Nuclear Physics 29

Residual Interactions in the Shell Model

RECAP

SHELL MODEL

$\Downarrow$

- magic numbers $(N = 2, 8, 20, 28, \dots)$
- separation energies

$\Downarrow$

1) MEAN FIELD POTENTIAL:

$$H = \sum_i \frac{p_i^2}{2m_i} + \sum_{ij} V_{ij} + \sum_{ijk} V_{ijk} + \dots$$

$$\Rightarrow H = \sum_i \left(\frac{p_i^2}{2m_i} + V_i^{MF} \right) + \Delta V$$

2) Nucleons are fermions

3) We fill shells

SHELL MODEL

→ * Explain JP near a closed shell ($^{16}O \rightarrow {}^{15}O/{}^{17}O$)

* Explain excited spectrum near a closed shell (^{41}Ca, ^{38}Ar)

* Pairing interaction (Example of $\underline{\Delta V}$)
(RESIDUAL INTERACTION)
(^{203}Tl, ^{205}Tl, ^{207}Pb)

→ $\boxed{\text{LESSON 26}}$

Yet this was merely the overview

TWO BIG PROBLEMS

1) How to define a V^{MF}?

$$V^{MF} = \frac{1}{2} m\omega r^2 - \xi \vec{\ell}\cdot\vec{s} - \kappa \ell^2$$

→ Easy option

→ Hartree-Fock, Skyrme, Gogny, etc

→ Advance options

2) How to deal w/ ΔV?

Pairing interaction

→ Easy example

→ But everything else?

1) Mean field → $\boxed{\text{NEXT LESSON}}$

2) Residual interaction

$\boxed{\text{THIS LESSON}}$

GENERAL SETUP

1) Mean field potential

$$H = \sum_i h_i^{MF} + \Delta V$$

2) Monoparticular wavefunctions

$$h \phi^{(n)} = \epsilon_n \phi^{(n)}$$

3) Total wf → antisymmetrize

$$\Psi_\alpha = \mathcal{A}\left[\prod_{i=1}^{A} \phi_i\right]$$

Notice this :

[The Φ_α's spawn an A-body Hilbert space]

$$\Phi_{\alpha} = \mathcal{A}\Big[\prod_i \phi_i^{(n_i)}\Big]$$

↳ Each α represents a possible combination of monoparticular energy levels / Example :

$$\alpha = \{ \text{2 protons in } 1s_{1/2},\ \text{1 neutron in } 1s_{1/2},\ \text{1 neutron in } 1p_{3/2} \}$$

(but $\exists$ infinite α's)

Easy to see for 1-particle in 1-dimension:

1) Standard Schrödinger equation

$$H|\psi\rangle = E|\psi\rangle$$

$$-\frac{1}{2m}\psi''(x) + V(x)\psi(x) = E\psi(x)$$

2) Oscillator basis:

$$V \to \underbrace{\frac{1}{2}m\omega^2 r^2}_{V^{MF}} + \underbrace{\left(V - \frac{1}{2}m\omega^2 r^2\right)}_{\Delta V}$$

$$\underbrace{H^{MF}|n\rangle = \omega\left(n+\frac{1}{2}\right)|n\rangle}_{\text{SM-like solution}}$$

3) Solve in oscillator basis:

$$H|\psi\rangle = E|\psi\rangle$$

$$|\psi\rangle = \sum_n |n\rangle\langle n|\psi\rangle$$

$$= \sum_n \psi_n |n\rangle$$

$$H_{mn} = \langle m|H|n\rangle$$

$$\boxed{\underbrace{H_{mn}}_{\text{matrix}}\,\underbrace{\psi_n}_{\text{eigenvector}} = \underbrace{E}_{\text{eigenvalue}}\,\psi_n}$$

