INTERMEZZO 3

Finding the characteristic scale in two-body systems

&

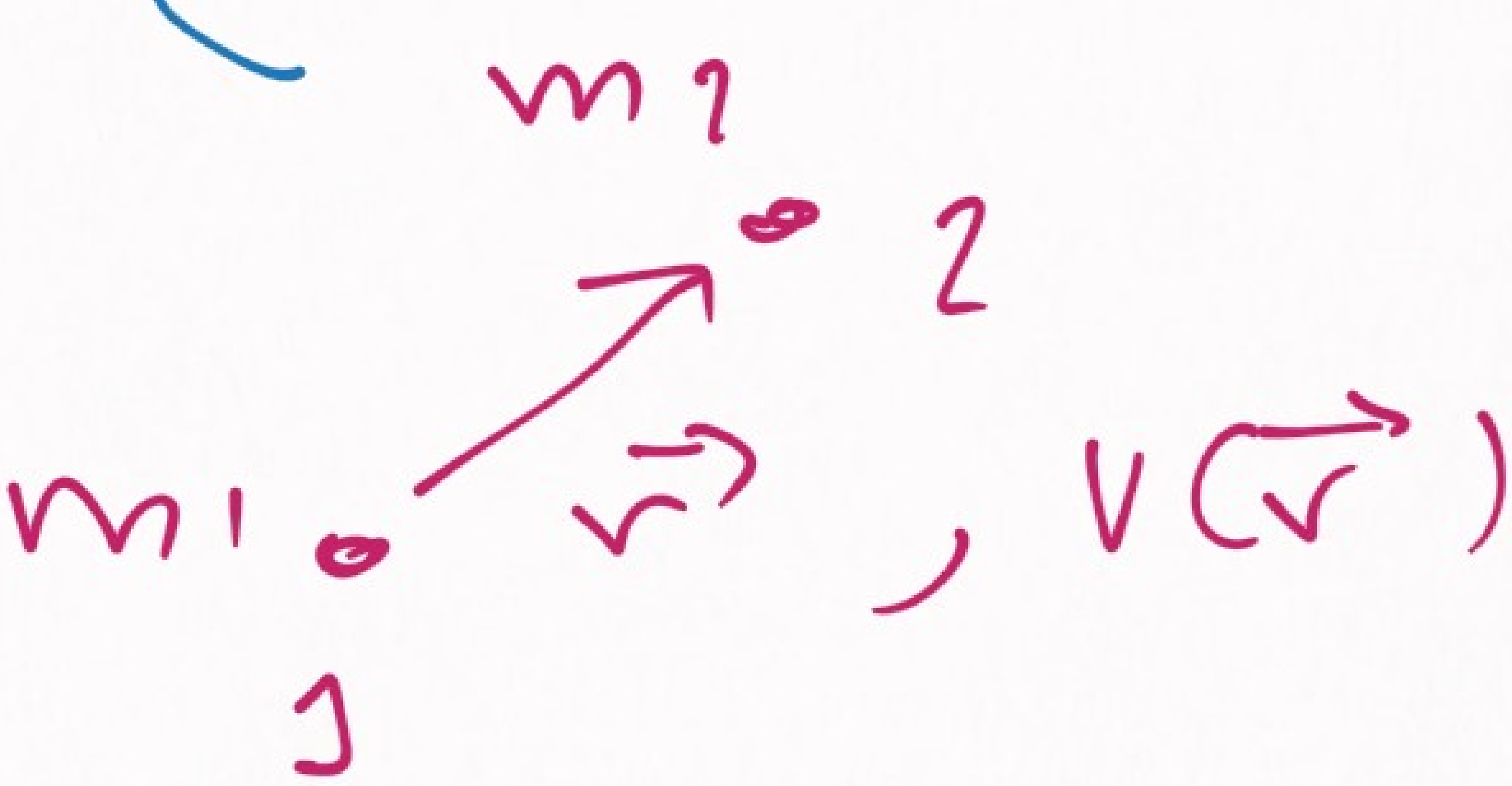
Naturalness PK fine-tuning

We begin with Schrödinger:

$$\left[-\frac{1}{2\mu}\vec{\nabla}^2 + V(\vec{r})\right]\psi(\vec{r}) = E_B\,\psi(\vec{r})$$

↳ reduced mass

$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}$$



Other reduced quantities include:

$$U(\vec{r}) = 2\mu V(\vec{r})$$

$$E_B = -\frac{\gamma^2}{2\mu}$$

Now we can rewrite things:

$$\left[-\nabla^2 + U(\vec{r})\right]\psi(\vec{r}) = -\gamma^2 \psi(\vec{r})$$

which is simpler

Notice that the reduced potential U has dimensions of $[\text{length}]^{-2}$

$$[U] = [L]^{-2}$$

$\rightarrow$ this constrains the definition of the scales

Example 1 : Coulomb

$$V(\vec{r}) = -\frac{\alpha}{r} \qquad [L]^{-1}$$

$$U(\vec{r}) = -2\mu\frac{\alpha}{r} \qquad [L]^{-2}$$

$$= -\frac{2}{a_B r}$$

2 → convention

↳ distance

this is the only length scale left

Coulomb → $\boxed{a_B}$

Example 2: van der Waals

$$V(\vec{r}) = -\frac{C_6}{r^6}$$

$$U(\vec{r}) = -2\mu \frac{C_6}{r^6}$$

$$= -\frac{R_{vdw}^4}{r^6} \quad [L]^{-2}$$

By dimensions we can define R_{vdw} as

$$R_{vdW} = (2\mu C_6)^{1/4}$$

$$\Rightarrow \left[\frac{R_{vdW}^4}{r^6}\right] = \frac{1}{L^2} \quad \checkmark$$

Example 3: inverse square potential

$$V(r) = -\frac{C_2}{r^2}$$

$$U(r) = -2\mu \frac{C_2}{r^2}$$

$$= -\frac{g}{r^2}$$ → pure number

pure numbers are dimensionless

→ there is no scale

In fact the $1/r^2$ potential is very special;

$$\underbrace{\left[-\nabla^2 - \frac{g}{r^2}\right]}\psi(\vec{r}) = -\gamma^2\psi(\vec{r})$$

why no scale? try to zoom in and see if there's a difference

$$\left.\begin{aligned} r &\to \lambda r \\ \vec{\nabla} &\to \frac{1}{\lambda}\vec{\nabla} \\ \gamma &\to \frac{1}{\lambda}\gamma \end{aligned}\right]$$

Dilatations do not change this system

This means that $^1/_{r^2}$ describes a system that looks identical when zooming in

like these

In physics, this is called scale invariance and there are two types:

continuous (boring type)

& discrete (interesting type)

