Nuclear Physics 12

The Two-Body Problem

Nuclear Physics → Nuclei

[Bound states of A nucleons]

A = 2 → Deuteron (氘核)

A = 3 → Triton (氚核)

^3He

A = 4 → Alpha particle (^4He nucleus)

and so on (∃ a few thousand)

A = 2 → well-known

A = 3, 4 → doable

A > 4 → increasingly more difficult

$\boxed{A=2} \rightarrow \boxed{\text{two-body problem}}$

(review)

1) Schrödinger equation:

$$\left[\frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + V(\vec{r}_1 - \vec{r}_2)\right]\Psi(\vec{r}_1, \vec{r}_2) = E_T \Psi(\vec{r}_1, \vec{r}_2)$$

CM (center of mass)

$$\vec{P} = \vec{p}_1 + \vec{p}_2 \quad , \quad \vec{p} = \frac{m_1 \vec{p}_2 - m_2 \vec{p}_1}{m_1 + m_2}$$

$$\vec{R} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2} \quad , \quad \vec{r} = \vec{r}_2 - \vec{r}_1$$

$$\left[\frac{p^2}{2\mu} + \frac{P^2}{2M} + V(\vec{r})\right]\Psi(\vec{r}; \vec{R}) = E_T \Psi(\vec{r}; \vec{R})$$

$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2} \quad , \quad M = m_1 + m_2$$

Then we remove CM movement:

$$\psi(\vec{r};\vec{R}) = \psi(\vec{r})\,\psi_{CM}(\vec{R})$$

$$\psi_{CM}(\vec{R}) = e^{i\vec{K}\cdot\vec{R}}$$

$$E_T = E_{cm} + \frac{\vec{K}^2}{2M}$$

And no everything is easier:

$$\left[\frac{p^2}{2\mu} + V(\vec{r})\right]\psi(\vec{r}) = E_{cm}\psi(\vec{r})$$

— ⊗ —

Next useful trick → partial waves

$$\frac{\vec{p}^2}{2\mu} = -\frac{\nabla^2}{2\mu} = -\frac{1}{2\mu}\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right) + \frac{\vec{L}^2}{2M}$$

l is a good quantum number

$$\Rightarrow \psi(\vec{r}) = \underbrace{\frac{u_\ell(r)}{r}}_{\text{reduced wavefunction}} \underbrace{Y_{\ell m}(\hat{r})}_{\text{spherical harmonic}}$$

球谐函数

Outcome → reduced Schrödinger equation

1) $E > 0$ $(E = k^2/2\mu)$

$$-u_\ell''(r) + \left[2\mu V + \frac{\ell(\ell+1)}{r^2}\right] u_\ell(r) = k^2 u_\ell(r)$$

2) $E < 0$ $(E = -\gamma^2/2\mu)$

$$-u_\ell''(r) + \left[2\mu V + \frac{\ell(\ell+1)}{r^2}\right] u_\ell(r) = -\gamma^2 u_\ell(r)$$

→ Easier than original version

Now we review → Asymptotic solutions

$\xrightarrow{\ }$ $(r \to \infty)$

Condition: finite-range potential

$$\lim_{r \to \infty} r^n V(r) \to 0$$

$$\left(\text{example } V(r) \sim \frac{e^{-mr}}{r^a} \right)$$

1) $E < 0$

$$r \to \infty \Rightarrow \left[-u_\ell'' + \frac{\ell(\ell+1)}{r^2} \right] u_\ell = -\gamma^2 u_\ell$$

$$\Rightarrow \quad u_\ell(r) \to A_\ell e^{-\gamma r} \left[1 + O\left(\frac{1}{\gamma r}\right) \right]$$

$$\ell = 0 \Rightarrow \boxed{u_s(r) \to A_s e^{-\gamma r}}$$

really simple

2) $E > 0$ & $r \to \infty$

$$\left[u_\ell(r) \to a_\ell(k)\, \hat{j}_\ell(kr) + b_\ell(k)\, \hat{y}_\ell(kr) \right]$$