1) Schrödinger in standard basis

$$\boxed{H|\psi\rangle = E|\psi\rangle}$$

2) Schrödinger in oscillator basis

$$\boxed{H_{mn}\psi_n = E\,\psi_n}$$

⇒ Equivalent ways to solve Schrödinger

⇒ Very straightforward for 1-particle & 1-dim

⇒ For the SM the only difference is: $\boxed{\text{A-particles, 3-dims}}$

BASIC IDEA

→ We can solve A-body Schrödinger in the mean-field basis

→ It becomes an infinite dimensional matrix

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1) ORIGINAL PROBLEM

$$H|\Psi\rangle = E|\Psi\rangle$$

$$H = \sum_i T_i + \sum_{ij} V_{ij} + \sum_{ijk} V_{ijk}$$

2) MEAN FIELD PROBLEM

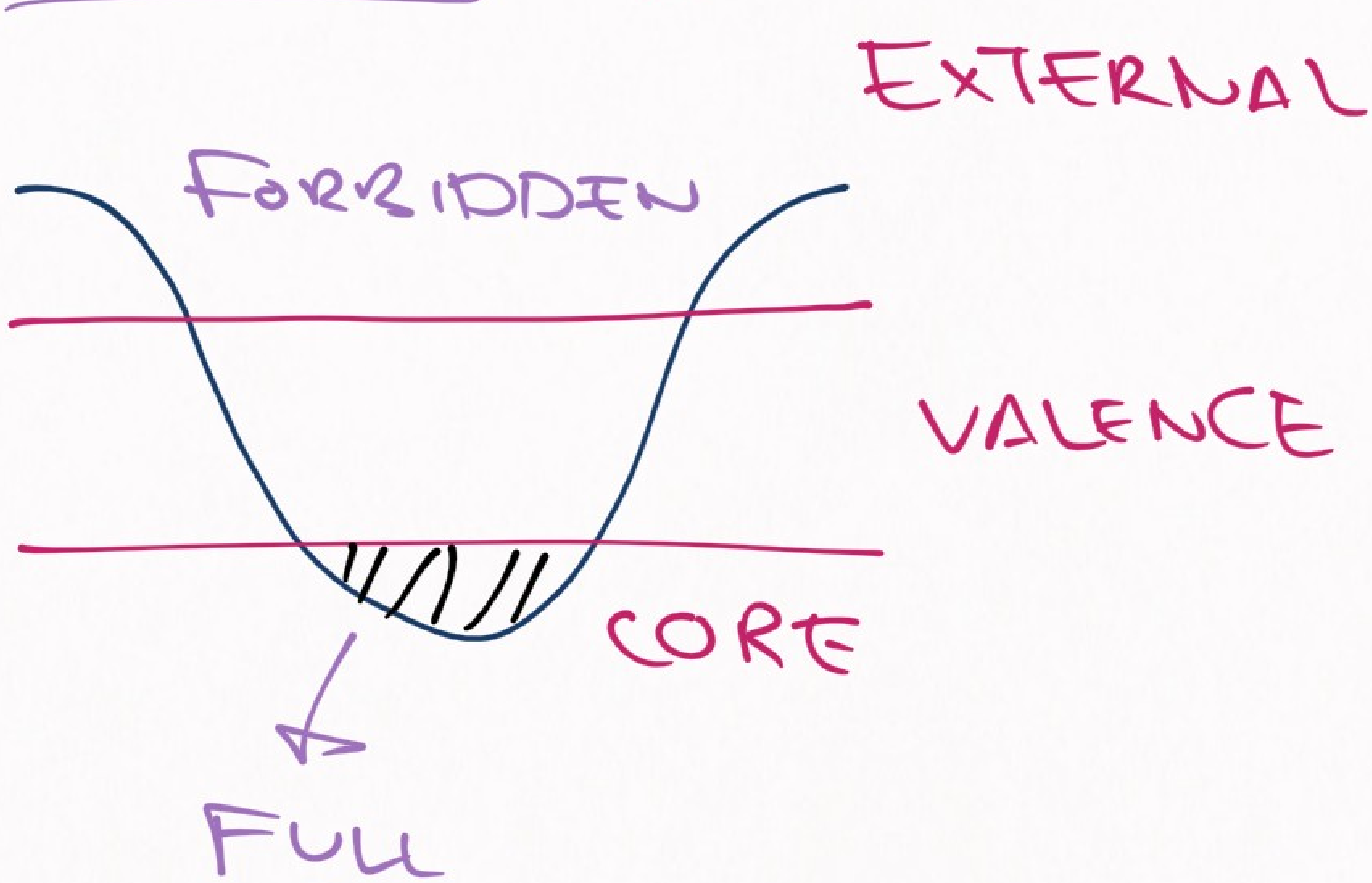
$$H_{\alpha\beta}\Psi_\beta = E\Psi_\alpha$$

