Nuclear Physics (14)

Formal Scattering Theory

(The T-matrix)

Now it's time for abstraction
$$\rightarrow$$
?

Two views on Quantum Mechanics:

1) Wave functions obeying differential equations

$$\left[-\frac{\nabla^2}{2m}+V(\vec{r})\right]\psi(\vec{r})=E\,\psi(\vec{r})$$

2) Operators acting on vectors on a Hilbert space

$$\hat{H}|\psi\rangle=E|\psi\rangle$$

+ other views (Path Integrals)

What does this imply for scattering theory?

View 1)

$$\psi(\vec{r}) \to e^{i\vec{k}\cdot\vec{r}} + f(\Omega)\frac{e^{ikr}}{r}$$

$$\frac{d\sigma}{d\Omega} = |f(\Omega)|^2$$

View 2)

$$|\psi(\vec{r})\rangle = |\vec{k}\rangle + T G_0 |\vec{k}\rangle$$

$$\langle \vec{r}| G_0 |\vec{k}\rangle = -\frac{\mu}{2\pi}\frac{e^{ikr}}{kr}$$

$$f(\Omega) = -\frac{\mu}{2\pi}\langle \vec{k}'|T|\vec{k}\rangle$$

$\boxed{T, G_0}$ ⟶ operators

Aim of today's lesson:

To define the T-matrix, the operator equivalent of the scattering amplitude

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A schematic derivation

1) $\underline{H}|\phi\rangle = E\,\underline{|\phi\rangle}$

↳ wave-function

↳ Hamiltonian

2) $H = \underline{H_0} + \underline{V}$

↳ Potential

↳ kinetic energy

1+2 ⇒ 3) $(E - H_0)|\phi\rangle = V|\phi\rangle$

We want to solve $(E-H_0)|\phi\rangle = V|\phi\rangle$

→ Green's function method

First, we go back to the language of wave functions & diff. eq.'s

$$|\phi\rangle \longrightarrow \phi(\vec{r}) = \langle \vec{r}|\phi\rangle$$

$$\phi(\vec{r}) = e^{i\vec{k}\cdot\vec{r}} + \int d^3\vec{r}\,' G_0(\vec{r}-\vec{r}\,') \times V(\vec{r}\,')\phi(\vec{r}\,')$$

→ this is actually an ansatz

(≡ proposal of a solution)

It will be a solution if:

$$(E-H_0)G_0(\vec{r}-\vec{r}\,') = \delta^{(3)}(\vec{r}-\vec{r}\,')$$

Now, solving this equation

$$(E-H_0)\,G_0(\vec{r},\vec{r}\,') = \delta^{(3)}(\vec{r}-\vec{r}\,')$$

does not look trivial

We try momentum space

$$G_0(\vec{p}) = \int \frac{d^3\vec{p}}{(2\pi)^3}\, G_0(\vec{r})\, e^{-i\vec{p}\cdot\vec{r}}$$

$$\left[\begin{array}{l} (E-H_0)\,G_0(\vec{p}) = 1 \\ H_0 = \dfrac{\vec{p}^{\,2}}{2\mu} \end{array}\right]$$

$$\boxed{G_0(\vec{p}) = \frac{1}{E-\frac{\vec{p}^{\,2}}{2\mu}}}$$

Equivalent

Operator language: $\boxed{G_0(E) = \frac{1}{E-H_0}}$

Still, the original objective is:

$$G_0(\vec{r}) = \int \frac{d^3\vec{\ell}}{(2\pi)^3} G_0(\vec{\ell})\, e^{i\vec{\ell}\cdot\vec{r}}$$

$$= \frac{1}{2\pi r^2} \int_0^\infty d\ell \, \frac{\ell \sin(\ell r)}{E - \frac{\ell^2}{2\mu}}$$

$$= \frac{1}{4\pi r^2} \operatorname{Im}\left[\int_{-\infty}^{+\infty} d\ell \, \frac{\ell e^{i\ell r}}{E - \frac{\ell^2}{2\mu}} \right]$$

Typical integral to be solved by the residue method

$$\int_{-\infty}^{\infty} \ell \, d\ell \, \frac{e^{i\ell r}}{E - \frac{\ell^2}{2\mu}}$$

