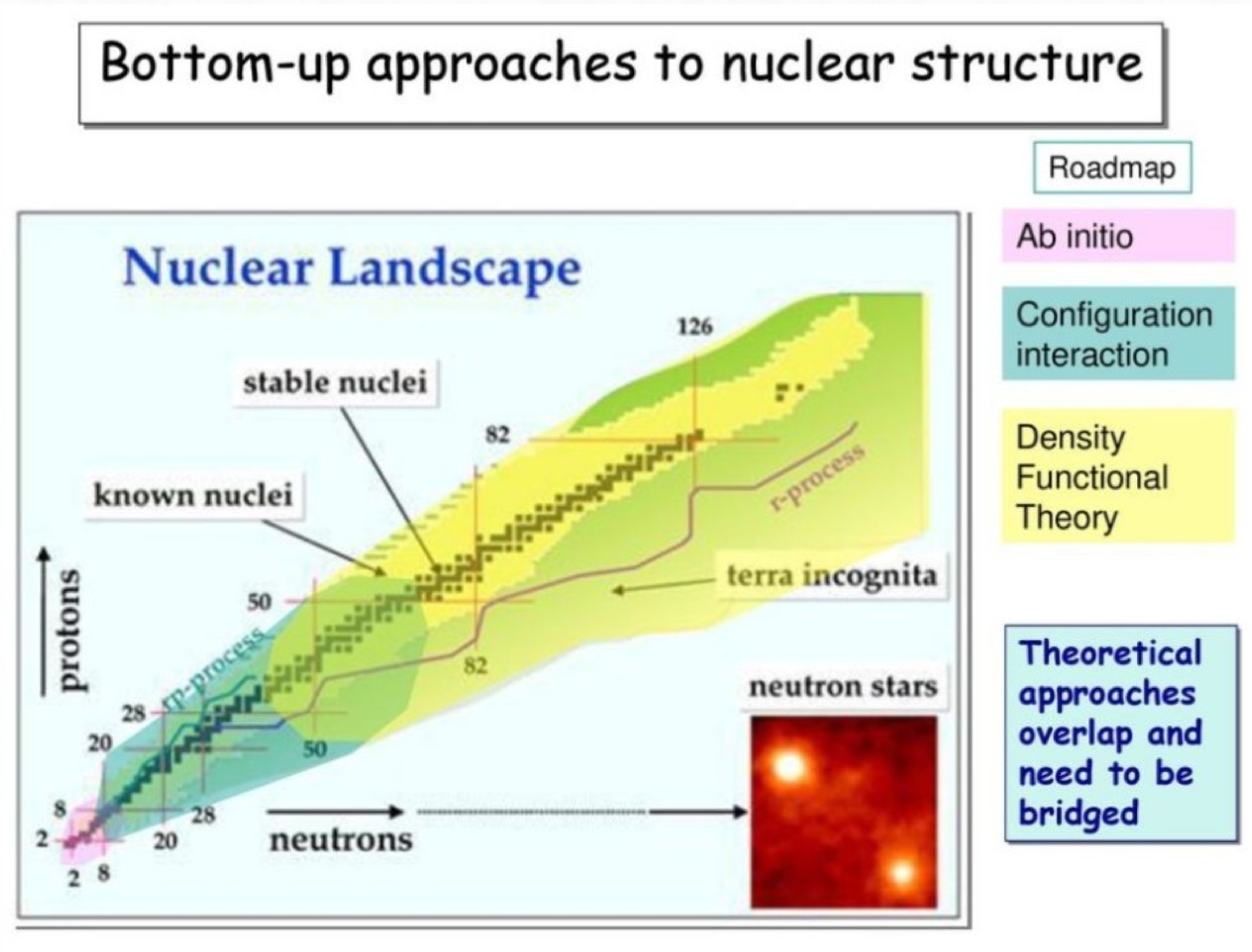
Nuclear Physics 21

The three-body problem
Part 1: Faddeev Equations

Warning: difficult stuff

A roadmap of nuclear physics



→ $\boxed{\text{too many nuclei}}$

A → # of nucleons

$\boxed{A \leq 10-12}$ → $\boxed{\text{ab-initio methods}}$

What does ab-initio mean?