$$\Psi_\alpha = \mathcal{A}\left[\prod_{i=1}^{A} \phi_i(r_i)\right]$$

PROBLEM → $\boxed{\text{INFINITE DIMENSIONAL}}$

We have to reduce the dimensionality if we want to solve the shell-model

$\boxed{\text{SOLUTION}}$ →



Normally we divide the shell space into 1) CORE

2) VALENCE

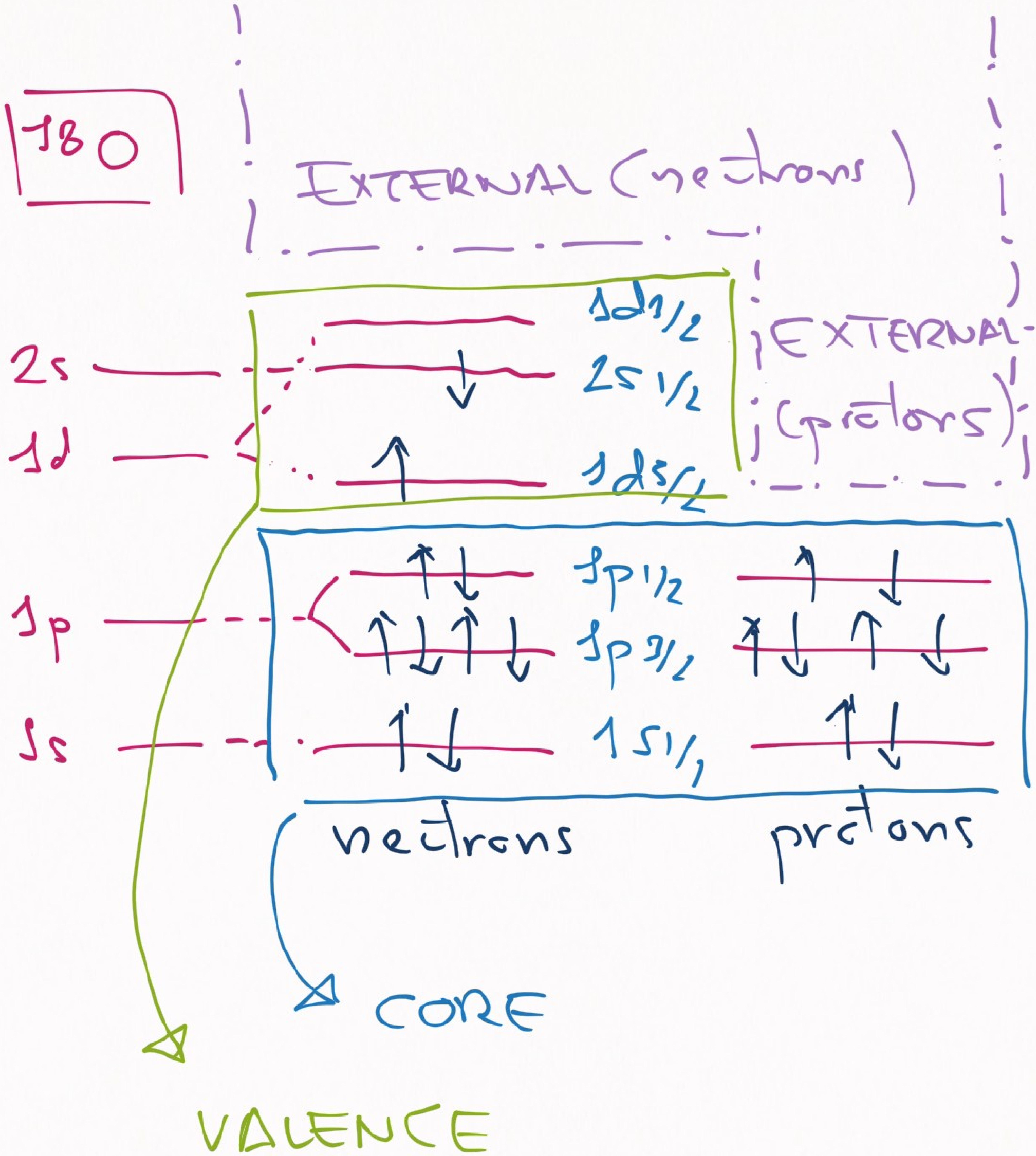
3) EXTERNAL

This is what matters

Reduces the dimension of the matrices a lot

Example → ^{18}O

^{16}O CORE + two s,d valence neutrons



Typical Interacting SM

How do we solve it?

1) We write all the Ψ_α

$$\Psi_\alpha = \Psi_{core} \times \Psi_{valence}(\alpha)$$

2) We find the dimensions

2s ......

1d ......

$1d_{3/2}$ (4)

$2s_{1/2}$ (2)

$1d_{5/2}$ (6)

Valence → a) 12 possible states

$6+2+4$

b) 2 particles

2 neutrons

$$\binom{12}{2} = \frac{11\cdot 10}{2} = 55 \text{ configurations}$$

$\Rightarrow$ $\boxed{55 \text{ dimensions}}$

3) We build the base:

Ψ_α, for $\alpha = 1, 2, \dots, 55$

Each α a different combination of two neutrons in 12 states

$$\alpha \rightarrow |1d_{5/2}(+5/2)\, 1d_{5/2}(+3/2)\rangle$$

$$|1d_{5/2}(+5/2)\, 1d_{5/2}(+1/2)\rangle$$

$$\vdots$$

$$\underbrace{|1d_{3/2}\underbrace{(-1/2)}_{m_1}\, 1d_{3/2}\underbrace{(-3/2)}_{m_2}\rangle}_{55 \text{ possibilities}}$$

4) We find the matrix elements of H

$$H \longrightarrow H_{\alpha\beta} = \underbrace{\begin{pmatrix} H_{11} & H_{12} & \cdots \\ H_{21} & H_{22} & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix}}_{55 \times 55 \text{ matrix}}$$

5) We diagonalize & find the eigenvectors & eigenvalues

$\rightarrow$ $\boxed{\text{GREAT!?}}$

$\rightarrow$ Can't wait till we go to heavier nuclei & we diagonalize $10^6 \times 10^6$ matrices !!

DIMENSIONALITY (BAD NEWS)

$$\text{Dim} \sim \underbrace{\binom{\Omega_p}{N_p}\binom{\Omega_n}{N_n}}_{\text{binomial coefficients}}$$

