Nuclear Physics TS

All you wanted to know about the T-matrix but were afraid to ask

PART 1

$\boxed{\text{RECAP}}$

1) $T = V + V G_0 T$

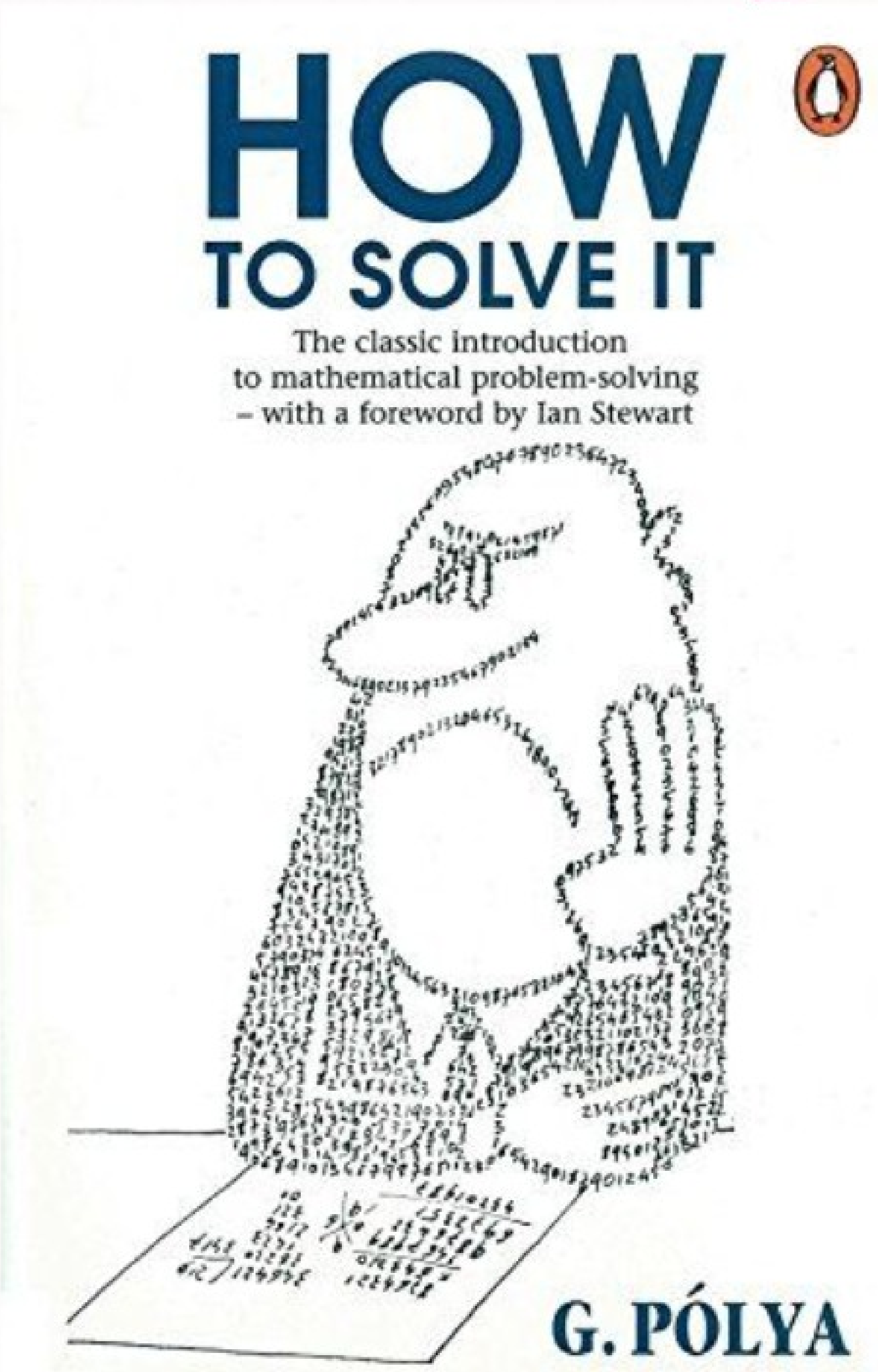
The Lippmann-Schwinger equation

2) $\langle \vec{p}\,' | T(E) | \vec{p} \rangle = \circledast$

$\circledast = \langle \vec{p}\,' | V | \vec{p} \rangle + \int \frac{d^3\vec{e}}{(2\pi)^3} \frac{\langle \vec{p}\,' | V | \vec{e} \rangle \langle \vec{e} | T | \vec{p} \rangle}{E - \frac{\vec{e}^{\,2}}{2\mu}}$

Explicit form of LS equation

→ $\boxed{\text{How to solve it?}}$



Which reminds me of a classic book

(very much recommended)

So going back to physics

$$\boxed{T = V + V G_0 T}$$

Let's begin w/ our favorite potential:

$$\boxed{V(\vec{q}) = C_0} \rightarrow \text{The Dirac delta}$$

But first we regularize:

$$\langle \vec{p}\,' | V | \vec{p} \rangle = C_0 f\left(\frac{p'}{\Lambda}\right) f\left(\frac{p}{\Lambda}\right)$$

w/ $f(x) \longrightarrow 1$ for $x \to 0$

$f(x) \longrightarrow 0$ for $x \to \infty$

$$\langle \vec{p}\,' | T | \vec{p} \rangle = C_0 f\left(\frac{p'}{\Lambda}\right) f\left(\frac{p}{\Lambda}\right) + C_0 f\left(\frac{p'}{\Lambda}\right) \int \frac{d^3\vec{\ell}}{(2\pi)^3} f\left(\frac{\ell}{\Lambda}\right) \frac{\langle \vec{\ell} | T | \vec{p}\,' \rangle}{E - \frac{\ell^2}{2\mu}}$$