$$\hat{j}_\ell(x) = x\, j_\ell(x)\,, \quad \hat{y}_\ell(x) = x\, y_\ell(x)$$

spherical Bessel functions

$$j_\ell(x) \xrightarrow[x \to \infty]{} \frac{1}{x} \sin\left(x - \ell \frac{\pi}{2}\right)$$

$$y_\ell(x) \xrightarrow[x \to \infty]{} -\frac{1}{x} \cos\left(x - \ell \frac{\pi}{2}\right)$$

normalization

$$\Rightarrow u_\ell(r) \to N_\ell \sin\left(kr - \ell \frac{\pi}{2} + \delta_\ell(k)\right)$$

(*)

(*) → $\boxed{\text{phase shift}}$

Phase shift → important concept in scattering theory

$\sum$

$\ell=0$ → $u_0(r)$ → $\sin(kr+\delta_0(k))$

3) $E=0$ case

$\delta_\ell(k) \to -a_\ell k^{2\ell+1} + O(k^{2\ell+3})$

$\boxed{\ell=0}$ $\delta_0(k) \to -\underbrace{a_0}k$

[scattering length]

another important guy

$\boxed{u_0(r) \to 1-\frac{r}{a_0}}$

important formula

$\boxed{\sigma \to 4\pi|a_0|^2}$

Next thing → what happens near the origin?

Conditions: $u_\ell(0) = 0$ → ⊛

(guarantees $\langle\psi|\psi\rangle = \int dr\, |u_\ell(r)|^2$ is finite for bound states)

⊛ → regularity condition

Case 1) Regular potential

$$\left[\lim_{r \to 0} r^2 V(r) = 0 \right]$$

⇓

$$\underbrace{u_\ell(r) \sim r^{\ell+1}}_{\substack{\text{regular solution} \\ (u_\ell(0)=0)}} \quad \text{or} \quad \underbrace{u_\ell(r) \sim \frac{1}{r^\ell}}_{\text{irregular solution}}$$

Case 2) $\underbrace{\text{Singular potential}}$

most general type of potential

natural result of long-range $\left(\frac{1}{r}\right)$ expansion

$$V(r) = g_3 \frac{e^{-mr}}{r^3} + g_4 \frac{e^{-mr}}{r^4} + \cdots$$

(expansion in $1/r$)

$$\boxed{\lim_{r \to 0} r^2 V(r) \neq 0} \rightarrow \text{singular potential}$$

→ for now we will concentrate on $\lim_{r \to 0} r^2 V(r) \rightarrow \pm\infty$

2.a) Repulsive singular potential

$$2\mu V(r) \longrightarrow + \underbrace{\frac{a^{n-2}}{r^n}}_{\downarrow} \quad w/\ n > 2$$

$$\left[2\mu V(r) \gg \frac{L^2}{r^2} \ (r \to 0) \right]$$

no ℓ-dependence for $r \to 0$

$$u_\ell(r) = c_+ \left(\frac{r}{a}\right)^{n/4} \underbrace{\exp\left[+\frac{2}{n-1}\left(\frac{a}{r}\right)^{\frac{n-2}{2}}\right]}_{1}$$

$$+ c_- \left(\frac{r}{a}\right)^{n/4} \underbrace{\exp\left[-\frac{2}{n-1}\left(\frac{a}{r}\right)^{\frac{n-2}{2}}\right]}_{2}$$

1 $\to$ irregular

2 $\to$ regular $(u(0) = 0)$

$$\boxed{u_\ell(0) = 0} \Rightarrow \boxed{c_+ = 0}$$

2-b) Attractive singular potential

$$2\mu V(r) \to -\frac{a^{n-2}}{r^n}$$

$$u_e(r) = c_1\left(\frac{r}{a}\right)^{n/4} \sin\left[\frac{2}{n-1}\left(\frac{a}{r}\right)^{\frac{n-2}{2}}\right] + c_2\left(\frac{r}{a}\right)^{n/4} \cos\left[\frac{2}{n-1}\left(\frac{a}{r}\right)^{\frac{n-2}{2}}\right]$$

