^2 - \frac{g}{r^2}\right]\psi(\vec{r}) = -\gamma^2\psi(\vec{r})$

~~Boring type~~ $(\vec{r} \rightarrow \lambda\vec{r}$, for any $\lambda)$

Anomaly : classical symmetry that is broken in the quantization process

Anomalies $\rightarrow$ most examples in QFT

$\uparrow$

$1/r^2$ simplest example & only requires

quantum mechanics

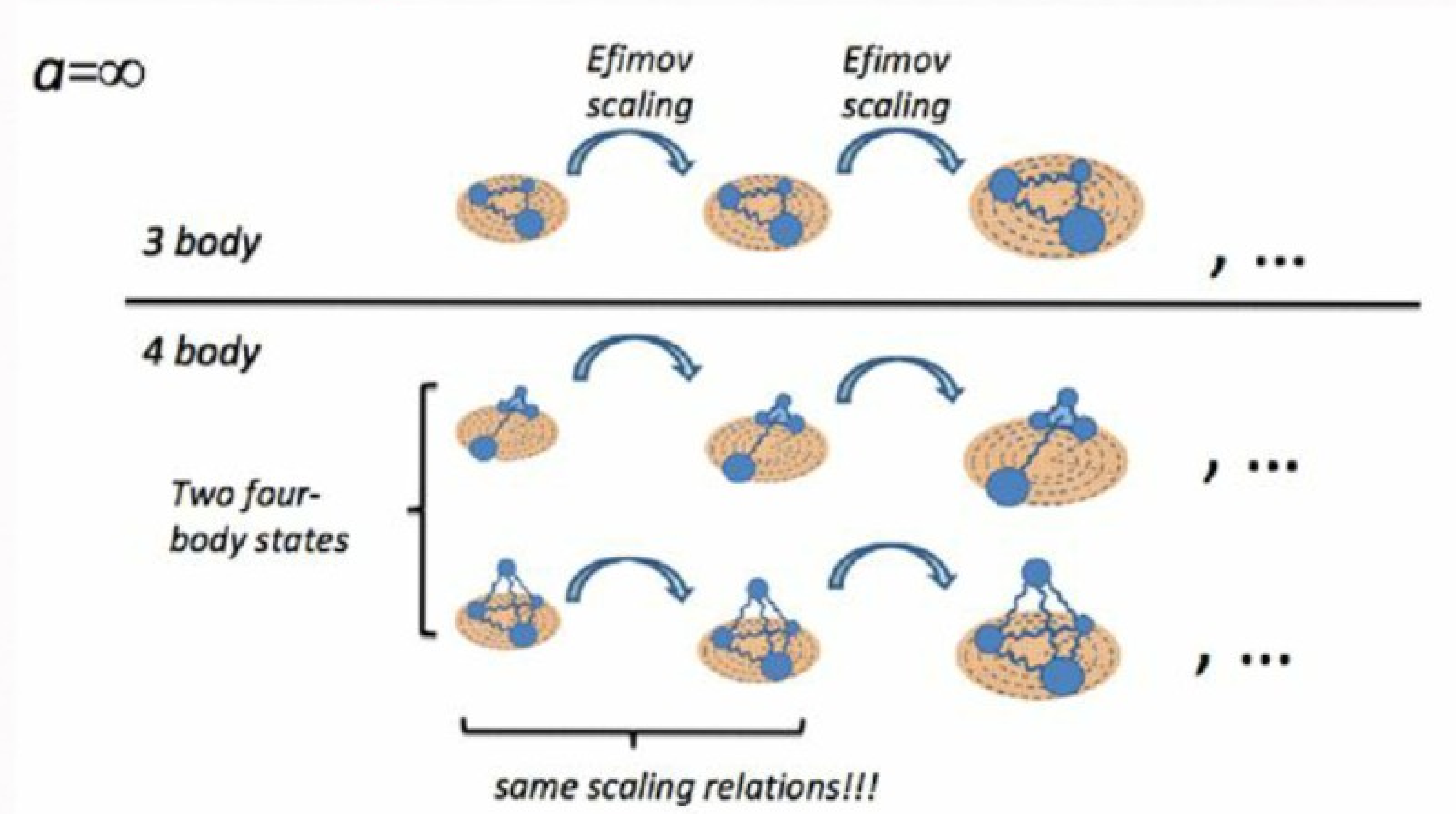
$\sum$

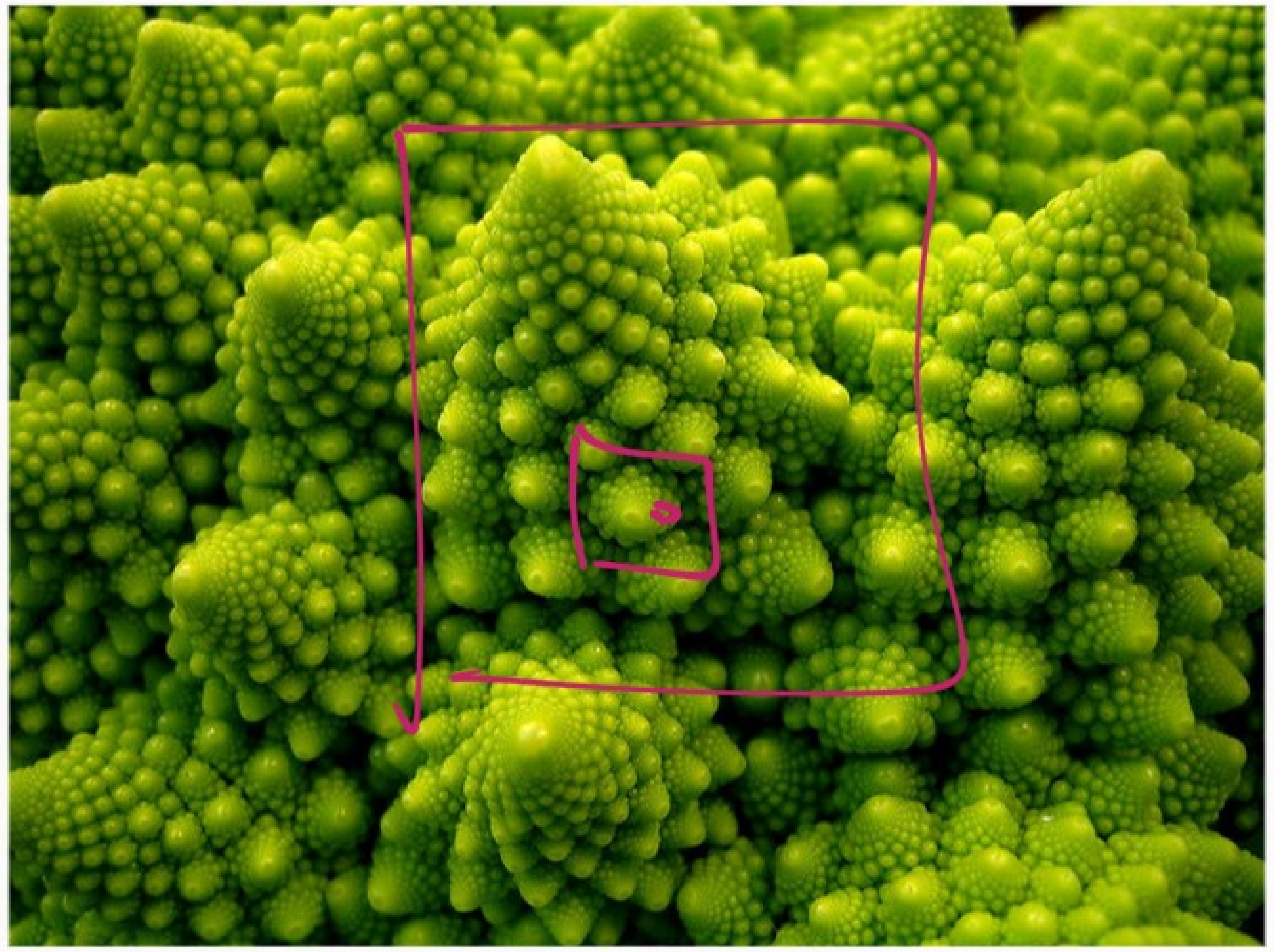
Discrete & continuous scale invariance:

$\vec{r} \to \lambda_0 \vec{r}$, $\lambda_0 \simeq 22.7$ (any other specific number)

3-body systems

fractal-like



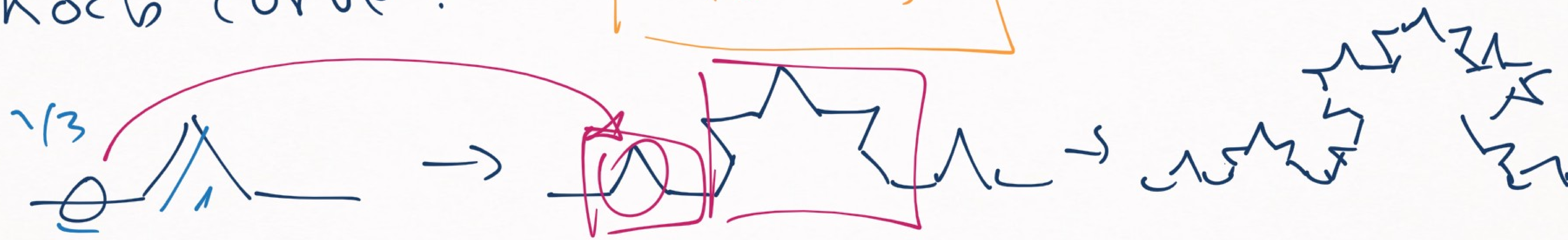


Repetition only happens at a specific value of $\lambda = \lambda_0$

$(\vec{r} \to \lambda_0 \vec{r})$

Koch curve:

$\boxed{\lambda_0 = 3}$



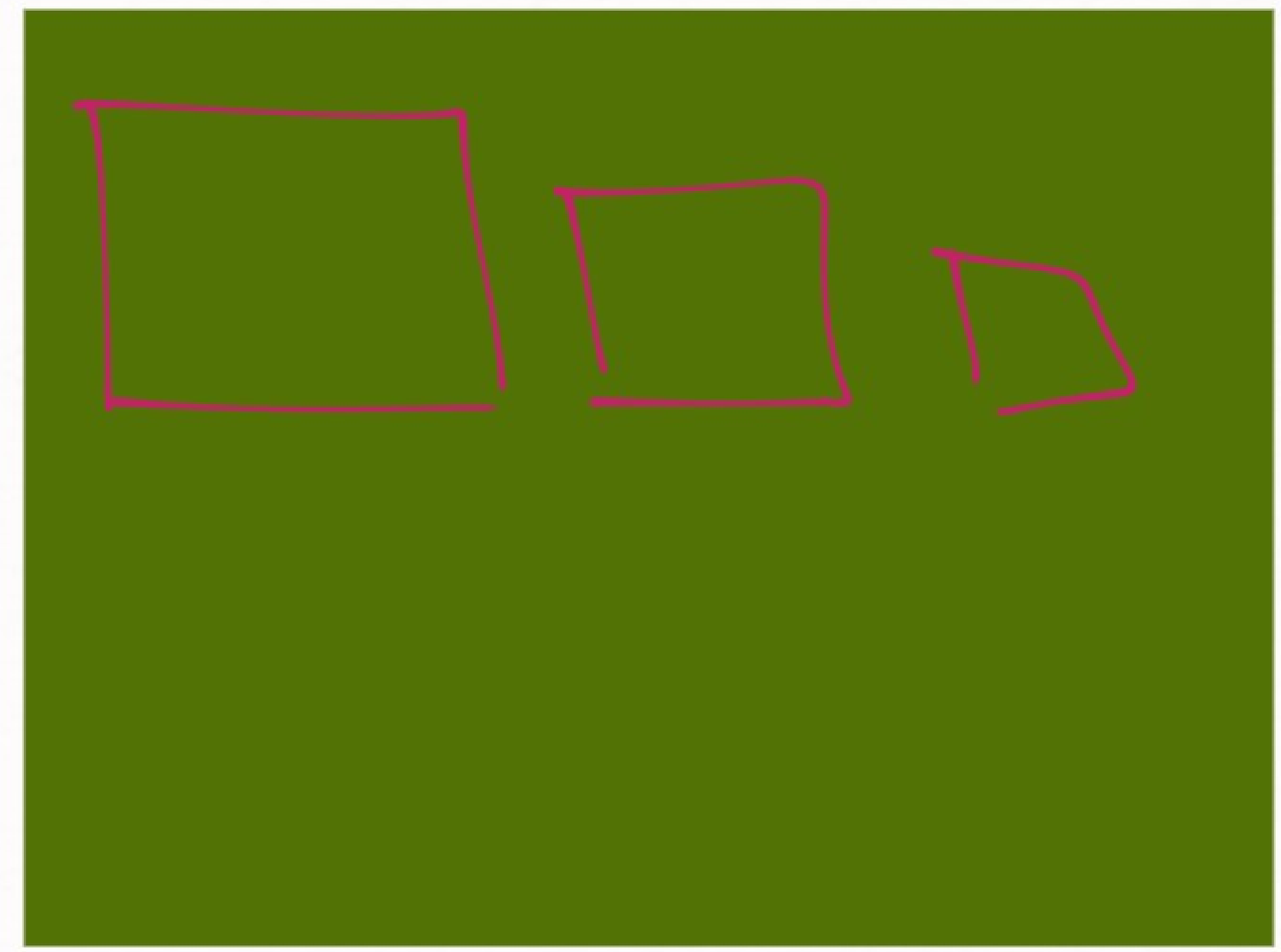
Continuous → $\forall \lambda$ → no structure