↳ this looks doable

$$\langle \vec{p}\,' | T | \vec{p} \rangle = \underbrace{\tau(E) f\left(\frac{p'}{\Lambda}\right) f\left(\frac{p}{\Lambda}\right)}_{\text{ansatz}}$$

⇒

If we use this ansatz:

$$T(E) = C_0 + C_0 T(E) I(E,\Lambda)$$

w/ $I(E,\Lambda) = \int \frac{d^3\vec{l}}{(2\pi)^3} \frac{f^2(l/\Lambda)}{E - \frac{l^2}{2\mu}}$

loop function

$$\boxed{T(E) = \frac{1}{\frac{1}{C_0} - I(E,\Lambda)}}$$

much more compact expression

Exactly solvable for a few choices of $f(x)$:

$$f(x) = \Theta(1-x)$$

$$\Downarrow$$

$$I(E,\Lambda) = \int \frac{d^3\vec{\ell}}{(2\pi)^3} \frac{\Theta(\Lambda - |\vec{\ell}|)}{E - \frac{\ell^2}{2\mu}}$$

$$\Downarrow \quad \text{(Exercise, 2pts)}$$

$$I(E \pm i\epsilon, \Lambda) = \circledast$$

$$\circledast = \frac{\mu}{\pi^2}\left[\mp i\frac{\pi}{2}k - \Lambda + \frac{k}{2}\log\left|\frac{\Lambda+k}{\Lambda-k}\right|\right]$$

w/ $k = \sqrt{2\mu E}$

So we indeed have a solution

$$T(E) = \frac{1}{\frac{1}{C_0(\Lambda)} - I(E,\Lambda)}$$

$$I(E+i\epsilon,\Lambda) = \frac{M}{\pi^2}\left[-i\frac{\pi}{2}k - \Lambda + \frac{k}{2}\log\left|\frac{\Lambda+k}{\Lambda-k}\right|\right]$$

Still, a bit to abstract

But remember that:

1) $\sigma(E) \xrightarrow[E\to 0]{} 4\pi|a_0|^2$

2) $\sigma = \int |f(\Omega)|^2 d\Omega$

3) $f(\Omega) = -\frac{M}{2\pi}\langle \vec{p}\,'|T(E)|\vec{p}\rangle$

From the previous equations we can deduce :

$$\sigma = 4\pi\left(\frac{\mu}{2\pi}\tau(E)\right)^2 \xrightarrow[E\to 0]{} 4\pi|a_0|^2$$

$$\Downarrow$$

$$\frac{\mu}{2\pi}\tau(E) \xrightarrow[E\to 0]{} \pm a_0$$

$$\Downarrow$$

$$\boxed{\tau(E) \xrightarrow[E\to 0]{} \frac{2\pi}{\mu}a_0}$$

And now we can renormalize

$\boxed{C_0(\Lambda)}$

1) $\tau(E) = \dfrac{1}{\frac{1}{C_0(\Lambda)} - I(E,\Lambda)}$

2) $I(0,\Lambda) = \displaystyle\int \frac{d^3\vec{p}}{(2\pi)^3} \frac{\Theta(\Lambda - |\vec{p}|)}{\left(-\frac{p^2}{2\mu}\right)} = (*)$

$(*) = -\dfrac{\mu}{\pi^2}\Lambda$

3) $\tau(0) = \dfrac{1}{\frac{1}{C_0(\Lambda)} + \frac{\mu}{\pi^2}\Lambda} = \dfrac{2\pi}{\mu} a_0$

4) $\dfrac{d}{d\Lambda}\tau(0) = 0$

$$\boxed{\frac{1}{C_0(\Lambda)} = \frac{\mu}{2\pi}\left(\frac{1}{a} - \frac{2}{\pi}\Lambda\right)}$$

RECAP

1) $T = V + V G_0 T$ solvable

for $\langle \bar{p}' | V | \bar{p} \rangle = \lambda g(p') g(p)$

(separable potential)

2) $\langle \bar{p}' | V | \bar{p} \rangle = C_0$ is the Dirac

delta potential in p-space

3) $\langle \bar{p}' | V | \bar{p} \rangle = C_0(\Lambda) f\left(\frac{p'}{\Lambda}\right) f\left(\frac{p}{\Lambda}\right)$

regularizes the δ-potential

4) $f(x) = \Theta(1-x)$ gives an

analytic solution

5) $\frac{d}{d\Lambda} \langle \bar{p}' | T | \bar{p} \rangle = 0$ give us

an RGE for $C_0(\Lambda)$

In short:

We found a really good toy model for:

1) Solving the LS equation
2) Playing w/ renormalization

Trivia about this solution:

1) $\langle \bar{p}' | V | \bar{p} \rangle = \lambda g(p') g(p)$

$\Rightarrow \langle \bar{p}' | T(E) | \bar{p} \rangle$

$= \tau(E) g(p') g(p)$

S-wave solution

2) $f(\omega) = \sum_{\ell} (2\ell+1) f_\ell \, P_\ell(\cos\theta)$

$= f_0$

$$f_0(k) = \frac{1}{k\cot\delta - ik}$$

$$\Rightarrow \boxed{T(E+i\epsilon) = -\frac{2\pi}{\mu} \frac{1}{k\cot\delta - ik}}$$

Another useful equation

$\Rightarrow$ This type of potential is easy to solve & models S-wave interactions