$\longrightarrow$ $\boxed{\text{input}: \mathcal{H}\ (V)}$

$\Downarrow$

$\boxed{\text{output}: |\psi\rangle, E_B}$

It means what we have always done:

1) use some potential

2) plug it in the Schrödinger equation (or an equivalent)

3) get binding energies & wave functions

1) Coordinate space :

→ Schrödinger equation

2) Momentum space :

$A=2$ → Lippmann-Schwinger

$A=3$ → Faddeev

$A \geq 4$ → Faddeev-Yakubowski

— ⊗ —

First, for $A=3$ we have :

a) Lippmann-Schwinger does not work (problem)

b) But Faddeev found a clever trick (solution)

Let's consider a 3-body system:

$$H = H_0 + V$$

$$= \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + \frac{p_3^2}{2m_3}$$

$$+ V_{12} + V_{13} + V_{23}$$

or in p-space:

$$T(z) = V + V G_0(z) T(z)$$

$$G_0(z) = \frac{1}{z - H_0}$$

or even better: p-space + bound state

$$|\Psi_{3B}\rangle = G_0(z) V |\Psi_{3B}\rangle$$

Two-body PK Three-body:

A=2) Remove total momentum, get nice equation

$$\left[\langle \vec{p}_1{}' \vec{p}_2{}' | V | \vec{p}_1 \vec{p}_2 \rangle = (2\pi)^3 \delta^{(3)}(\vec{P}' - \vec{P}) \langle \vec{p}\,' | V | \vec{p} \rangle \right]$$

$$\vec{P} = \vec{p}_1 + \vec{p}_2 \quad (\vec{P}' = \vec{p}_1{}' + \vec{p}_2{}')$$

$$\vec{p} = \frac{1}{2}(\vec{p}_1 - \vec{p}_2) \quad (\vec{p}\,' = \dots)$$

$$\left[\langle \vec{p}_1{}' \vec{p}_2{}' | G_0(z) | \vec{p}_1 \vec{p}_2 \rangle = (2\pi)^3 \delta^{(3)}(\vec{P}' - \vec{P}) \frac{(2\pi)^3 \delta^{(3)}(\vec{p} - \vec{p}\,')}{\underbrace{E - \frac{P^2}{2M}}_{E_{CM}} - \frac{p^2}{2\mu}} \right]$$

From this: easy equation $\boxed{\text{without } \delta^{(3)}\text{'s}}$

$\Delta = 3$) Remove total momentum

... there are still δ's!?

$\longrightarrow$ $\boxed{\text{PROBLEM}}$

$$\left[\begin{array}{l} \langle \bar{p}_1' \bar{p}_2' \bar{p}_3' | V_{12} | \bar{p}_1 \bar{p}_2 \bar{p}_3 \rangle = \\ (2\pi)^3 \delta^{(3)}(P_{12}' - P_{12}) \, (2\pi)^3 \delta^{(3)}(\bar{p}_3' - \bar{p}_3) \\ \times \langle \bar{p}_{12}' | V | \bar{p}_{12} \rangle \end{array}\right]$$

Δ Repeat this w/ V_{13}, V_{23}

Δ In each case a different

δ combination

Δ $\boxed{\text{Can't get rid of singularities!}}$

$\boxed{\text{PROBLEM}}$ → $\boxed{\text{Lippmann-Schwinger for } A=3 \text{ full of Dirac deltas}}$

↓

In general ($\exists$ exceptions: $V_{12} = V_{23} = V_{31} = 0$ but $V_{123} \neq 0$)

↓

But Faddeev thought of a clever trick:

$\boxed{\text{Faddeev components}}$

↓

the name of the trick

Let's see how this is done:

$$T(z) = V + V G_0(z) T(z)$$

(original)

$$T(z) = T^{(1)}(z) + T^{(2)}(z) + T^{(3)}(z)$$

(decomposition)

w/ $T^{(k)}(z) = V_{ij} + V_{ij} G_0(z) T(z)$

$(ijk = 123, 231, 312)$

(even permutation)

But we have to go further

$\Rightarrow$ [Consider the two-body T-matrices]

$$T_{ij}(z) = V_{ij} + V_{ij} G_0(z) T_{ij}(z)$$

Equivalent two-body equation:

$$(1 - V_{ij} G_0(z))^{-1} T_{ij}(z) = V_{ij}$$

And we use it to get rid of $T(z)$

$$T^{(k)}(z) = V_{ij} + V_{ij} G_0(z) T(z)$$

$$(1 - V_{ij} G_0(z)) T^{(k)}(z) =$$

$$V_{ij} + V_{ij} G_0(z) [T^{(i)}(z) + T^{(j)}(z)]$$

$$T^{(k)}(z) = (1 - V_{ij} G_0(z))^{-1} [\dots]$$

FADDEEV EQUATIONS

$$\boxed{T^{(k)}(z) = T_{ij}(z) + T_{ij}(z) G_0(z) [T^{(i)}(z) + T^{(j)}(z)]}$$

$\delta^{(3)}(P_{ij}' - P_{ij}) \delta^{(3)}(p_k' - p_k)$ which can be removed

1) Lippmann–Schwinger (A=3)