$\vec{r} \to 2\vec{r}$

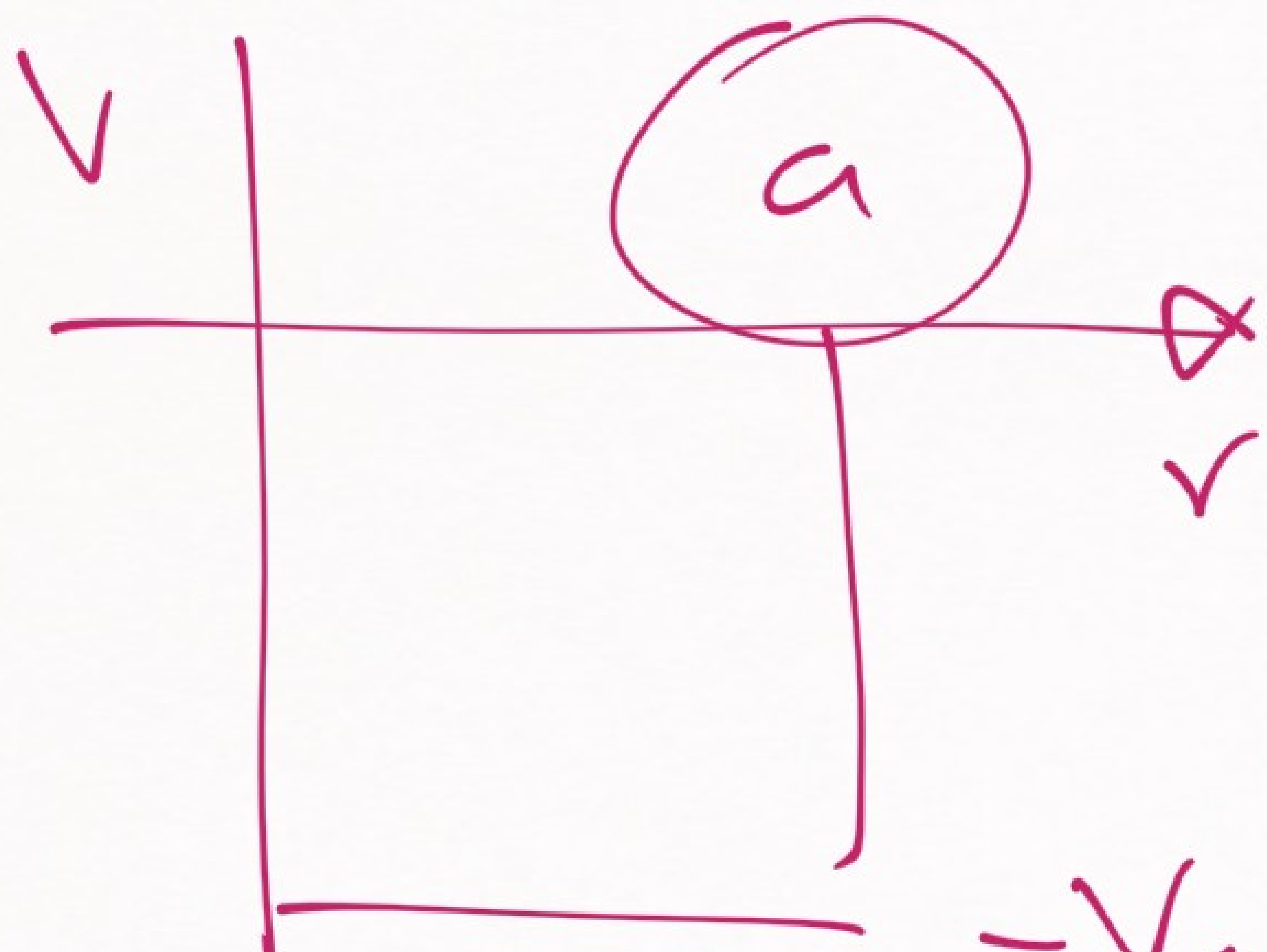
$\vec{r} \to 3\vec{r}$

$\vec{r} \to 2\pi\vec{r}$

$\forall$ the same



EXAMPLE OF A TWO-SCALE SYSTEM



→ "Square well"

$-V_0 = -\frac{1}{2\mu}\frac{1}{R_{sg}^2}$

$R_{sg} \to R_s$

(shorter to write)

Two scales $\rightarrow$ a (the range)

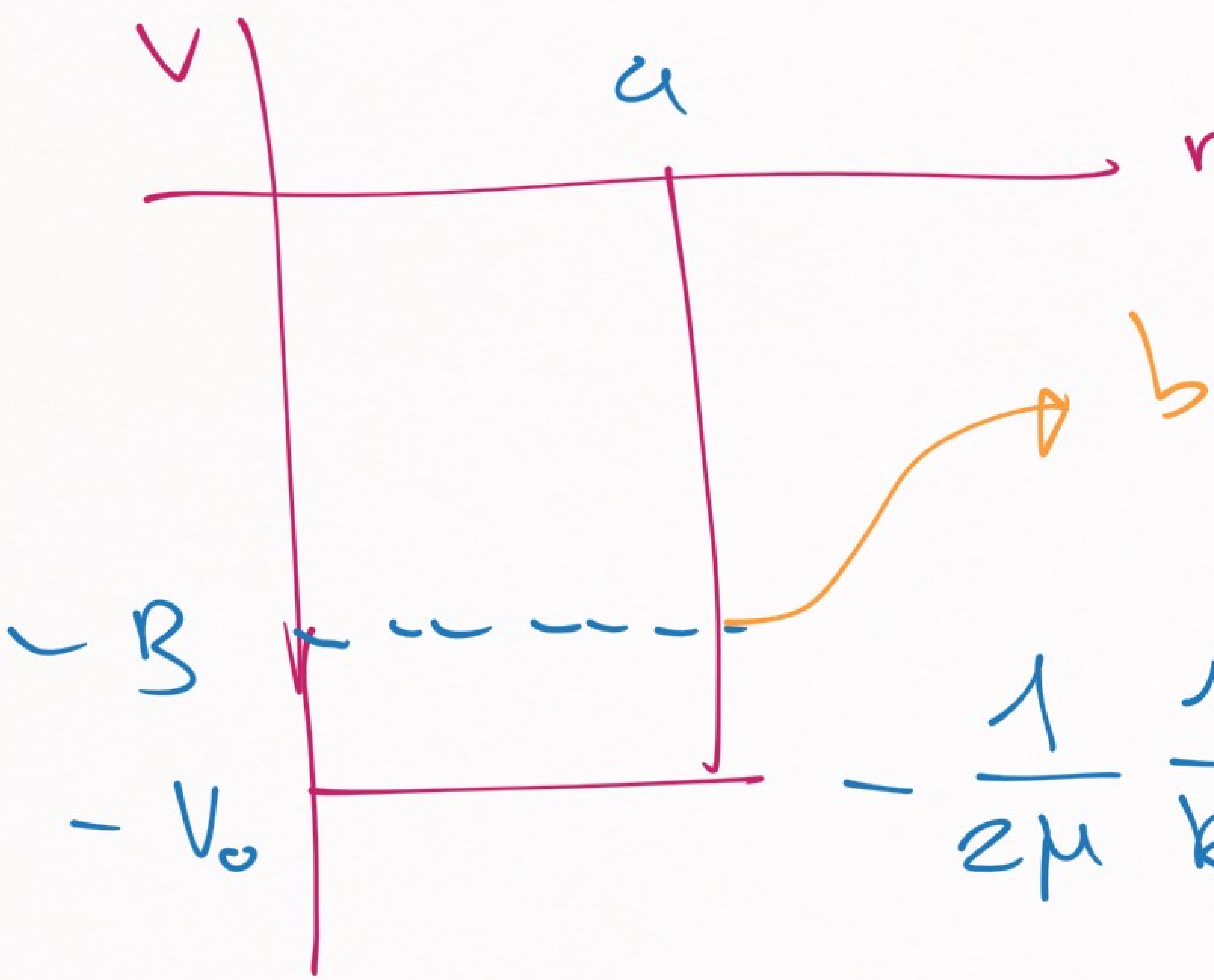
R_s $\left(1/\sqrt{2\mu V_0} \right)$

Why R_s? $\longrightarrow$ $\left[-\nabla^2 - U(\vec{r})\right]\psi(\vec{r}) = -\gamma^2\psi(\vec{r})$

$$U(\vec{r}) = 2\mu V(\vec{r}) = -2\mu V_0\,\theta(a-r)$$

$$= -\frac{1}{R_s^2}\,\theta(a-r)$$

What does "natural" means in SW?



bound state energy

$$-\frac{1}{2\mu}\frac{1}{R_s^2}$$

$$0 \le B \le V_0$$

(if bound state exists)

Imagine B is random : $\boxed{B \sim \mathcal{O}(V_0)}$

$\downarrow$

natural case

most probable outcome

$\boxed{B \ll V_0}$

$\downarrow$

unnatural case

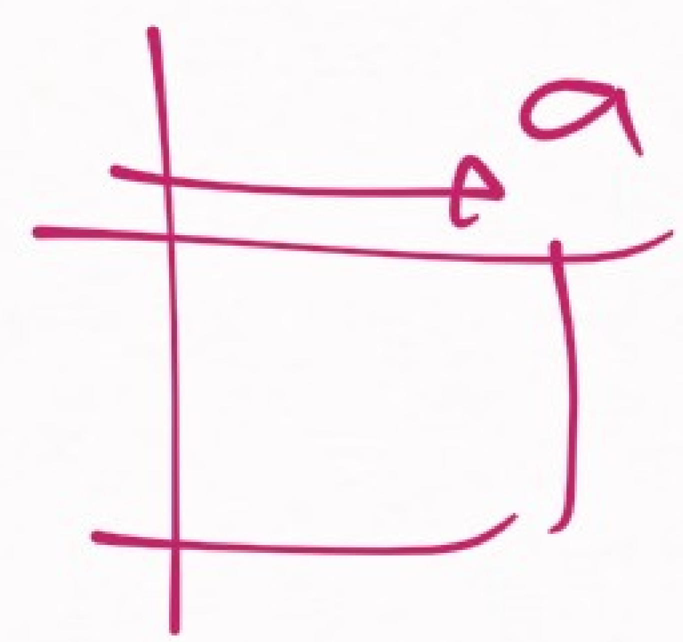
this can only happen very rarely

Study the natural case:

Reminder QM $\rightarrow$ $\kappa \cot(\kappa a) = -\gamma$

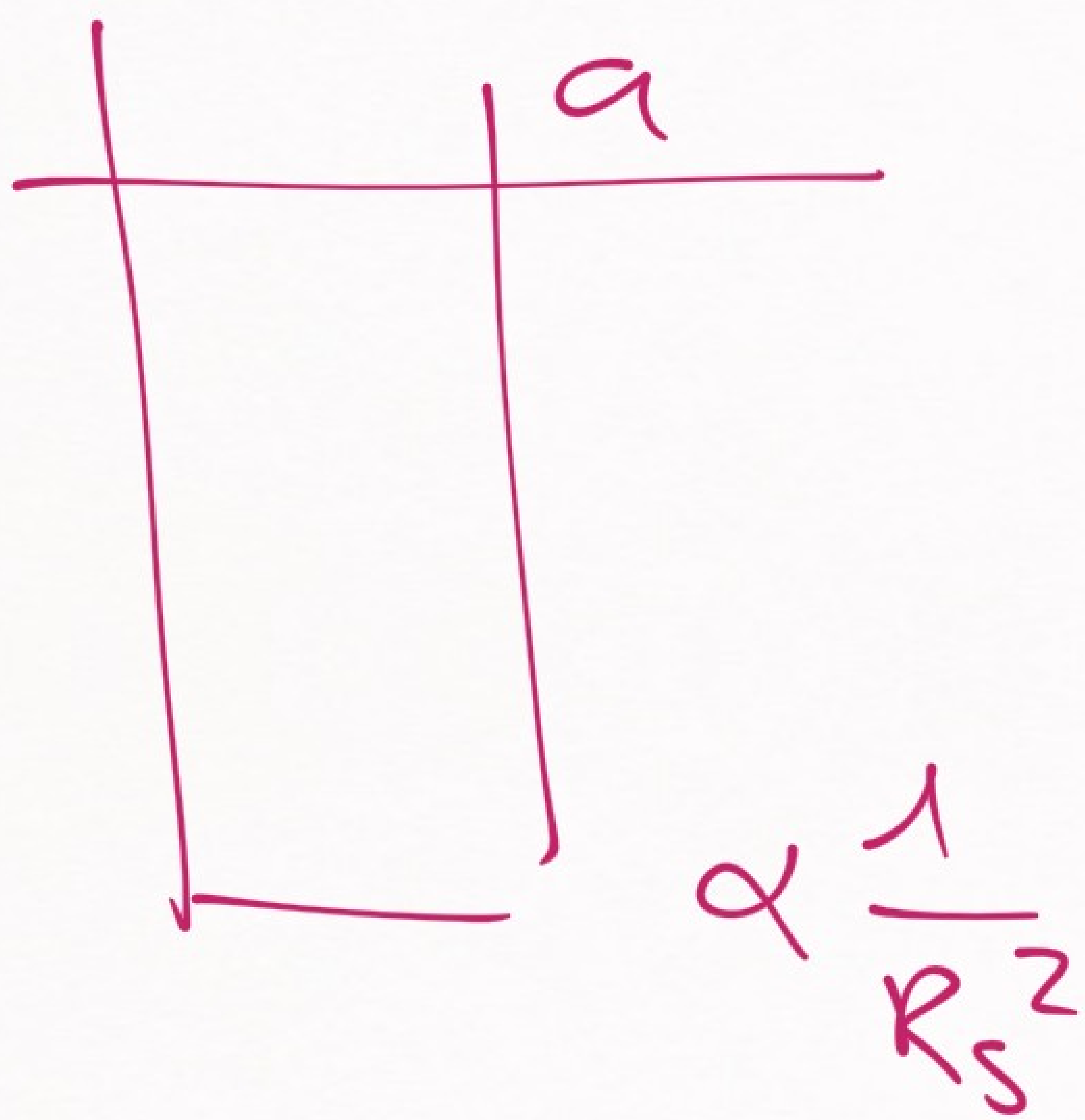
(Eigenvalue equation of the square well)

$$K = \sqrt{\frac{1}{R_S^2} - \gamma^2}$$



$$B = \frac{\gamma^2}{2M}$$

$\boxed{\text{Check}}$



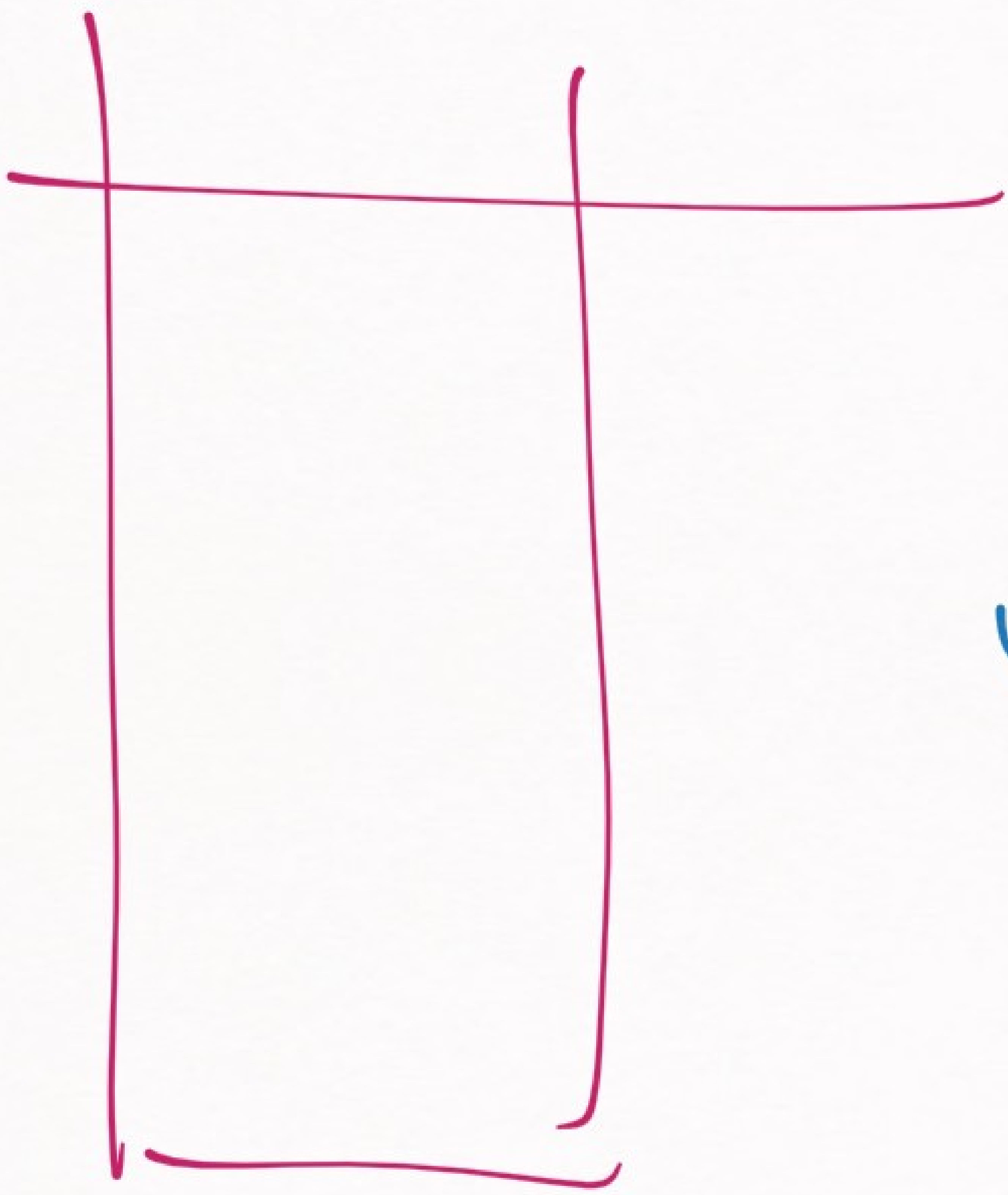
Separation of scales:

$$\boxed{R_s \ll a}$$

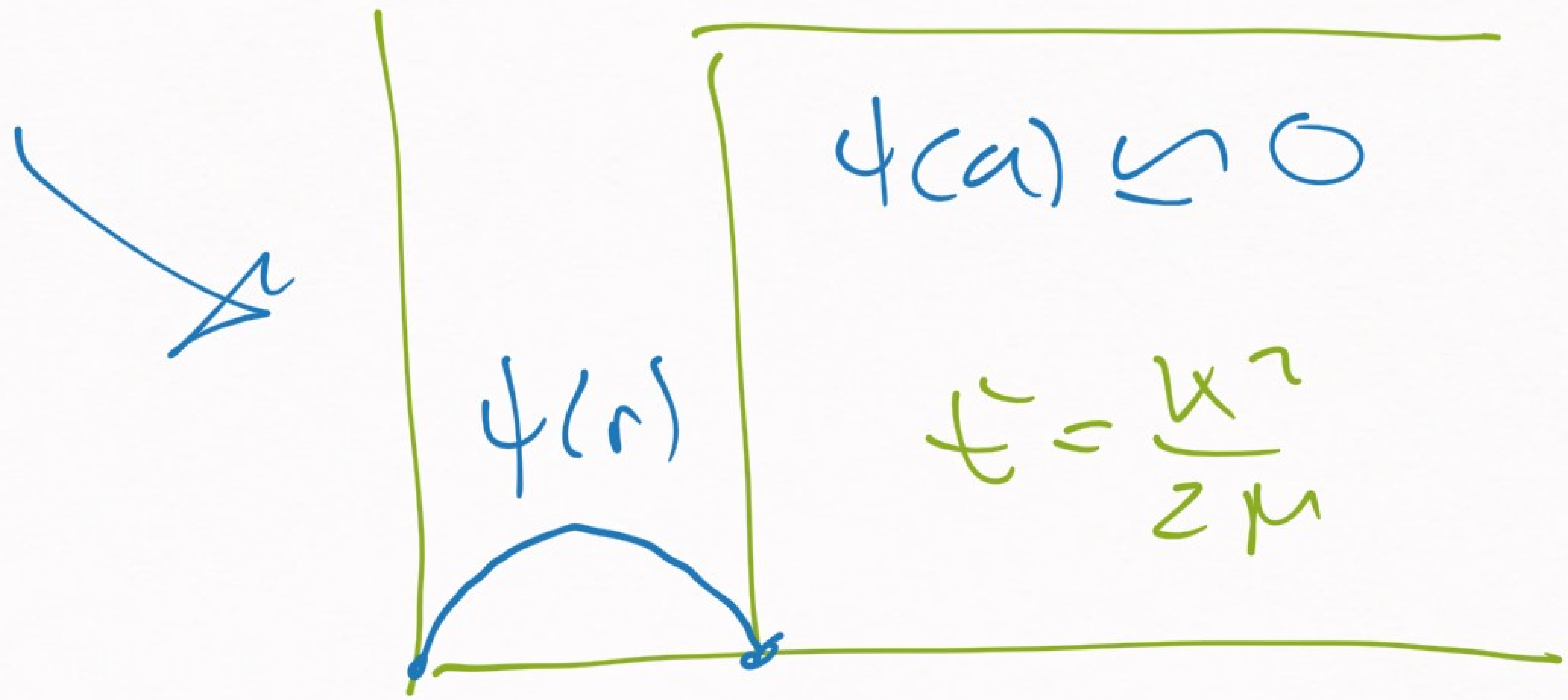
$$\Downarrow$$

$$\boxed{\begin{aligned} \beta &\sim \mathcal{O}(V_0) \\ \gamma &\sim \mathcal{O}(1/R_s) \end{aligned}}$$

if $R_s \ll a \Rightarrow$ well will be deep



$$E = -\frac{\gamma^2}{2\mu}$$



(change in origin of energy)

$$\left.\begin{array}{l}\psi(r) \sim \sin(kr) \\ \psi(a) \sim 0\end{array}\right] \rightarrow ka = n\pi + \underbrace{\delta}_{\text{correction } O\left(\frac{R_s}{a}\right)}$$

$$k\cot(ka) = -\gamma \quad , \quad \gamma = \sqrt{\frac{1}{R_s^2} - k^2}$$

$$\gamma = \sqrt{\frac{1}{R_s^2} - \left(\frac{n\pi + \delta}{a}\right)^2}$$

$k \cot(ka) \simeq -\gamma$ → Trick: expand in $\frac{R_s}{a}$

↳ check it & do it by yourselves

Ansatz (proposal of a solution)

$ka = n\pi + \delta$ → check $\delta \sim O\left(\frac{R_s}{a}\right)$

(δ is small)

$$ka = n\pi + \delta \Leftrightarrow \cot(ka) = \cot\delta$$

$$\gamma = \sqrt{\frac{1}{R_s^2} - \left(\frac{n\pi+\delta}{a}\right)^2} = \frac{1}{R_s}\sqrt{1 - \left(\frac{R_s}{a}\right)^2 (n\pi+\delta)^2}$$

$$\simeq \frac{1}{R_s}\left(1 - \frac{1}{2}\left(\frac{R_s}{a}\right)^2 (n\pi+\delta)^2 + O\left(\left(\frac{R_s}{a}\right)^4\right)\right)$$

$$\sqrt{1+x} \simeq 1 + \frac{x}{2} + O(x^2)$$

Combine everything:

$$\frac{(n\pi+\delta)}{a}\,\underbrace{\cot\delta}_{\sim \frac{1}{\delta}} = -\frac{1}{R_s}\left(1 - \frac{1}{2}\left(\frac{R_s}{a}\right)^2 (n\pi+\delta)^2 + \dots\right)$$

($\delta \sim O\left(\frac{R_s}{a}\right) \to$ small, $\cot x \simeq 1/x$)

$$\frac{n\pi}{\delta a} = \frac{1}{R_s} - \frac{1}{2}\,\frac{R_s}{a}\left(\frac{n\pi}{a}\right) + O\left(\left(\frac{R_s}{a}\right)^3\right)$$

$$\delta = -n\pi \left(\frac{R_s}{a}\right) + O\left(\left(\frac{R_s}{a}\right)^2\right)$$

$$\gamma = \frac{1}{R_s}\sqrt{1 - \frac{(n\pi + \delta)^2}{a^2} R_s^2} \simeq \frac{1}{R_s}\left(1 + O\left(\left(\frac{R_s}{a}\right)^2\right)\right)$$

$\boxed{\text{RECAP}}$ 1) Scale separation $R_s \ll a$

2) Ansatz $(ka = n\pi + \delta)$ for $R_s \ll a$

3) Calculated corrections $O\left(\frac{R_s}{a}\right)$

$$\Rightarrow \quad \gamma \simeq \frac{1}{R_s}\Big(1 + \underbrace{O\left(\frac{R_s^2}{a^2}\right)}_{}\Big)$$

small corrections

Naturalness for the Square well

$\gamma \simeq \mathcal{O}(1/R_s) \to$ [confirmed w/ calculations]