$\frac{1}{r^2}$ potential $\to$ interesting type

also, the most simple example of an anomaly

$\downarrow$

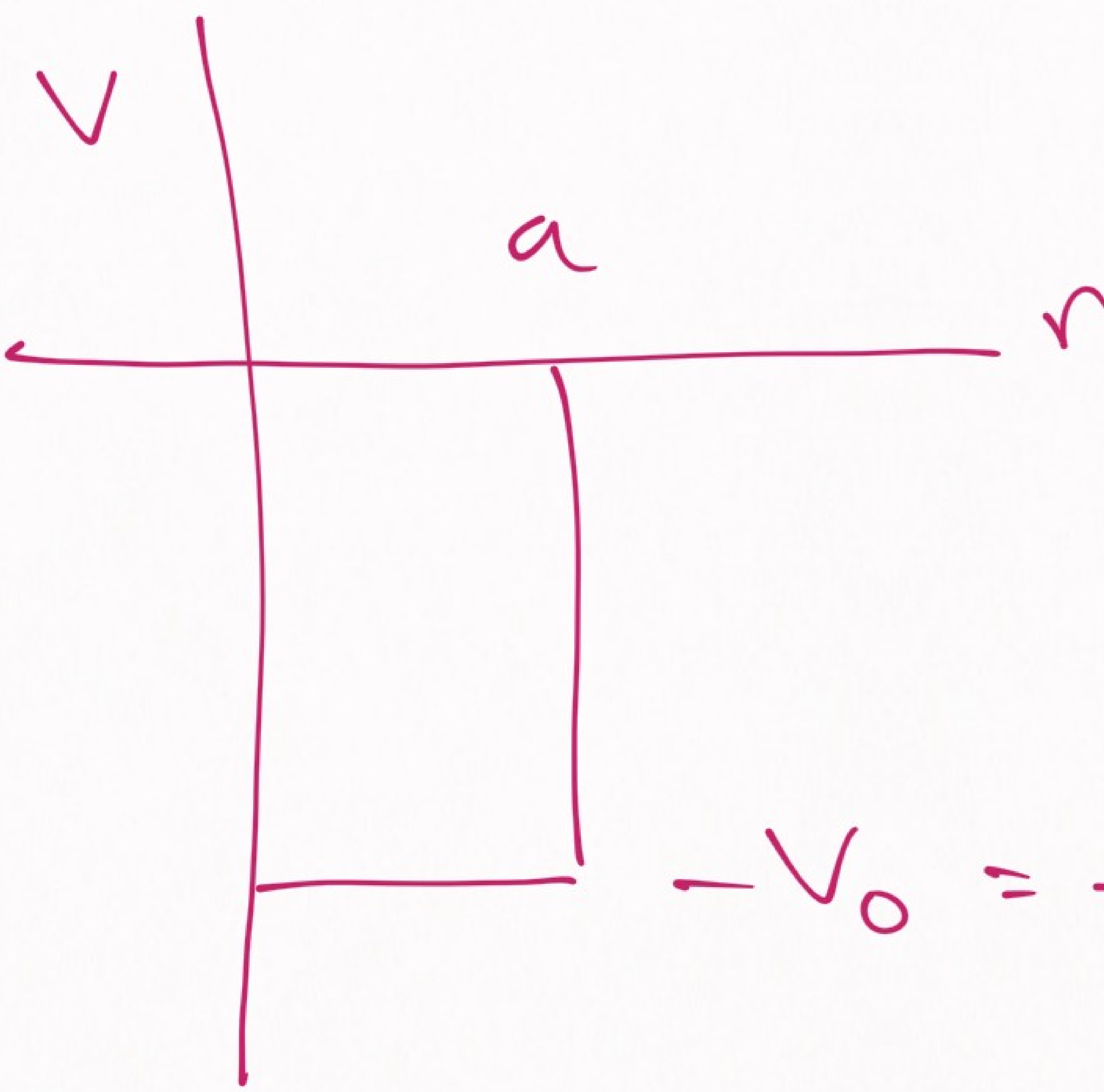
a classical symmetry that is broken by the quantization process

Example 4 : square well

$$V(r) = -V_0 \text{ for } r < a$$

$$U(r) = -U_0 \text{ for } r < a$$

$$\left(= -2mV_0 = -\frac{1}{R_{sq}^2}\right)$$



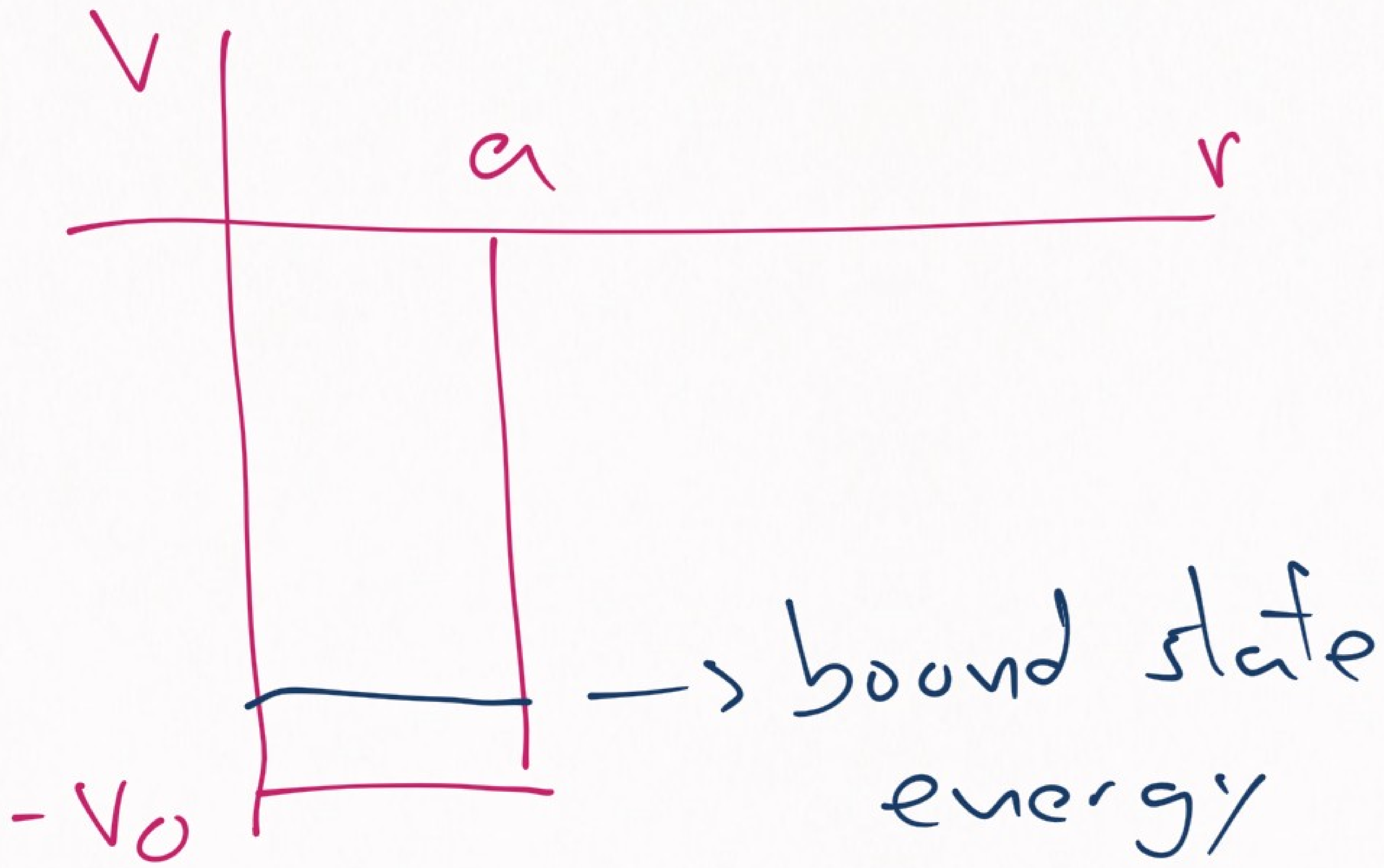
$$-V_0 = -\left[\frac{1}{2m}\frac{1}{R_{sq}^2}\right]$$

Actually there's an ambiguity

Strictly speaking we have two scales $\boxed{a \ , \ R_{sg}}$

So why did I choose R_{sg} ?

Let's see ...



Natural situation:

* Deep square well
* Binding energy similar to depth:

$$E_B = -B \,, \quad B \leq V_0$$

but usually $B \simeq V_0$

Let's check that:

Expectation: $B \sim V_0$

$$\frac{\gamma^2}{2\mu} \sim \frac{1}{2\mu}\left(\frac{1}{R_{SG}}\right)^2$$

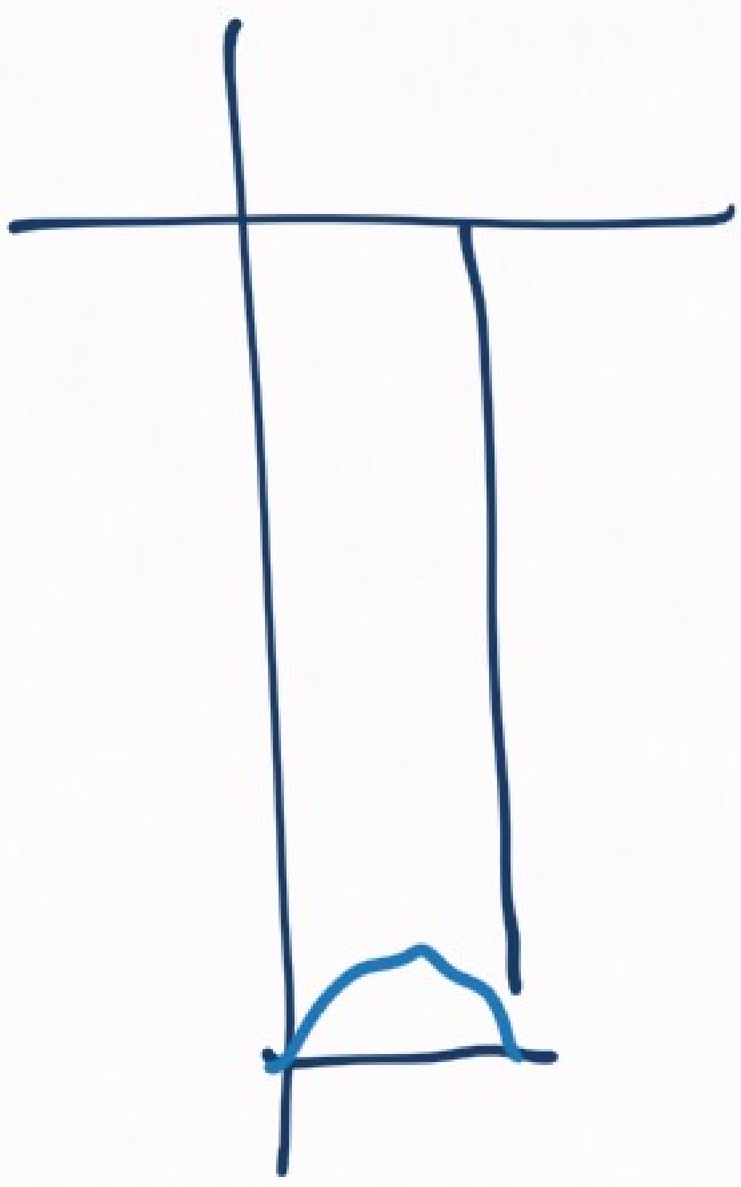
$$\Rightarrow \boxed{\gamma \sim \frac{1}{R_{SG}}} \quad (\text{for } R_{SG} \ll a)$$

Reality: we have to solve the eigenvalue equation

$$\kappa \cot \kappa a = -\gamma$$

$$\text{w/ } \kappa^2 = -\gamma^2 + \frac{1}{R_{SG}^2}$$

for $R_s \ll a$ the well is really deep



$$\kappa a = n\frac{\pi}{2} + \delta$$

small

why? because it is like an infinite well

$$\frac{1}{a}\left(n\frac{\pi}{2}+\delta\right)\cot\left(n\frac{\pi}{2}+\delta\right) = -\gamma$$

$$\cot\left(n\frac{\pi}{2}+\delta\right) = (-1)^n \cot\delta$$

$\Rightarrow$ we take $n=1$

$(R_s \to R_{sg})$ (but R_s shorter to write)

$$\frac{1}{a}(\pi+\delta)\cot\delta = \gamma = \sqrt{\frac{1}{R_s^2} - \frac{1}{a^2}(\pi+\delta)^2}$$

now we notice that $a \gg R_s$

$$\Rightarrow \gamma = \frac{1}{R_s}\left(1 - \frac{1}{2}\left(\frac{R_s}{a}\right)^2(\pi+\delta)^2 + O\left(\left(\frac{R_s}{a}\right)^4\right)\right)$$

$$\simeq \frac{1}{R_s}$$

$$\Rightarrow \frac{1}{a}(\pi+\delta)\cot\delta \simeq \frac{1}{R_s}$$

If $\delta \to 0$, $\cot\delta \to \frac{1}{\delta}$

$$\Rightarrow \frac{\pi}{\delta} = \frac{a}{R_s} + O\left(\frac{a}{R_s}\right)^?$$

Or equivalently:

$$\delta = \pi \frac{R_s}{a} + \mathcal{O}\left(\left(\frac{R_s}{a}\right)^2\right)$$

$$ka = \pi\left(1 + \frac{R_s}{a}\right) + \mathcal{O}\left(\left(\frac{R_s}{a}\right)^2\right)$$

$$\boxed{\gamma = \frac{1}{R_s} + \mathcal{O}\left(\frac{R_s}{a^2}\right)}$$

Reality matches expectation

However, something different happens when $R_s \sim a$

Eigenvalue equation $\boxed{\kappa\cot(\kappa a) = -\gamma}$

New assumptions:

$$\frac{1}{\gamma} \gg R_s, a \qquad R_s \sim a$$

$$\Rightarrow \kappa\cot(\kappa a) = 0 + \mathcal{O}(\gamma)$$

$$\kappa = \frac{1}{R_s} + \mathcal{O}(\gamma)$$

$$\Rightarrow \frac{1}{R_s}\cot\left(\frac{a}{R_s}\right) = 0$$

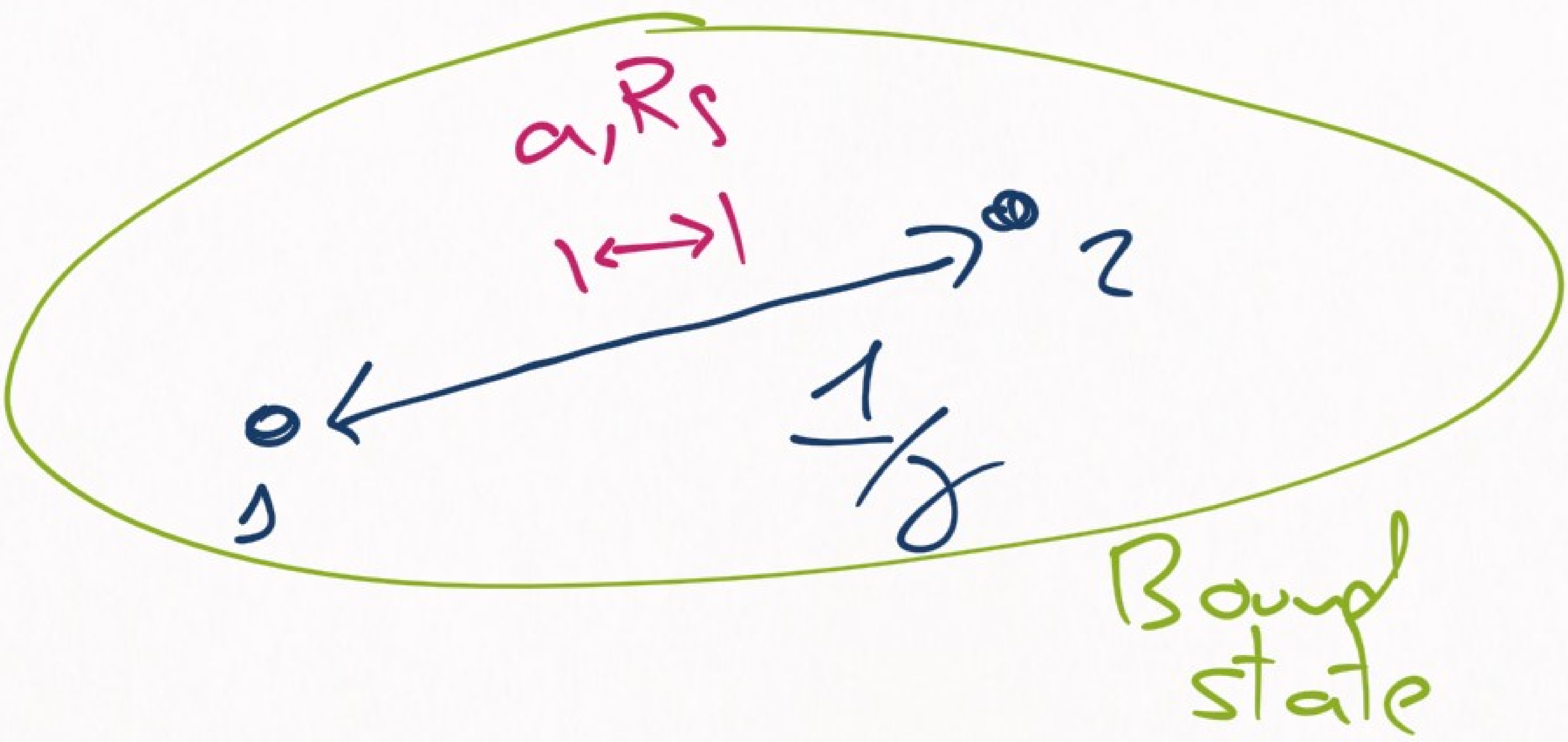
$$\Rightarrow \boxed{\frac{a}{R_S} = \frac{\pi}{2} + \mathcal{O}(\gamma R_s, \gamma a)}$$

This case is different

→ not natural

$$\boxed{\frac{1}{\gamma} \gg a, R_S}$$

↳ this defines the size of the system



If $\frac{a}{R_s} \simeq \frac{\pi}{2}$, the bound state will be bigger than expected