1.a) Only 1 equation

1.b) 3 types of δ-terms:

$$\delta^{(3)}(\underline{p}'_j - \underline{p}_j)\,\delta^{(3)}(\underline{p}'_k - \underline{p}_k)$$

$jk = 123, 231, 312$

$\longrightarrow$ $\boxed{\text{non-removable}}$

2) Faddeev (A=3)

2.a) 3 equations (more)

2.b) Each equation only 1-type of δ-term (each equation $\rightarrow$ 1 permutation)

$\longrightarrow$ $\boxed{\text{removable}}$

Next step: $\boxed{\text{FADDEEV FOR BOUND STATE}}$

1) $$T(z) \xrightarrow[z \to E_{3B}]{} G_0^{-1}(z) \frac{|\psi_{3B}\rangle\langle\psi_{3B}|}{z - E_{3B}} G_0^{-1}(z)$$

2) Faddeev trick:

$$|\psi_{3B}\rangle = |\psi^{(1)}\rangle + |\psi^{(2)}\rangle + |\psi^{(3)}\rangle$$

$$T(z) = T^{(1)}(z) + T^{(2)}(z) + T^{(3)}(z)$$

$$T^{(k)}(z) \xrightarrow[z \to E_{3B}]{} G_0^{-1} \frac{|\psi^{(k)}\rangle\langle\psi_{3B}|}{z - E_{3B}} G_0^{-1}$$

3) Rearrange equations around the pole

$$\boxed{|\psi^{(x)}\rangle = G_0(z)\, T_y(z) \times \left[|\psi^{(z)}\rangle + |\psi^{(y)}\rangle \right]}$$

FADEEV EQUATIONS FOR BOUND STATES

Or in matrix form:

$$\begin{pmatrix} |\psi^{(1)}\rangle \\ |\psi^{(2)}\rangle \\ |\psi^{(3)}\rangle \end{pmatrix} = G_0(z) \begin{pmatrix} 0 & T_{23} & T_{23} \\ T_{31} & 0 & T_{31} \\ T_{12} & T_{12} & 0 \end{pmatrix} \times \begin{pmatrix} |\psi^{(1)}\rangle \\ |\psi^{(2)}\rangle \\ |\psi^{(3)}\rangle \end{pmatrix}$$

→ Only a complicated version of the two-body BS equation

Next step $\rightarrow$ find explicit expressions for the Faddeev equations

$\boxed{\text{Jacobi coordinates}}$

A=2) $\vec{k}_1, \vec{k}_2$

$$\vec{p}_{12} = \frac{m_2 \vec{k}_1 - m_1 \vec{k}_2}{m_1 + m_2}, \quad \vec{P}_{12} = \vec{k}_1 + \vec{k}_2$$

A=3) $\vec{k}_1, \vec{k}_2, \vec{k}_3$

Jacobi momenta

$$\vec{P} = \vec{k}_1 + \vec{k}_2 + \vec{k}_3$$

$$\vec{p}_1 = \frac{1}{M}\{(m_2+m_3)\vec{k}_1 - m_1 \times (\vec{k}_2 + \vec{k}_3)\}$$