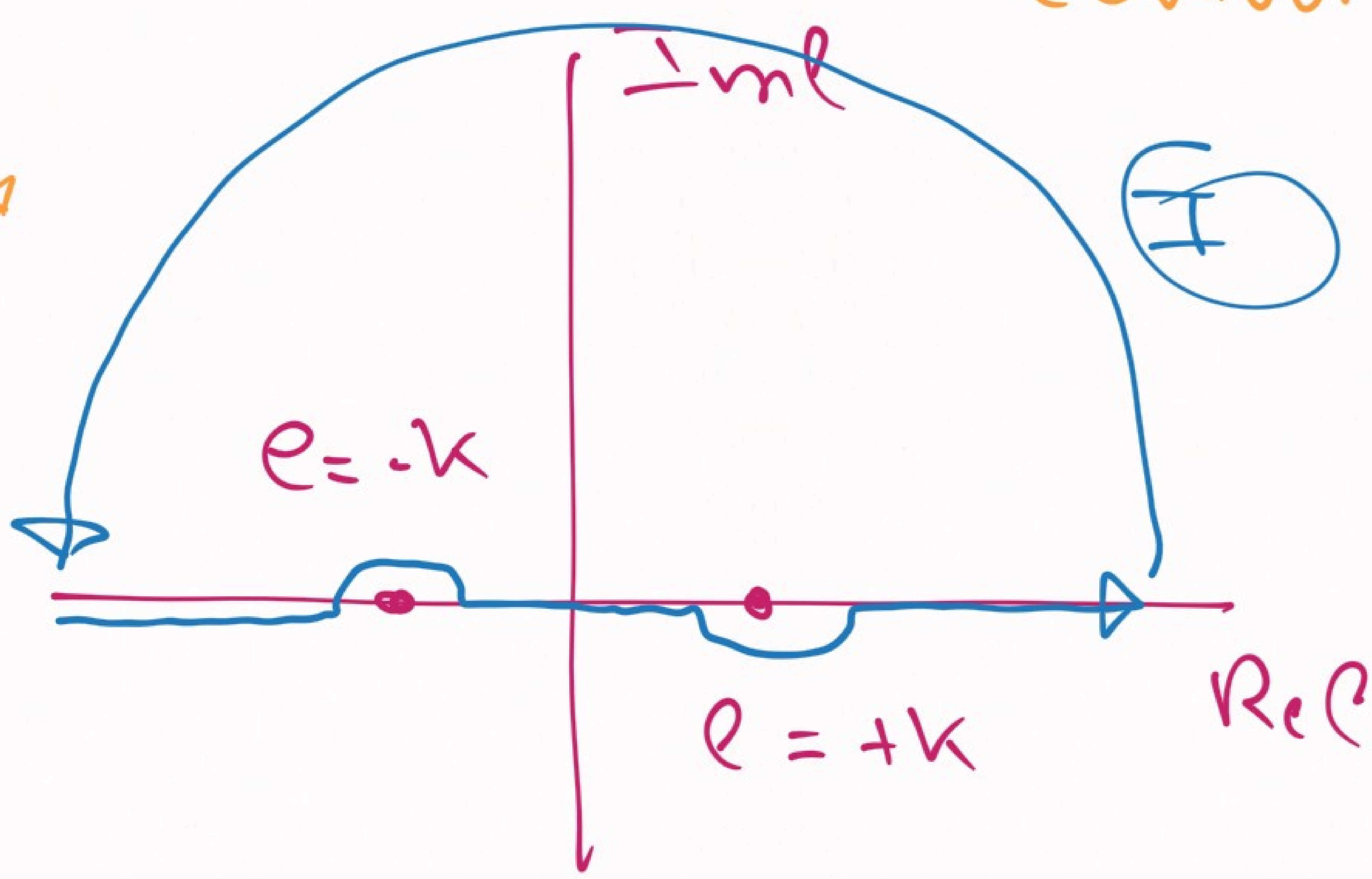
Poles:

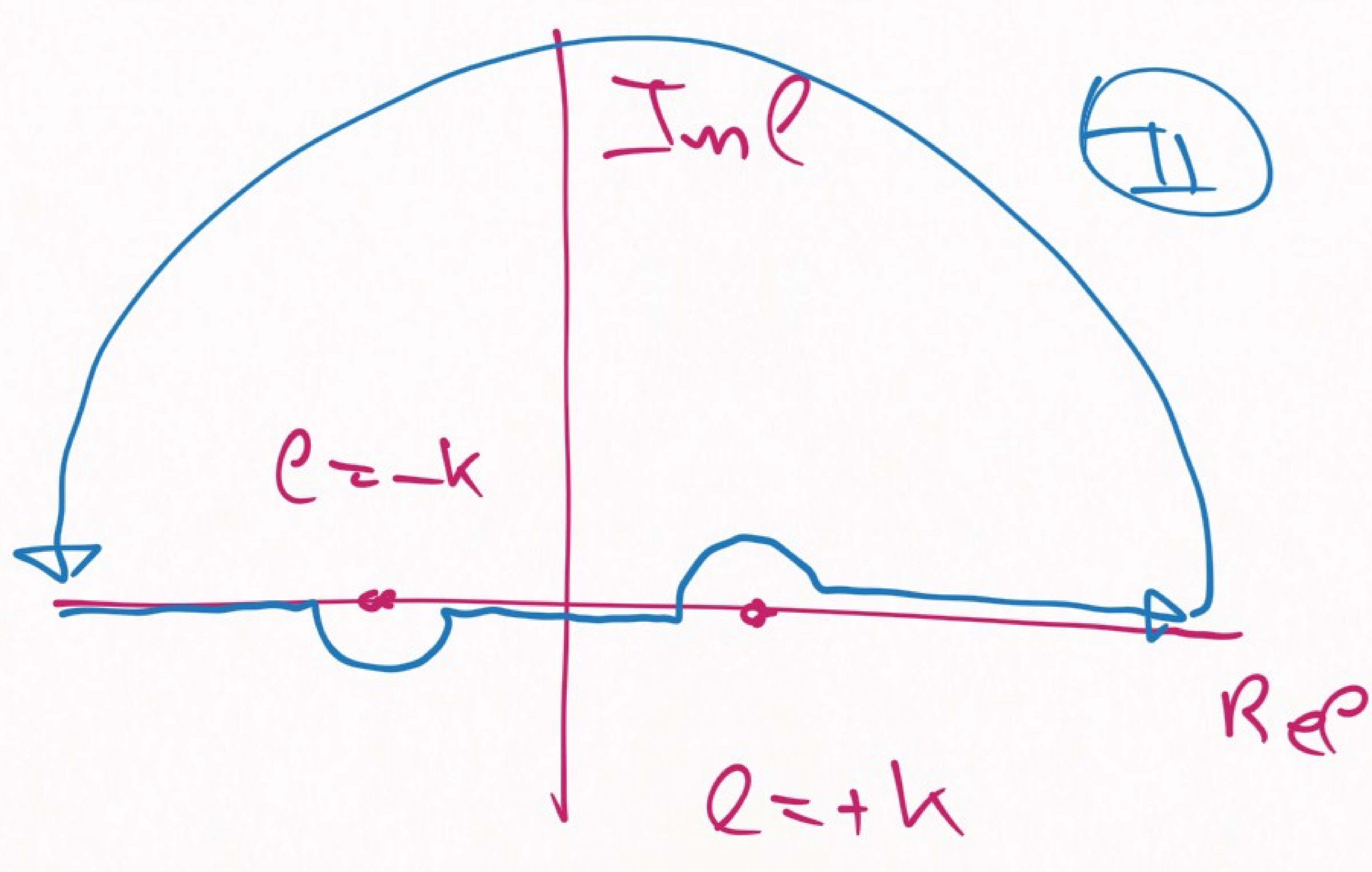
$$\Rightarrow \ell = \pm\sqrt{2\mu E}$$

$$= \pm K$$

$$\int_{-\infty}^{+\infty} \ell\, d\ell \frac{e^{i\ell r}}{E - \frac{\ell^2}{2\mu}} = \oint \ell\, d\ell \frac{e^{i\ell r}}{E - \frac{\ell^2}{2\mu}}$$

Over a suitable integration contour





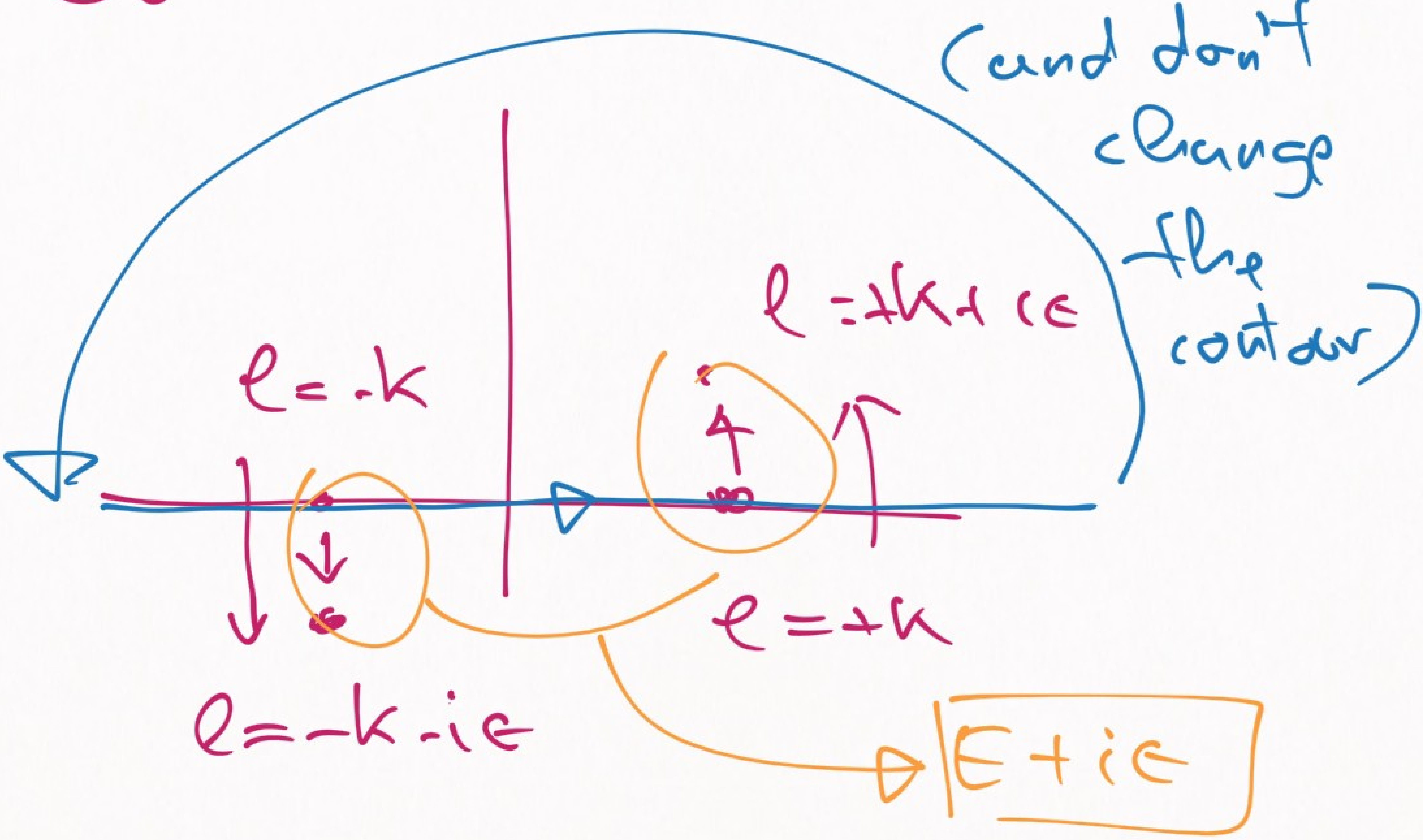
Then we go back to review complex analysis and find that:

$\boxed{\text{Contour I}}$ $\quad G_0^{(I)}(\vec{r}) = -\frac{\mu}{2\pi}\frac{e^{+ikr}}{r}$

$\boxed{\text{Contour II}}$ $\quad G_0^{(II)}(\vec{r}) = -\frac{\mu}{2\pi}\frac{e^{-ikr}}{r}$

