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Chiral Effective Lagrangians [1]

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Contents

1 Generalities

These lectures concern the properties of field theories at low energies and related issues such as the behaviour of the partition function at low temperatures. The low energy structure is particularly simple if the spectrum of the theory contains a mass gap - let us start the discussion with this case, assuming that the mass M of the lightest particle is different from zero. Consider, e.g., the elastic scattering amplitude T of this particle. Assuming, for simplicity, that the particle is spinless, the partial wave expansion takes the form

$$T(s,t) = 32\pi \sum_l (2l+1) P_l(\cos\theta) t_l(q) \tag{1.1}$$

where q and θ are the momentum and the scattering angle in the center of mass system, related to the invariants s, t by

$$s = 4(M^2 + q^2) \; ; \; t = -4q^2 \sin^2 \frac{\theta}{2}. \tag{1.2}$$

For small momenta, the partial wave amplitudes t_l can be expanded as

$$\mathrm{Re} t_l(q) = q^{2l}\{a_l + b_l q^2 + O(q^4)\}. \tag{1.3}$$

The first two coefficients are referred to as the scattering length a_l and the effective range b_l. If the momenta are small compared to the mass gap M of the theory, the scattering is purely elastic and is dominated by the S-wave, such that $d\sigma/d\Omega \simeq \mid a_0 \mid^2$.

Similarly, the electromagnetic form factor of a spinless particle of unit charge

$$< p' \mid j_\mu \mid p > = (p'_\mu + p_\mu) F(t) \tag{1.4}$$

can be expanded in powers of the momentum transfer $t = (p' - p)^2$,

$$F(t) = 1 + \frac{1}{6} < r^2 > t + O(t^2) \tag{1.5}$$

where $< r^2 >$ is the mean square charge radius.

These examples illustrate the fact that, in the presence of a mass gap, the low energy structure of the theory can be analyzed in terms of an ordinary Taylor series expansion. The expansion parameters can be identified with the ratio of the momenta to the mass gap and the physical significance of the coefficients occurring in the expansion is well known.

The properties of the partition function at low temperatures are determined by the lightest excitations of the system. If the lightest particle carries mass M, the pressure, e.g., is proportional to $\exp(-M/T)$ and therefore becomes exponentially small if the temperature falls below M. Likewise, if the system is enclosed in a box of size L, the finite size effects generated by the walls of the box are of order $\exp(-ML)$ and are invisibly small if the box is large compared to the Compton wavelength $1/M$.

In the absence of a gap - i.e. if the theory contains massless particles - the low energy structure is quite different. The exchange of massless particles then generates poles at $p^2 = 0$ and cuts starting there, such that the Taylor series expansion in powers of the momenta fails. A well-known example is Coulomb scattering, which to lowest order in the electromagnetic interaction is described by the exchange of a photon. The corresponding scattering amplitude is proportional to e^2/t where $t = (p' - p)^2$ is the momentum transfer between the two charged particles. Clearly, the scattering is not dominated by the S-wave here, even if the momenta are very small. In the presence of massless particles, the thermal properties at low T are also quite different; the pressure then only tends to zero with a power of the temperature. In the case of the finite size effects generated by the walls of a box, the qualitative difference between theories which contain a mass gap and theories which don't is even more striking: for massless theories, some of the finite size effects even persist if the size of the box is sent to infinity.

The absence of a gap is the exception rather than the rule. For an analysis of the low energy structure of theories without mass gap, it is essential to understand why these theories contain massless particles in the first place. Three different mechanisms which can insure the absence of a gap are well-known:

(i) Local gauge invariance

$$A_\mu \to A_\mu + D_\mu \alpha$$

can protect spin 1 particles from picking up mass. The mass of the photon vanishes for this reason.

(ii) Global chiral symmetry

$$\psi \to e^{i\alpha\gamma_5}\psi$$

can protect fermions from becoming massive (possibly, neutrinos are massless for this reason).