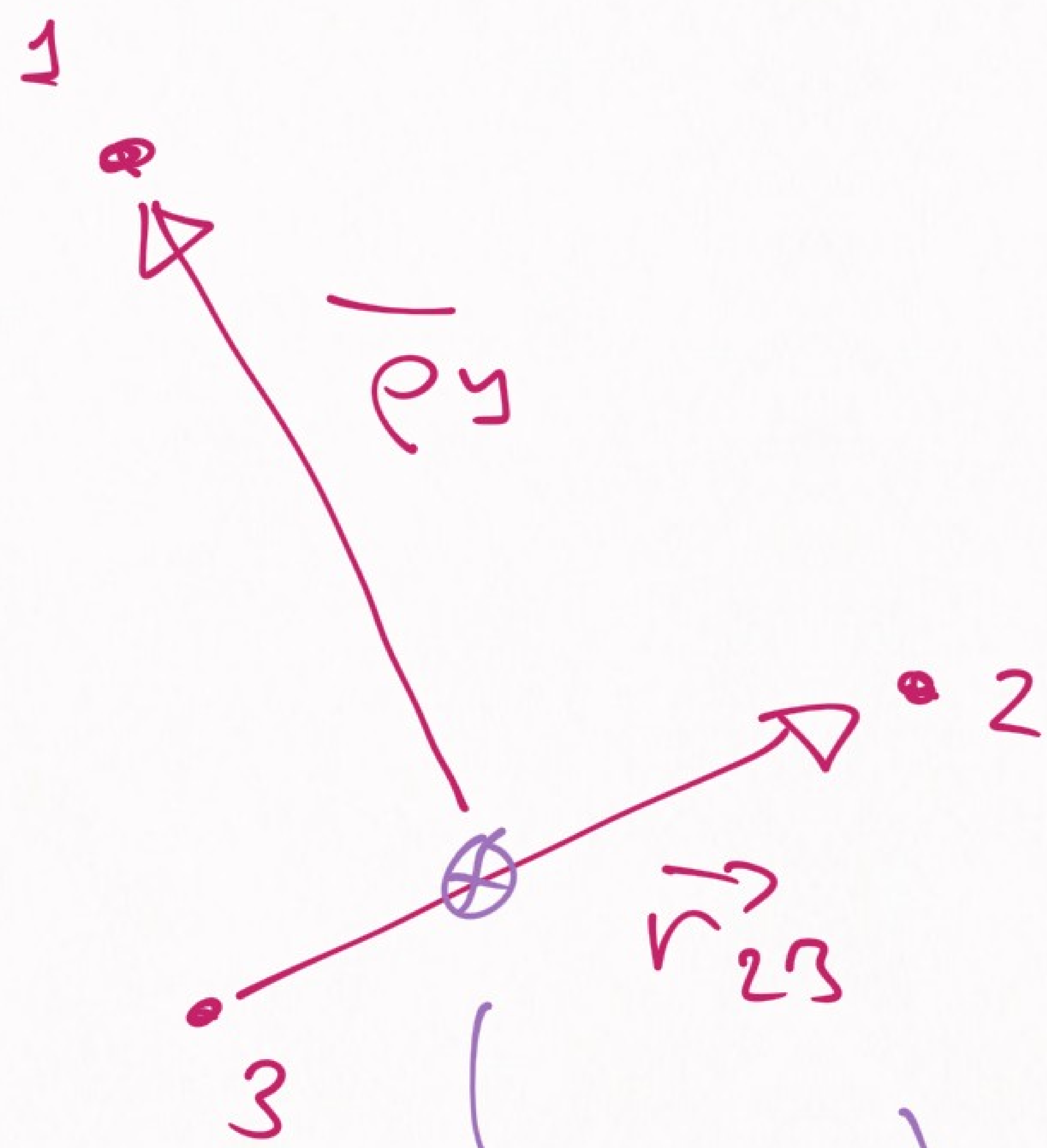
$$\vec{k}_{23} = \frac{m_3 \vec{k}_2 - m_2 \vec{k}_3}{m_2 + m_3}$$

+ permutations

Easier to understand in r-space:

$$\vec{r}_1, \vec{r}_2, \vec{r}_3 \quad \left[\begin{array}{l} \vec{\rho}_1 = \vec{r}_1 - \dfrac{m_2 \vec{r}_2 + m_3 \vec{r}_3}{m_2 + m_3} \\[2ex] \vec{r}_{23} = \vec{r}_2 - \vec{r}_3 \end{array}\right]$$

+ permutations



center-of-mass of particles 2 & 3

The point → we can rewrite Faddeev equations w/ Jacobi momenta

$$\frac{k_1^2}{2m_1}+\frac{k_2^2}{2m_2}+\frac{k_3^2}{2m_3}=\frac{P^2}{2M}+\frac{\vec{k}_{23}^2}{2\mu_{23}}+\frac{\vec{p}_1^{\,2}}{2\mu_1}$$

$$\frac{1}{\mu_{23}}=\frac{1}{m_2}+\frac{1}{m_3}\,,\quad \frac{1}{\mu_1}=\frac{1}{m_1}+\frac{1}{m_2+m_3}$$

$$\psi^{(1)}(\vec{k}_{23},\vec{p}_1)=\left(z-\frac{k_{23}^2}{2\mu_{23}}-\frac{p_1^2}{2\mu_1}\right)^{-1}\times$$

$$\times\int\frac{d^3\vec{k}_{23}'}{(2\pi)^3}\langle\vec{k}_{23}|t_{23}\left(z-\frac{k_{23}^2}{2\mu_{23}}\right)|\vec{k}_{23}'\rangle$$

$$\times\left[\psi^{(2)}(\vec{k}_{31}',\vec{p}_2')+\psi^{(3)}(\vec{k}_{12}',\vec{p}_3')\right]$$