$\Omega_p, \Omega_n \rightarrow$ possible valence states for protons & neutrons

$N_p, N_n \rightarrow$ number of protons & neutrons in the valence

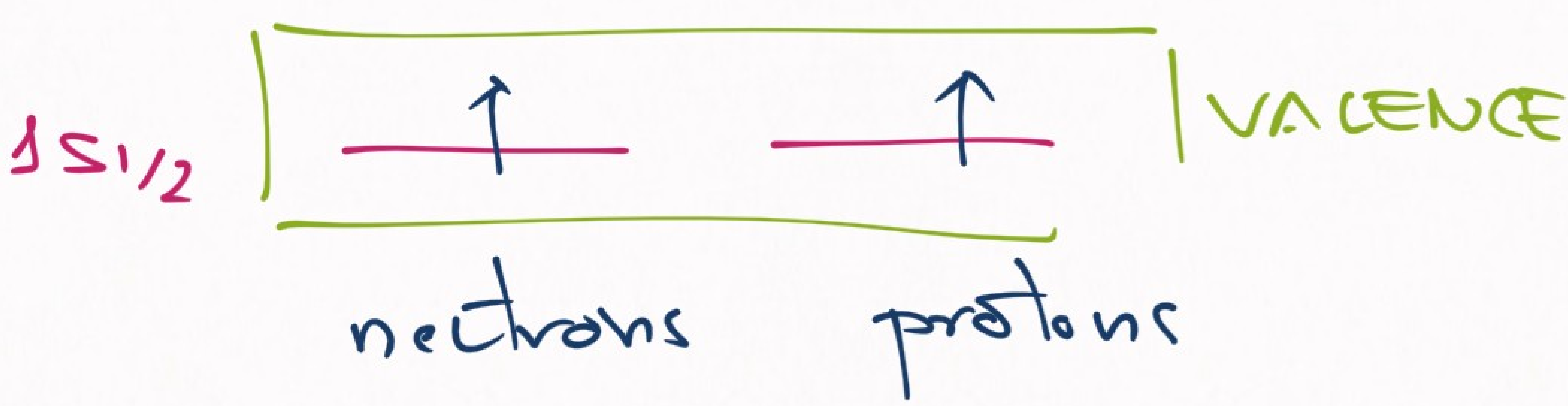
DIMENSIONALITY (GOOD NEWS)