$\to \boxed{\text{Square well} \,/\, R_s \ll a \text{ is natural}}$

UNNATURAL SQUARE WELL

$a \sim R_s$ (poor separation of scales)

$$\kappa \cot(\kappa a) = -\gamma \quad / \quad \gamma^2 = \frac{1}{R_s^2} - \kappa^2$$

$$\underline{\underline{\frac{1}{\gamma} \gg a, R_s}} \rightarrow \kappa \simeq 1/R_s \rightarrow \frac{1}{R_s} \cot\left(\frac{a}{R_s}\right) \simeq O(1)$$

$$\frac{\gamma}{R_s}, \frac{\gamma}{a} \ll 1$$

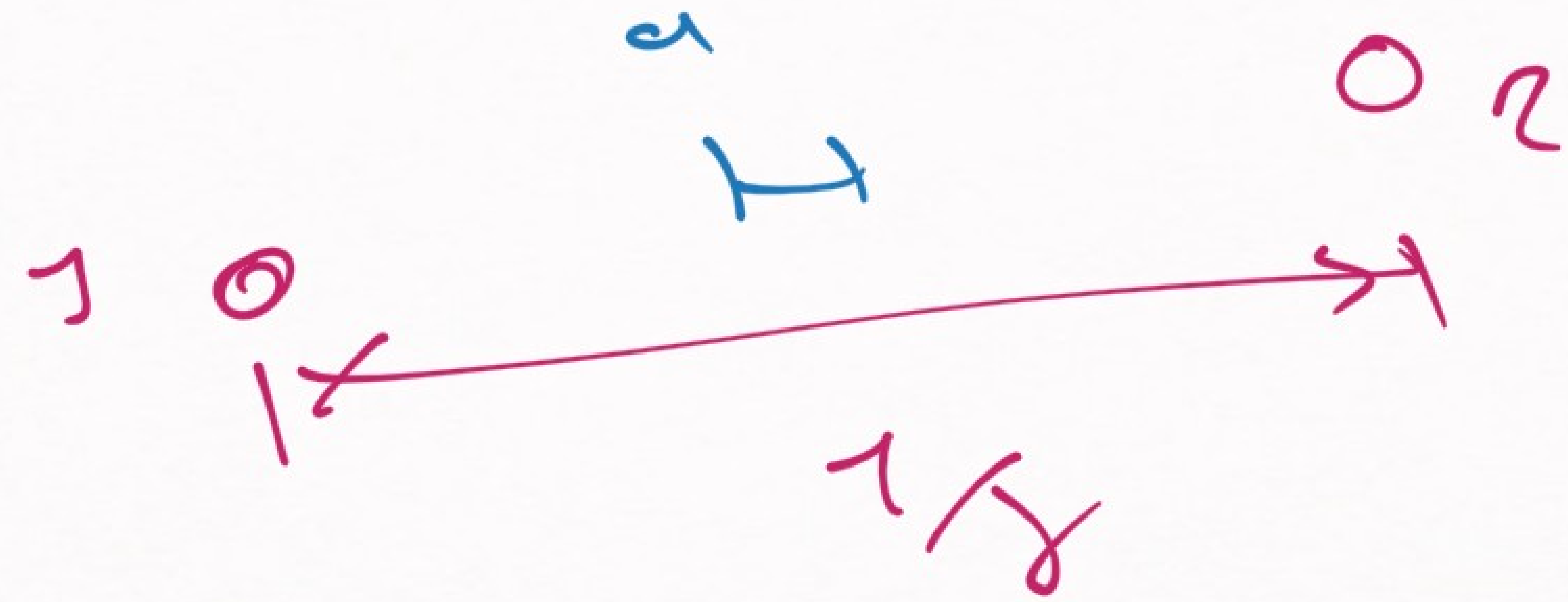
Eigenvalue equation for $\frac{1}{\gamma} \gg a, R_S$ $(a \sim R_S)$

$(\gamma \ll 1/a, 1/R_S)$

$$\cot\left(\frac{a}{R_S}\right) = \mathcal{O}(\gamma R_S)$$

$$\boxed{\frac{a}{R_S} \simeq \frac{\pi}{2}}$$

$\frac{1}{\gamma} \gg a, R_s$



Bound state is much, much bigger than the interaction range

(similar to deuteron)

→ Unnatural system only happens under very specific conditions:

$$\boxed{\frac{a}{R_s} \simeq \frac{\pi}{2}} \longrightarrow \text{fine-tuning}$$

(if you change a/R_s

⇒ normal bound state w/ $\gamma \simeq O(1/R_s)$)

$$\frac{a}{R_s} = \frac{\pi}{2}(1+x) \rightarrow x \text{ is small}$$

1) $x = 0\,,\ \gamma = 0$ (unnatural bound state)

2) $x \simeq 0.1\,,\ \gamma \simeq \frac{0.16}{R_s}$ (still unnatural)

3) $x \simeq 0.2\,,\ \gamma \simeq 0.37/R_s$ (limit)

4) $x \simeq 0.3\,,\ \gamma \simeq 0.51/R_s$ (natural)

5) $x \simeq 0.4\,,\ \gamma \simeq 0.73/R_s$

Fine-tuning: $\frac{a}{R_S} = \frac{\pi}{2}(1+x)$

$x \lesssim 0$ to have a large bound state

$\boxed{\text{RECAP}}$ SQUARE WELL

1) $\frac{a}{R_s} \ll \frac{\pi}{2}$ → no bound state

2) $\frac{a}{R_s} \simeq \frac{\pi}{2}$ → gigantic bound state
(fine-tuned)

3) $\frac{a}{R_s} > \frac{\pi}{2}$ → natural bound state
($\frac{1}{\gamma} \simeq R_s$)

$\cot\left(\frac{3}{2}\pi\right) = 0$

$\cot\left(\frac{\pi}{2}\right) = 0$

→ Think about Question 1

$$\boxed{\frac{a}{R_s} = \frac{\pi}{2}(2n+1)} \qquad \kappa \cot(\kappa a) = -\gamma$$

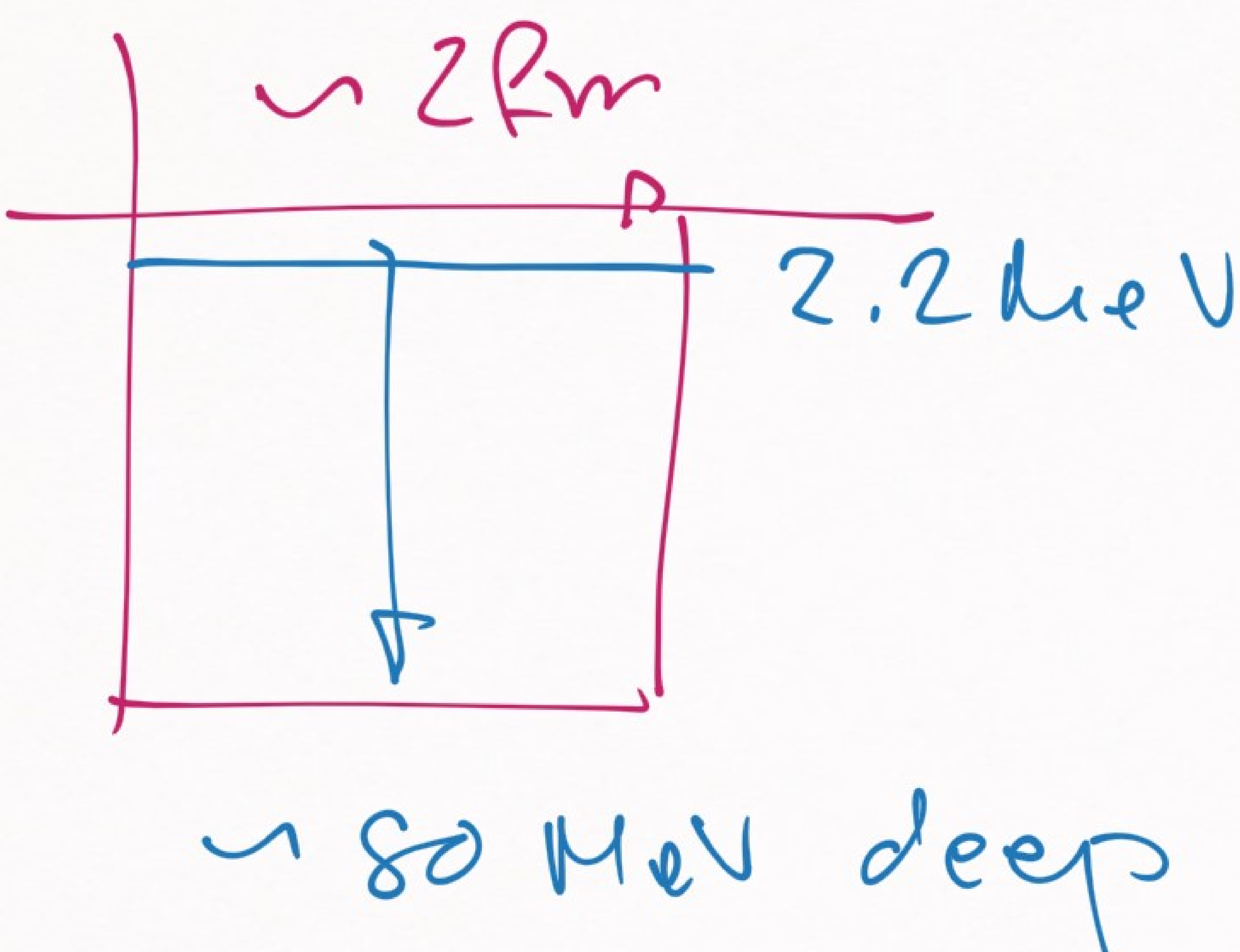
$$\cot\left(\frac{a}{R_s}\right) = 0$$

$$\gamma = 0 \rightarrow \cot(\kappa a) = 0 \qquad \gamma^2 = \frac{1}{R_s^2} - \kappa^2$$

