Nuclear Physics (17)

The tensor force:

an unexpected nuisance

Let's go back a few lessons:

[Origin of the nuclear force]

1) Yukawa: exchange of a scalar meson

$$V_Y(\vec{r}) = -g^2 \frac{e^{-mr}}{4\pi r}$$

2) But deuteron has a quadrupolar moment

$$Q_d = 0.286 \text{ fm}^2$$

3) We need a pseudoscalar meson (the pion)

One pion exchange:

$$V_{OPE}(\vec{q}) = -\frac{g_A^2}{4f_\pi^2}\,\vec{\tau}_1\cdot\vec{\tau}_2\,\frac{\vec{\sigma}_1\cdot\vec{q}\;\vec{\sigma}_2\cdot\vec{q}}{m_\pi^2+|\vec{q}|^2}$$

$g_A \simeq 1.26$, $f_\pi \simeq 92.3$ MeV,

$m_\pi \simeq 138$ MeV

Fourier-transforms into:

$$V_{OPE}(\vec{r}) = \frac{g_A^2 m_\pi^3}{48\pi f_\pi^2}\,\vec{\tau}_1\cdot\vec{\tau}_2 \times (*)$$

$$(*) \times \left[\vec{\sigma}_1\cdot\vec{\sigma}_2\, W_C(m_\pi r) + S_{12}(\hat{r})\, W_T(m_\pi r)\right]$$

$$W_C(x) = \frac{e^{-x}}{x}$$

$$W_T(x) = \frac{e^{-x}}{x}\left(1+\frac{3}{x}+\frac{3}{x^2}\right)$$

Bottom-line:

$$V_{OPE}(\vec{r}) \propto \underbrace{\vec{\sigma}_1 \cdot \vec{\sigma}_2 \, W_C}_{\text{central (or spin-spin) term}} + \underbrace{S_{12}(\hat{r}) \, W_T}_{\text{tensor term}}$$

central (or spin-spin) term → 1

tensor term → 2

1 → $\vec{\sigma}_1 \cdot \vec{\sigma}_2 = \begin{cases} 1 & S=1 \\ -3 & S=0 \end{cases}$

(easy)

2 → $S_{12}(\hat{r}) = 3\, \vec{\sigma}_1 \cdot \hat{r} \; \vec{\sigma}_2 \cdot \hat{r} - \vec{\sigma}_1 \cdot \vec{\sigma}_2$

(difficult)

this is what we want to understand today

Tensor force → Why difficult?

Because it mixes partial waves

$$\underbrace{[S_{12}(\hat{r}), \vec{L}^2] \neq 0}_{\text{here lies the problem}}$$

$[S_{12}(\hat{r}), \vec{S}^2] = 0$ $\underbrace{\text{(by chance)}}_{\text{Only bc } S < 2}$

But luckily:

$$[S_{12}(\hat{r}), \vec{J}^2] = 0$$

$$\underbrace{\vec{J} = \vec{L} + \vec{S}}$$

↳ total angular momentum still conserved

Actually the problem lies only in $\underbrace{\text{triplet}}_{S=1}$ waves

1) Singlet waves $(S=0)$

$$ {}_S\langle 00 | 3\,\vec{\sigma}_1\cdot\hat{r}\,\vec{\sigma}_2\cdot\hat{r} - \vec{\sigma}_1\cdot\vec{\sigma}_2 | 00\rangle_S = 0 $$

$$ S_{12}(\hat{r}) = \sigma_{1i}\sigma_{2j}\left(3\hat{r}_i\hat{r}_j - \delta_{ij}\right) $$

$$ = \underbrace{\left(3\sigma_{1i}\sigma_{2j} - \vec{\sigma}_1\cdot\vec{\sigma}_2\,\delta_{ij}\right)}_{\substack{\text{antisymmetric}\\ \text{for } S=0}}\underbrace{\hat{r}_i\hat{r}_j}_{\text{symmetric}} $$

$$ \left(|00\rangle = \frac{1}{\sqrt{2}}\left\{ |+-\rangle - |-+\rangle \right\} \right) $$

2) Triplet waves ($S=1$)

