Nuclear Physics 16

All you wanted to know about the T-matrix but were afraid to ask

part 2

RECAP

1) $T = V + V G_0 T$ seems difficult

2) Separable potentials:

$$\langle \vec{p}' | V | \vec{p} \rangle = \lambda\, g(p)\, g(p')$$

→ easy solution of LS

$$\langle \vec{p}' | T | \vec{p} \rangle = \tau(E)\, g(p)\, g(p')$$

$$\tau(E) = \frac{1}{\frac{1}{\lambda} - I(E)}$$

w/ $I(E) = \int \frac{d^3\vec{\ell}}{(2\pi)^3} \frac{g^2(\ell)}{E - \frac{\ell^2}{2\mu}}$

Separable potentials:

1) $\langle \vec{p}\,' | V | \vec{p} \rangle = C_0(\Lambda) f\left(\frac{p'}{\Lambda}\right) f\left(\frac{p}{\Lambda}\right)$

w/ $f(x) \xrightarrow[x \to 0]{} 1$

$\xrightarrow[x \to \infty]{} 0$

$\Rightarrow$ Renormalization example

$$\frac{d}{d\Lambda} T(E;\Lambda) \simeq 0 \Rightarrow \boxed{C_0 = C_0(\Lambda)}$$

2) Yamaguchi potential:

$$\langle \vec{p}\,' | V | \vec{p} \rangle = \lambda g(p') g(p)$$

$$g(x) \propto \frac{1}{x^2 + \beta^2}$$

$\to$ historically important

T-matrix & bound states:

1) $T(E) \to$ poles for $E \to E_B$

$$T(E) \xrightarrow[E \to E_B]{} \frac{\mathrm{Res}\,T(E)}{E - E_B}$$

$$\mathrm{Res}\,T(E) = V|B\rangle\langle B|V$$

$$= G_0^{-1}(E_B)|B\rangle\langle B|G_0^{-1}(E_B)$$

2) $\underbrace{|B\rangle = G_0(E)V|B\rangle}_{\text{Bound state equation}}$

3) $\langle \vec{p}'|V|\vec{p}\rangle = \lambda\, g(p)\, g(p')$

$$\left[\Rightarrow \langle \vec{p}|B\rangle = N \frac{g(p)}{p^2 + \gamma^2} \right]$$

$$\left(E_B = -\frac{\gamma^2}{2\mu} \right)$$

$\boxed{\text{PRO TIP}}$

→ The vertex function

$$|B\rangle = G_0(E_B)\,|\phi_B\rangle$$

Has a few really good properties

1) $|\phi_B\rangle = G_0^{-1}(E_B)\,|B\rangle$

$$\Rightarrow \boxed{T(E) \xrightarrow[E \to E_B]{} \frac{|\phi_B\rangle\langle\phi_B|}{E - E_B}}$$

→ $|\phi_B\rangle$ is what goes into the residue of the T-matrix

2) $\underbrace{|\phi_B\rangle = V G_0\,|\phi_B\rangle}$

Bound state equation for the vertex function

Why is this bound state equation more convenient?

1) $|B\rangle = G_0 V |B\rangle$

$$\underbrace{\psi_B(\vec{p}) = \frac{1}{E_B - \frac{p^2}{2\mu}}}_{\text{annoying detail}} \int \frac{d^3\vec{\ell}}{(2\pi)^3} \langle \vec{p}|V|\vec{\ell}\rangle \times \psi_B(\vec{\ell})$$

2) $|\phi_B\rangle = V G_0 |\phi_B\rangle$

$$\phi_B(\vec{p}) = \int \frac{d^3\vec{\ell}}{(2\pi)^3} \frac{\langle \vec{p}|V|\vec{\ell}\rangle \phi_B(\vec{\ell})}{E_B - \frac{p^2}{2\mu}}$$

a bit easier to solve

My recommendation $\rightarrow$ 2

$\boxed{\text{RECAP}}$

1) Separable potentials

→ T-matrix & BS Equation

2) Vertex function

→ Simplification of BS equation

NEXT:

$\boxed{\text{PARTIAL WAVE EXPANSION}}$

Partial wave expansion:

1) $f(\Omega) = \sum_{\ell} (2\ell+1) f_\ell(k) P_\ell(\cos\theta)$

2) $\langle \vec{p}' | T(E) | \vec{p} \rangle = \circledast$

$$\circledast = \sum_{\ell} (2\ell+1) \langle p' | T_\ell(E) | p \rangle \times P_\ell(\hat{p} \cdot \hat{p}')$$