$$\kappa = 1/R_s \leftarrow 0 = \frac{1}{R_s^2} - \kappa^2$$

[UNNATURALNESS EXAMPLE 1]

The deuteron $\to$ np bound state (2.2 MeV):



$$2.2 \ll 50$$

$$-B = \langle T \rangle + \langle V \rangle$$

$$-2 = 48 - 50$$

The binding energy of the deuteron requires a very precise cancellation between the kinetic & potential energy

$$\leftarrow -50 + 48 \simeq -2 \rightarrow \frac{2}{50} \sim \frac{1}{25}$$

Deuteron $\rightarrow$ $\boxed{\text{fine-tuning of } 1/25}$

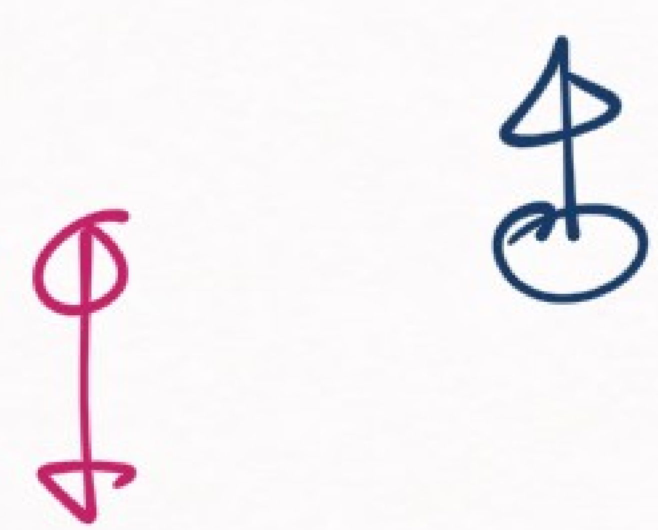
$\rightarrow$ More extreme example in nuclear physics

Two-nucleon system ($N = p$ or n)

spin-1/2 particles

$S=1 \rightarrow$ deuteron

(triplet $\rightarrow |1\, m_s\rangle$, $m_s = -1, 0, +1$)



$S=0 \rightarrow$ singlet configuration

$|0\,0\rangle \Rightarrow$ one state

1) np singlet → no bound state → ⊛

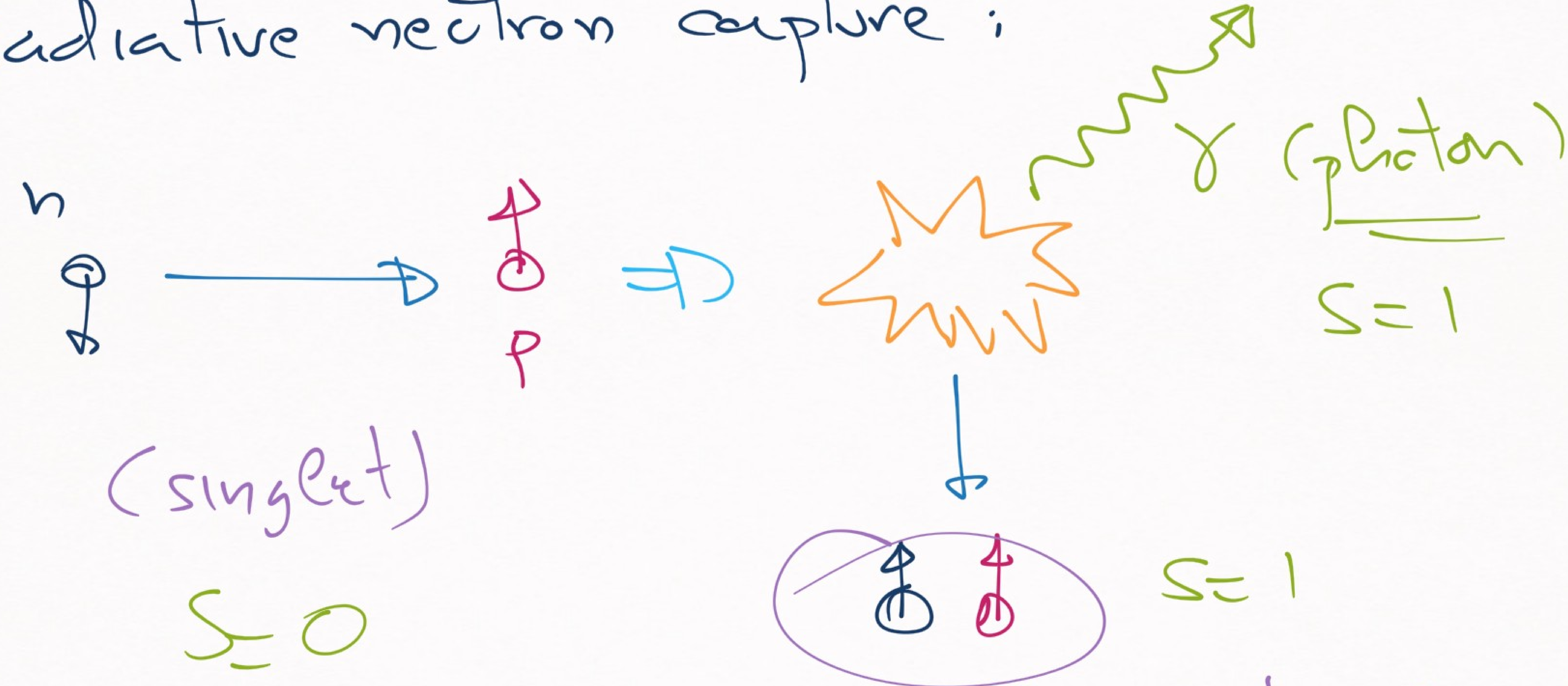
2) np triplet → deuteron (1/25 fine-tuning)

⊛ → it almost binds

(it doesn't bind by a bit)

↳ how do we know this?