$\rightarrow$ no simplification possible

But still: $[S_{12}(\hat{r}), \vec{J}^2] = 0$

$\rightarrow$ we begin by constructing states with good J

2.a) $|s m_s\rangle Y_{\ell m_\ell}(\hat{r})$

2.b) $|jm\rangle \rightarrow$ what we want

2.c)

$$|jm\rangle = \sum_{\substack{s m_s \\ \ell m_s}} Y_{\ell m_\ell}(\hat{r}) |s m_s\rangle \times \underbrace{\langle \ell m_\ell s m_\ell | jm\rangle}_{\text{Clebsch-Gordan coefficient}}$$

Finally, we compute the matrix elements:

$$\langle (s'\ell')j'm' | S_{12}(\hat{r}) | (s\ell)jm \rangle$$

$$= \underbrace{S^{j}_{s'\ell' s\ell}}_{\hookrightarrow \text{ general case}} \delta_{jj'} \delta_{mm'}$$

For triplets $s = s' = 1$

$$\langle (1\ell')j'm' | S_{12}(\hat{r}) | (1\ell)jm \rangle$$

$$= S^{j}_{\ell'\ell} \delta_{j'j} \delta_{s's}$$

Let's begin w/ the deuteron:

np $\longrightarrow$ $S=1, J=1$ ($\underline{L=0,2}$)

1) No tensor force:

$$\psi_d(\vec{r}) = \frac{u(r)}{r} \underbrace{Y_{00}(\hat{r})}_{\text{angular momentum}} \underbrace{|1 m_d\rangle}_{\text{spin}}$$

2) Tensor force:

S-wave

$$\psi_d(\vec{r}) = \frac{u(r)}{r} Y_{00}(\hat{r}) |1 m_d\rangle$$

$$+ \frac{w(r)}{r} \sum_{m_s m_\ell} Y_{2m_\ell}(\hat{r}) |1 m_s\rangle \times$$

$$\times \langle 2 m_\ell 1 m_s | 1 m_d\rangle$$

D-wave

How does this change the Schrödinger equation:

$$V_{OPE}(r) = \vec{\sigma}_1 \cdot \vec{\sigma}_2 V_C + S_{12}(\hat{r}) V_T$$

$$|S(1 m_d)\rangle = Y_{00}(\hat{r}) |1 m_d\rangle$$

$$|D(1 m_d)\rangle = \sum Y_{2 m_\ell}(\hat{r}) |1 m_s\rangle \times \langle 2 m_\ell 1 m_s | 1 m_d \rangle$$

$$S_{12} = \begin{pmatrix} 0 & 2\sqrt{2} \\ 2\sqrt{2} & -2 \end{pmatrix}$$

these are the matrix elements

And we end up with:

$$\left[-u''(r) + 2\mu\left[V_c u + 2\sqrt{2}\, V_T w\right] = -\gamma^2 u(r) \right]$$

$$\left[-w''(r) + \frac{6}{r^2} w(r) + 2\mu\left[2\sqrt{2}\, V_T u + (V_c - 2V_T) w\right] = -\gamma^2 w(r) \right]$$

$\longrightarrow$ Coupled-channel equations

Asymptotic behavior: $u(r) \rightarrow A_S e^{-\gamma r}$

$$w(r) \rightarrow A_D e^{-\gamma r}\left(1 + \frac{3}{\gamma r} + \frac{3}{(\gamma r)^2}\right)$$

Analogous to single-channel case, but a bit more complex

1) Normalization:

$$\int_0^\infty dr\,(u^2(r) + w^2(r)) = 1$$

2) Important observables:

$$r_m^2 = \frac{\langle r^2 \rangle}{4} = \frac{1}{4}\int_0^\infty r^2 (u^2 + w^2)\,dr$$

(matter radius)

$$Q_d = \frac{1}{20}\int_0^\infty r^2 w\,(2\sqrt{2}\,u - w)\,dr$$

(quadrupole moment)

not observable

$$P_D = \int_0^\infty dr\, w^2(r)$$ (D-wave probability)

Values: $\sqrt{r_m^2} = 1.9754(9)$ fm

$Q_d = 0.2859(3)$ fm^2

$P_D \sim \underbrace{3\%?\ 5\%?}_{\text{depends on the model}}$

Comes from deuteron magnetic moment (will be explained later)

