Nuclear Physics 13

Scattering theory

RECAP:

$$\boxed{\text{Cross section}}$$

$$\sigma = \frac{N_S}{N_A N_B} S$$

σ → cross section

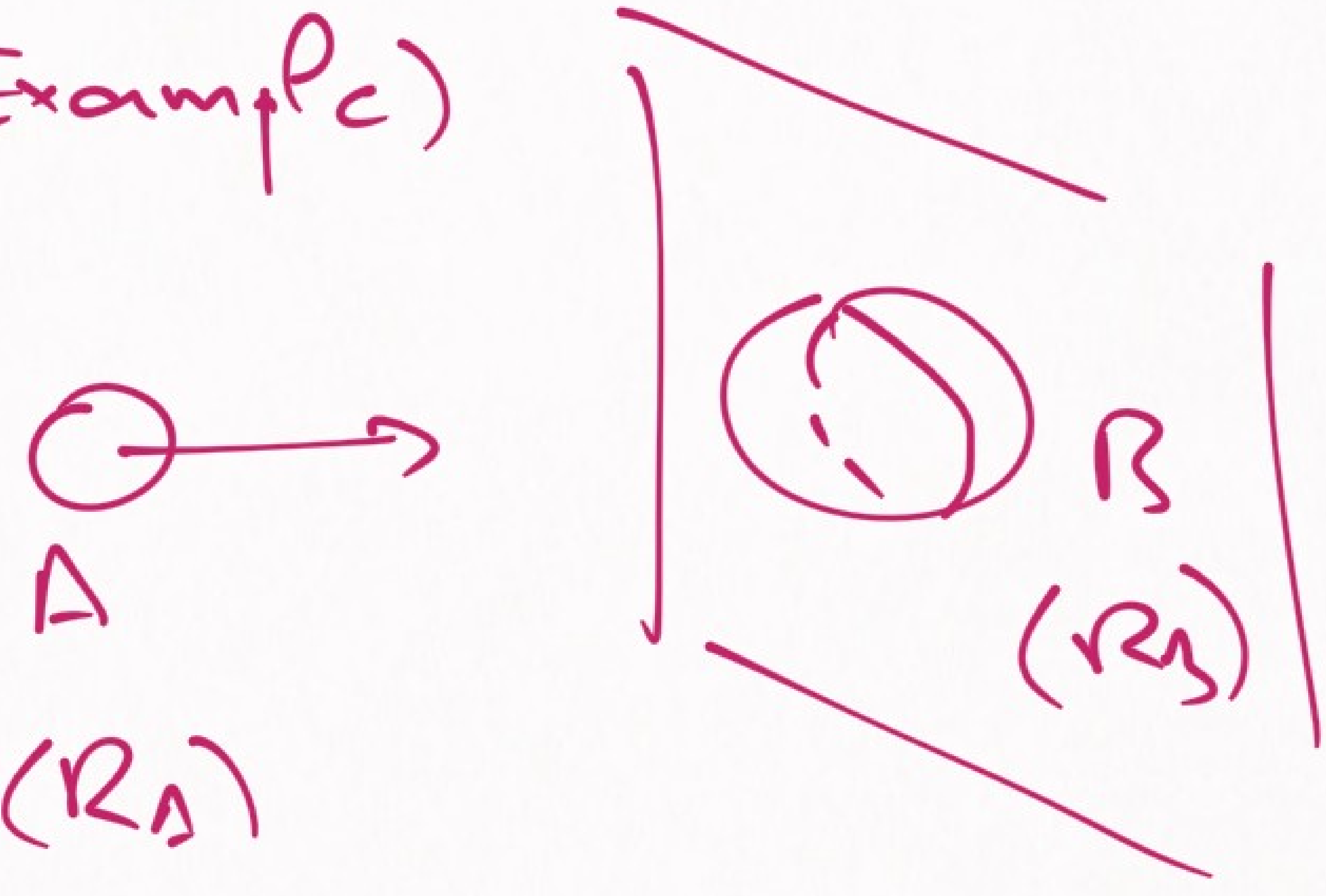
N_A → # of incoming particles

N_B → # of target particles

N_S → # of scattered particles

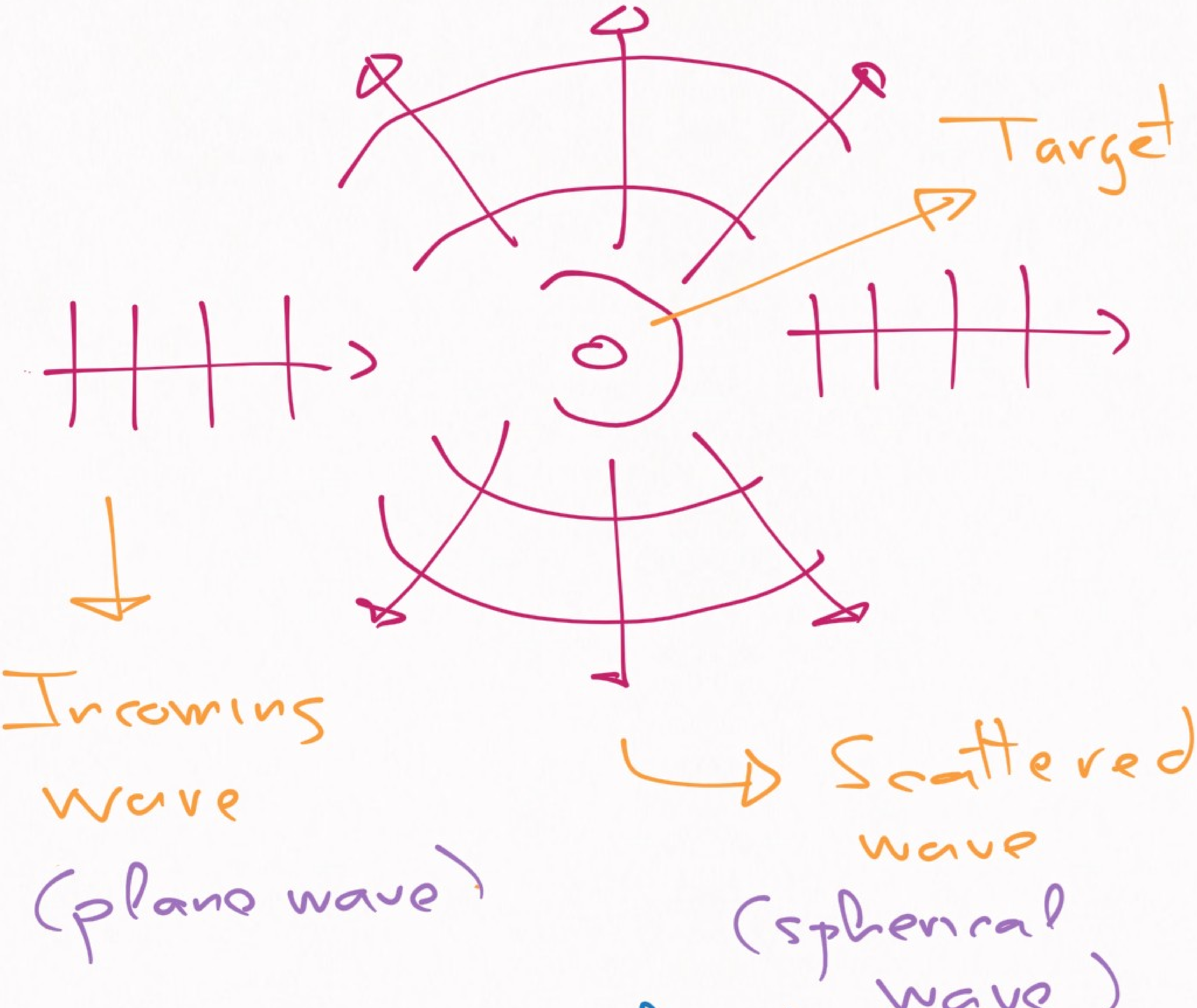
S → surface of beam/target

(Example)



$$\boxed{\sigma = \pi (R_A + R_B)^2}$$

Now we want the QM version:



We write a wave function describing this:

$$\psi_{\vec{k}}(\vec{r}) \rightarrow \underbrace{e^{i\vec{k}\cdot\vec{r}}}_{\text{incoming}} + \underbrace{f(\Omega)\frac{e^{ikr}}{r}}_{\text{scattered}}$$