+ permutations

Next thing: Faddeev equations are easily solvable for separable potentials

$$\langle \vec{k}' | V_j | \vec{k} \rangle = \lambda_j \, g_i(k) \, g_j(k)$$

$$\langle \vec{k}' | t_j(z) | \vec{k} \rangle = \tau_j(z) \, g_i(k) \, g_j(k)$$

$$\psi^{(i)}(\vec{k}, \vec{p}) = N \frac{g_i(k) \, a_i(p)}{-z + \frac{k^2}{2\mu_j} + \frac{p^2}{2\mu_k}}$$

The Faddeev equations will only involve $\boxed{a_i(p)}$

We end up w/ this type of equations: (easy to discretize)

$$a_k(p_k) = \tau_{ij}(z_{ij}) \times \Big[\int \frac{d^3\vec{p}_i}{(2\pi)^3} B_{ki}(\vec{p}_k, \vec{p}_i)\, a_i(p_i) + \int \frac{d^3 \vec{p}_j}{(2\pi)^3} B_{kj}(\vec{p}_k, \vec{p}_j)\, a_j(p_j) \Big]$$

with: $z_{ij} = z - \frac{p_k^2}{2\mu_k}$

$$B_{ij}(\vec{p}_i, \vec{p}_j) = \frac{g_i(\vec{q}_i)\, g_j(\vec{q}_j)}{z - \frac{p_1^2}{2m_1} - \frac{p_2^2}{2m_2} - \frac{p_3^2}{2m_3}}$$

$\vec{p}_1 + \vec{p}_2 + \vec{p}_3 = 0$ & $\vec{q}_i = \frac{m_k \vec{p}_j - m_j \vec{p}_k}{m_k + m_j}$

$(ijk = 123, 231, 312)$

Application → $\boxed{\text{3-BOSON SYSTEM w/ RESONANT INTERACTIONS}}$

& $\boxed{\text{THE EFIMOV EFFECT}}$

→ Curious phenomenon that happens in systems of identical particles w/ large scattering lengths

$\boxed{\text{BOSONS}}$ → $\boxed{\text{IDENTICAL PARTICLES}}$

$$\Rightarrow \Psi_{3B} = \psi^{(1)} + \psi^{(2)} + \psi^{(3)}$$

$$\Rightarrow \boxed{\begin{array}{l} \psi^{(1)} = \psi^{(2)} = \psi^{(3)} \\ \psi(-\vec{k}, \vec{p}) = \psi(\vec{k}, \vec{p}) \end{array}}$$

$$\boxed{\text{RESONANT INTERACTIONS}} \Rightarrow \boxed{a_0 \to \infty}$$

The scattering length diverges

$$\tau(z) = \frac{2\pi}{\mu}\frac{1}{ik} = \frac{2\pi}{\mu}\frac{1}{i\sqrt{mz}}$$

(for $z>0$)

$$= \frac{2\pi}{\mu}\frac{1}{-\sqrt{-mz}}$$ (for $z<0$)

— ⊗ —

What happens in this limit?
(Simplified version)

$$\psi(\bar{k},\bar{p}) \to \mathcal{N}\,\frac{a(\bar{p})\,g(k)}{k^2+\frac{3}{4}p^2+\gamma^2}$$

w/ $g(k) \to 1$ $\quad (m_1 = m_2 = m_3 = m)$

$\boxed{a_0 \to \infty}$ $\Rightarrow$ problem w/ no characteristic scale

Just like this

$$\psi(\vec{k},\vec{p}) \to \frac{a(p)}{k^2+\frac{3}{4}p^2}$$

$$\boxed{a(p) \sim p^{s-2}}$$

pure power-law dependence

It happens that by solving the Faddeev equations we can obtain the equation for s:

$$1 = \frac{4}{\pi\sqrt{3}} \int_0^\infty dx \, x^{s-1} \log\left(\frac{1+x+x^2}{1-x+x^2}\right)$$

Mellin transformation

$$1 = \frac{8}{\sqrt{3}\, s} \frac{\sin(\pi s/6)}{\cos(\pi s/2)}$$

$$\boxed{s = \pm i s_0 \,, \quad s_0 = 1.00624}$$

Imaginary solution

$$x^{i s_0} = e^{i s_0 \log(x)} = \cos(s_0 \log x) + i \sin(s_0 \log x)$$

(oscillatory solution)