Alternatively, we can modify the propagator (G_0)

$$G_0(E) \longrightarrow G_0(E \pm i\epsilon)$$



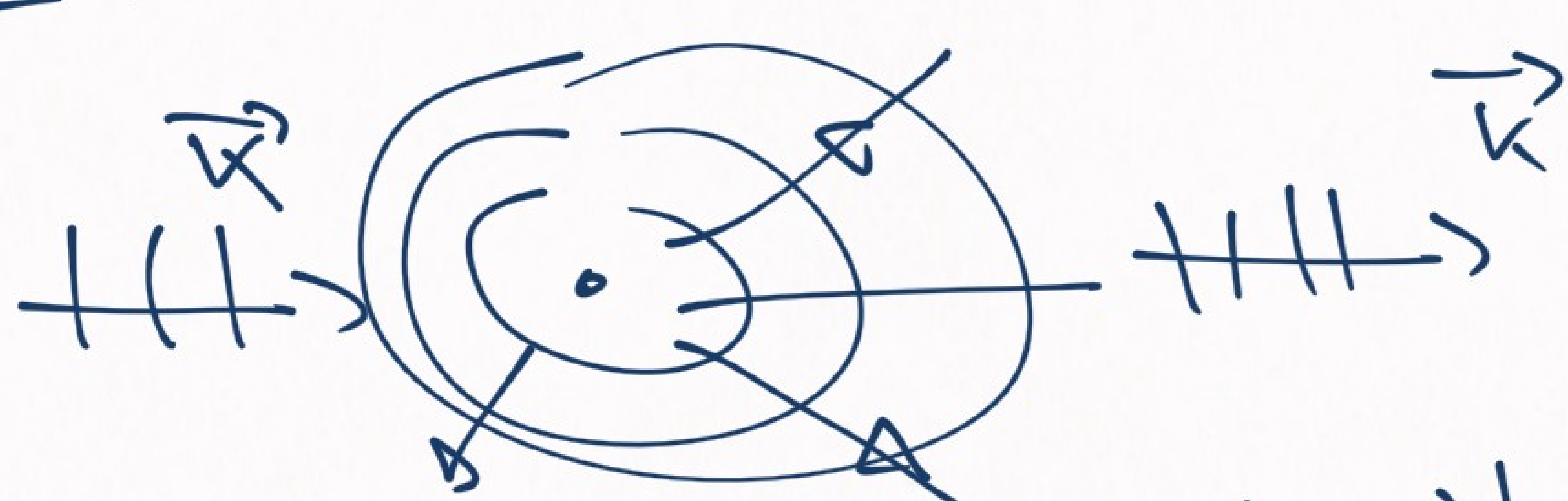
The result of $E \to E \pm i\epsilon$

$$G_0(\vec{r}; E \pm i\epsilon) = -\frac{\mu}{2\pi} \frac{e^{\pm ikr}}{r}$$

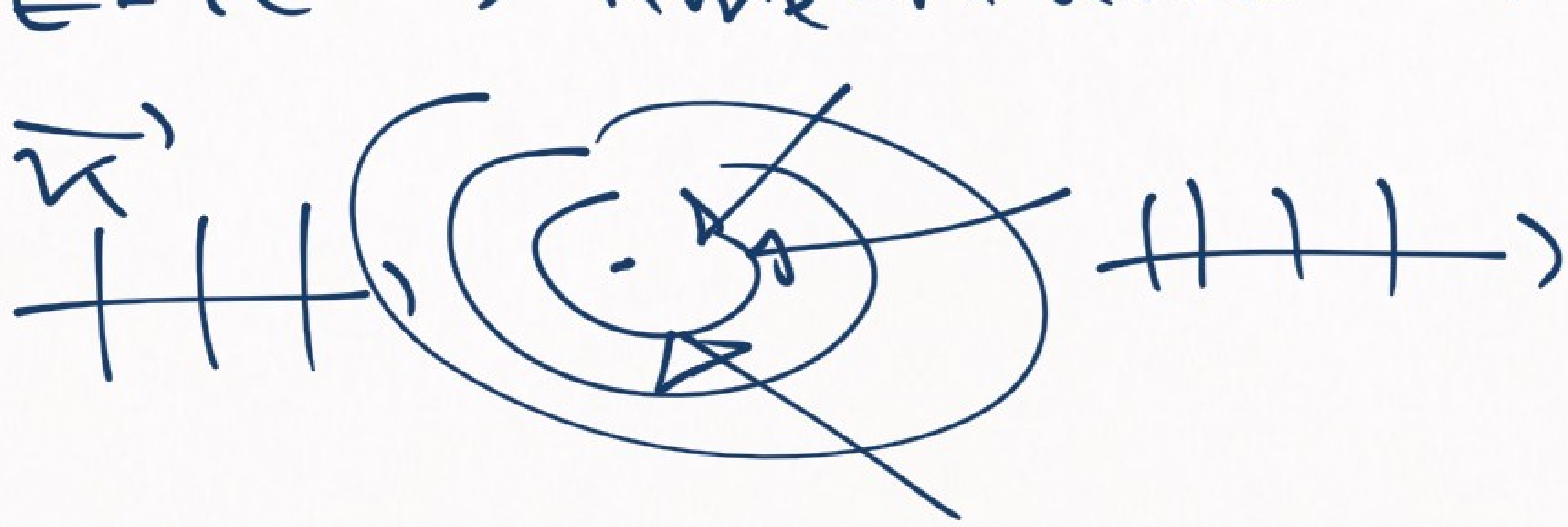
↳ more compact, this is indeed the standard procedure

$$|\phi\rangle \longrightarrow |\phi^{\mp}\rangle$$ (depending on whether we use $E \pm i\epsilon$)