Next step → translate the classical terms into QM

$$\psi_k(\vec{r}) \to e^{i\vec{k}\cdot\vec{r}} + f(\omega)\frac{e^{ikr}}{r}$$

$$\to \psi_{in}(\vec{r}) + \psi_{out}(\vec{r})$$

$$\left.\begin{array}{l} N_A N_B \sim \psi_{in}(\vec{r}) \\ N_S S \sim \psi_{out}(\vec{r}) \end{array}\right\} \exists \text{ some relation}$$

But we have to figure out

1) $N_B = 1$ (simplification)

$\Rightarrow \frac{N_A}{T} \propto \Phi_{in}$ (incoming flux)

A per time unit

$S \frac{N_C}{T} \propto \int d\vec{S} \cdot \vec{\Phi}_{out}$

(Surface integral of outgoing flux)

2) $J_{in} = |\vec{J}_{in}|$

$$\vec{J}_{in} = -\frac{i}{2m}\left[\psi_{in}^* \vec{\nabla} \psi_{in} - \psi_{in} \vec{\nabla} \psi_{in}^*\right]$$

$$\Rightarrow \boxed{J_{in} = \frac{k}{m}}$$

3) $$\vec{J}_{out} = -\frac{i}{2m}\left[\psi_{out}^* \hat{r}\cdot\vec{\nabla} \psi_{out} - \psi_{out} \hat{r}\cdot\vec{\nabla} \psi_{out}^*\right]\hat{r}$$

$$\Rightarrow \vec{J}_{out} = \frac{k}{m}|f(\Omega)|^2 \frac{\hat{r}}{R^2}$$

distance from target

4) $$\lim_{R\to\infty} \int d\vec{S}\cdot\vec{J}_{out} \quad \left[d\vec{S} = R^2 d\Omega\, \hat{r} \right]$$

$$\lim_{R\to\infty} \int d\vec{S}\cdot\vec{J}_{out} = \frac{k}{m}\int |f(\Omega)|^2 d\Omega$$

Putting the pieces together:

$$\boxed{\sigma = \int |f(\Omega)|^2 \, d\Omega}$$

(total cross section)

Or if we want angular dependence:

$$\boxed{\frac{d\sigma}{d\Omega} = |f(\Omega)|^2}$$

(differential cross section)

But we can still look for more details $\Rightarrow$ Partial Wave Expansion

Partial Wave Expansion

→ We separate the partial waves (i.e. ℓ):

$$\psi(\vec{r}) = \sum_{\ell m} \psi_\ell(r) Y_{\ell m}(\hat{r})$$

→ But we can also include $\vec{k}$:

$$e^{i\vec{k}\cdot\vec{r}} = 4\pi \sum_{\ell m} i^\ell j_\ell(kr) Y^*_{\ell m}(\hat{k}) Y_{\ell m}(\hat{r})$$

$$= \sum_\ell (2\ell+1) i^\ell j_\ell(kr) P_\ell(\hat{k}\cdot\hat{r})$$

$$f(\Omega) = \sum_\ell (2\ell+1) f_\ell(k) P_\ell(\hat{k}\cdot\hat{r})$$

→ But we can do it w/ the full wf:

$$\psi_k(\vec{r}) = 4\pi \sum_{\ell m} i^\ell \frac{u_\ell(r;k)}{r} Y^*_{\ell m}(\hat{k}) Y_{\ell m}(\hat{r})$$

$$= \sum_\ell (2\ell+1)\, i^\ell \frac{u_\ell}{r} P_\ell(\hat{k}\cdot\hat{r})$$

we can plug here
the asymptotic form of wf:

$$\frac{u_\ell}{r} \to e^{i\delta_\ell}\left[\cos\delta_\ell(k)\, j_\ell(kr) - \sin\delta_\ell(k)\, y_\ell(kr)\right]$$

and after some elaboration:

$$f_\ell(k) = \frac{e^{i\delta_\ell}\sin\delta_\ell}{k} = \frac{1}{k\cot\delta_\ell - ik}$$

Note about language:

$f(\Omega) \to$ scattering amplitude

—— ⊗ ——

By the way, if we go back here:

$$\sigma = \int |f(\Omega)|^2 d\Omega = \frac{4\pi}{k^2} \sum_l \sin^2 \delta_l$$

We can also study $k \to 0$

$$\delta_l(k) \to \underbrace{-a_l}_{\text{scattering length}} k^{2l+1} + \mathcal{O}(k^{2l+3})$$

$$\Rightarrow \boxed{\sigma = 4\pi |a_0|^2 + \mathcal{O}(k^2)}$$

A few comments:

1) Experiments measure the cross section

$\boxed{\sigma \,,\, \frac{d\sigma}{d\Omega}}$ $\rightarrow$ exp. input

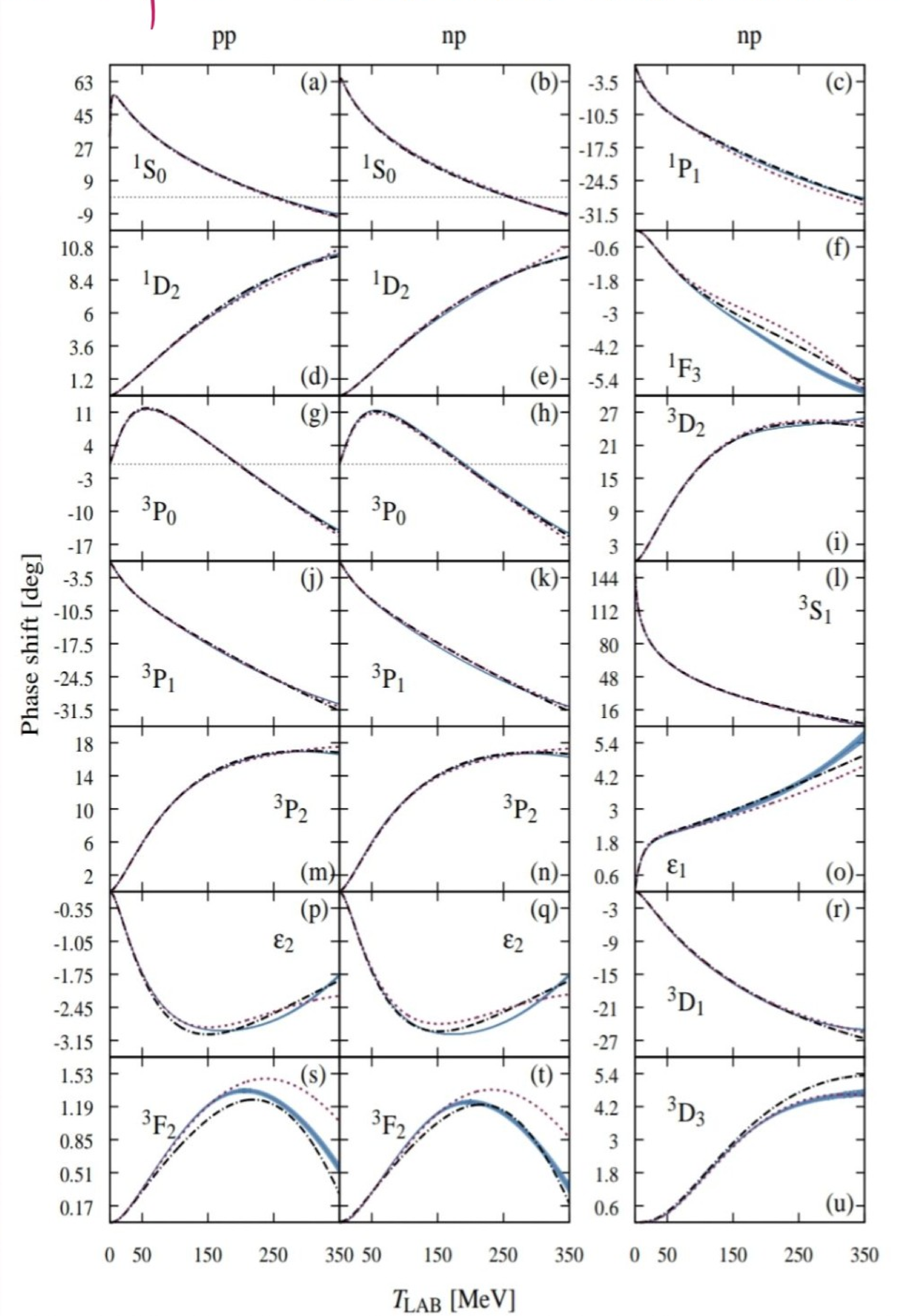
2) Phase shifts are extracted by means of theoretical models

$\rightarrow$ $\boxed{\text{PWA}}$ (partial wave analysis)