The $\Lambda \to \infty$ limit

$$T(E) = \frac{1}{\frac{1}{C_0(\Lambda)} - I(E,\Lambda)}$$

$$I(E,\Lambda) \xrightarrow[\Lambda \to \infty]{} \frac{\mu}{\pi^2}\left[-i\frac{\pi}{2}k - \Lambda + \cancel{\mathcal{O}\left(\frac{k}{\Lambda^2}\right)} \right]$$

$$T(E) \to \frac{1}{\frac{1}{C_0(\Lambda)} + \frac{\mu}{\pi^2}\Lambda + \frac{\mu}{2\pi} ik}$$

$$= \frac{2\pi}{\mu} \frac{1}{\frac{1}{a_0} + ik}$$

$$\Rightarrow \boxed{T(E) = \frac{2\pi}{\mu} \frac{1}{\frac{1}{a_0} + ik}}$$

$\Rightarrow$ The $\Lambda \to \infty$ limit of a contact-range theory

$$\boxed{a_0 \neq 0\ ,\ r_0 = 0}$$

$\Rightarrow$ no effective range

[T-matrix & bound states]

$$T = V + V G_0 T$$

$\Updownarrow$ Two equivalent expressions

$$T = V + V G V$$

$$G = G_0 + G_0 V G_0 + G_0 V G_0 V G_0 + \dots$$

Now, let's look at the second:

$$T(E) = V + V\,G(E)\,V$$

with $\rightarrow$ $G(E) = \dfrac{1}{E - H}$

the full Hamiltonian

$$\left[\ \underline{\mathbb{1}} = \sum_{i=1}^{n_B} |B_i\rangle\langle B_i| + \int \frac{d^3\vec{k}}{(2\pi)^3} |\phi^{\pm}_{k}\rangle\langle\phi^{\pm}_{k}|\ \right]$$

the identity for $H = H_0 + V$

for $V \to 0$, no $\cancel{|B\rangle}$, $|\phi^{+}_{k}\rangle = |\vec{k}\rangle$

$$\Rightarrow \quad \underline{\mathbb{1}} = \int \frac{d^3\vec{k}}{(2\pi)^3} |\vec{k}\rangle\langle\vec{k}|$$

The bottom-line:

$$\mathbb{1} = \underbrace{\sum_i |B_i\rangle\langle B_i|}_{\text{Bound states}} + \underbrace{\int \frac{d^3\vec{k}}{(2\pi)^3} |\Phi^+_{\vec{k}}\rangle\langle\Phi^+_{\vec{k}}|}_{\text{continuum}}$$

$$T = V + VGV$$

$\hookrightarrow$ $$G(E) = \sum_i |B_i\rangle \frac{1}{E - B_i} \langle B_i| + \text{continuum}$$

$$\boxed{T(E \to B_i) \longrightarrow V \frac{|B_i\rangle\langle B_i|}{E - B_i} V}$$

$\hookrightarrow$ T-matrix has a pole if there is a bound state

→ This is really important

$$T(E) \xrightarrow[E \to B_i]{} V \frac{|B_i\rangle\langle B_i|}{E - B_i} V$$

because now we can calculate bound states ✓

$$\left[\operatorname*{Res}_{E = B_i} T(E) = V |B_i\rangle\langle B_i| V \right]$$

for example, let's go back to the contact-theory:

$$\underbrace{T(E+ik) = \frac{2\pi}{\mu} \frac{1}{\frac{1}{a} + ik}}$$

↳ does it have a bound state?

Let's analyze it:

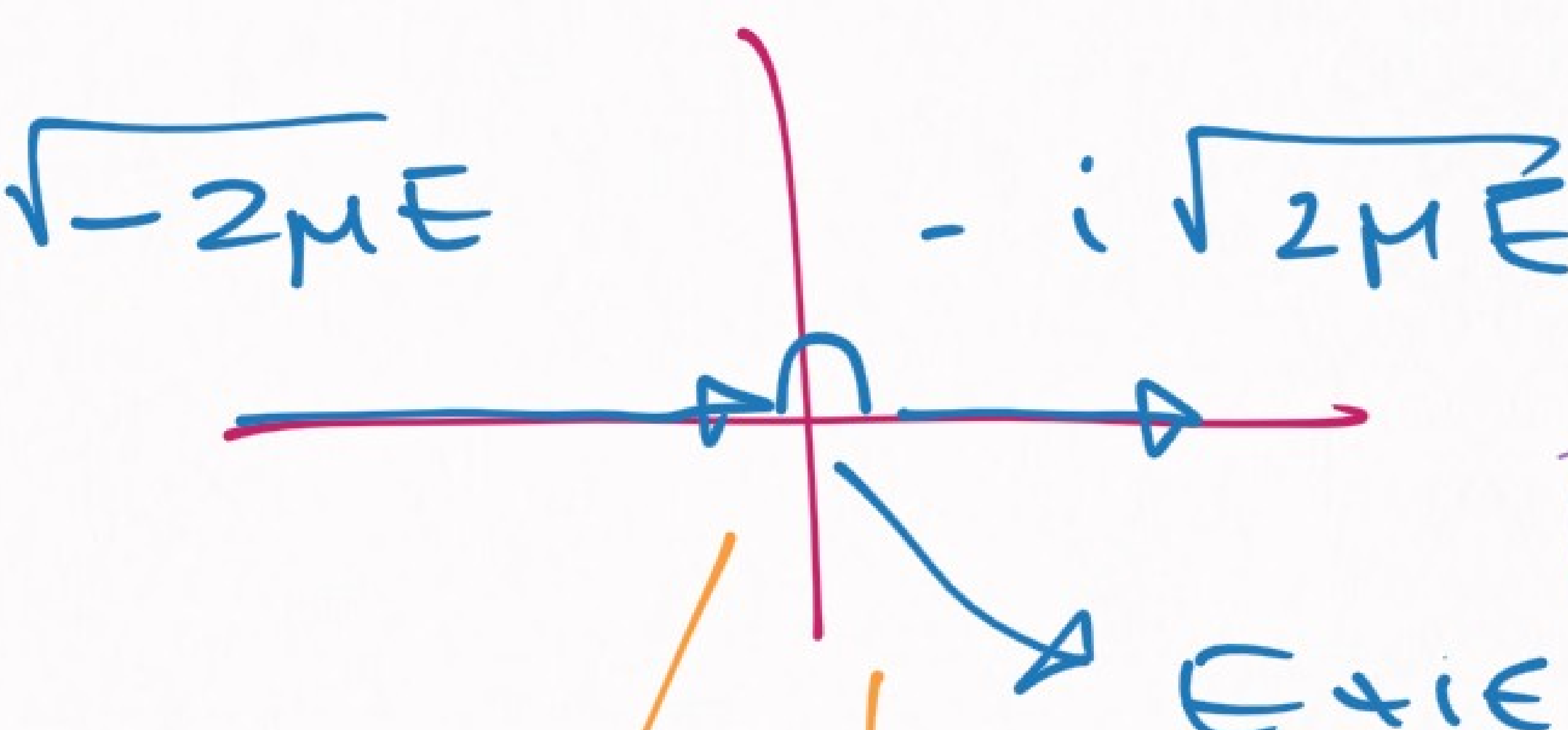
$$\tau(E+i\epsilon) = \frac{2\pi}{\mu}\frac{1}{1/a + ik}$$