1) $E + i\epsilon \to$ standard scattering



2) $E - i\epsilon \to$ time-reversed scattering



But we still need to see what happens w/ ϕ :

$\boxed{\text{RECAP}}$ 1) $(E-H_0)|\phi\rangle = V|\phi\rangle$

2) $\phi(\vec{r}) = e^{i\vec{k}\cdot\vec{r}} + \int d^3\vec{r}\,' G_0(\vec{r}-\vec{r}\,') \times V(\vec{r}\,')\phi(\vec{r}\,')$

or: $\underbrace{|\phi\rangle = |\vec{k}\rangle + G_0 V|\phi\rangle}_{\text{In operator language}}$

3) $G_0(\vec{r}; E \pm i\epsilon) = -\frac{\mu}{2\pi}\frac{e^{\pm ikr}}{r}$

or: $G_0(E) = \frac{1}{E-H_0}$

$\boxed{\text{NEXT}}$

4) $\phi^+(\vec{r}) \to e^{i\vec{k}\cdot\vec{r}} + f(\Omega)\frac{e^{ikr}}{r}$?

Well, let's see ...

$$\phi^+(\vec{r}) = e^{i\vec{k}\cdot\vec{r}} - \frac{\mu}{2\pi}\int d^3\vec{r}\,' \frac{e^{ik|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|}$$

$$\times V(\vec{r}\,')\,\phi^+(\vec{r}\,')$$

$\lim r \to \infty$

$$\phi^+(\vec{r}) \to e^{i\vec{k}\cdot\vec{r}} - \underbrace{\frac{\mu}{2\pi}\frac{e^{ikr}}{r}\int d^3\vec{r}\,' e^{-i\vec{k}\,'\cdot\vec{r}\,'} \times V(\vec{r}\,')\,\phi^+(\vec{r}\,')}_{+ f(\Omega)\frac{e^{ikr}}{r}}$$

$$f(\Omega) = -\frac{\mu}{2\pi}\int d^3\vec{r}\,' e^{-i\vec{k}\,'\cdot\vec{r}\,'} V(\vec{r}\,')\,\phi^+(\vec{r}\,')$$

$$= -\frac{\mu}{2\pi}\langle \vec{k}\,'|V|\phi^+\rangle$$

So we arrived at this:

$$f(\Omega) = -\frac{\mu}{2\pi} \langle \vec{k}' | V | \phi^+ \rangle$$

which is a bit puzzling.

→ we still need $|\phi^+\rangle$

$$|\phi^+\rangle = |\vec{k}\rangle + G_0(E+i\epsilon) V |\phi^+\rangle$$

$$= |\vec{k}\rangle + G_0 V |\vec{k}\rangle$$

$$+ G_0 V G_0 V |\phi^+\rangle = \circledast$$

$$\circledast = |\vec{k}\rangle + G_0 V |\vec{k}\rangle + G_0 V G_0 V |\vec{k}\rangle$$

$$+ G_0 V G_0 V G_0 V |\vec{k}\rangle + \dots$$

↳ This is an iterative equation

So let's go iterative...

$$f(\Omega) = -\frac{\mu}{2\pi} \langle \vec{k}' | V | \phi^+ \rangle$$

$$\langle \vec{k}' | V | \phi^{(+)} \rangle = \langle \vec{k}' | V | \vec{k} \rangle + \langle \vec{k}' | V G_0 V | \phi^{(+)} \rangle$$

$$= \langle \vec{k}' | V + V G_0 V | \vec{k}' \rangle + \langle \vec{k}' | V G_0 V G_0 V | \phi^{(+)} \rangle$$

$$= \ldots$$

$$\langle \vec{k}' | V | \phi^+ \rangle = \circledast$$

$$\circledast = \langle \vec{k}' | V + V G_0 V + V G_0 V G_0 V + \ldots | \vec{k} \rangle$$

$$= \underbrace{\langle \vec{k}' | T | \vec{k} \rangle}$$

Here enters the T-matrix

And we can now rewrite $f(\Omega)$ as:

$$f(\Omega) = -\frac{\mu}{2\pi} \langle \vec{k}' | V | \Phi^+ \rangle$$

$$= -\frac{\mu}{2\pi} \langle \vec{k}' | T | \vec{k} \rangle$$

with: $\boxed{T = V + V G_0 T}$

$\Downarrow$

[The Lippmann-Schwinger equation]

$\Downarrow$

So... what can we do w/ this equation?

First, it's an integral equation:

$$\boxed{T = V + V G_0 T}$$

$\langle \vec{k}'|$ $|\vec{k}\rangle$

$$\langle \vec{k}'|T|\vec{k}\rangle = \langle \vec{k}'|V|\vec{k}\rangle + \langle \vec{k}'|V G_0 T|\vec{k}\rangle$$

$$\int \frac{d^3\vec{e}}{(2\pi)^3} |\vec{e}\rangle\langle \vec{e}|$$

$$\boxed{\langle \vec{k}'|T|\vec{k}\rangle = \langle \vec{k}'|V|\vec{k}\rangle + \int \frac{d^3\vec{e}}{(2\pi)^3} \frac{\langle \vec{k}'|V|\vec{e}\rangle\langle \vec{e}|T|\vec{k}\rangle}{E - \frac{e^2}{2\mu}}}$$

Notes:

1) $G_0|\vec{p}\rangle = \frac{1}{E-H_0}|\vec{p}\rangle = \circledast$

$$\circledast = \frac{1}{E-\frac{\vec{p}^2}{2\mu}}|\vec{p}\rangle$$

2) $\langle\vec{k}'|\vec{k}\rangle = (2\pi)^3\delta^{(3)}(\vec{k}'-\vec{k})$

$\hookrightarrow \mathbb{1} = \int \frac{d^3\vec{k}}{(2\pi)^3}|\vec{k}\rangle\langle\vec{k}|$

(the identity)

Connections:

1) Perturbative expansion

2) Feynman diagram

Connection 1)

$$T = V + V G_0 T$$

$$= \underbrace{V}_{\substack{\downarrow \\ 1^{st} \text{ order perturbation theory (PT)} \\ \text{(Born approximation)}}} + \underbrace{V G_0 V + V G_0 V G_0 V}_{2^{nd},\, 3^{rd} \text{ order PT}} + \cdots$$

Born approximation:

$$\left[T = V + \mathcal{O}(V^2) \right]$$

$$f(\Omega) = -\frac{\mu}{2\pi} \langle \vec{k}' | V | \vec{k} \rangle + \mathcal{O}(V^2)$$

$$= -\frac{\mu}{2\pi} \int d^3\vec{r}\, e^{i(\vec{k}' - \vec{k}) \cdot \vec{r}}\, V(\vec{r}) + \mathcal{O}(V^2)$$

With this we can reproduce
Rutherford scattering:

$$f(\Omega) = -\frac{\mu}{2\pi}\langle \bar{k}'|V_c|\bar{k}\rangle + O(V_c^2)$$

$$\langle \bar{k}'|V_c|\bar{k}\rangle = V_c(\bar{k}'-\bar{k})$$

(local potential)

$$V_c(\bar{q}) = 4\pi\frac{\alpha}{|\bar{q}|^2}$$

$$f(\Omega) = -2\mu\frac{\alpha}{|\bar{k}-\bar{k}'|^2} + O(\alpha^2)$$

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{4\mu^2\alpha^2}{|\bar{k}'-\bar{k}|^4} + O(\alpha^3)}$$

Or Yukawa scattering in the Born approximation:

$$f(\Omega) = -\frac{\mu}{2\pi} \langle \bar{k}' | V_Y | \bar{k} \rangle + \mathcal{O}(V_Y^2)$$

$$= -\frac{\mu}{2\pi} \frac{g^2}{m^2 + (\bar{k} - \bar{k}')^2} + \mathcal{O}(g^4)$$

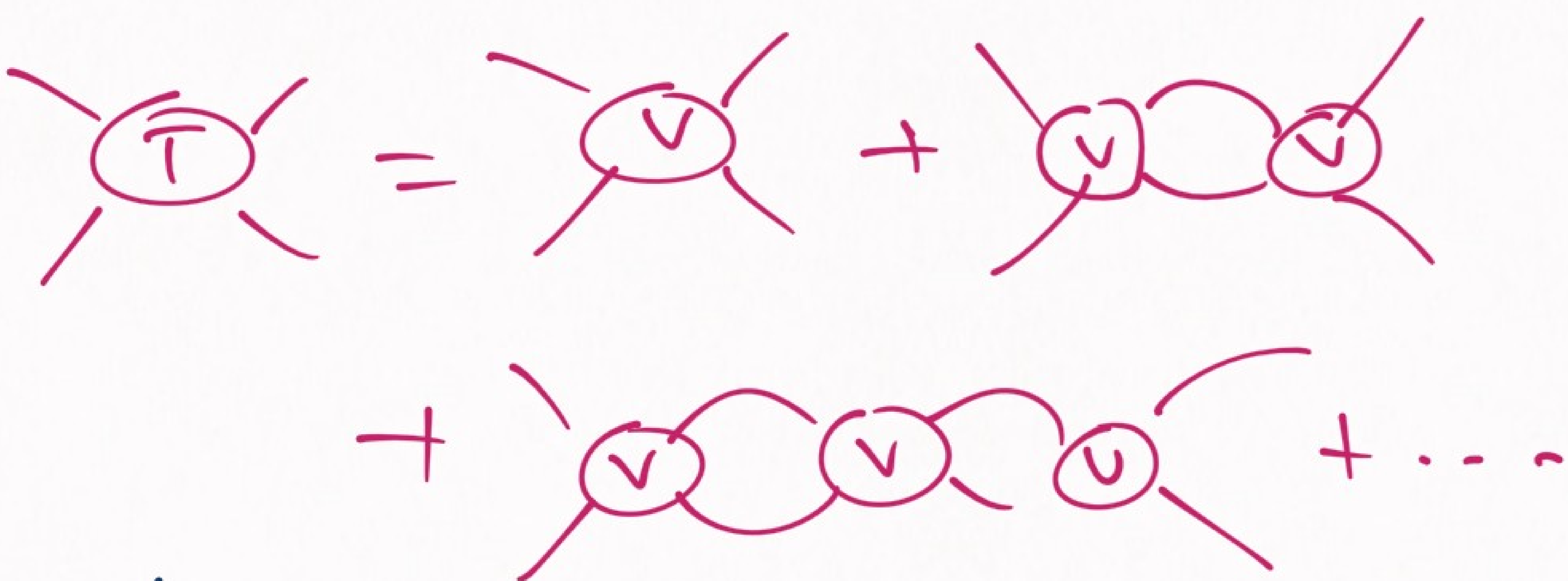
$$\frac{d\sigma}{d\Omega} = \left[-\frac{\mu}{2\pi} \frac{g^2}{m^2 + (\bar{k} - \bar{k}')^2} \right]^2$$