Very sophisticated, but they are $\boxed{\text{NOT}}$ experimental data

$\exists$ some model dependence

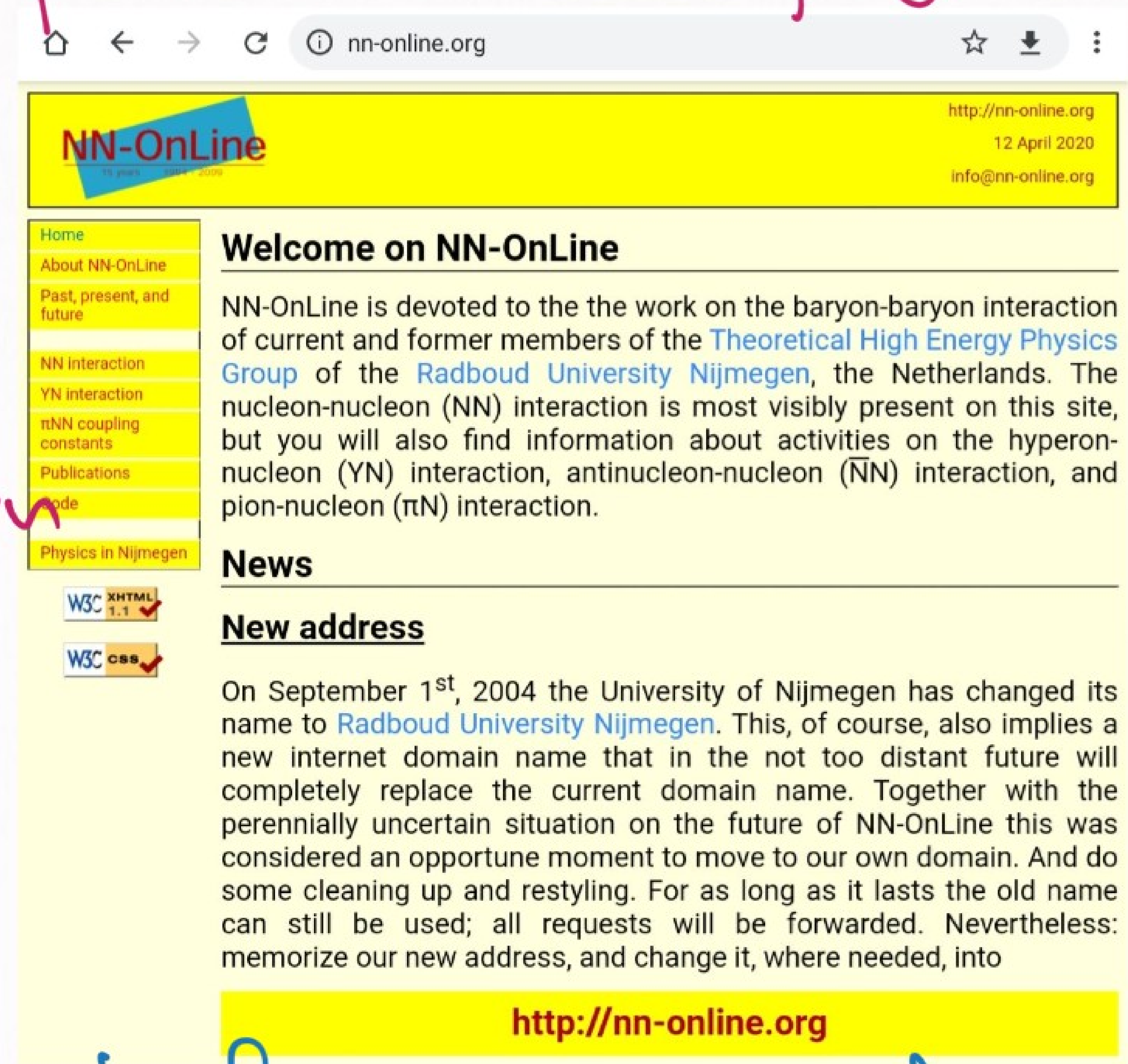
Example: (navarro, amaro, arriola (arxiv: 1304.0895)



$^{2S+1}L_J$ notation: $S \to$ spin, $L \to$ orbital, $J \to$ total (angular momentum)

You can also consult some specialized webpages:

Nijmegen group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NN-OnLine

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Welcome on NN-OnLine

NN-OnLine is devoted to the the work on the baryon-baryon interaction of current and former members of the Theoretical High Energy Physics Group of the Radboud University Nijmegen, the Netherlands. The nucleon-nucleon (NN) interaction is most visibly present on this site, but you will also find information about activities on the hyperon-nucleon (YN) interaction, antinucleon-nucleon ($\overline{N}N$) interaction, and pion-nucleon (πN) interaction.

News

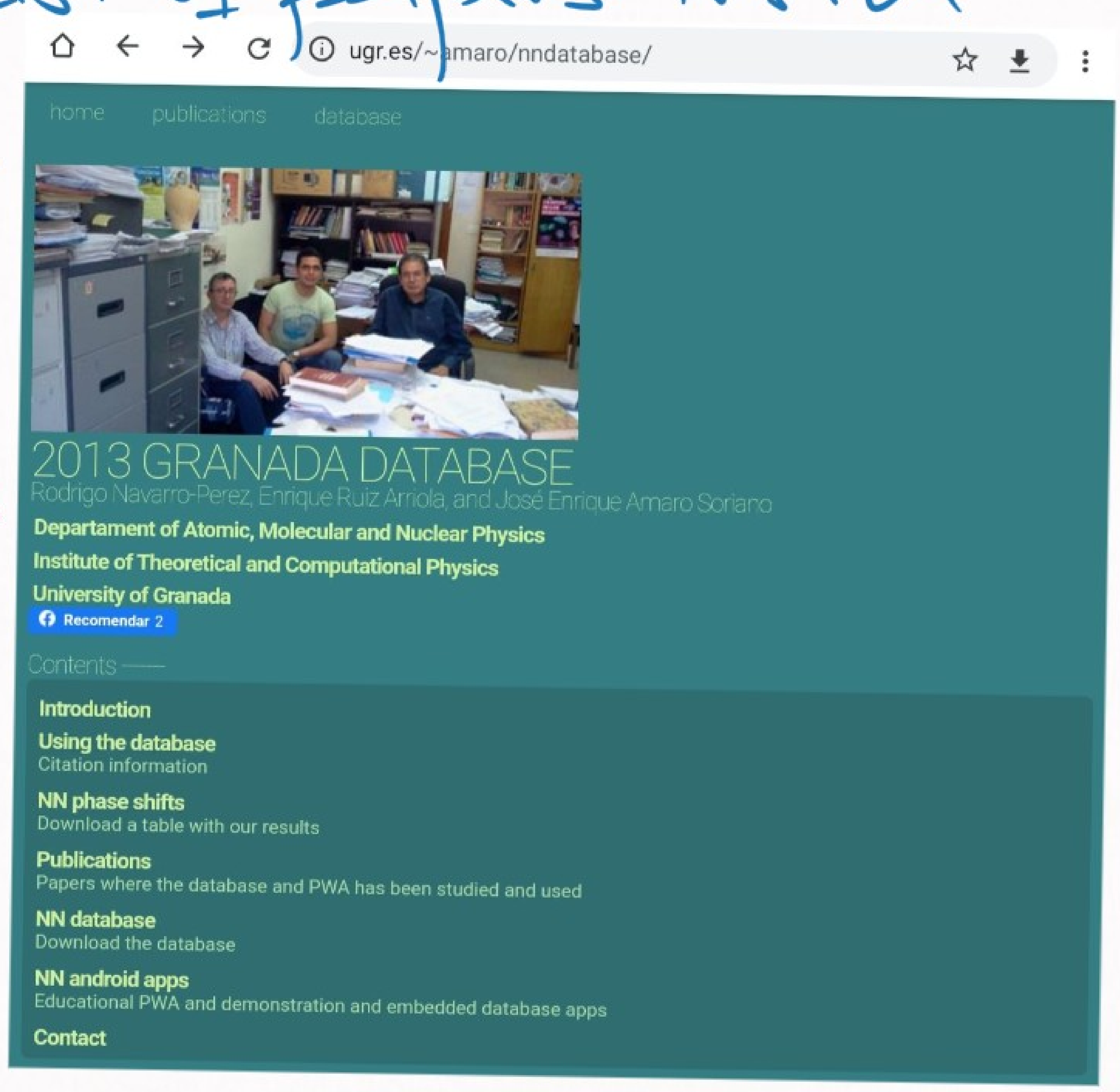
New address

On September 1st, 2004 the University of Nijmegen has changed its name to Radboud University Nijmegen. This, of course, also implies a new internet domain name that in the not too distant future will completely replace the current domain name. Together with the perennially uncertain situation on the future of NN-OnLine this was considered an opportune moment to move to our own domain. And do some cleaning up and restyling. For as long as it lasts the old name can still be used; all requests will be forwarded. Nevertheless: memorize our new address, and change it, where needed, into

http://nn-online.org

List of papers inside

Granada group


ugr.es/~amaro/nndatabase/

home publications database

2013 GRANADA DATABASE

Rodrigo Navarro-Perez, Enrique Ruiz Arriola, and José Enrique Amaro Soriano

Departament of Atomic, Molecular and Nuclear Physics
Institute of Theoretical and Computational Physics
University of Granada