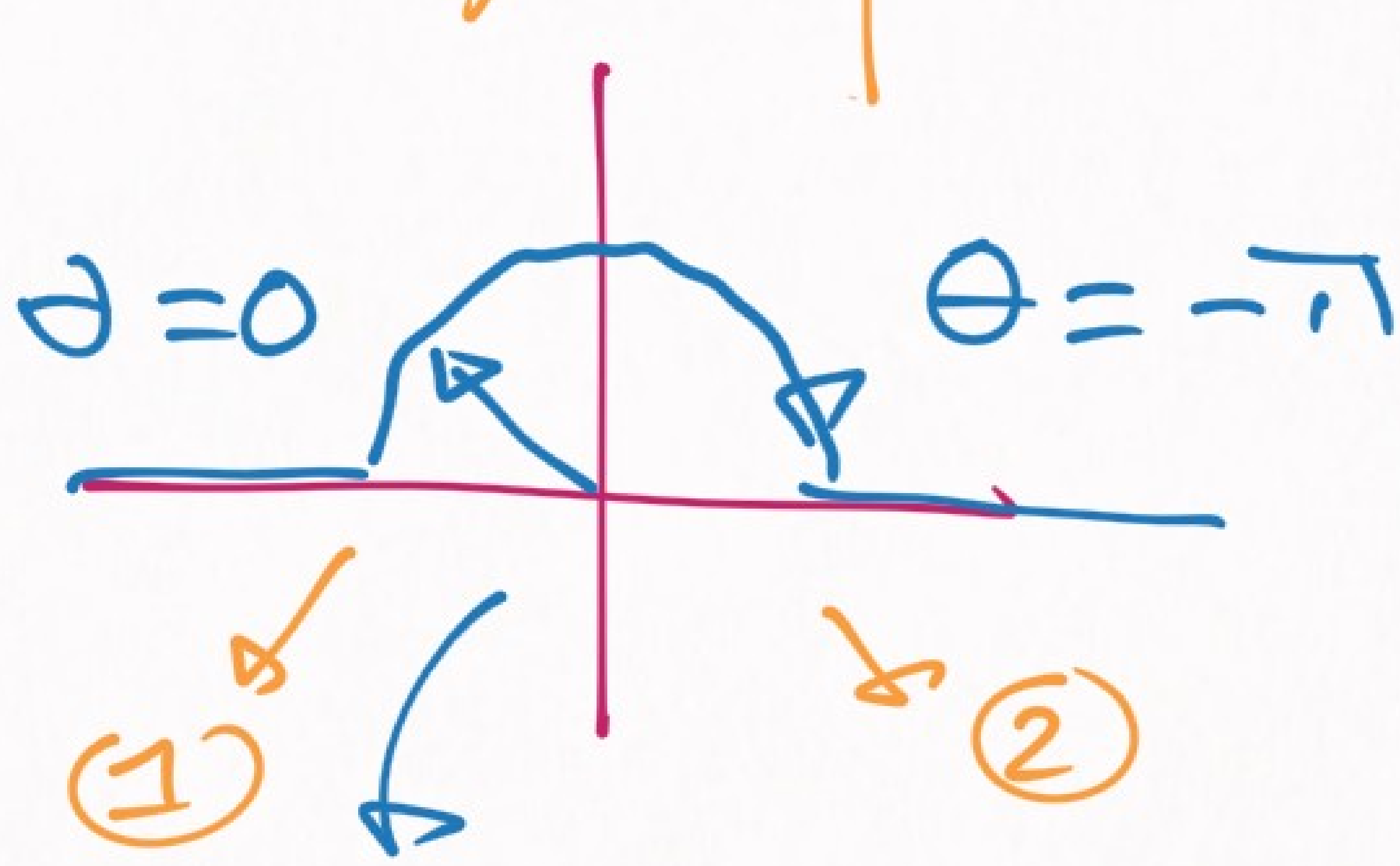
$E > 0$

but we need $E<0$

$\boxed{\text{THE POINT}}$

$ik \to -\sqrt{-2\mu E}$





(clock-wise

$\equiv$ negative

angle)

$$-2\mu E = |2\mu E| e^{i\theta}$$

① : $\sqrt{|2\mu E|}$

② $\sqrt{|2\mu E|}\, e^{i\frac{\theta}{2}} = -i\sqrt{|2\mu E|}$

So we plug this in the T-matrix

$$T(E) = \frac{2\pi}{\mu} \frac{1}{1/a + ik}$$

$$= \frac{2\pi}{\mu} \frac{1}{1/a - \sqrt{-2\mu E}} \rightarrow ①$$

$$= \frac{2\pi}{\mu} \left(\frac{1/a + \sqrt{-2\mu E}}{1/a^2 + 2\mu E} \right)$$

① → If $a > 0 \Rightarrow \exists$ a pole

$$E = -\frac{1}{2\mu}\frac{1}{a^2}$$

② → The residue of the pole is:

$$\boxed{\operatorname{Res} T(E) = \frac{\pi}{2\mu^2}\frac{1}{a^2}}$$

From this residue we can find the wave function:

$$\text{Res}\,T(E) = V|B\rangle\langle B|V$$

$$T = V + V G_0 T$$

$$E \to B\,, \quad T \to \frac{\text{Res}\,T}{E-B}$$

$$\text{Res}\,T = \cancelto{0}{(E-B)V} + V G_0 \,\text{Res}\,T$$

$$V|B\rangle\cancel{\langle B|}V = V G_0 \left(V|B\rangle\cancel{\langle B|}V\right)$$

$$V|B\rangle = V G_0 V|B\rangle$$

$$\boxed{|B\rangle = G_0 V|B\rangle}$$

So we found something new:

$$\boxed{|B\rangle = G_0 V |B\rangle}$$

bound state equation

$$\mathrm{Res}\,T = V|B\rangle\langle B|V$$

$$V|B\rangle = G_0^{-1}(B)\,|B\rangle$$

$$\langle \vec{p}|V|B\rangle = \left(B - \frac{p^2}{2\mu}\right)\underbrace{\langle \vec{p}|B\rangle}_{\psi_B(\vec{p})}$$

the wave function

$$\langle \vec{p}\,'|\mathrm{Res}\,T|\vec{p}\rangle = \left(B - \frac{p^2}{2\mu}\right)\left(B - \frac{p'^2}{2\mu}\right) \times \psi_B(\vec{p})\,\psi_B(\vec{p}\,')$$

Putting the pieces together:

1) $\mathrm{Res}\,T(E) = \frac{\pi}{\mu}\frac{2}{a}$

2) $\langle \bar{p}' | \mathrm{Res}\,T(E) | \bar{p} \rangle =$

$$\left(B - \frac{p^2}{2\mu}\right)\left(B - \frac{p'^2}{2\mu}\right) \psi_B(\bar{p}) \psi_B(\bar{p}')$$

3) $$\psi_B(\bar{p}) = \frac{\sqrt{8\pi/a}}{p^2 - 2\mu B}$$

$2\mu B = -\gamma^2 \quad \left(B = -\frac{\gamma^2}{2\mu}\right)$

4) $\Rightarrow \gamma = \frac{1}{a}$

$$\Rightarrow \boxed{\psi_B(\bar{p}) = \frac{\sqrt{8\pi\gamma}}{p^2 + \gamma^2}}$$

And we learn the following:

1) Pure contact-range-theory:

$a_0 \neq 0 \; , r_0 = 0 \quad (\Lambda \to \infty)$

2) If $a_0 > 0$, $\exists$ bound state

2.a) $B = -\frac{1}{2\mu}\left(\frac{1}{a_0}\right)^2$

$\Rightarrow \gamma = 1/a_0$

2.b) $\Psi_B(\vec{p}) = \frac{\sqrt{8\pi\gamma}}{p^2+\gamma^2}$

$$\Psi_B(\vec{r}) = \frac{\sqrt{2\gamma}}{\sqrt{4\pi}} \frac{e^{-\gamma r}}{r}$$

(if we Fourier-transform)

$\hookrightarrow$ [Exercise: 1 point]

Today's lesson might feel a bit like reading Wittgenstein (on acid)