$\to$ [There's something fishy in this solution]

$$\boxed{u_e(0) = 0} \to \text{does not determine the solution}$$

[all solutions are regular]

→ this is the particular feature of attractive singular potentials

Physical meaning:

They are incomplete: need to be supplemented w/ short-range physics

they appear everywhere in EFTs

(b/c EFTs contain contact range potentials)

Recommended lecture:

1. (108) **Singular potentials and limit cycles**
S.R. Beane, Paulo F. Bedaque, L. Childress, A. Kryjevski, J. McGuire (Washington U., Seattle), U. van Kolck (Arizona U. & RIKEN BNL & Caltech, Kellogg Lab). Oct 2000. 8 pp.
Published in **Phys.Rev. A64 (2001) 042103**
NT-UW-00-023, DOE-ER-41132-102-INT-00, RBRC-140, KRL-MAP-271
DOI: 10.1103/PhysRevA.64.042103
e-Print: **quant-ph/0010073** | **PDF**
References | BibTeX | LaTeX(US) | LaTeX(EU) | Harvmac | EndNote
ADS Abstract Service; OSTI.gov Server
Registro completo - Citado por 108 registros 100+

where you can make practical use of what we have just learned ✓

↳ You have to grok singular potentials to be a good EFT theorist

Case 3) Inverse square potential

$$\left[2\mu V(r) = \frac{g}{r^2} \right] \rightarrow \text{really special}$$

$[\ell = 0 \text{ case}] + [k=0]$

$$\hookrightarrow \; -u_0'' + \frac{g}{r^2} u_0(r) = 0$$

$\hookrightarrow$ Invariant under:

$$\boxed{r \rightarrow r' = \lambda r}$$

$\hookrightarrow$ Dilatations

aka "Scale invariance"

Reminder → $\boxed{\text{Lesson 3}}$

$\frac{1}{r^2}$ shows scale invariance,

two types:

This depends on the coupling:

$$2\mu V(r) = \frac{g}{r^2}$$

$\boxed{g > g_{crit}} \Rightarrow u_0(r) = c_+ r^{1/2+\nu} + c_- r^{1/2-\nu}$

(boring type)

$$\nu = \nu(g)$$

$\boxed{g < g_{crit}}$

$$u_0(r) = r^{1/2} \sin(\alpha \log r + \varphi)$$

$$= \underbrace{r^{1/2} \sin\left(\alpha \log \frac{r}{R^*}\right)}_{}$$

discrete scale invariance

Let's check the wave function

$u(r) \propto r^{1/2} \sin(\alpha \log \frac{r}{R^*})$

$r \to \lambda r$

$\sin(\alpha \log \frac{r}{R_0}) \to \sin(\alpha \log \frac{r}{R_0} + \alpha \log \lambda)$

(Question: why I don't consider the $r^{1/2}$ piece of the wave function?) 1pt

$\Rightarrow$ if $\alpha \log \lambda = \pi$

$\Rightarrow \left[\sin(\alpha \log \frac{r}{R_0} + \pi) = \sin(\alpha \log \frac{r}{R_0}) \right]$

$\Rightarrow$ no change!

Thus there is a special λ_0:

$\lambda_0 = e^{\pi/\alpha}$ / $r \to \lambda_0 r$ is a symmetry

discrete scale invariance

$r \to \lambda r$

λ arbitrary

$r \to \lambda_0 r$

λ_0 fixed

Case 4) Delta-shell potential

$$2\mu V(r;R_c) = \frac{C_0(R_c)}{4\pi R_c^2}\,\delta(r-R_c)$$

→ Extremely useful

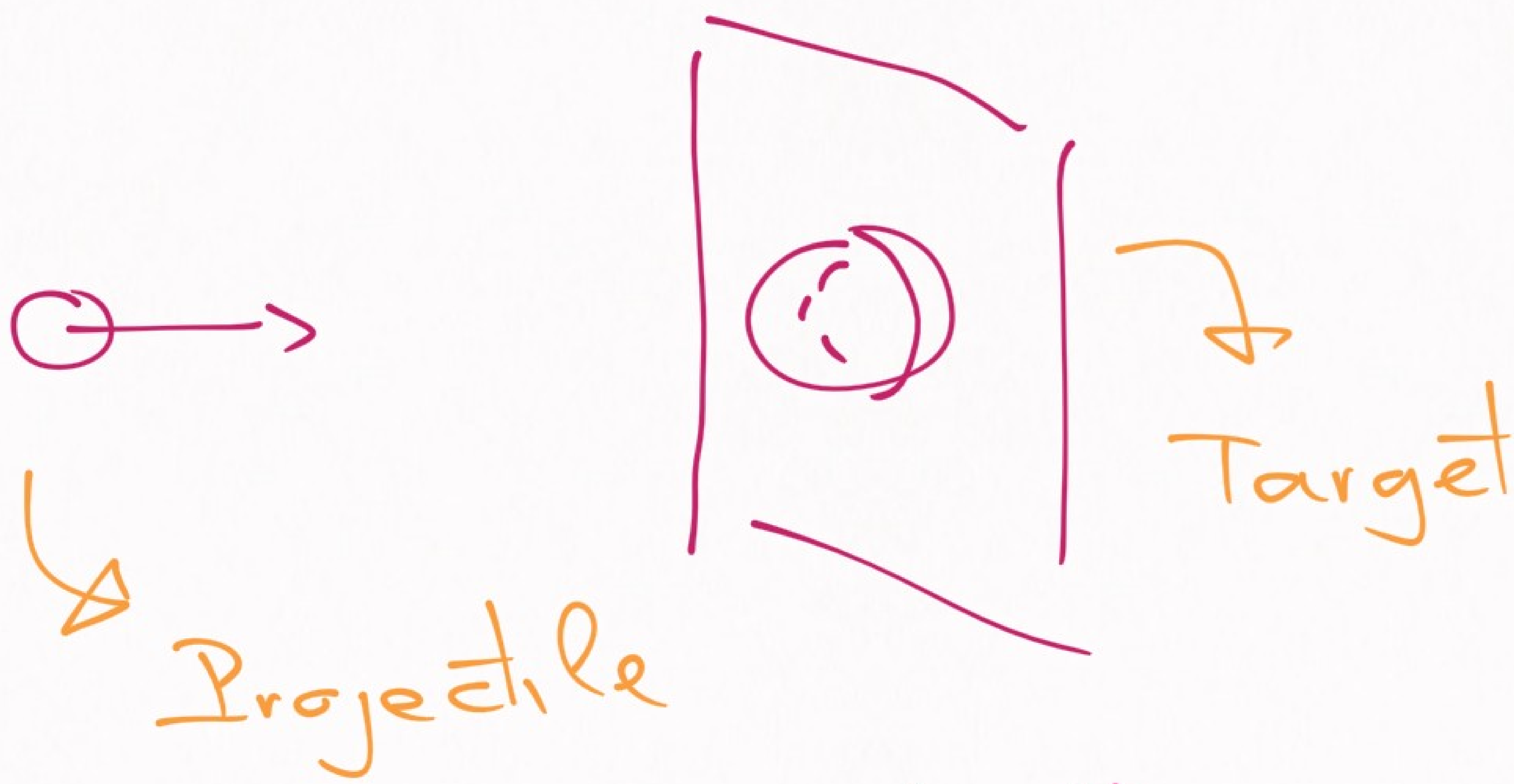
Wonderful choice for analyzing RGEs / EFTs

→ Solution left as exercise

Though it has been discussed in previous lectures

Now: $\boxed{\text{SCATTERING THEORY}}$

Classical setting:



We want to compute this:

[the cross section σ]