Recomendar 2

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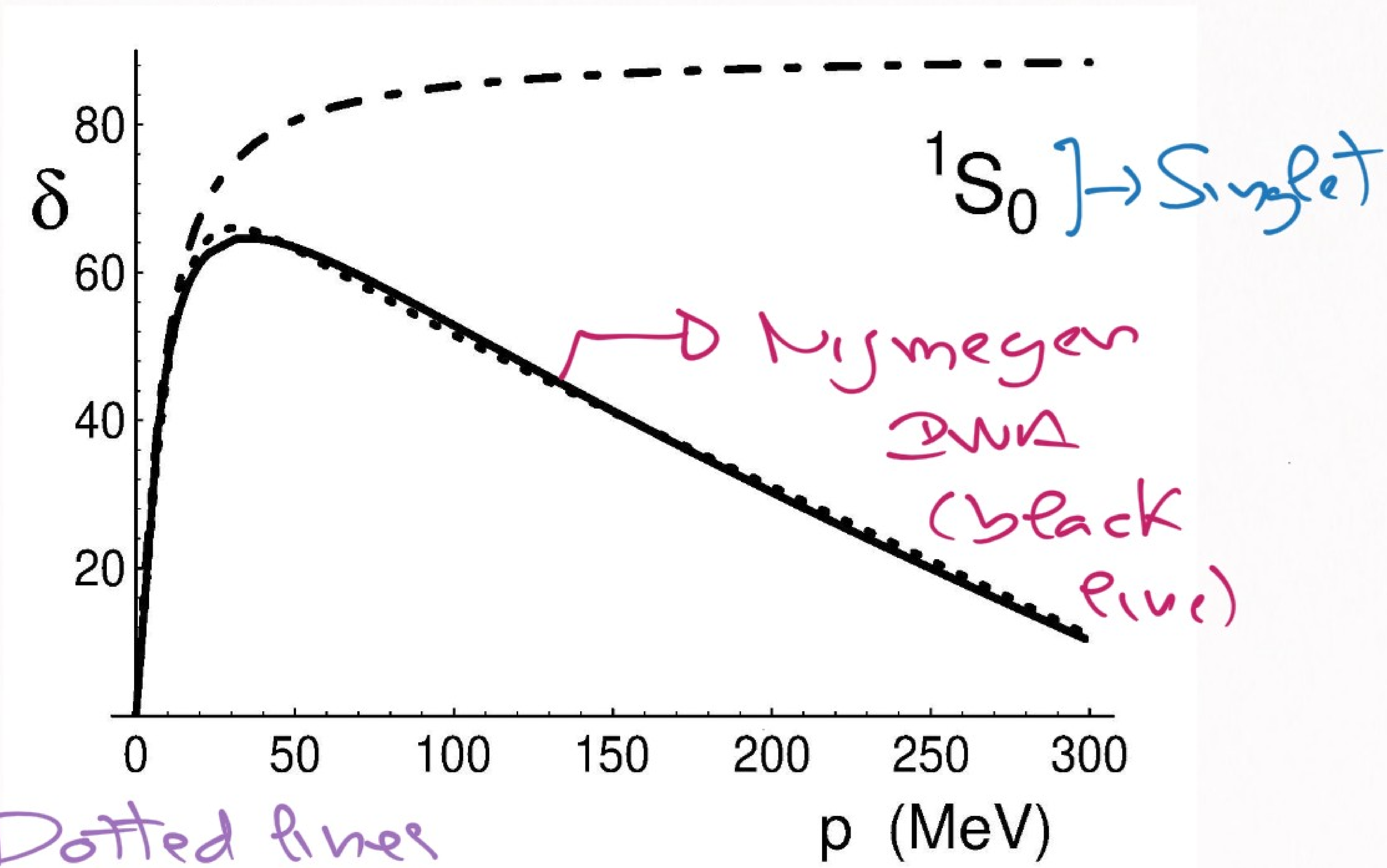
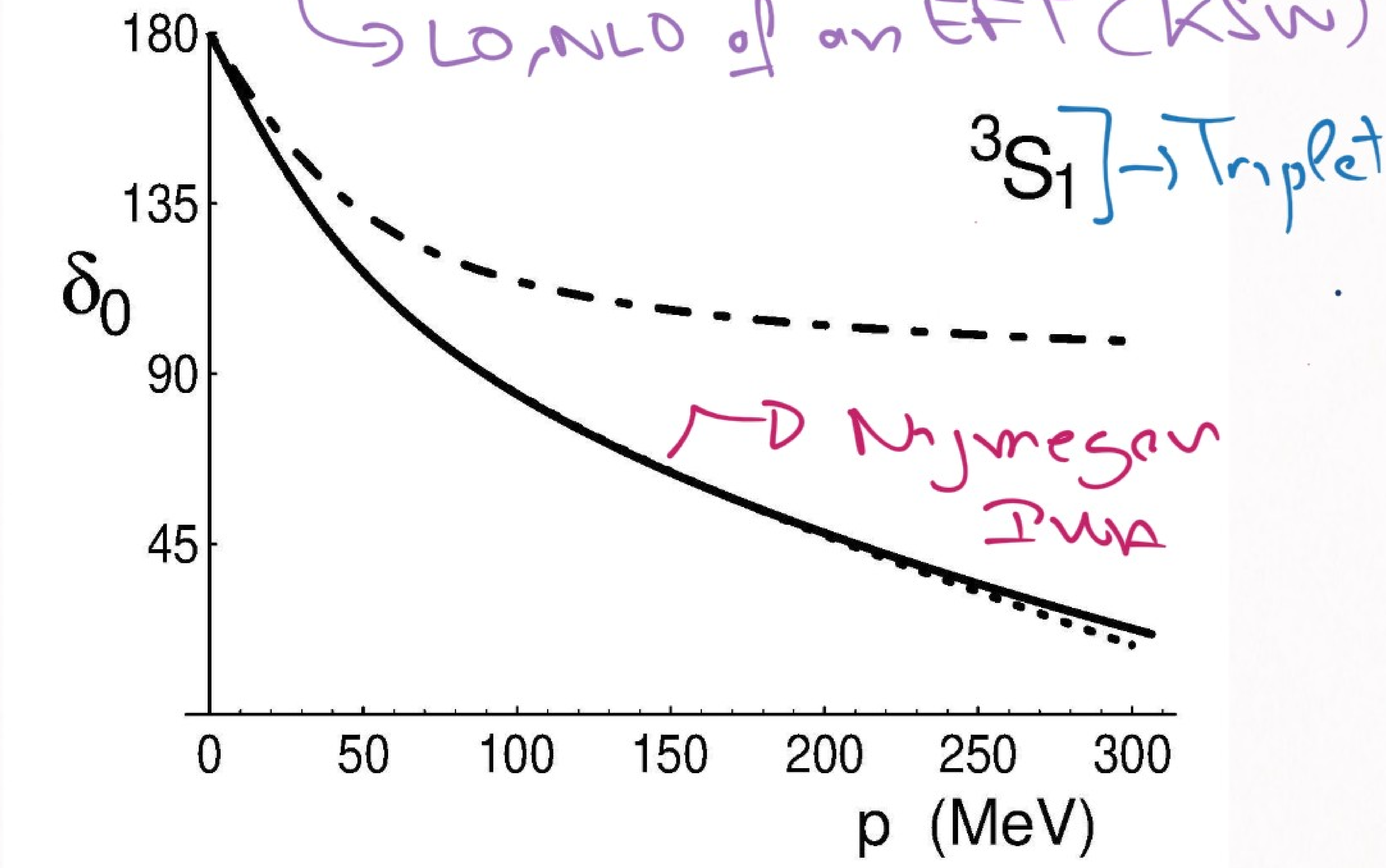
[A look at the S-waves]



Dotted lines
↳ LO, NLO of an EFT (KSW)



BTW, the previous is extracted from these papers:

Two nucleon systems from effective field theory #3

David B. Kaplan (Washington U., Seattle), Martin J. Savage (Washington U., Seattle), Mark B. Wise (Caltech) (Feb 25, 1998)

Published in: *Nucl.Phys.B* 534 (1998) 329-355 • e-Print: nucl-th/9802075 [nucl-th]

pdf DOI cite 657 citations

A New expansion for nucleon-nucleon interactions #4

David B. Kaplan (Washington U., Seattle), Martin J. Savage (Washington U., Seattle), Mark B. Wise (Caltech) (Jan 20, 1998)

Published in: *Phys.Lett.B* 424 (1998) 390-396 • e-Print: nucl-th/9801034 [nucl-th]

pdf DOI cite 687 citations

Nucleon - nucleon scattering from effective field theory #5

David B. Kaplan (Washington U., Seattle), Martin J. Savage (Carnegie Mellon U.), Mark B. Wise (Caltech) (May 3, 1996)

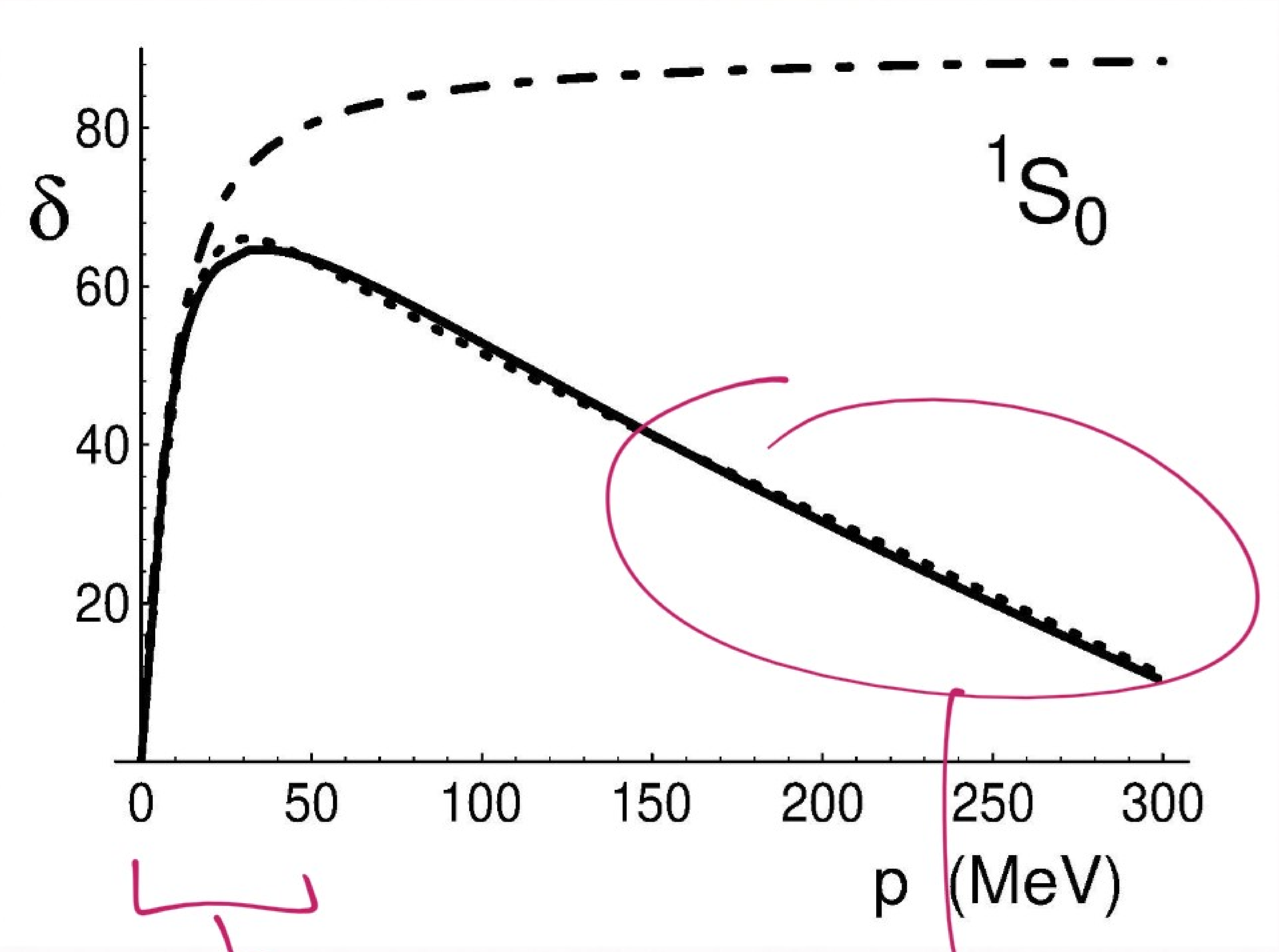
Published in: *Nucl.Phys.B* 478 (1996) 629-659 • e-Print: nucl-th/9605002 [nucl-th]

pdf DOI cite 334 citatio

Feedback

which you should be able to understand (except maybe for the heavy use of momentum space)

$^1S_0$ $\longrightarrow$ a few comments



①
②

① Low energy : $\boxed{\delta_0 \to -a_0 p}$

$\rightarrow$ $a_0 \simeq -23.7$ fm

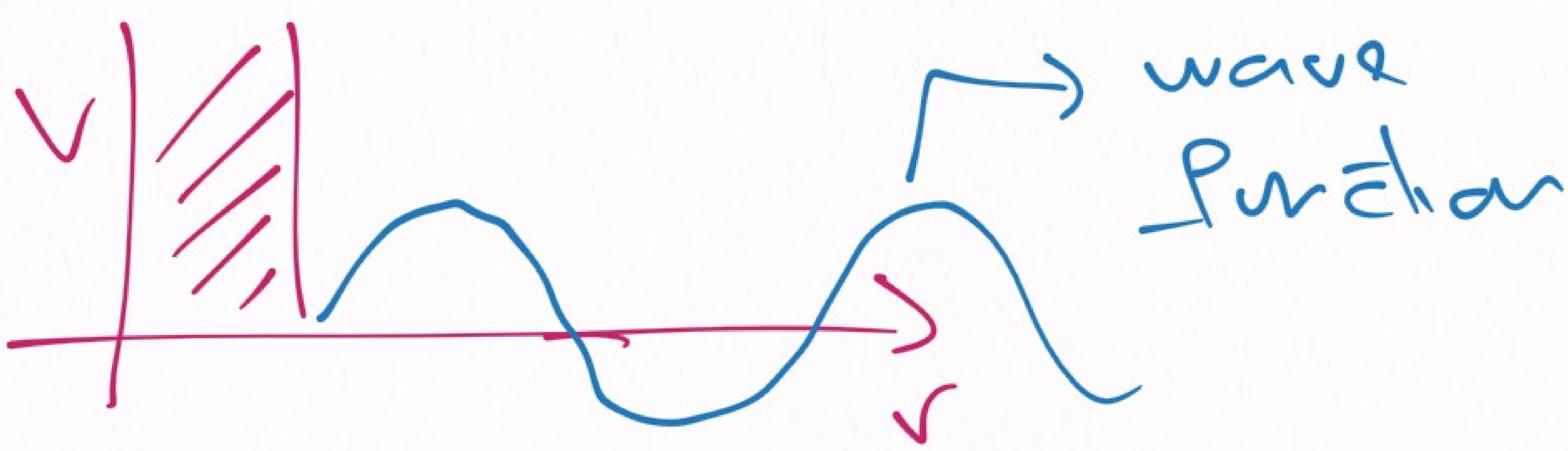
② Higher energies : $\delta_0 \sim -Rp$

$\rightarrow$ Indication of the existence of a repulsive core

Let's see why a repulsive core:

$$V(r) = \begin{cases} 0 & \text{for } r > R \\ \infty & \text{for } r < R \end{cases}$$

How to solve this potential?