In short, we end up with:

$$\psi(k,p) = \mathcal{N}\,\frac{1}{p^2}\,\frac{\sin\left(s_0 \log \frac{p}{\Lambda_*}\right)}{k^2 + \frac{3}{4}p^2}$$

discrete scale invariance

$$\psi(\lambda_0 k, \lambda_0 p) = \frac{1}{\lambda_0^4}\,\psi(k,p)$$

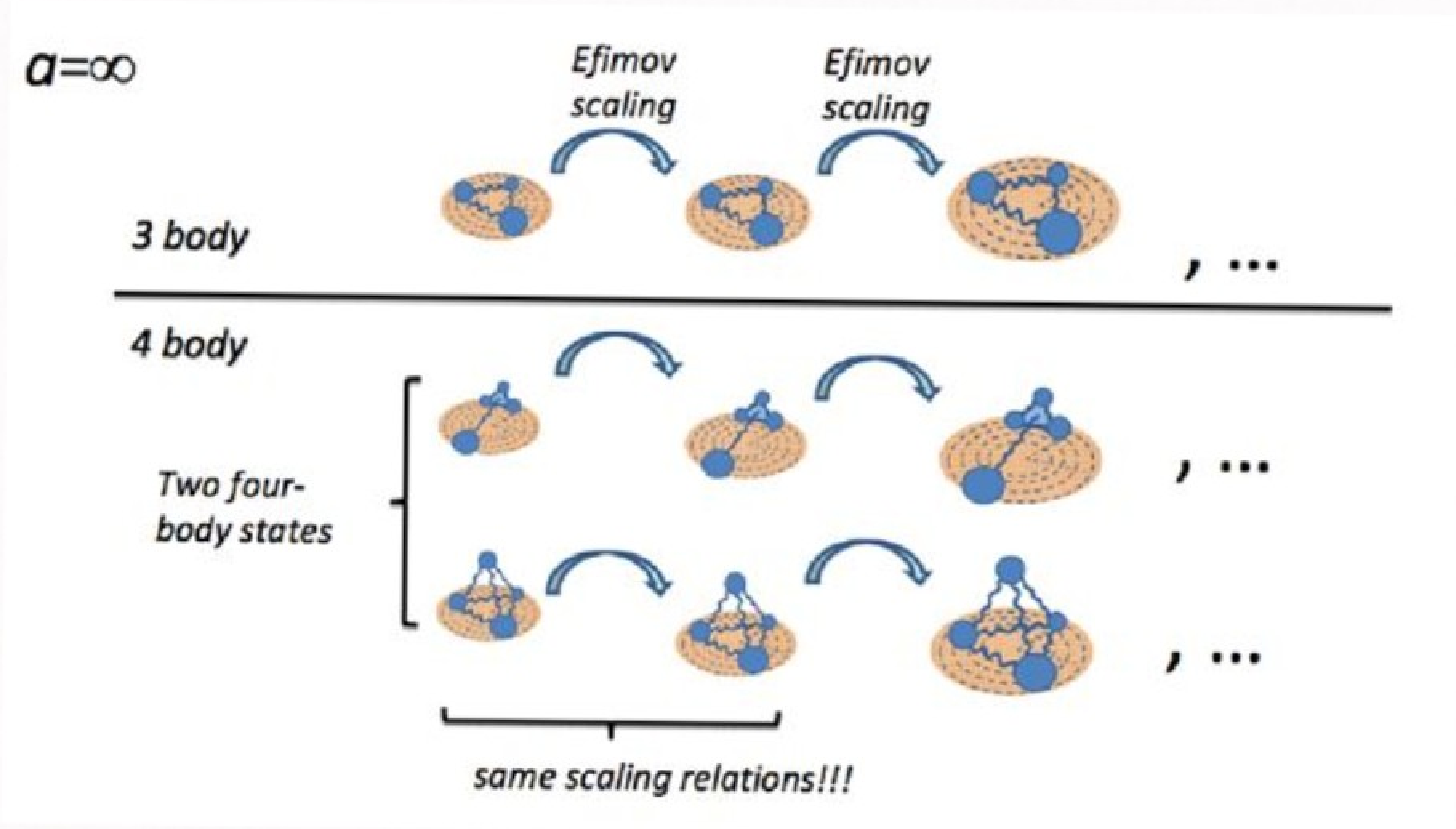
w/ $\lambda_0 = e^{\pi/s_0} \simeq 22.69$