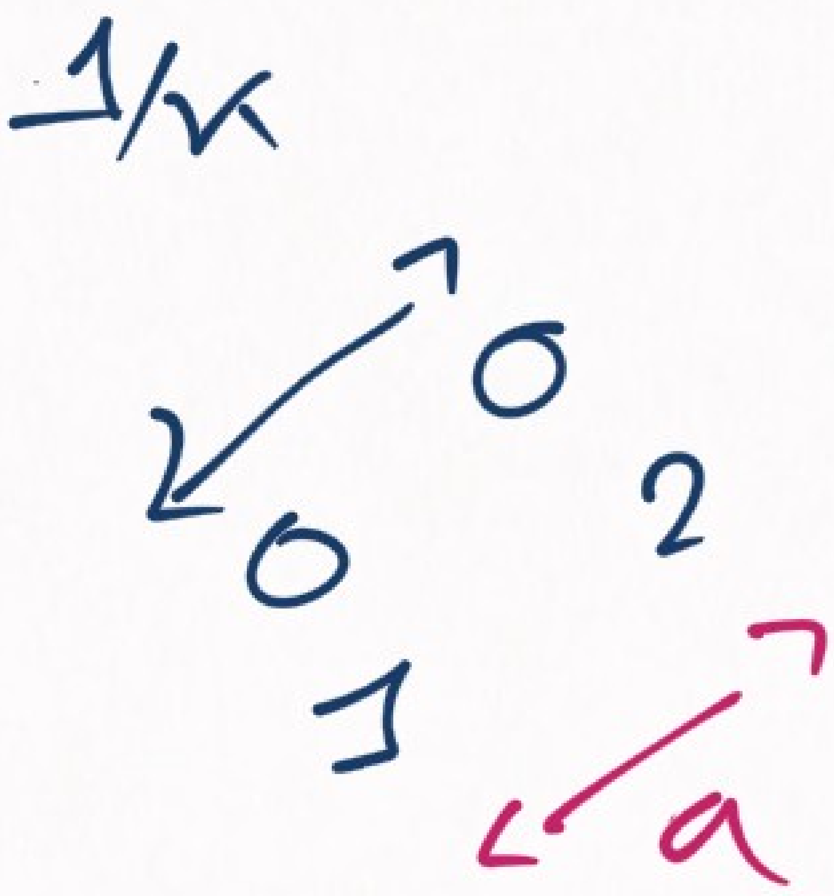
But notice:

1) $\frac{a}{R_s} \ll \frac{\pi}{2}$ → no bound state

2) $\frac{a}{R_s} \simeq \frac{\pi}{2}$ → "ginormous" bound state

3) $\frac{a}{R_s} \gg \frac{\pi}{2}$ → natural bound state

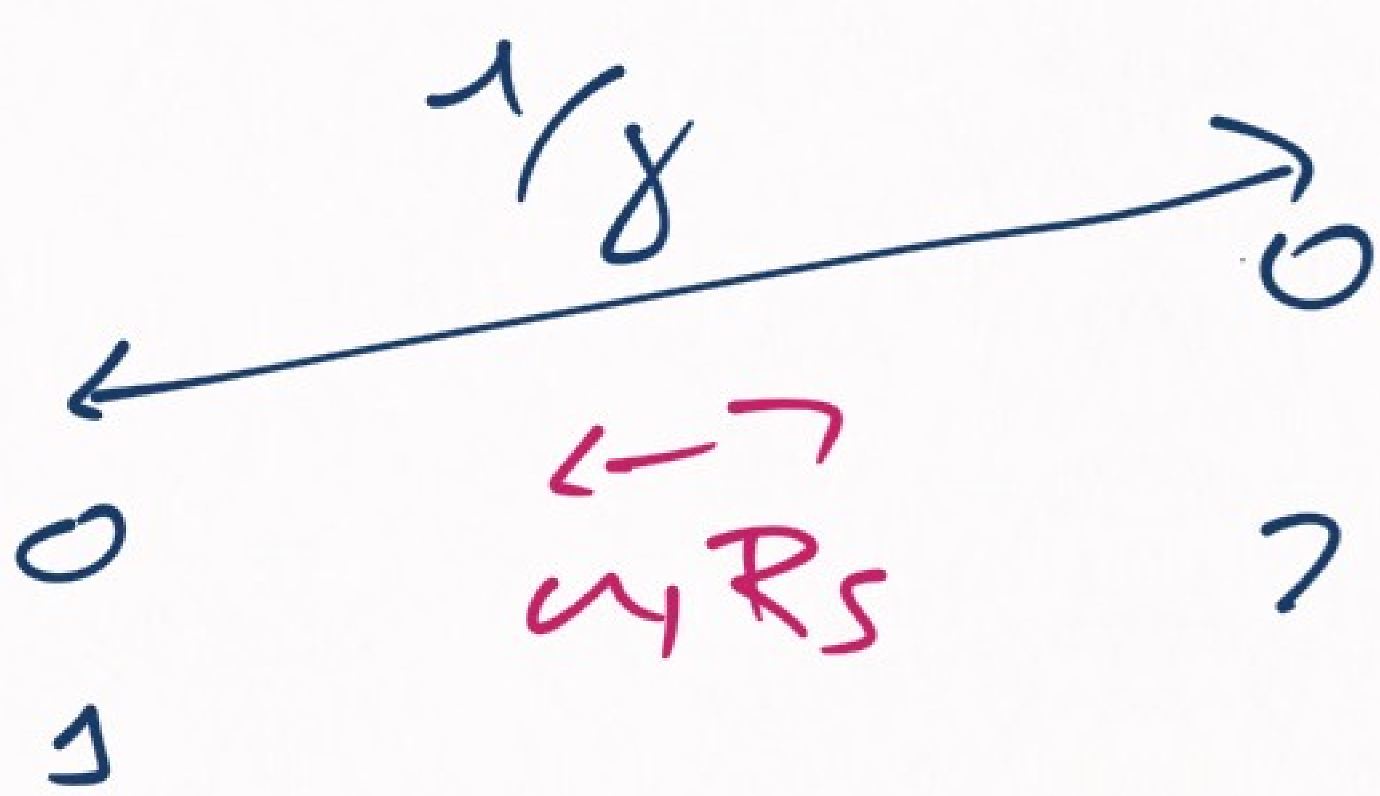
Natural case:



→ size: $1/\kappa \simeq a$

→ binding

$B \simeq V_0$



Unnatural case

→ size: bigger than range of potential

→ requires special condition

$$\frac{a}{R_s} \simeq \frac{\pi}{2}$$

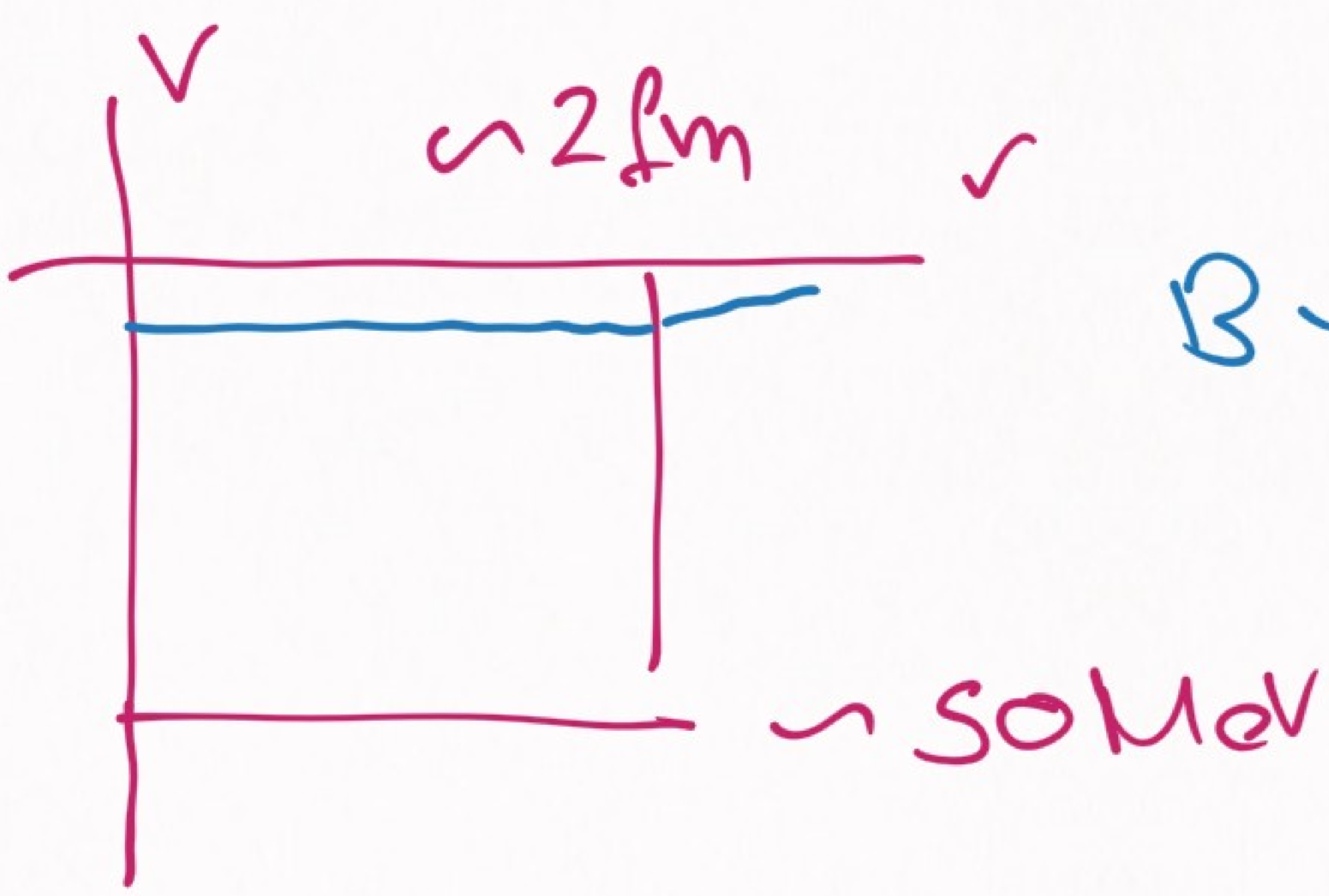
Unnatural case :

$$\boxed{\frac{a}{R_s} \simeq \frac{\pi}{2}} \rightarrow \text{fine tuning}$$

requires a very specific condition (that's why is called unnatural)

Unnaturalness example 1

→ the deuteron



Now $B = \langle T \rangle + \langle V \rangle$

$$= \underbrace{+48 - 50}_{} = 2$$

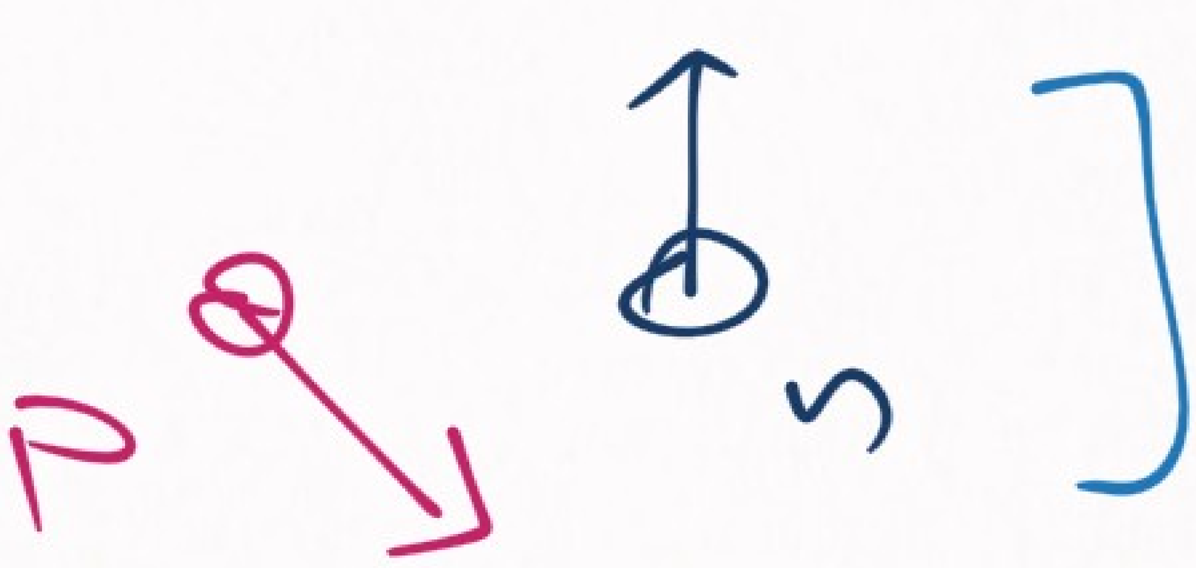
Two big numbers cancel out to make a small number

→ like a miracle

The fine tuning of the deuteron is about $1/25$

But there are more extreme examples

The neutron-proton system



$n, p \rightarrow$ spin-$1/2$

$$np \begin{cases} S = 0 & \uparrow \downarrow \quad \boxed{\text{SINGLET}} \\ S = 1 & \uparrow \uparrow \quad \boxed{\text{TRIPLET}} \end{cases}$$

1) SINGLET : $S=0$ (one spin state)

→ no bound state

2) TRIPLET : $S=1$ (three spin states)

→ $\boxed{\text{the deuteron}}$

↓

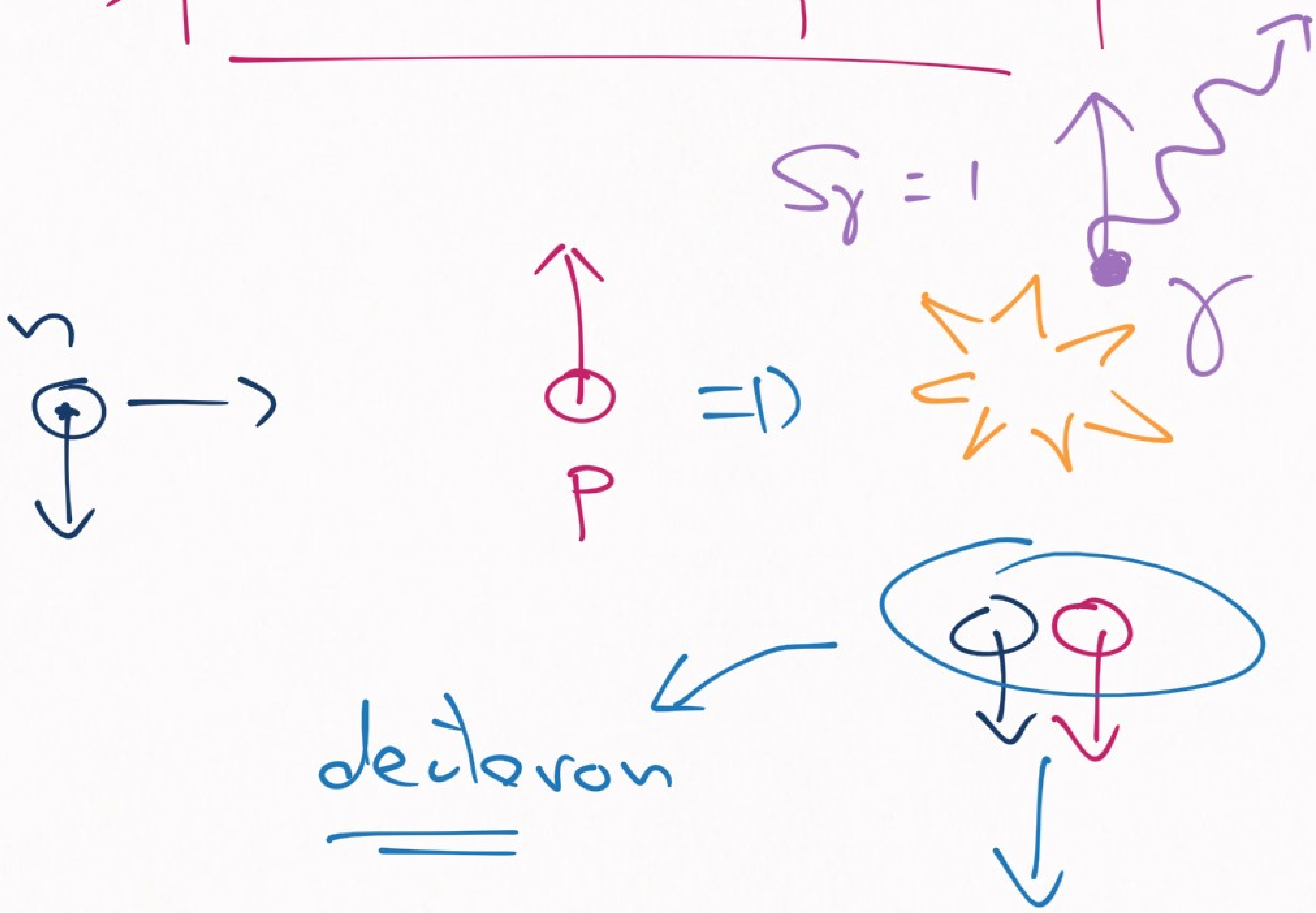
fine tuning of $1/25$

— ⊗ —

But the singlet almost binds!