Problem $\rightarrow$ find the expressions for $T_\ell(E)$ & the equations

$$\langle p' | T_\ell(E) | p \rangle = \int \frac{d^2\hat{p}'}{4\pi} \int \frac{d^2\hat{p}}{4\pi} \times \langle \vec{p}' | T | \vec{p} \rangle Y^*_{\ell m}(\hat{p}') Y_{\ell m}(\hat{p})$$

$\hookrightarrow$ [PW Projection]

Actually, it's really simple:

$$f(\vec{r}) = \sum_{\ell m} f_\ell(r)\, Y_{\ell m}(\hat{r})$$

$$f_\ell(r) = \int d^2\hat{r}\; Y^*_{\ell m}(\hat{r})\, f(\vec{r})$$

$\Rightarrow \left[\int d^2\hat{r}\, Y^*_{\ell m}(\hat{r}) \text{ projects onto } |\ell m\rangle \right]$

$$\langle \vec{p}\,' | T | \vec{p} \rangle = 4\pi \sum_{\ell m} T_\ell\, Y_{\ell m}(\hat{p}')\, Y^*_{\ell m}(\hat{p})$$

→ we have to project twice (for $\langle \vec{p}\,'|$ & for $|\vec{p}\rangle$)

→ there is also a 4π (different normalization)

$$\int \frac{d^2\hat{p}'}{4\pi} Y^*_{\ell m}(\hat{p}')$$

$$\langle \vec{p}\,' | T(E) | \vec{p} \rangle$$

$$\int \frac{d^2\hat{p}}{4\pi} Y_{\ell m}(\hat{p})$$

We have to include both projections

$$\langle p' | T_\ell(E) | p \rangle = \int \frac{d^2\hat{p}'}{4\pi} \int \frac{d^2\hat{p}}{4\pi} \langle \vec{p}\,' | T | \vec{p} \rangle \times Y^*_{\ell m}(\hat{p}') Y_{\ell m}(\hat{p})$$

Same for the potential:

$$\langle p' | V_\ell | p \rangle = \int \frac{d^2\hat{p}'}{4\pi} \int \frac{d^2\hat{p}}{4\pi} \langle \vec{p}\,' | V | \vec{p} \rangle \times Y^*_{\ell m}(\hat{p}') Y_{\ell m}(\hat{p})$$

Putting $\forall$ the pieces together:

$$\langle p'|T_\ell(E)|p\rangle = \langle p'|V_\ell|p\rangle + \frac{\mu}{\pi^2}\int_0^\infty \frac{\ell^2 d\ell}{k^2-\ell^2}\langle p'|V_\ell|\ell\rangle\langle \ell|T_\ell|p\rangle$$

[→ Basically the same equation as before] (*)

(*) → derive this for 1 point

(Exercise)

Other possible excercise (2 pts):

$$V_\chi(\vec{q}) = \frac{-g^2}{m^2+\vec{q}^{\,2}}$$ and find its S-wave projection

$$\Rightarrow \boxed{\langle p'|(V_\chi)_{L=0}|p\rangle \;?}$$

And now there is another open problem ...

1) We know how to solve

$T = V + V G_0 T \, , \; |\phi_B\rangle = V G_0 |\phi_B\rangle$

for a type of potential
(separable potential)

2) But how do we solve it for a generic potential?

$\Downarrow$

$\boxed{\text{How do we solve integral equations?}}$

Basic idea:

Integral equation

$\Big\downarrow$ Discretization

Linear equation

$$\int \frac{d^3\vec{k}}{(2\pi)^3} f(|\vec{k}|) = \int \frac{k^2\,dk}{2\pi^2} f(k)$$

$$\simeq \sum_i w_i \frac{k_i^2}{2\pi^2} f(k_i)$$

Gauss points

The idea is to use some quadrature method for the integrals

$$\int_a^b p(x)\, f(x)\, dx \simeq \sum_{i=1}^{N} \omega_i f(x_i)$$

$x = \{x_1, x_2, \dots, x_N\}$ → Quadrature points

$\omega = \{\omega_1, \omega_2, \dots, \omega_N\}$ → Quadrature weights

Choice depends on interval of integration ($[a,b]$) & weight function ($p(x)$)

Examples:

1) Gauss-Legendre Quadrature

$$\int_{-1}^{+1} f(x)\,dx \simeq \sum_{i=1}^{N} \omega_i f(x_i)$$

$$N=5 \left\{ \begin{array}{l} x = \{\pm 0.538469\ldots, \\ \quad \pm 0.90618\ldots, 0\} \\ \omega = \{0.478629\ldots, \\ \quad 0.236927\ldots, \\ \quad 0.568889\ldots\} \end{array} \right.$$

But there are for any N

(N=5 just an example)