$\eta = \frac{A_D}{A_S} \rightarrow$ asymptotic D/S-ratio

$\eta = 0.0256(4)$

↳ we know from its indirectly relation w/ scattering

Now we go to scattering:

→ Singlet & triplet channels

1) No-spin scattering:

$$\psi_k(\vec{r}) \to e^{i\vec{k}\cdot\vec{r}} + f(\Omega)\frac{e^{ikr}}{r}$$

2) If we add spin:

2.a) Singlets

$$\psi_k(\vec{r}) \to \left[e^{i\vec{k}\cdot\vec{r}} + f(\Omega)\frac{e^{ikr}}{r}\right]|00\rangle$$

→ No change at all

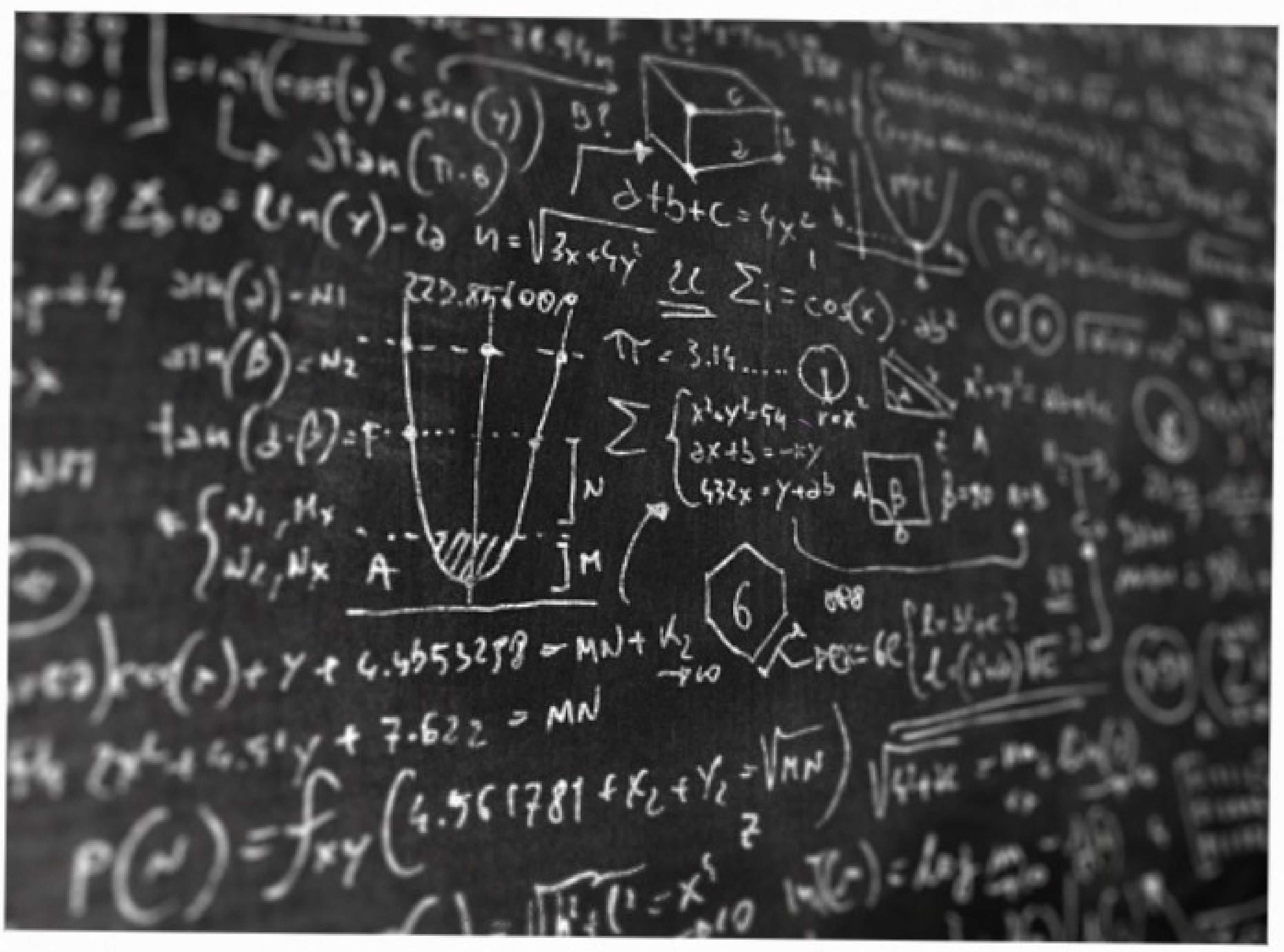
2.b) Triplets

$$\psi_k(\vec{r}) \to e^{i\vec{k}\cdot\vec{r}}\,|1 m_s\rangle$$

$$+ \sum_{m_s'} f_{m_s m_s'}(\Omega)\frac{e^{ikr}}{r}\,|1 m_s'\rangle$$

Bottom line: $f(\Omega) \to f_{m_s m_s'}(\Omega)$

$\Rightarrow$ a matrix in spin-space ✓



There is a bit of elaboration involved here

$$f^{(st)}_{m_s m_s'} = \sum_{j\ell\ell'} \underbrace{[\dots]}_{\text{(some coefficients)}} f^{sj}_{\ell\ell'}$$

$$(1-(-)^{\ell+s+t}) \sum_m i^{\ell-\ell'} \langle \ell\, m-m_s\; s\, m_s | j m\rangle$$

$$\langle j m | \ell'\, m-m_s'\; s\, m_s' \rangle\, Y_{\ell\, m-m_s}(\hat{k})\, Y_{\ell\, m-m_s'}(\vec{\hat{k}}')$$

And the PW expansion goes from:

$$f_\ell(k) = \frac{e^{2i\delta_\ell} - 1}{2ik}$$

to this:

$$f^{sj}_{\ell\ell'}(k) = \underbrace{\frac{1}{2ik}\left(S^{sj}_{\ell\ell'} - \delta_{\ell\ell'}\right)}_{\text{matrix in } \ell\text{-space}}$$