→ the effective area that a target offers to a projectile

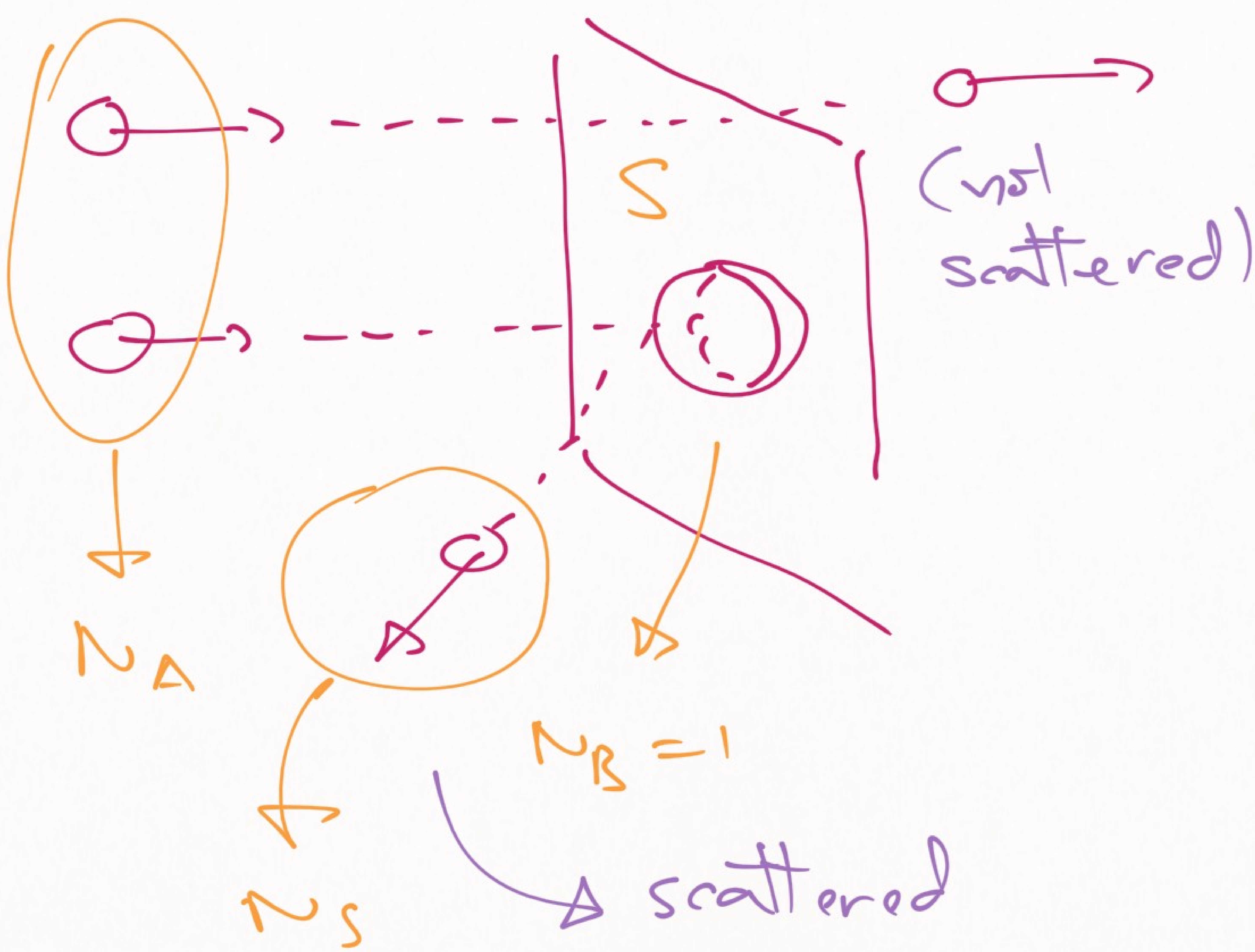
Definition:

$$\sigma = \frac{N_s}{N_A N_B} S$$

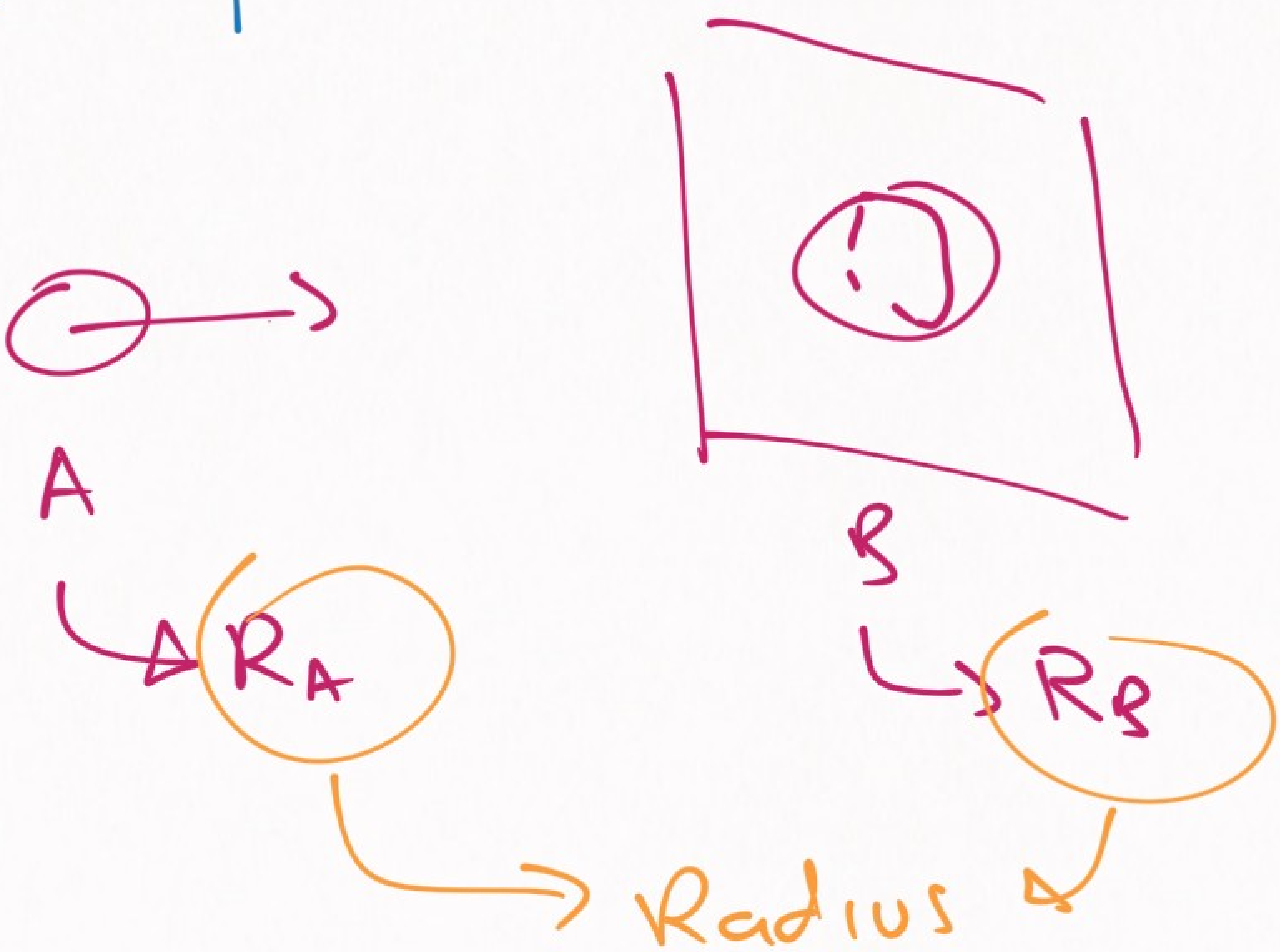
N_s → scattered particles

S → area in which projectile/target can collide

$N_A N_B$ → # of projectiles (A) or targets (B)