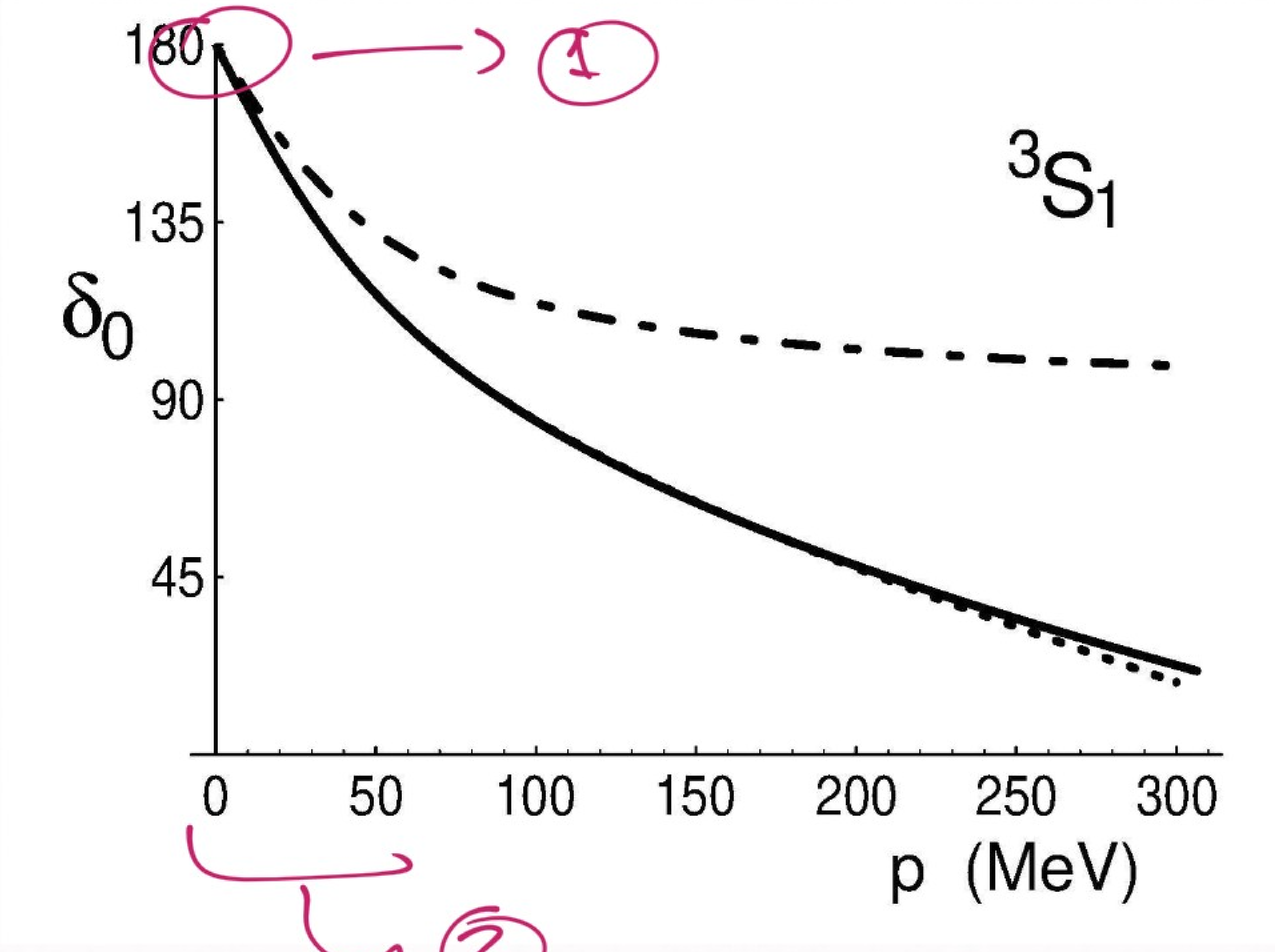
→ $u(R) = 0$

→ $u(r > R) \propto \sin(kr - kR)$

→ $\boxed{\delta(k) \propto -kR}$

→ that's why $^1S_0$ shows the core

$^3S_1$ → a closer look



① $\delta(p=0) = \pi$ → $\boxed{\text{Why?}}$

→ Levinson's theorem:

$\delta(p=0) - \delta(p\to\infty) = n_B \pi$

n_B → # of bound states

② $\delta(p\to 0) \to -a_0 p$

$\boxed{a_0 \simeq 5.4\,\text{fm}}$

$^1S_0$ & $^3S_1$ comparison:

1) $^1S_0 \rightarrow \boxed{a_0 < 0} \rightarrow$ *

↓

no bound state

(but still, very large ⇒ almost binds)

2) $^3S_1 \rightarrow \boxed{a_0 > 0} \rightarrow$ +

↓

one bound state

(deuteron)

* → only true for attractive interaction

→ two bound states: $\boxed{a_0 < 0 \text{ again}}$

→ repulsive interaction $\boxed{a_0 > 0}$

Partial Wave Expansion $\boxed{\text{RECAP}}$

1) $\sigma = \int |f(\Omega)|^2 d\Omega$

1.a) $f(\Omega) = \sum_\ell (2\ell+1) f_\ell(k) P_\ell(\cos\theta)$

1.b) $f_\ell(k) = \dfrac{1}{k\cot\delta_\ell - ik}$

1.c) $\sigma = \dfrac{4\pi}{k^2} \sum \sin^2\delta_\ell$

2) $\delta_\ell \to -a_\ell k^{2\ell+1} + O(k^{2\ell+3})$

2.a) $\sigma \to 4\pi |a_0|^2$

2.b) NN system

$^1S_0 \to a_0 \simeq -23.7\ \text{fm}$

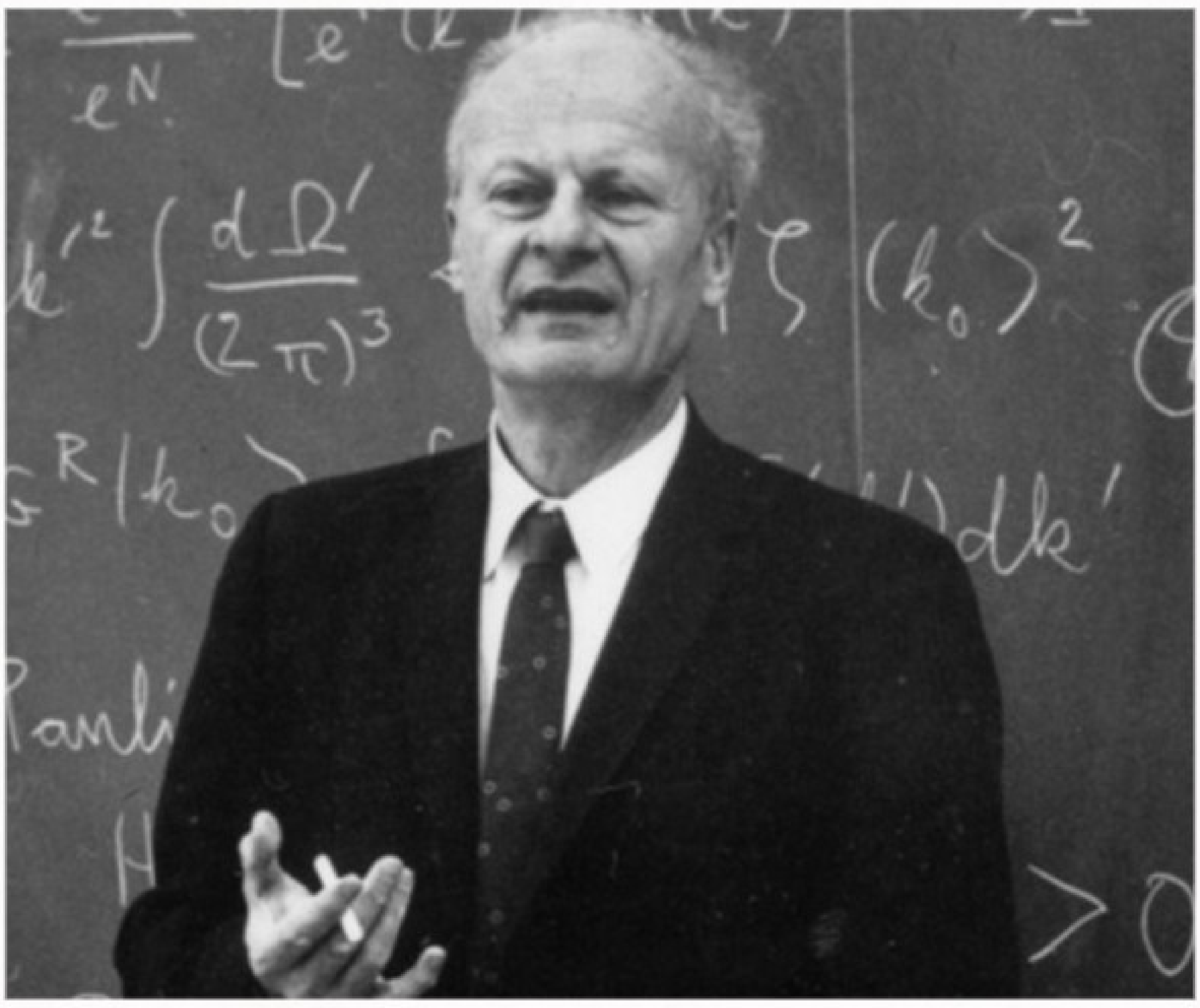
$^3S_1 \to a_0 \simeq 5.4\ \text{fm}$

Next: EFFECTIVE RANGE EXPANSION

Hans Bethe

+

Schwinger /
Landau &
Smorodinsky



Strong point: translate physical problems into ordinary differential equations (ERE, G-matrix)