1) $s = 1$, $\ell = j \pm 1$

$$S^{sj} = \begin{pmatrix} \cos\epsilon_j & -\sin\epsilon_j \\ \sin\epsilon_j & \cos\epsilon_j \end{pmatrix} \times \begin{pmatrix} e^{2i\delta_{j-1}} & 0 \\ 0 & e^{2i\delta_{j+1}} \end{pmatrix} \times \begin{pmatrix} \cos\epsilon_j & \sin\epsilon_j \\ -\sin\epsilon_j & \cos\epsilon_j \end{pmatrix}$$

2) $s=1,\ j=l$

$$S^{sj} = e^{2i\delta_j} \to \text{simple case}$$

— ⊗ —

→ For the moment we will concentrate on the coupled channel case

1) $S=1,\ l=j\pm 1 \to$ coupled

2) $s=1,\ l=j \to$ uncoupled

— ⊗ —

Uncoupled → Coupled

$$e^{2i\delta_l} \to \underbrace{R(\epsilon_j)}_{\text{rotation matrix}} \begin{pmatrix} e^{2i\delta_{j-1}} & 0 \\ 0 & e^{2i\delta_{j+1}} \end{pmatrix} R^{-1}(\epsilon_j)$$

(1 phase → 3 phases)

How do we compute the phases?

1) Uncoupled channels:

$$-u_\ell''(r) + \left[2\mu V_\ell(r) + \frac{\ell(\ell+1)}{r^2}\right] u_\ell(r) = k^2 u_\ell(r)$$

$e^{2i\delta_\ell}$

$$\frac{u_\ell(r)}{r} \to h_\ell^{(-)}(kr) - h_\ell^{(+)}(kr) S_\ell(k)$$

$$h_\ell^{(\pm)}(x) = x^\ell \left(-\frac{1}{x}\frac{d}{dx}\right)^\ell \frac{e^{\pm ix}}{x}$$

Spherical Hänkel functions

$$\frac{u_\ell(r)}{r} \to \cot\delta_\ell(k)\, j_\ell(kr) - y_\ell(kr)$$

(Equivalent to the previous)