$+$ corrections

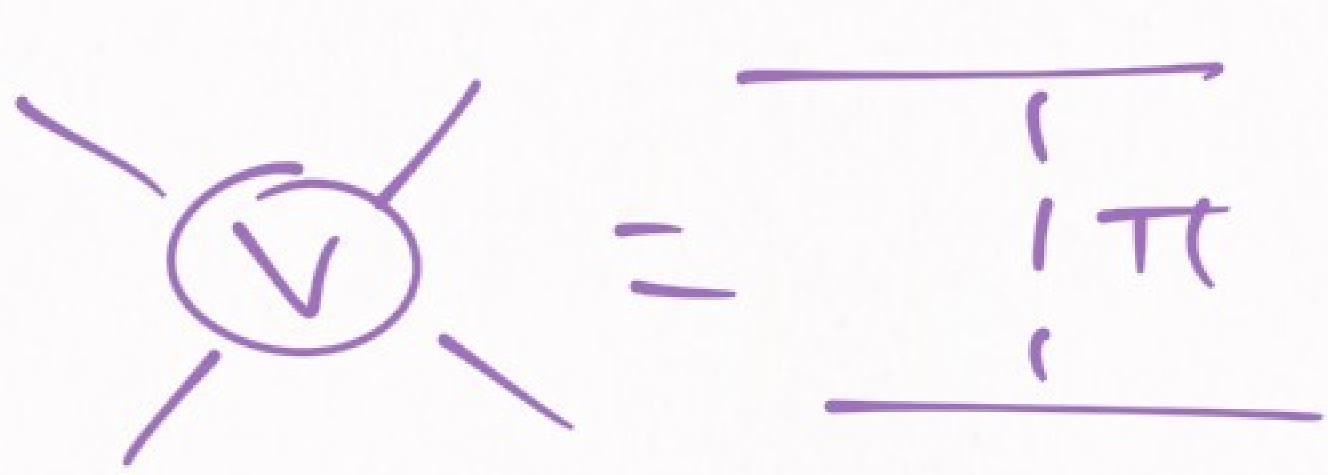
With $T = V + V G_0 V + \mathcal{O}(V^3)$ we could easily do 2^{nd} order perturbation theory etc.

Connection 2)

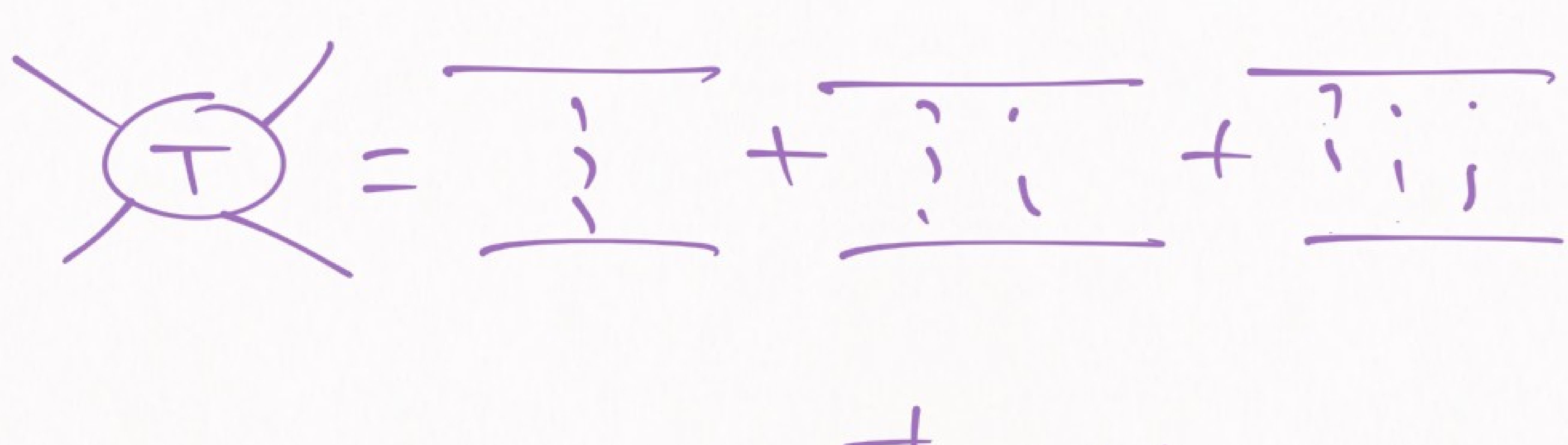
$$T = V + V G_0 V + V G_0 V G_0 V + \dots$$