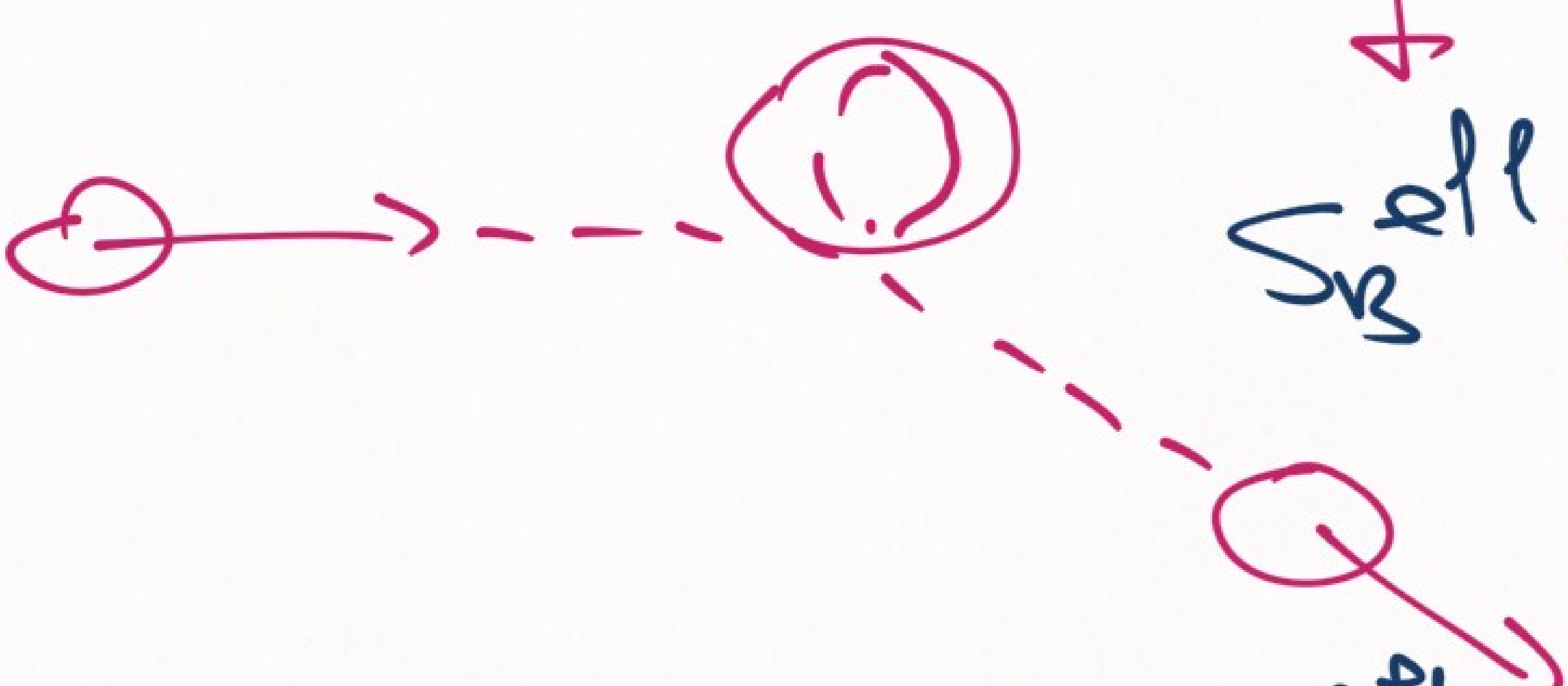


Example → Hard balls



$\Rightarrow N_A \rightarrow$ input $(S_A = \pi R_A^2)$

$N_B \rightarrow$ input $(S_B = \pi R_B^2)$



$$S_B^{eff} = \pi (R_A + R_B)^2$$

$$N_S = N_A \times \left(N_B \frac{S_B^{eff}}{S} \right)$$

$$= \frac{N_A N_B}{S} \pi (R_A + R_B)^2$$

Putting the pieces together:

$$\sigma = \frac{N_s}{N_A N_B} S$$

$$N_s = N_A N_B \frac{S_B^{eff}}{S}$$

$$\boxed{\sigma = S_B^{eff} = \pi (R_A + R_B)^2}$$

you can see that this is indeed a "cross section"

NEXT LESSON:

THE QUANTUM VERSION