Radiative neutron capture :



$1 \otimes 1 = 0 \oplus 1 \oplus 2$

(triplet ↳ deuteron)

RNC $\rightarrow$ the probability is high

$\downarrow$

if singlet is close to binding

In terms of Square well:

1) $\frac{a}{R_s} = \frac{\pi}{2} + \epsilon \rightarrow$ binding

2) $\frac{a}{R_s} = \frac{\pi}{2} - \epsilon \rightarrow$ no binding

(but fine-tuned)

$\downarrow$

a virtual state

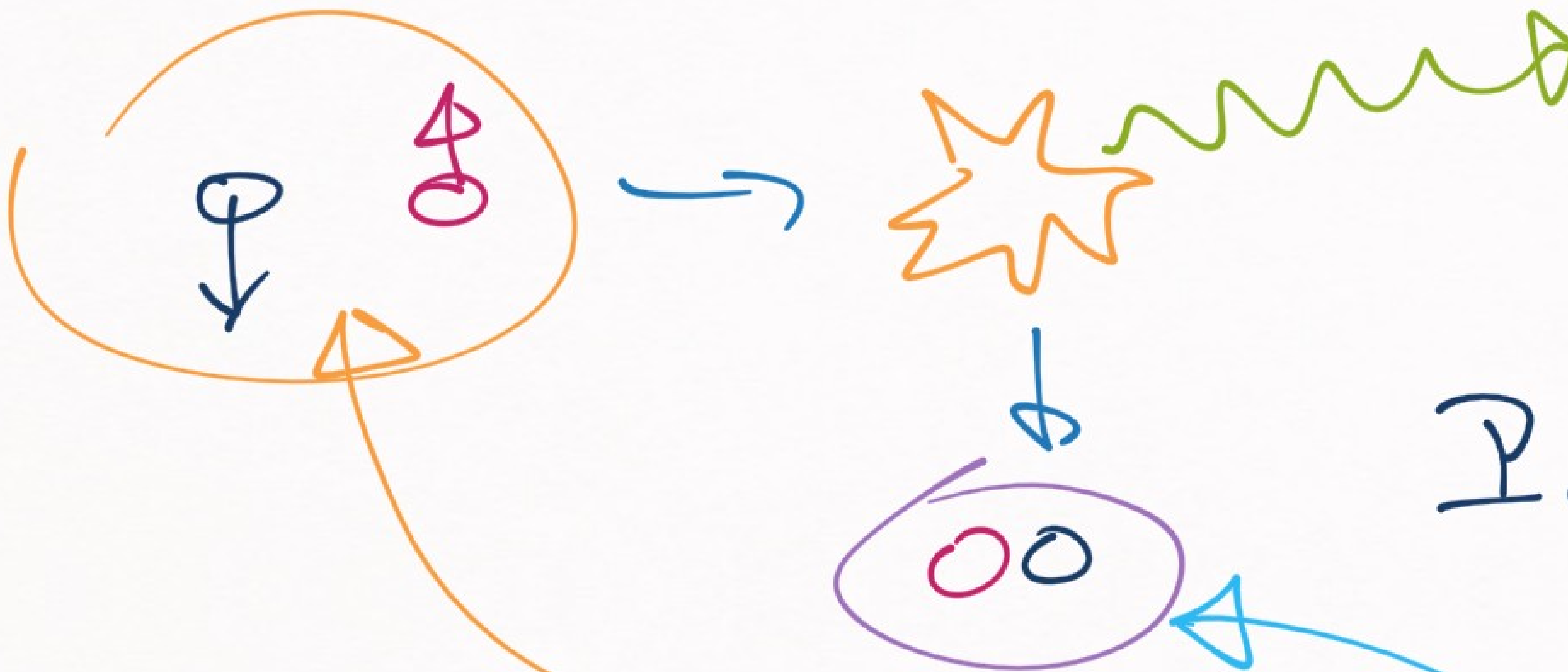
1) bound state solution:

$$\psi_B(\vec{r}) \longrightarrow \frac{e^{-\gamma_B r}}{r} \qquad E_B = -\frac{\gamma_B^2}{2\mu}$$

2) virtual state solution:

$$\psi_v(\vec{r}) \rightarrow \frac{e^{+\gamma_v r}}{r} \qquad E_v = -\frac{\gamma_v^2}{2\mu}$$

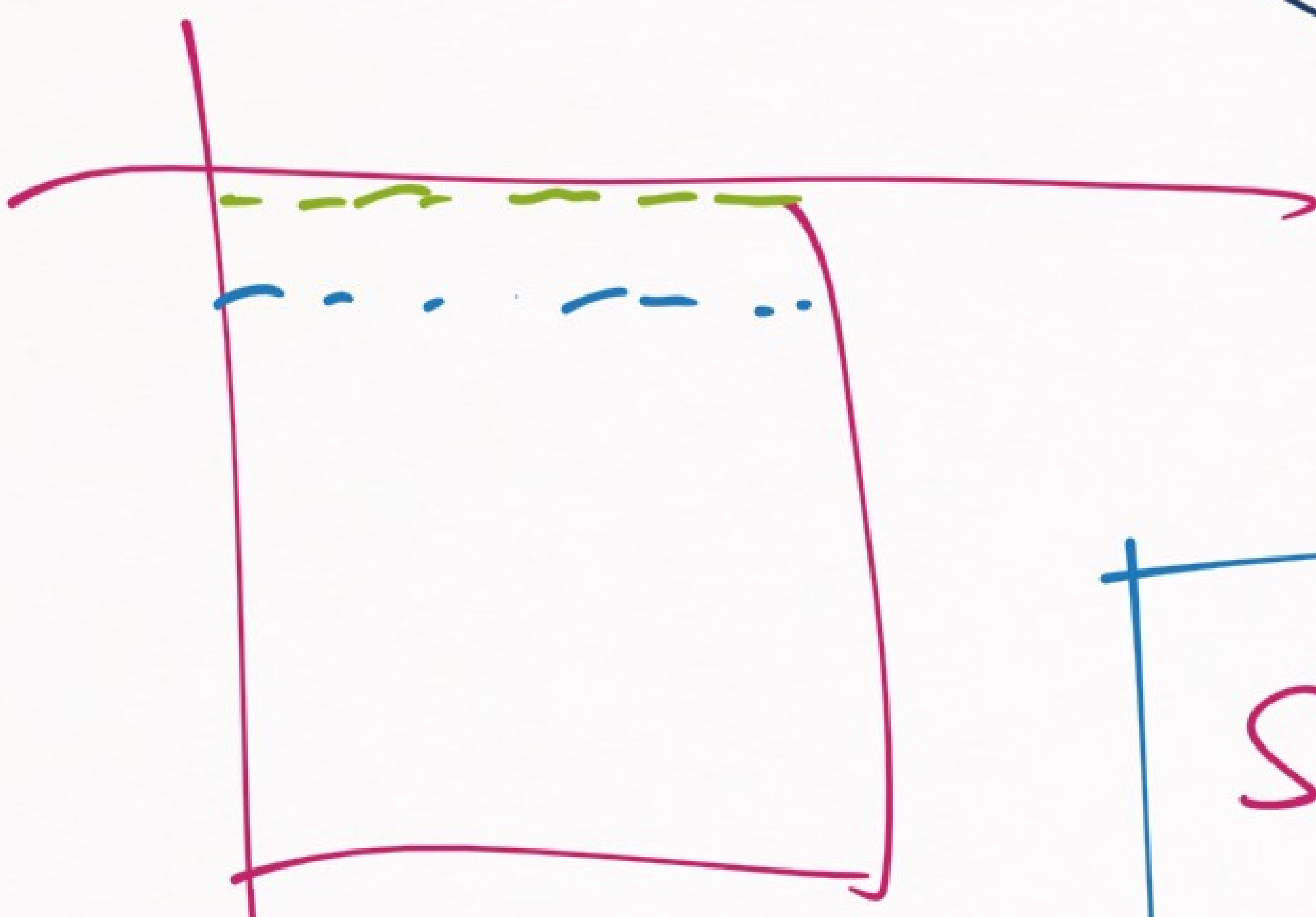
$$(\ \gamma_v + \gamma_v^2 r + \cdots)$$



$$P_{capture} \propto |\langle \psi_\nu | \psi_B \rangle|^2$$

$$|\langle \psi_\nu | \psi_B \rangle|^2 \sim \left| \int dr \, e^{-\gamma_B r} \left(r + \frac{1}{\gamma_\nu} \right) \right|^2 \underset{\gamma_\nu \to 0}{\sim} \left| \frac{1}{\gamma_\nu \gamma_B} \right|^2$$

$$\Rightarrow \boxed{E_\nu \sim -0.07\ \text{MeV}} \qquad (E_B \sim -2.2\ \text{MeV})$$



$$\frac{0{,}07}{50} \sim \frac{1}{800} \sim \frac{1}{1000}$$

Singlet is fine-tuned at about 1/1000

What does fine-tuning imply?

1) fortuitous (chance) $\rightarrow$ nuclear physics

2) "conspiracy" $\rightarrow$ there is a deeper reason for the fine-tuning to happen

$\hookrightarrow$ next lesson

Questions ?

See you on Thursday