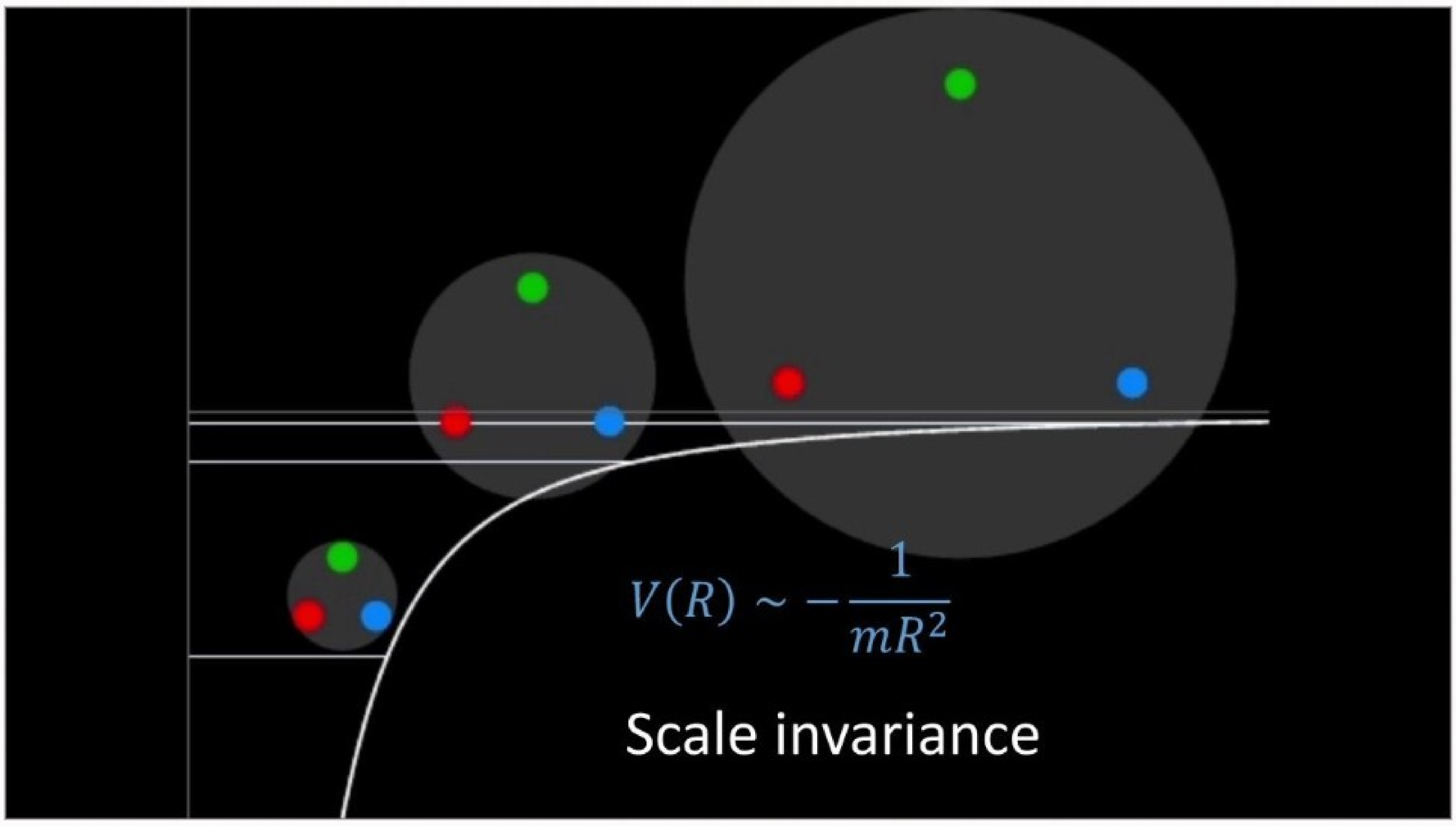
Just like this type of system

Repeats itself

→ This is called the Efimov effect

Matrioska-like series of 3-body bound states





Efimov spectrum:

$$E_n = \frac{E_0}{\lambda_0^n}$$

($\lambda_0 \simeq 521$ for 3-bosons)

$\boxed{\text{Geometric}}$

Hydrogen atom:

$$E_n = \frac{E_0}{n+1}$$

$\boxed{\text{Power-law}}$

Efimov's original manuscript:

Energy levels arising form the resonant two-body forces in a three-body system

V. Efimov (Ioffe Phys. Tech. Inst.)
Dec, 1970

2 pages
Published in: *Phys.Lett.B* 33 (1970) 563-564
DOI: 10.1016/0370-2693(70)90349-7
View in: OSTI Information Bridge Server

cite

Experimental confirmation:

Published: 16 March 2006

Evidence for Efimov quantum states in an ultracold gas of caesium atoms

T. Kraemer, M. Mark, P. Waldburger, J. G. Danzl, C. Chin, B. Engeser, A. D. Lange, K. Pilch, A. Jaakkola, H.-C. Nägerl & R. Grimm

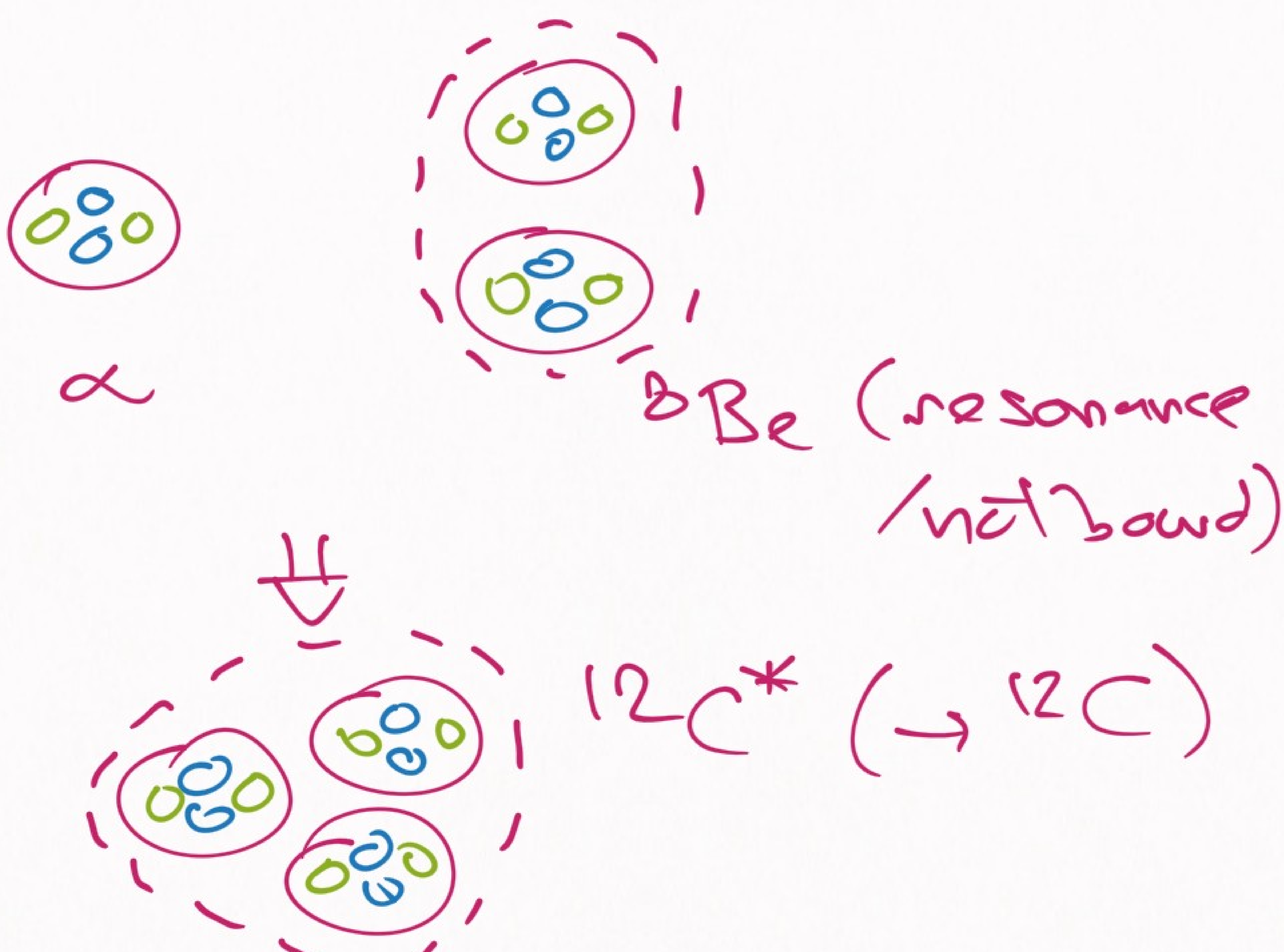
Nature **440**, 315–318(2006) | Cite this article

Possible examples in Nuclear Physics:

1) Triton & ^3He

(we will see why)

2) Triple-α process (^{12}C)



3) Halo nuclei (heteronuclear Efimov effect)

Next lesson :

→ $\boxed{\text{the triton}}$