$$\mathrm{Dim} \ll \binom{\Omega_p}{N_p}\binom{\Omega_n}{N_n}$$

Once we restrict the J^π in which we are interested

$$\Rightarrow$$

EXAMPLE 1 $\longrightarrow$ DEUTERON



$$\mathrm{Dim} = \binom{2}{1}\binom{2}{1} = 4$$

less

$$(\mathrm{Dim} / J = 1) = 3$$

much less

$$(\mathrm{Dim} / J, M = 1, 1) = 1$$

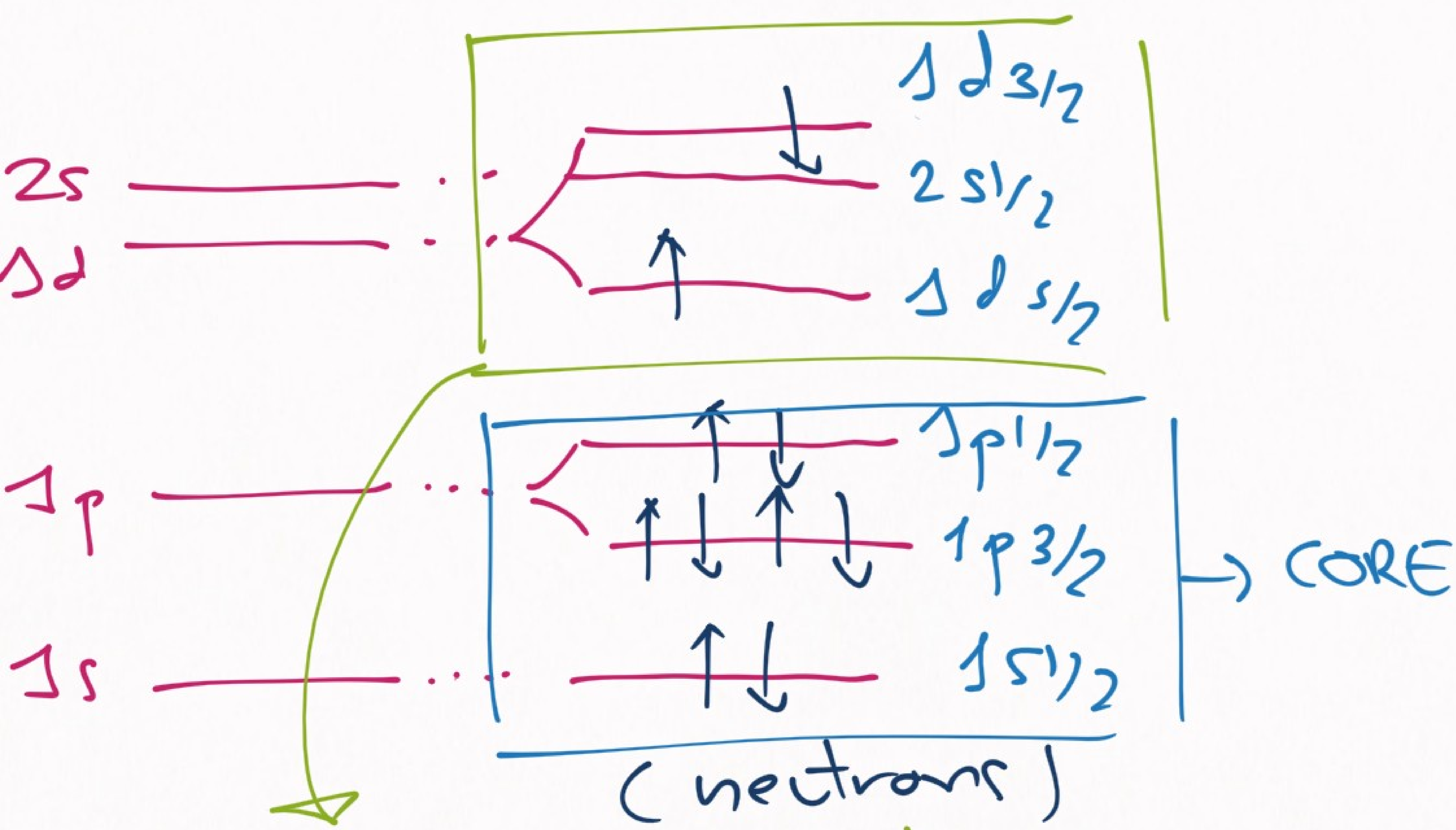
$$\Rightarrow$$

But the deuteron is not good for using the shell-model

→ too shallow, too big

(E_B too small)

EXAMPLE (2) → ^{18}O, $J^\pi = 0^+$ states



valence: 2 neutrons coupled to $J^\pi = 0^+$

$^{18}O \to {}^{16}O$ core

$\boxed{+\ 2n \text{ in } (s,d)\text{-shell as valence}}$

How many 0^+ configurations?

$d_{3/2}$

$s_{1/2}$

$d_{5/2}$

$\frac{1}{2} \otimes \frac{1}{2} = \underbrace{0}_{A} \oplus \underbrace{\cancel{1}}_{S}$ ✓

$\frac{3}{2} \otimes \frac{3}{2} = \underbrace{0}_{A} \oplus \ldots$ ✓

$\frac{5}{2} \otimes \frac{5}{2} = \underbrace{0}_{A} \oplus \ldots$ ✓

$\cancel{\frac{5}{2} \otimes \frac{1}{2} = 2 \oplus 3}$ $\qquad$ $\cancel{\frac{5}{2} \otimes \frac{3}{2} = 1 \oplus 2 \oplus 3 \oplus 4}$