How do we know that the singlet almost binds?

→ $\boxed{\text{neutron capture}}$



1) neutron hits proton in singlet

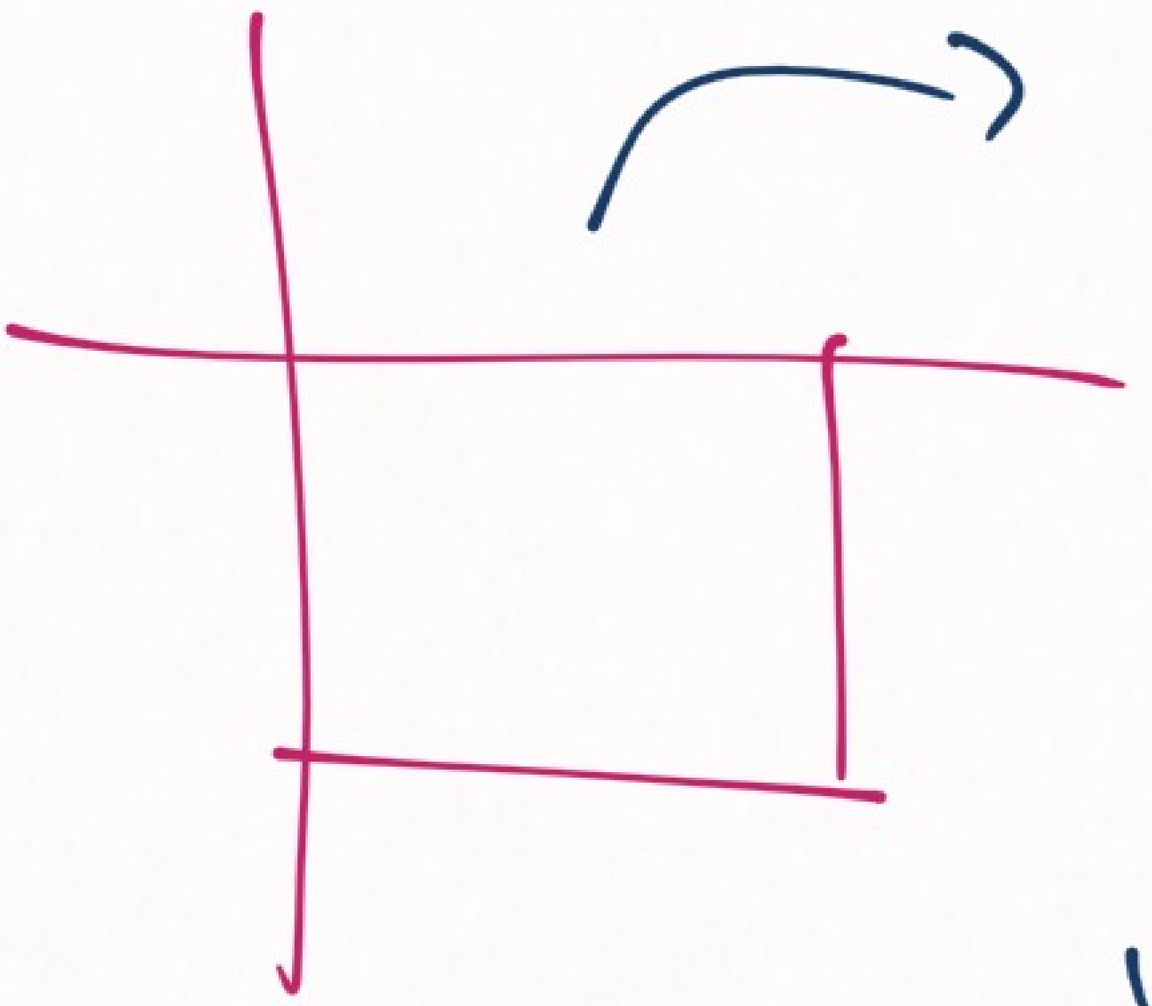
2. a) neutron-proton ends in triplet, form deuteron

2. b) a photon is emitted

The probability of a slow neutron to be captured is crazy high!

Why? Singlet forms a virtual state

What's a virtual state?



Square well

$$\boxed{\frac{a}{R_s} = \frac{\pi}{2}}$$

↓

Bound state at $E_B = 0$

1) $\frac{a}{R_S} = \frac{\pi}{2} + \epsilon \longrightarrow$ system binds

2) $\frac{a}{R_s} = \frac{\pi}{2} - \epsilon \rightarrow$ system does not bind, but...

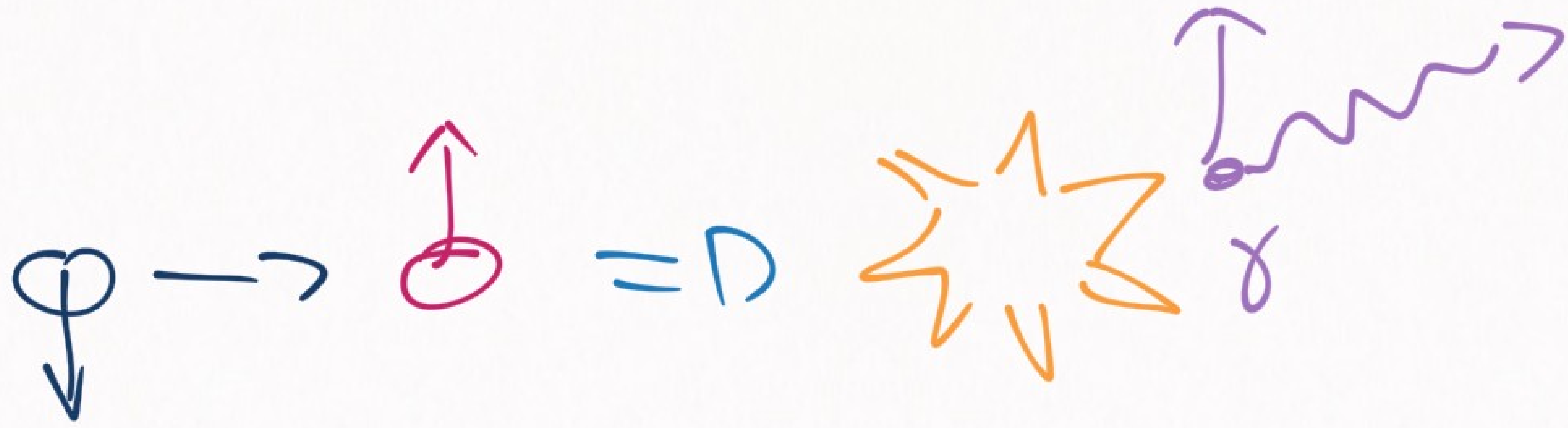
1) $\psi_B(\vec{r}) \longrightarrow \frac{e^{-\gamma_B r}}{r}$, $E_B = -\frac{\gamma_B^2}{2\mu}$