$\boxed{\text{Starting point}}$

$$\delta_0(k) \to -a_0 k + \mathcal{O}(k^3)$$

$$\Rightarrow k\cot\delta_0(k) = -\frac{1}{a_0} + \mathcal{O}(k^2)$$

$\to$ we want to extend this

Extension:

$$k\cot\delta = -\frac{1}{a_0} + \underbrace{\frac{1}{2}r_0k^2}_{\text{effective range}} + \underbrace{\sum_{n=2}^{\infty} v_n k^{2n}}_{\text{shape parameters}}$$

[Validity]

For $V(r) \propto \frac{e^{-mr}}{r^a} \Rightarrow \boxed{k < \frac{m}{2}}$

Derivation → Wronskian identity

Eq 1) $-u_k'' + 2\mu V u_k(r) = k^2 u_k(r)$

$u_k(r) \to \sin(kr+\delta)/\sin\delta \quad (r\to\infty)$

Eq 2) $-u_0'' + 2\mu u_0(r) = 0$

$u_0(r) \to 1 - r/a_0 \quad (r\to\infty)$

Building a Wronskian identity:

$$[Eq.1)] \times u_0 - [Eq 2)] \times u_k$$

$$\Downarrow$$

$$-(u_k'' u_0 - u_k u_0'') = k^2 u_k u_0$$

$$\Downarrow$$

$$\boxed{\text{Notice that}}$$

$$\Downarrow$$

$$(u_k'' u_0 - u_k u_0'') = (u_k' u_0 - u_k u_0')'$$

$$\Downarrow$$

$$-(u_k' u_0 - u_k u_0')' = k^2 u_k u_0$$

$$\Downarrow$$

$$-\underbrace{(u_k' u_0 - u_k u_0')}_{\text{this is a Wronskian ;)}}\Big|_{r_c}^{R} = k^2 \int_{r_c}^{R} u_k u_0 \, dr$$

Now we do the same w/
the potential:

Eq3) $-v_k'' = k^2 v_k$

$$v_k = \frac{\sin(kr+\delta)}{\sin\delta}$$

Eq4) $-v_0'' = 0\,,\quad v_0 = 1 - \frac{r}{a_0}$

$$\Downarrow$$

$$[\text{Eq}3] \times v_0 - [\text{Eq}4)] \times v_k$$

$$\Downarrow$$

$$-\left(v_k' v_0 - v_k v_0'\right)\Big|_{r_c}^{R} = k^2 \int_{r_c}^{R} v_k v_0 \, dr$$

Finally we compute the difference between the two:

$$\left.(u_k' u_0 - u_0 u_k')\right|_{r_c}^{R} - \left.(v_k' v_0 - v_k v_0')\right|_{r_c}^{R}$$

$$= k^2 \int_{r_c}^{R} (v_k(r) v_0(r) - u_k(r) u_0(r))$$

and take $r_c \to 0$, $R \to \infty$

$$k \cot\delta = -\frac{1}{a_0} + k^2 \int_0^\infty (v_k v_0 - u_k u_0) dr$$

$\Rightarrow u_k, v_k$ good k^2 expansion

$$\Rightarrow \int_0^\infty (v_k v_0 - u_k u_0) dr = \circledast$$

$$\circledast = \frac{1}{2} r_0 + v_2 k^2 + v_3 k^4 + \dots$$

Putting the pieces together:

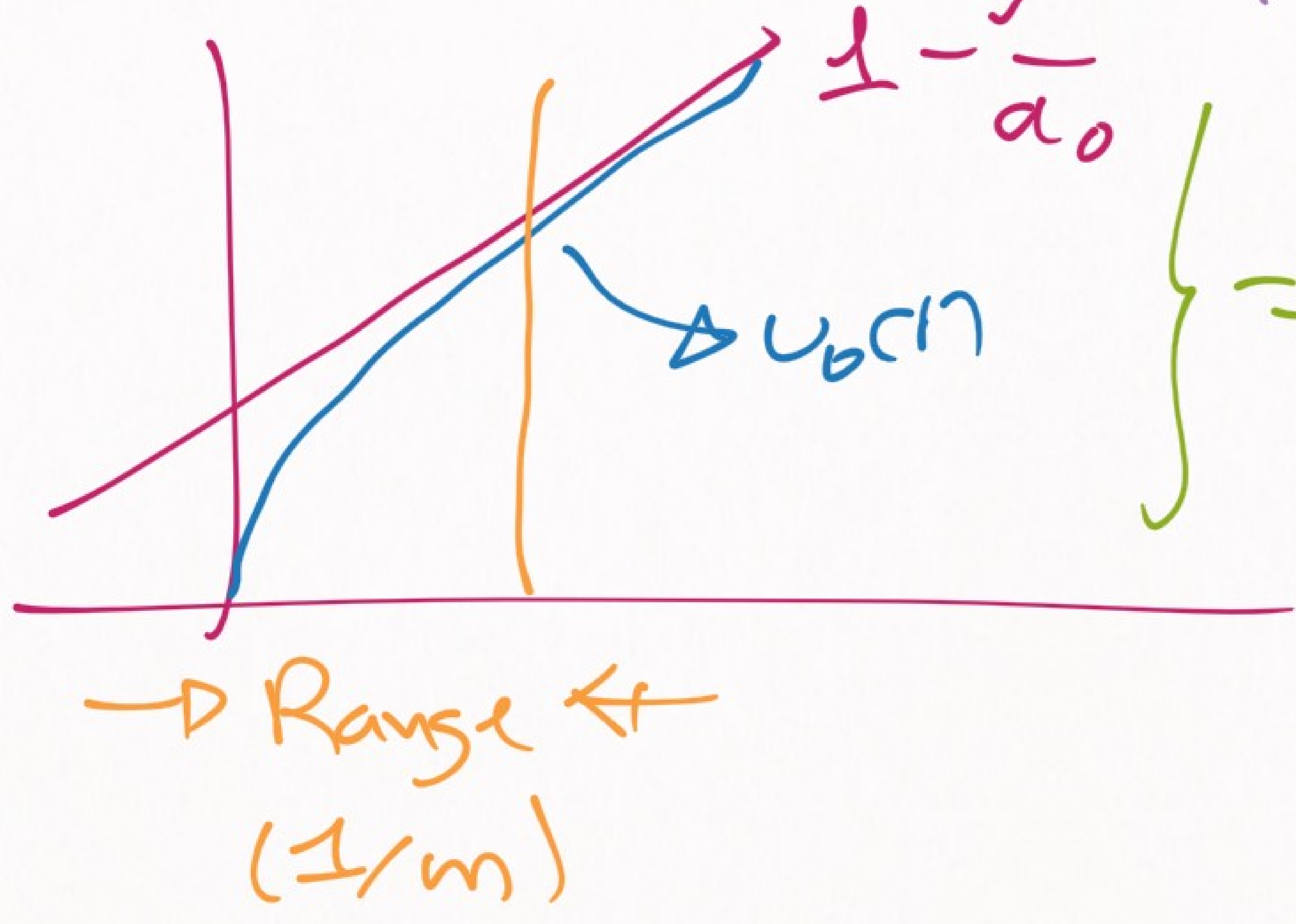
$$k \cot \delta = -\frac{1}{a_0} + \frac{1}{2} r_0 k^2 + \sum_{n=2}^{\infty} v_n k^{2n}$$

Why effective range?

$$r_0 = 2\int_0^{\infty} \left[\left(1 - \frac{r}{a_0}\right)^2 - u_0^2(r) \right] dr$$

but $u_0(r) \to \left(1 - \frac{r}{a_0}\right)$ for $r \to \infty$

$\to$ actually $u_0(r) \simeq \left(1 - \frac{r}{a_0}\right)$ for $r >$ (the range of the potential)