Gauss-Legendre useful for PW Projection

Let's see how this is done:

$$\langle k' | V_\ell | k \rangle = \int \frac{d^2\hat{k}'}{4\pi} \int \frac{d^2\hat{k}}{4\pi}$$

$$\langle \vec{k}' | V | \vec{k} \rangle \, Y^*_{\ell m}(\hat{k}') \, Y_\ell(\hat{k})$$

But this can be simplified for $\langle \vec{k}' | V | \vec{k} \rangle = V(\vec{k}' - \vec{k})$

$$= \int_{-1}^{+1} \frac{du}{2} \langle \vec{k}' | V | \vec{k} \rangle \, P_\ell(u)$$

where $u = \hat{k} \cdot \hat{k}' = \cos\theta$

Usually $N = 5$ is more than enough

(I normally use $N = 5$ or 8)

2) Gauss-Laguerre Quadrature

$$\int_0^\infty e^{-x} f(x)\, dx \simeq \sum_{i=1}^{N} w_i f(x_i)$$

3) Gauss-Hermite Quadrature

$$\int_{-\infty}^{\infty} e^{-x^2} f(x)\, dx \simeq \sum_{i=1}^{N} w_i f(x_i)$$

$\rightarrow$ $\boxed{\text{Always the same idea}}$

Lippmann-Schwinger equation:

$\rightarrow$ Best choice depends on choice of regulator

$f(x) = e^{-x^2} \rightarrow$ Gauss-Hermite

Example : bound state equation for the vertex function

$$|\phi_B\rangle = V G_0 |\phi_B\rangle$$

$$\phi_B(\vec{p}) = \int \frac{d^3\vec{e}}{(2\pi)^3} \frac{\langle \vec{p}|V|\vec{e}\rangle \phi_B(\vec{e})}{E - \frac{e^2}{2\mu}}$$

we assume PW projection

$$\phi_B(p) = \int_0^\infty \frac{e^2 de}{2\pi^2} \frac{\langle p|V_L|e\rangle \phi_B(e)}{E - \frac{e^2}{2\mu}}$$

now we discretize

$$\phi_i = \sum_{j=1}^N w_j \frac{e_j^2}{2\pi^2} \frac{V_{ij}\phi_j}{E - \frac{e_j^2}{2\mu}}$$

$\phi_i = \phi_B(p_i)$, $V_{ij} = \langle p_i|V_L|p_j\rangle$

The point is that this is now a linear system:

$$\boxed{\phi_i = \sum_{j=1}^{N} F_{ij}(E)\,\phi_j}$$

w/ $F_{ij}(E_B) = \omega_j \dfrac{\ell_j^2}{2\pi^2} \dfrac{V_{ij}}{E - \frac{\ell_j^2}{2\mu}}$

$$\boxed{\sum_j \underbrace{\left(\delta_{ij} - F_{ij}(E)\right)}_{A_{ij}(E)} \phi_j = 0}$$

$$\boxed{\sum_j A_{ij}(E)\,\phi_j = 0}$$

or simply $\Rightarrow$ $\boxed{A\,\phi = 0}$

What does this mean?

$$\underbrace{A(E_B)\phi = 0 \quad w/\ \phi \neq 0}$$

the solution we want is a particular case of

$$\boxed{A\phi = \lambda\phi}$$

→ Eigenvalue equation

$$A \rightarrow \underbrace{\{\lambda_1, \lambda_2, \dots, \lambda_N\}}_{\text{Eigenvalues of } A}$$