no 0^+

$\cancel{\frac{3}{2} \otimes \frac{1}{2} = 1 \oplus 2}$

$S \to$ symmetric

$A \to$ antisymmetric

$\boxed{^{18}O} \longrightarrow {}^{16}O$ core + $2n\ (s,d)$ valence

$\boxed{J^P = 0^+} \Longrightarrow \boxed{\mathrm{Dim} = 3}$

A doable calculation

$\boxed{\text{How to do it?}}$

1) Single particle levels

$h^{MF}\phi_a = \epsilon_a \phi_a$ for $a = 1d_{5/2}$, $2s_{1/2}$, $1d_{3/2}$

$$\epsilon(1d_{5/2}) = E_B(^{17}O) - E_B(^{16}O)$$

$$\epsilon(2s_{1/2}) = \epsilon(1d_{5/2}) + E_{ex}(^{17}O; \tfrac{1}{2}^+)$$

$$\epsilon(1d_{3/2}) = \epsilon(1d_{5/2}) + E_{ex}(^{17}O; \tfrac{3}{2}^+)$$

Results:

$$\epsilon(1d_{5/2}) = -4.1 \text{ MeV}$$
$$\epsilon(2s_{1/2}) = -3.3 \text{ MeV}$$
$$\epsilon(1d_{3/2}) = 0.9 \text{ MeV}$$

2) Matrix elements of H

$$\alpha = \{ (1d_{5/2})^2, (2s_{1/2})^2, \underbrace{(1d_{3/2})^2}\}$$
$$= \{1, 2, 3\}$$

$()^2 \Rightarrow$ two nucleous in this shell

$$H_{ij} = 2\delta_{ij}\epsilon_j + V_{ij}$$

$$V = \begin{pmatrix} -2.8 & -1.3 & -3.2 \\ -1.3 & -2.1 & -1.1 \\ -3.2 & -1.1 & -2.2 \end{pmatrix}$$

3) Diagonalize H

$H_y \rightarrow \boxed{-12.6,\ -8.1,\ +0.6}$

4) Get excitation energies

$E(0^+_1) = -8.1 + 12.6 = 4.4$ MeV

$E(0^+_2) = +0.6 + 12.6 = 13.1$ MeV

→ these are not absolute binding energies
(core not included)

5) Repeat for $1^+, 2^+, 3^+, 4^+$ to obtain full spectrum

(Example taken from lectures by Nadia Smyrnova)

EXAMPLE ③ $\rightarrow$ ^{18}O, $J^P = 3^+$

Again $\rightarrow$ ^{16}O core + 2n in (s, d) valence shell

NOW: COUPLED TO $\underline{\underline{3^+}}$

$1d_{3/2}$

$2s_{1/2}$

$1d_{5/2}$

$\frac{3}{2} \otimes \frac{3}{2} = 0 \oplus 1 \oplus 2 \oplus 3$ Nope

Symmetric

$\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$ nope

$\frac{5}{2} \otimes \frac{5}{2} = 0 \oplus 1 \oplus 2 \oplus 3 \oplus 4 \oplus 5$ nope

Symmetric

$\frac{1}{2} \otimes \frac{3}{2} = 1 \oplus 2$ nope

$$\boxed{\begin{array}{c} \frac{1}{2} \otimes \frac{5}{2} \\ \frac{3}{2} \otimes \frac{5}{2} \end{array}}$$

only two options

$\boxed{DIM = 2}$

1) Possible states:

$$\alpha = \{(1d5/2)(2s1/2), (1d5/2)(1d3/2)\}$$
$$= \{1, 2\}$$

2) Hamiltonian

$$H_{11} = \epsilon(1d5/2) + \epsilon(2s1/2) + V_{11}$$
$$H_{12} = H_{21} = V_{12} = V_{21}$$
$$H_{22} = \epsilon(1d5/2) + \epsilon(1d3/2) + V_{22}$$

$$V = \begin{pmatrix} 0.8 & 0.7 \\ 0.7 & 0.6 \end{pmatrix} \text{ MeV}$$

3) Diagonalize

$\rightarrow \boxed{-6.6, -2.5}$

$\rightarrow$ Just like before

And this gives us the basic mechanics of the SM

There is also the $\boxed{\text{NO-CORE SM}}$

Remove the core, which now becomes valence

$\boxed{\text{BIGGER MATRICES!}}$

NEXT LESSON:

$\boxed{\text{THE MEAN FIELD}}$