So we are rewriting
the uncoupled phase shifts
in a more convenient form:

$$\cot\delta = i\,\frac{e^{2i\delta}+1}{e^{2i\delta}-1}$$

$$h_\ell^{(-)}(x) = -y_\ell(x) - i j_\ell(x)$$

$$h_\ell^{(+)}(x) = -y_\ell(x) + i j_\ell(x)$$

which will be useful
in a little while

2) Coupled channels:

$$-u_J'' + \left[2\mu V_{J-1,J-1} + \frac{(J-1)J}{r^2}\right]u_J + 2\mu V_{J-1,J+1}\, w_J = k^2 u_J$$

$$-w_J'' + \left[2\mu V_{J+1,J+1} + \frac{(J+1)(J+2)}{r^2}\right]w_J + 2\mu V_{J+1,J-1}\, u_J = k^2 w_J$$

$\hookrightarrow$ like the Schrödinger equation for the deuteron but more complex

Asymptotic form:

$$\frac{u_J}{r} \to a_{J-1}\, h^{(-)}_{J-1}(kr) - b_{J-1}\, h^{(+)}_{J-1}(kr)$$

$$\frac{w_J}{r} \to a_{J+1}\, h^{(-)}_{J+1}(kr) - b_{J+1}\, h^{(+)}_{J+1}(kr)$$

↳ arbitrary (input)

(two linearly independent solutions)

S-matrix:

$$\boxed{\begin{pmatrix} b_{J-1} \\ b_{J+1} \end{pmatrix} = S^{J}(k) \begin{pmatrix} a_{J-1} \\ a_{J+1} \end{pmatrix}}$$

$$S^{J}(k) = R(\epsilon_J) \begin{pmatrix} e^{2i\delta_{J-1}} & 0 \\ 0 & e^{2i\delta_{J+1}} \end{pmatrix} R^{-1}(\epsilon_J)$$

RECAP

1) Include spin $f(\Omega) \to f_{m_s m_s'}(\Omega)$

$$\frac{d\sigma}{d\Omega} = |f(\Omega)|^2 \to \frac{d\sigma(m_s \to m_s')}{d\Omega} = |f_{m_s m_s'}|^2$$

2) Do PW decomposition

$$f_{m_s m_s'} = \sum (\dots) f_{\ell\ell'}^{(sj)}$$

3) Define phases

$$f_\ell = \frac{1}{2ik}\left(e^{2i\delta_\ell} - 1\right) \to f_{\ell\ell'}^{(sj)} = \frac{S_{\ell\ell'} - \delta_{\ell\ell'}}{2ik}$$

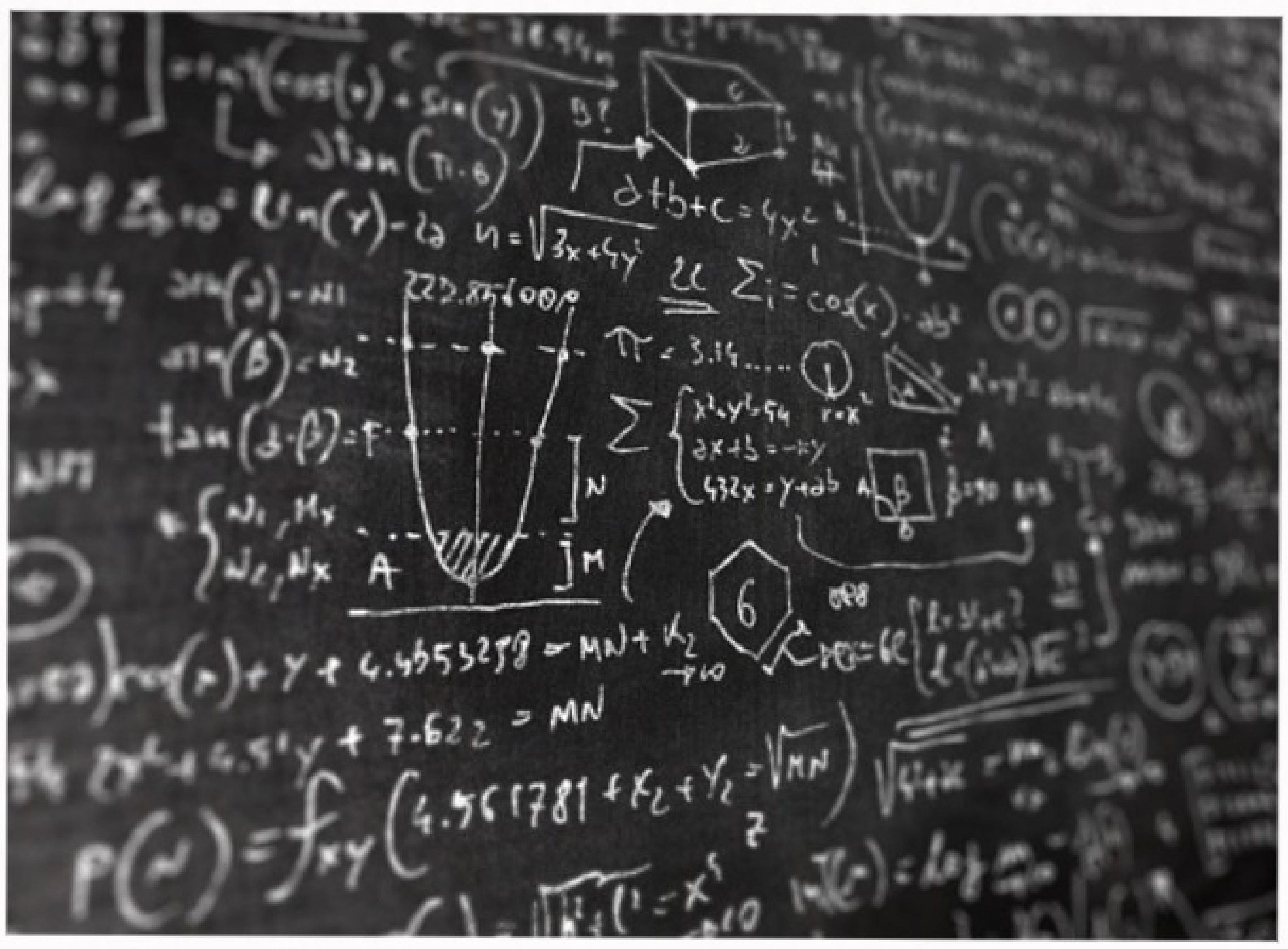
$$S = R \,\mathrm{Diag}(e^{2i\delta_\ell})\, R^{-1}$$