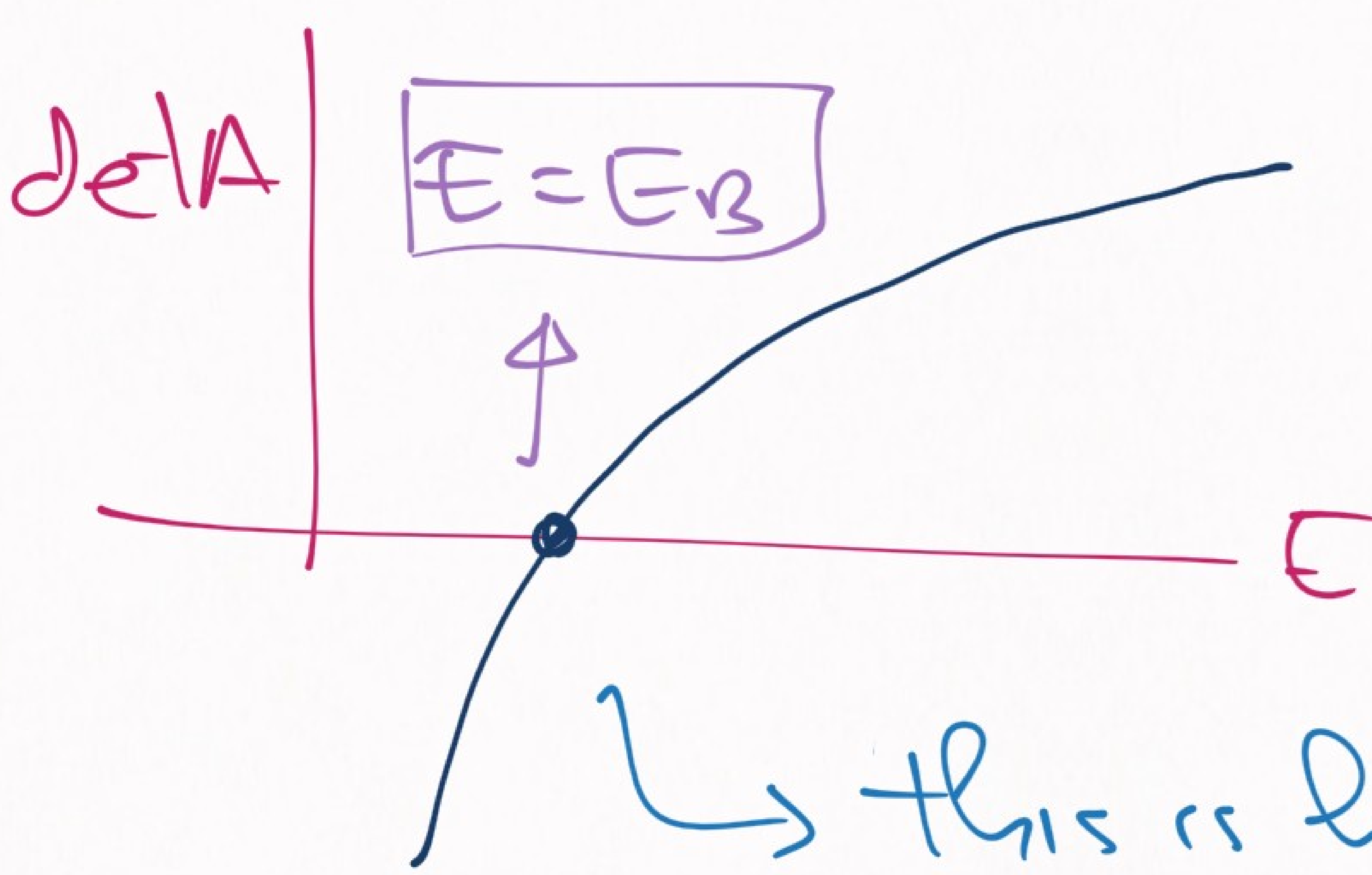
⇒ We need $\lambda_i = 0$ for $i \in \{1, N\}$

Actually there's a trick:

$$\det A = \prod_{i=1}^{N} \lambda_i$$

$$D(E_B) = \det A(E)$$

→ we compute this



↳ this is how we compute E_B

$\boxed{\text{RECAP}}$

1) $|\phi_B\rangle = V G_0 |\phi_B\rangle$

2) Discretize

3) Compute $\det(1 - V G_0)$

(det A in previous slide)

4) $\det(1 - V G_0(E)) = 0$

for $E = E_B$

Still one problem:

$\boxed{\text{how to compute } \phi_B(p)\text{ ?}}$

How do we compute $\phi_B(p)$?

1) $\det A = 0$ for $E = E_B$

2) $A \longrightarrow \{\lambda_1, \lambda_2, \ldots, \lambda_N\}$

$\{|\lambda_1\rangle, |\lambda_2\rangle, \ldots, |\lambda_N\rangle\}$

we find $\boxed{\lambda_K = 0}$

$\Rightarrow$ its eigenvector $\boxed{|\lambda_K\rangle}$ is the vertex function

$$\phi = |\lambda_K\rangle = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_N \end{pmatrix}$$

with $\phi_j = \phi_B(e_j)$ Gauss points

Correspondingly:

$$\psi_B(\ell_i) = \frac{\phi_B(\ell_i)}{E_B - \frac{\ell_i^2}{2\mu}}$$

and this is how we solve the bound state equation in p-space

Enjoy the trip!

For any of you who is good at coding..

CHALLENGE!

1) WRITE A CODE TO FIND THE BINDING ENERGY IN P-SPACE (that is, $\det(1 - V G_0)$)

2) EXTEND THAT CODE SO IT CAN COMPUTE THE VERTEX/WAVE FUNCTION

(*) NO COPYING!

1) → 8 points (first solver) / 6 points (rest of people) (*)

2) → 4 points (first) / 3 pts (rest)

NEXT LESSON:

→ Tensor force &

Coupled channels

(very heavy too)