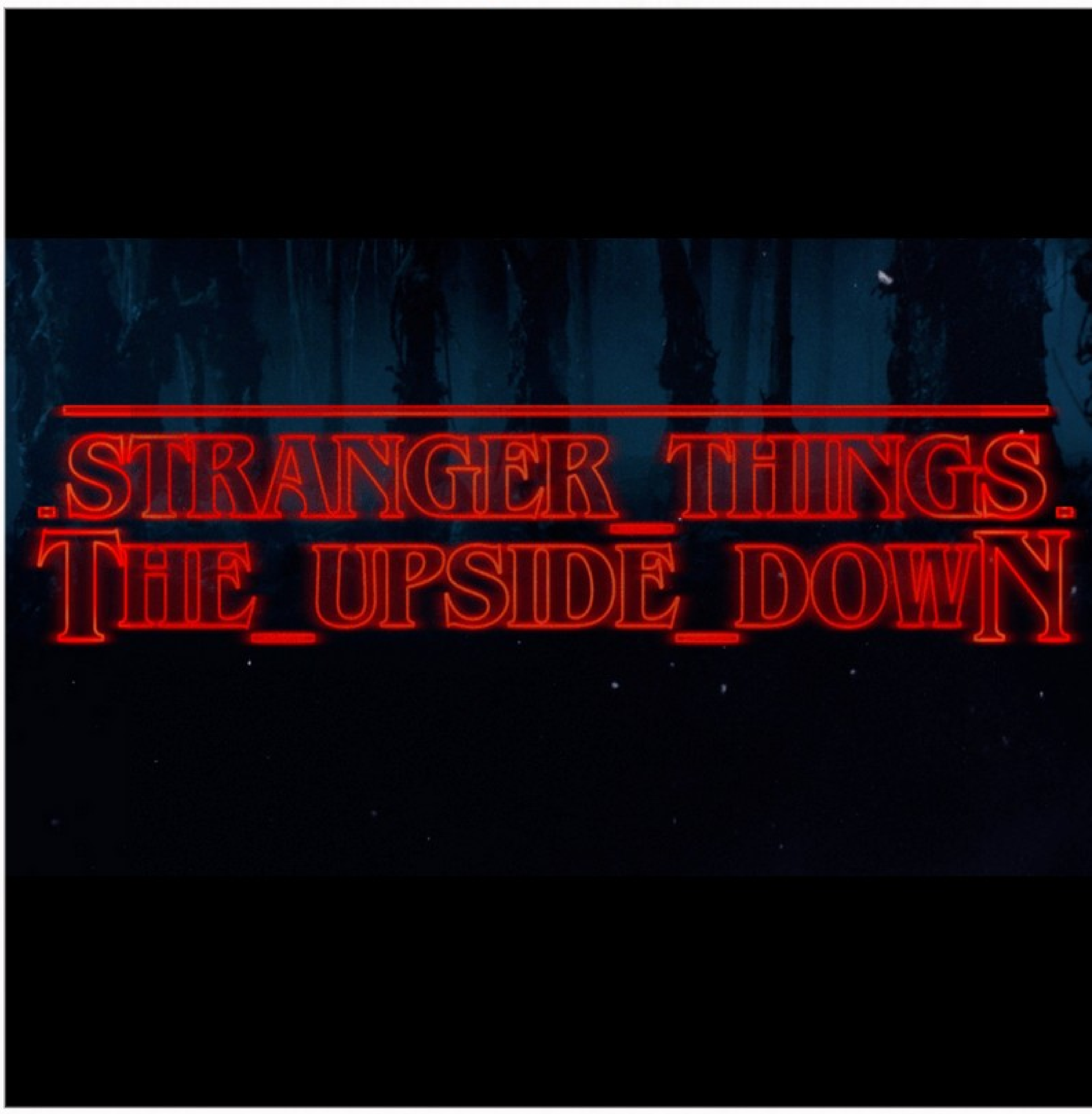
but this is just the everyday life of a theoretical physicist

$$T = V + V G_0 T$$

$$T(E) \xrightarrow[E \to B]{} V \frac{|B\rangle\langle B|}{E - B} V$$

Enjoy!

Theoreticians: you need to know all this upside down



...no matter how strange it is

$\boxed{\text{WELCOME TO P-SPACE!}}$

Next lesson → more about the T-matrix