Bound state solution

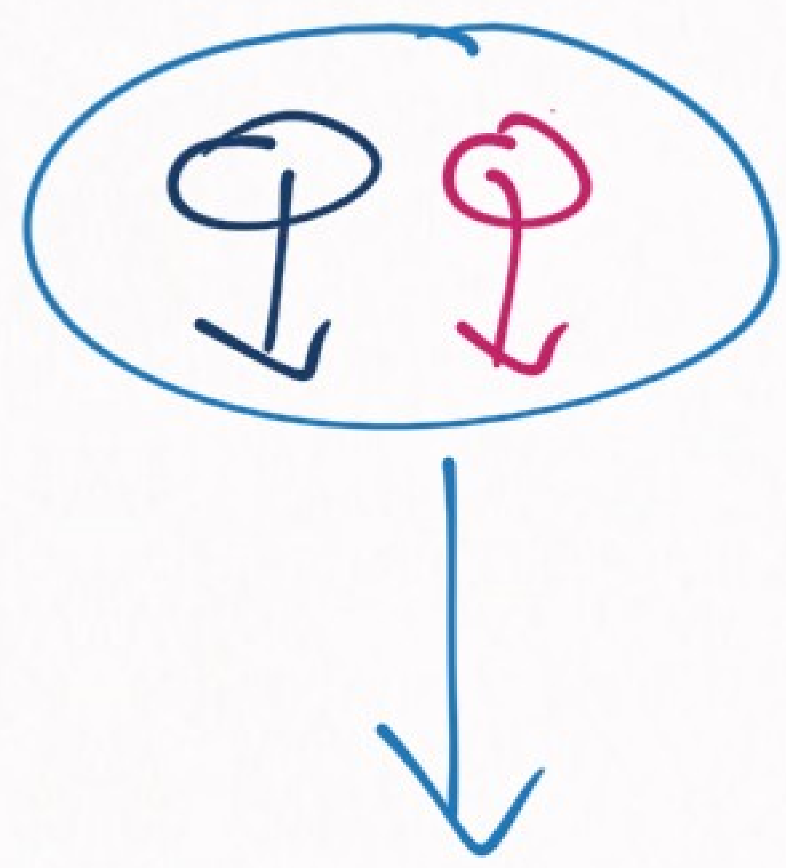
2) $\psi_v(\vec{r}) \longrightarrow \frac{e^{+\gamma_v r}}{r}$, $E_v = +\frac{\gamma_v^2}{2\mu}$

Virtual state solution

This looks like a math trick, but it's important



Neutron capture



The probability of this capture to happen is:

$$P_{capture} \propto |\langle \psi_v | \psi_B \rangle|^2$$

$$\sim \left| \int dr\, e^{-\gamma_B r} \left(r + \frac{1}{\gamma_v}\right) \right|^2$$

$$\sim \left| \frac{\gamma_B + \gamma_v}{\gamma_B^2 \gamma_v} \right|^2 \xrightarrow[\gamma_v \to 0]{} \left| \frac{1}{\gamma_B \gamma_v} \right|^2$$

From making more detailed calculations & comparing to experiments:

$$\boxed{E_\nu \simeq 0.04 \text{ MeV}}$$

(deuteron: $E_B \sim 2.2$ MeV)

Fine-tuning:

$$E_\nu = \langle T \rangle + \langle V \rangle$$

$$\langle V \rangle \simeq -50 \text{ MeV}$$

$$\boxed{\frac{E_\nu}{\langle V \rangle} \sim \frac{1}{1000}}$$ Wow!

(deuteron: $\frac{E_B}{\langle V \rangle} \sim \frac{1}{25}$)

Comment : this is the reason why neutrons are bad for your health

[→ they love all these hydrogen atoms in all the water in your body]

Fine-tuning in nuclear physics:

1) SINGLET : $1/1000$

2) TRIPLET : $1/25$

Fine-tuning can be:

1) fortuitous → happens by chance

e.g. nuclear physics, eclipses

2) or a conspiracy

→ happens because new physics we have not discover yet

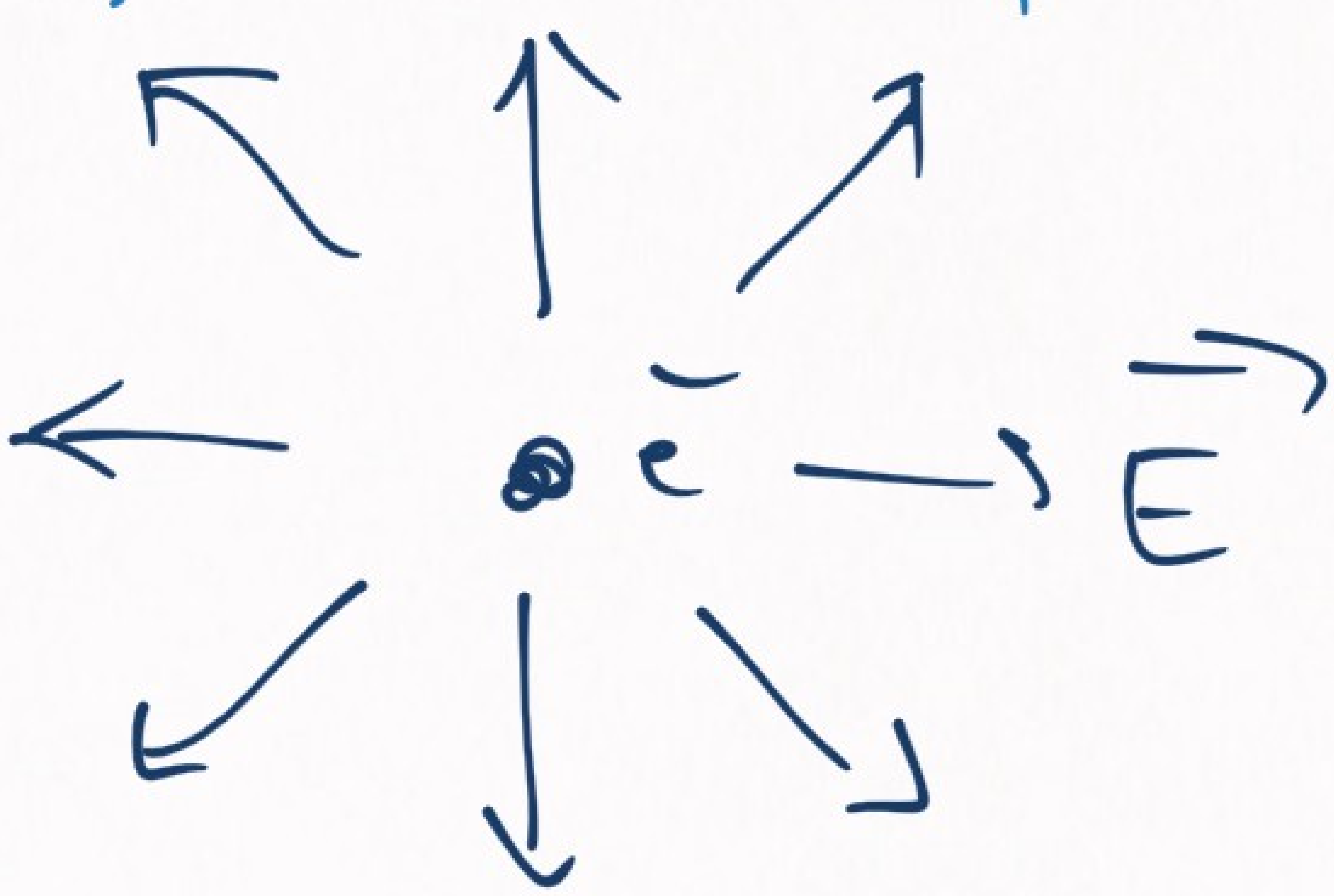
e.g. classical radius of the electron

↓

points to Quantum mechanics

CLASSICAL ELECTRON RADIUS:

1) If electron point-like



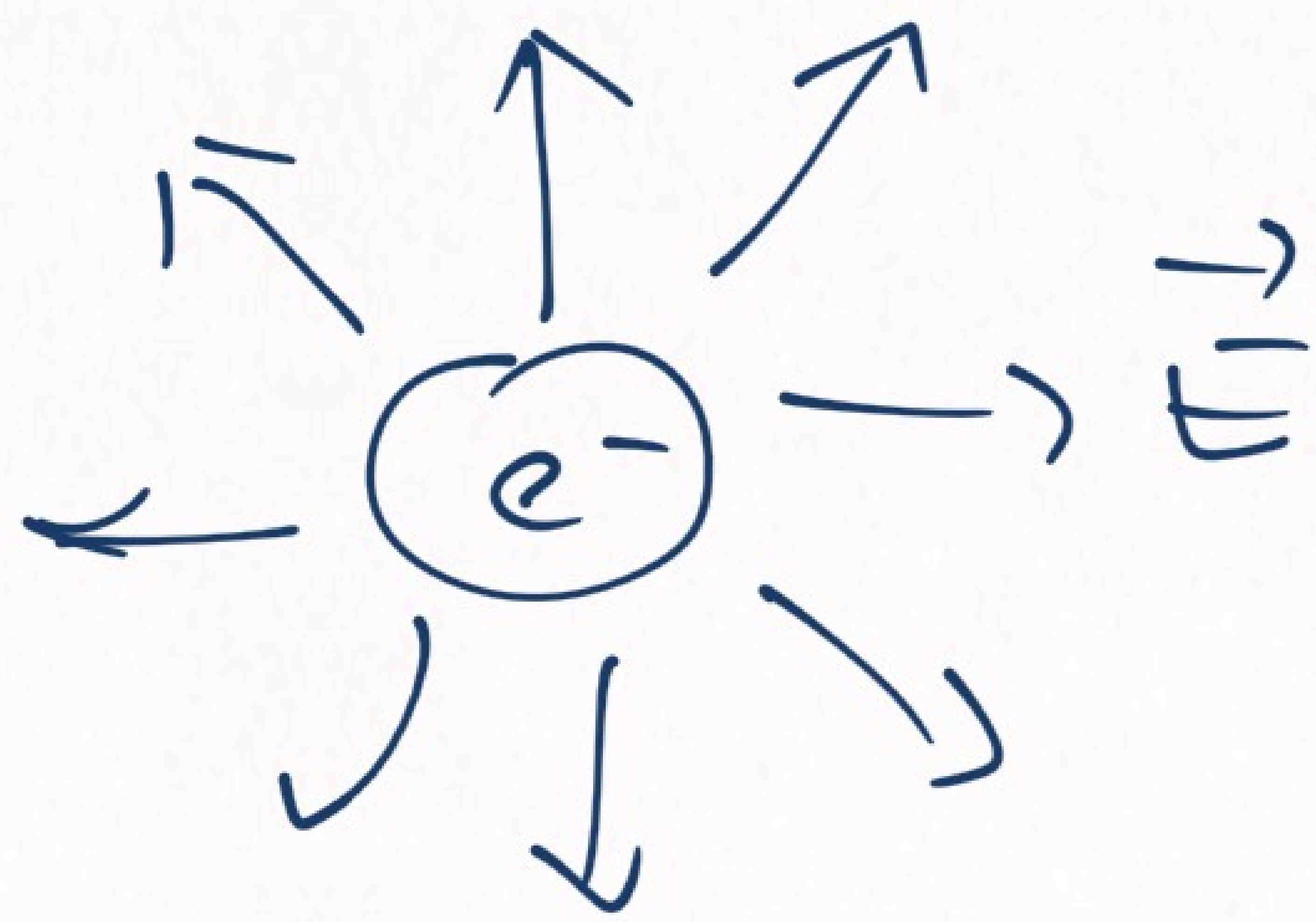
$\Rightarrow$ Energy in its electric field will be infinite

$m_{phys}(e^-) = m(\vec{E}) + m_{intrinsic}(e^-)$

$0.5\,\text{MeV} = \infty - \infty$

Infinite fine-tuning!

2) But if electron has finite size



then the energy of $\vec{E}$ is finite

If $m(e^-) = m(\vec{E})$

$$\Rightarrow \boxed{r_{cl} = \frac{\alpha}{m_e} \simeq 2.8\,\text{fm}}$$

$\Rightarrow$ If new physics happens for $r > r_e$, no fine-tuning required

[$\Rightarrow$ Natural physics]

3) Reality: quantum mechanics & quantum field theory happen

→ Quantum mechanics $r \sim a_B \sim 0.5\,\text{Å}$

→ Quantum field theory

$$r \sim \frac{1}{m_e} \sim 400\ \text{fm}$$

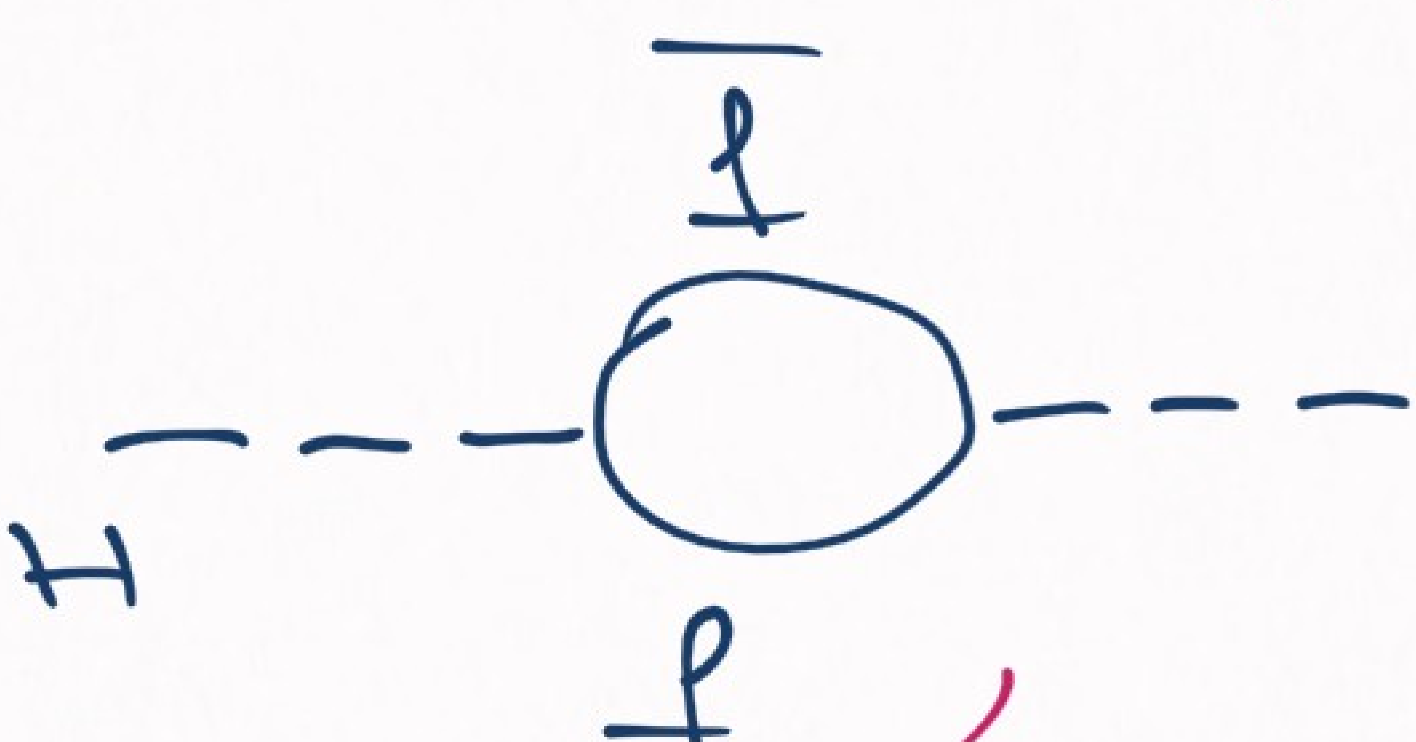
Naturalness & Fine-tuning are powerful ideas

?

Naturalness & fine-tuning:

THE HIGGS MASS

$$m_H^2 = m_{Bare}^2 + \underbrace{\Delta m_H^2}_{\text{Loop corrections}}$$



fermion loop

$$\Delta m_H^2 \propto -\frac{|\lambda_f|^2}{8\pi^2}\left[\Lambda_{UV}^2 + \dots\right]$$

energy at which we expect the standard model to fail

If $\Lambda_{UV} \sim M_{Pl} \simeq 2.5 \cdot 10^{18}$ GeV

$$\Rightarrow \boxed{\frac{m_H}{M_{Pl}} \sim 10^{-16}}$$

that's a lot of fine-tuning

→ there might be new physics

→ that's why particle physicists believe in SUPERSYMMETRY and other solutions to this

Naturalness & Fine-tuning
most extreme example

→ The cosmological constant

1) Naturalness:

$$\Lambda_c \sim M_{Pl}^4 \sim (10^{27}\,eV)^4 \sim 10^{108}\,eV^4$$

2) Reality: $\Lambda_c \sim (10^{-3}\,eV)^4$

3) Fine-tuning

$$\boxed{\frac{\Lambda_c}{M_{Pl}^4} \sim \frac{1}{10^{120}}}$$

We are a bit off w/ this one!

Cosmological constant problem

$$\boxed{\frac{\Lambda_c}{M_{pl}^4} \sim 10^{-120}}$$

which is humongous!

Solutions:

→ Anthropic principle & the multiverse?

→ points to new physics