$\Rightarrow r_0$ similar to range of potential

Effective range expansion for NN

$^1S_0 \longrightarrow a_0 \simeq -23.7$ fm

$r_0 \simeq 2.7$ fm

$v_2 \simeq -0.5$ fm

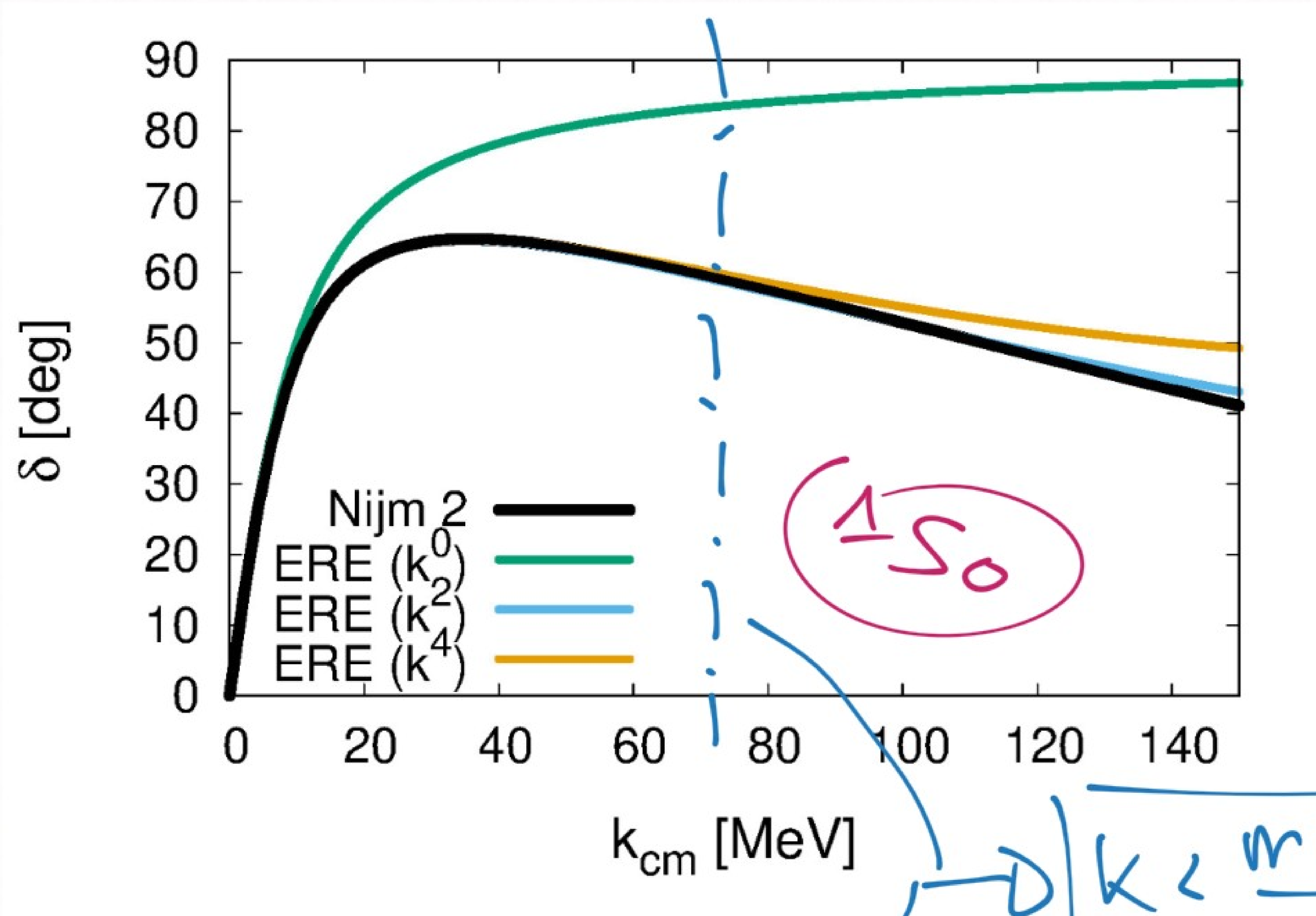
$^3S_1 \longrightarrow a_0 \simeq 5.4$ fm

$r_0 \simeq 1.8$ fm

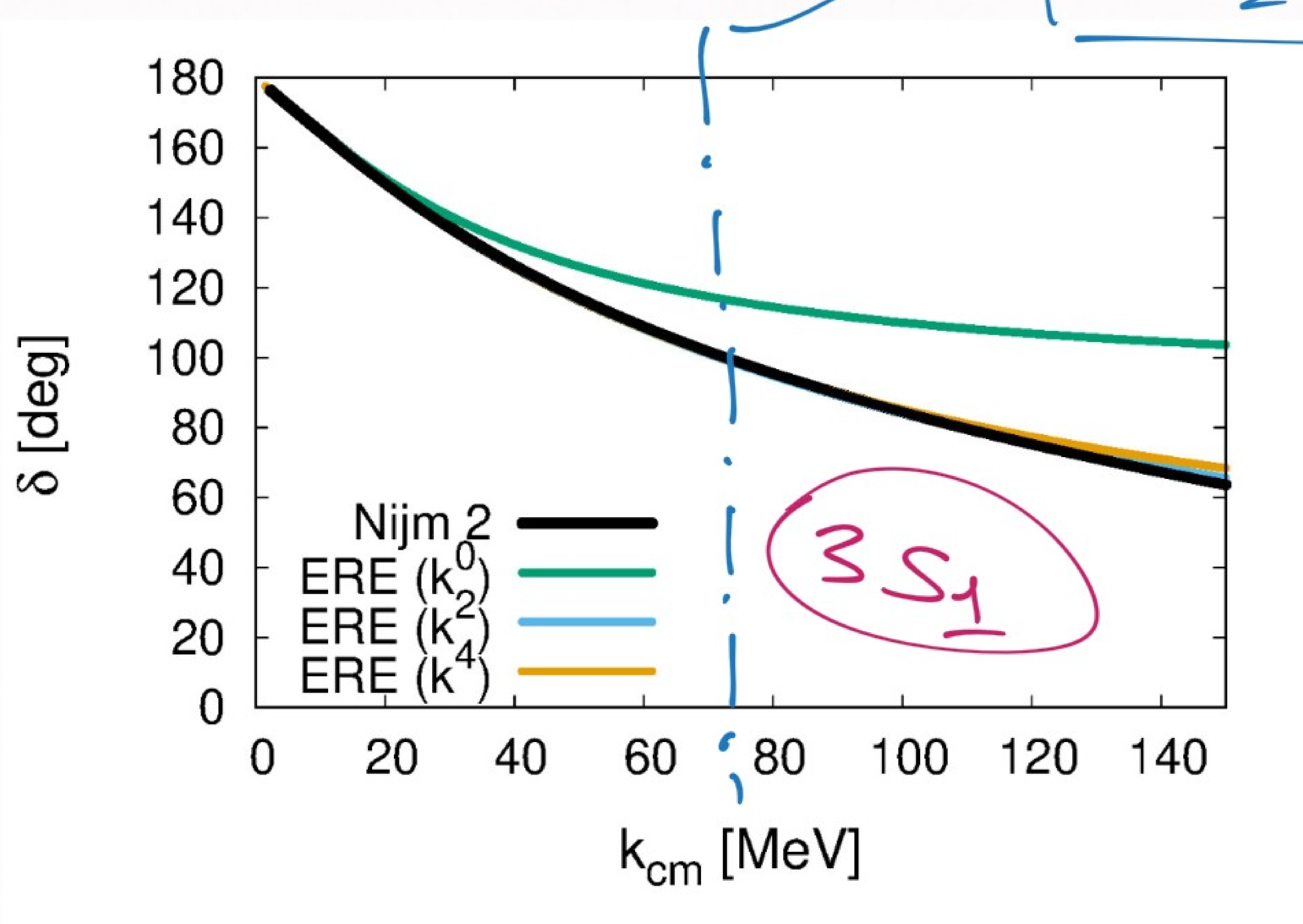
$v_2 \simeq 0.05$ fm

Alternatively ...

This is how it looks in practice:



$$\boxed{k < \frac{m_\pi}{2}}$$



Range of validity $k < \frac{m_\pi}{2} \sim 70$ MeV

Remember:

For $V(r) \sim \frac{e^{-mr}}{r^a}$ (exponential decay),

$$k\cot\delta = -\frac{1}{a_0} + \frac{1}{2}r_0 k^2 + \sum_{n=2}^{\infty} v_n k^{2n}$$

only converges for $k < \frac{m}{2}$

$2^{\underline{nd}}$ EXERCISE CHALLENGE!

1) First person to prove it 8 points! (half the course)

2) Second person to prove it: 5 points

3) Otherwise 3 points

NEXT LESSON :

$$\boxed{\text{The T-matrix}}$$

ƺ