4) Make connection w/ wave functions, potentials, etc

$$\frac{1}{r}\begin{pmatrix} u \\ w \end{pmatrix} \to \begin{pmatrix} a_{j-1} h_{j-1}^{(-)} \\ a_{j+1} h_{j+1}^{(-)} \end{pmatrix} - \begin{pmatrix} b_{j-1} h_{j-1}^{(+)} \\ b_{j+1} h_{j+1}^{(+)} \end{pmatrix}$$

$$b = Sa$$

And we are missing a few technical details...



but this is the problem of not having a real blackboard

Additional complication:

$\boxed{\text{multiple phase shift defs}}$

1) Eigenphases (or Blatt – Biedenharn)

$$S = \begin{pmatrix} \cos\epsilon_J & -\sin\epsilon_J \\ \sin\epsilon_J & \cos\epsilon_J \end{pmatrix} \begin{pmatrix} e^{2i\delta_{J-1}} & 0 \\ 0 & e^{2i\delta_{J+1}} \end{pmatrix} \begin{pmatrix} \cos\epsilon_J & \sin\epsilon_J \\ -\sin\epsilon_J & \cos\epsilon_J \end{pmatrix}$$

→ Easy conceptually

2) Nuclear-bar (or Stapp-Ipsilantis-Metropolis)

$$S = \begin{pmatrix} \cos(2\bar{\epsilon}_J)\, e^{2i\bar{\delta}_{1,J}} & i\sin(2\bar{\epsilon}_J)\, e^{i(\bar{\delta}_{1,J}+\bar{\delta}_{2,J})} \\ i\sin(2\bar{\epsilon}_J)\, e^{i(\bar{\delta}_{1,J}+\bar{\delta}_{2,J})} & \cos(2\bar{\epsilon}_J)\, e^{2i\bar{\delta}_{2,J}} \end{pmatrix}$$

→ most commonly used

Transformation rules:

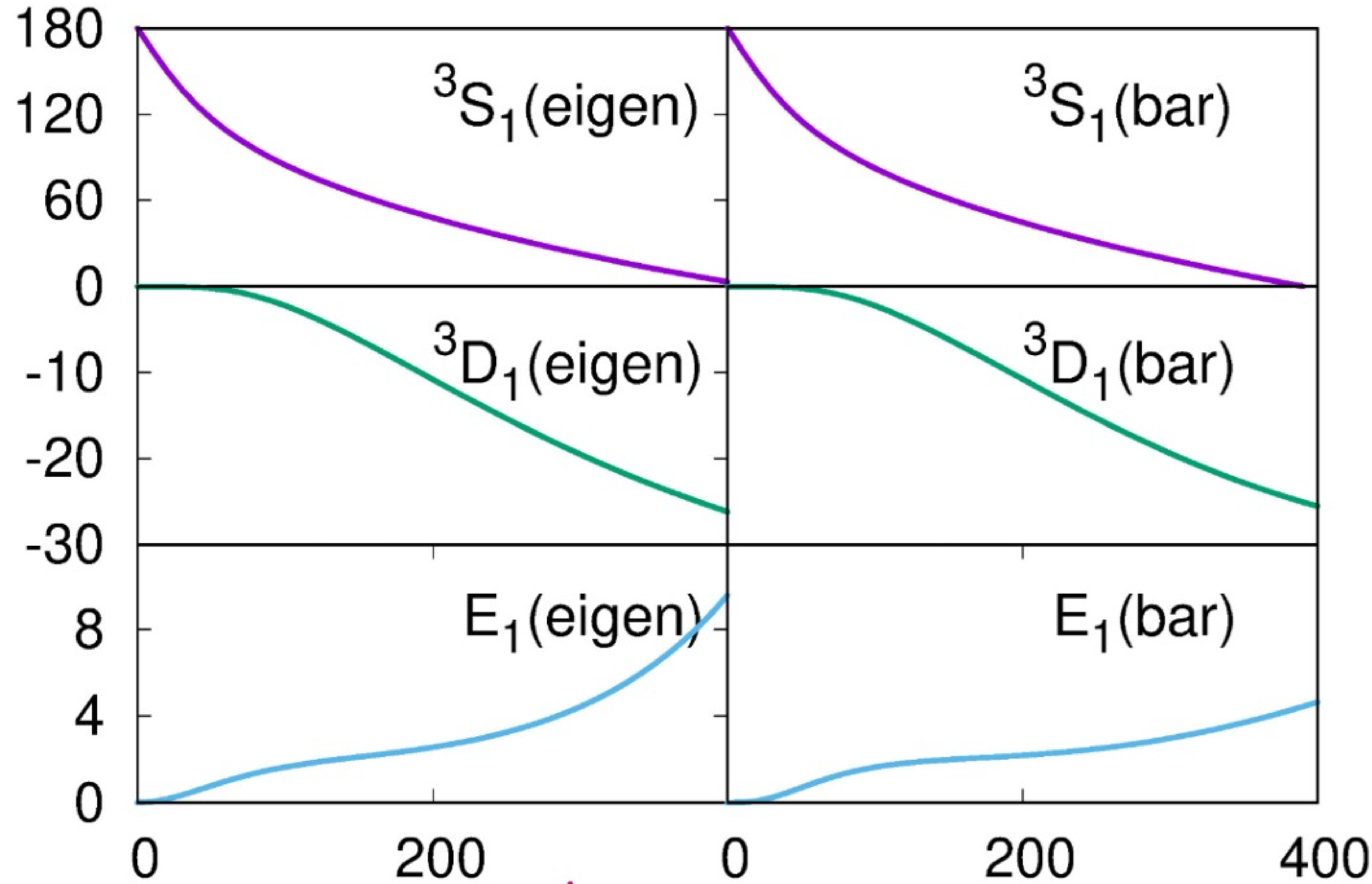
$$\left[\begin{aligned} \bar{\delta}_{1J} + \bar{\delta}_{2J} &= \delta_{J-1} + \delta_{J+1} \\ \sin(\bar{\delta}_{1J} - \bar{\delta}_{2J}) &= \frac{\tan(2\bar{\epsilon}_J)}{\tan(2\epsilon_J)} \end{aligned}\right]$$

→ All this simply adds more confusion, but nuclear physics is difficult

→ You can also try your own phase shift definition

Example : $^3S_1 - ^3D_1$ channel

(deuteron channel)



most important change :

mixing angle

(Nymegen II potential)

Next lesson:

The one boson exchange
(OBE) model