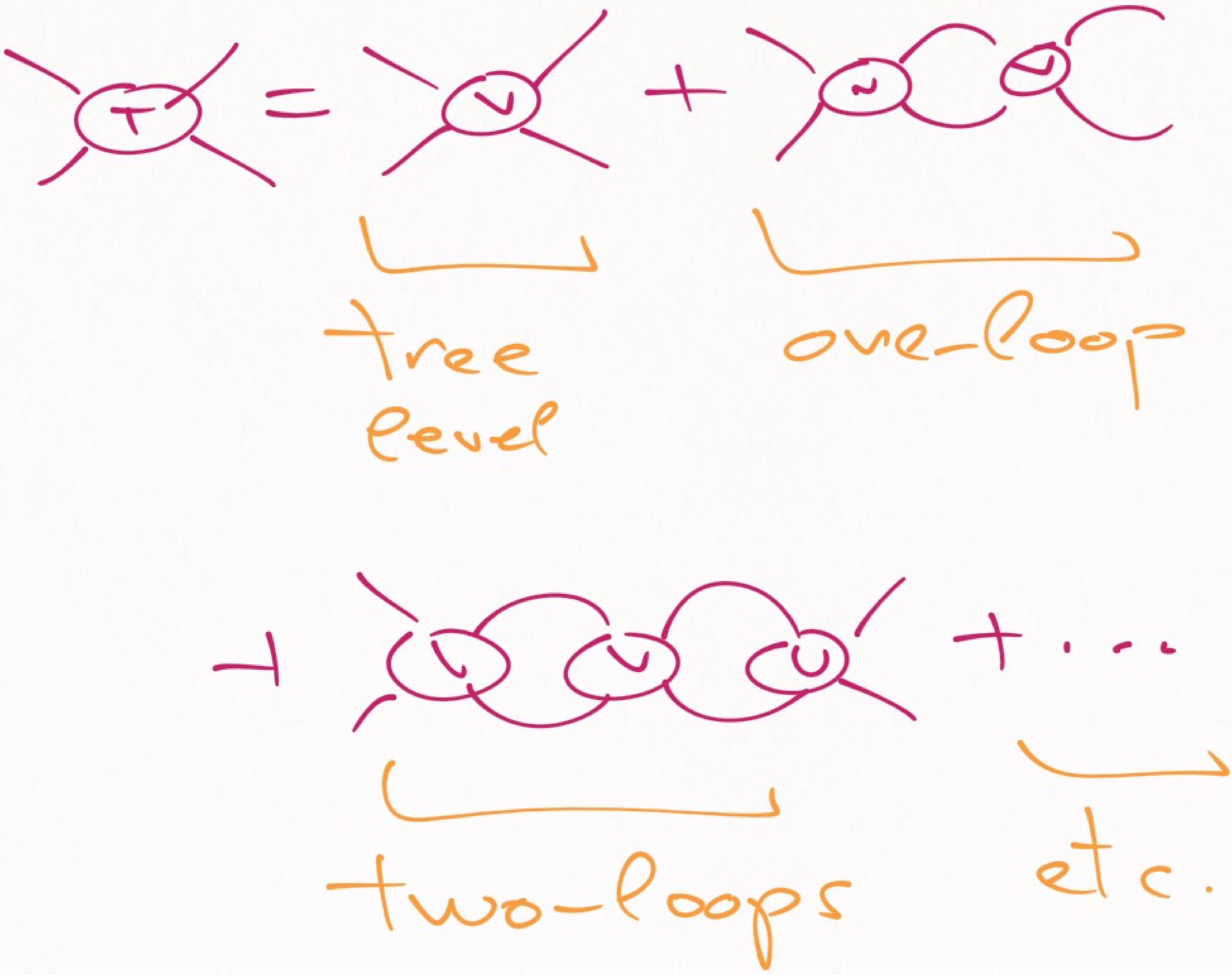


↳ these are simply Feynman diagrams as applied to QM

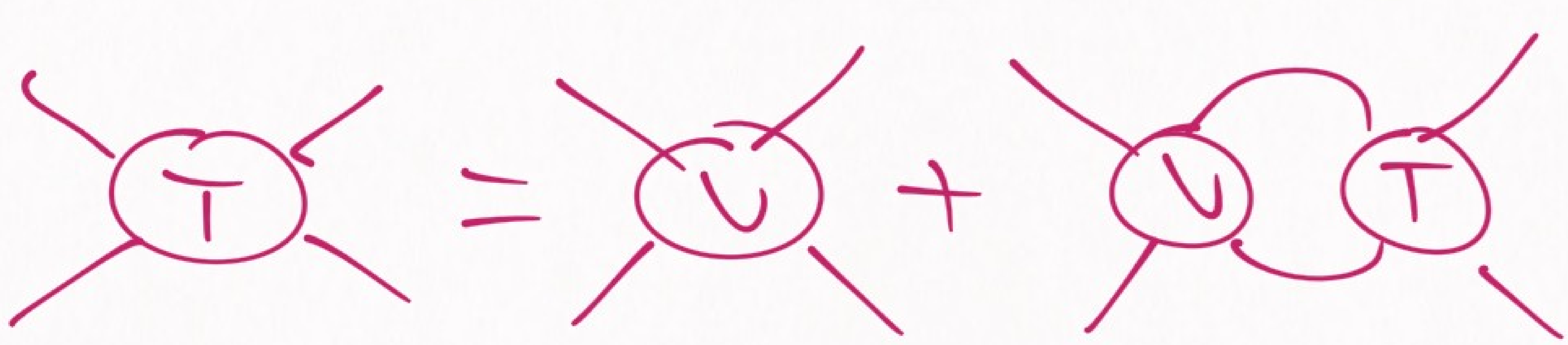
↳ If 

then





This also connects w/ the diagrammatic representation of Lippmann-Schwinger



[Why do we use $T = V + V G_0 T$?]

→ it's an integral equation (difficult), so why?

Advantage → more general

non-local potentials

$\langle \vec{k}' | V | \vec{k} \rangle \neq V(\vec{k}' - \vec{k})$

$\boxed{H|\psi\rangle = E|\psi\rangle}$ → difficult

(Schrödinger)

$\boxed{T = V + V G_0 T}$ → easier

(Lippmann-Schwinger)

$\boxed{\text{NEXT LESSON}}$

A few solutions of the LS equation