(iii) Spontaneously broken symmetries give rise to massless particles - Goldstone bosons (phonons, spin waves, pions,...).

In these lectures, I focus on (iii), i.e. discuss the low energy structure of theories which contain Goldstone bosons.

2 Goldstone bosons

I first give a rudimentary version of the Goldstone theorem [1]. Suppose the Hamiltonian of the model is invariant under some Lie group G and denote the generators of this group by Q_i, such that

$$[Q_i, \mathrm{H}] = 0. \tag{2.1}$$

The symmetry is called spontaneously broken if the ground state of the theory is not invariant under G. Suppose, therefore, that, for some of the generators

$$Q_i \mid 0 > \neq 0. \tag{2.2}$$

This immediately implies that the vacuum is not the only state of zero energy: since H commutes with Q_i, the vector $Q_i \mid 0 >$ describes a state with the same energy as the vacuum. The subset formed by those generators which do leave the

ground state invariant is a subalgebra: if Q_i and Q_k annihilate the vacuum, then this is also true of the commutator $[Q_i, Q_k]$. These operators therefore generate a subgroup $H \subset G$. Spontaneous symmetry breakdown thus involves two groups - the symmetry group G of the Hamiltonian and the symmetry group H of the vacuum. Denote the number of parameters required to label the elements of G by n_G such that there are n_G generators and suppose that $n_H < n_G$ is the number of parameters occurring in H. The $n_G - n_H$ generators which belong to the quotient G/H of the two groups do not annihilate the ground state. The corresponding vectors $Q_i \mid 0 >$ are linearly independent, because, otherwise, a suitable linear combination of these generators would leave the vacuum invariant and hence belong to H. Accordingly, spontaneous breakdown of the group G to the subgroup H requires the occurrence of $n_G - n_H$ independent states of zero energy - the spectrum of the theory contains $n_G - n_H$ different flavours of Goldstone bosons.

This "proof" of the Goldstone theorem requires mathematical massage, because the states $Q_i \mid 0 >$ occurring in the above argument have infinite norm (the vacuum is the only normalizable state invariant under translations). You can find a proper discussion of the theorem in the Coleman lectures on "Secret Symmetry" [2], where you can also see what happens if G is not an ordinary symmetry of the Hamiltonian, but a local gauge group. I do not discuss these phenomena here, but consider the spontaneous breakdown of ordinary symmetries where the Goldstone theorem applies and where the Goldstone bosons do occur as asymptotic states. For the following, it is essential that the spectrum does not contain other massless asymptotic states. In the case of QCD, it is crucial that the gauge group (colour) commutes with the symmetry group G (flavour) and that confinement of colour prevents gluons and quarks from showing up as massless physical particles.

3 Linear σ-model

As an illustration, consider the linear σ-model which involves n scalar fields $\vec{\phi} = (\phi^0, \ldots, \phi^{n-1})$, coupled through a $\lambda\phi^4$ interaction,

$$\mathcal{L} = \frac{1}{2}\partial_\mu\vec{\phi}\partial^\mu\vec{\phi} - \frac{\lambda}{4}(\vec{\phi}^2 - v^2)^2 \tag{3.1}$$

where v is a constant of dimension $(\text{mass})^{\frac{1}{2}(d-2)}$. Since this Lagrangian is invariant under rotations of the vector $\vec{\phi}$, the symmetry group of the model is $G = O(n)$. With $n = 4$, the model describes the Higgs sector of the Standard Model.

In four dimensions, the field theory specified in eq. (3.1) presumably describes a free field, the renormalized coupling constant being equal to zero. This indicates that the Higgs model only represents an effective theory, valid at low energies and that the basic degrees of freedom associated with mass generation yet need to be discovered. In our context, the problems afflicting $\lambda\phi^4$ in $d = 4$ are however not of central interest. Although the model only makes sense in the presence of an ultraviolet cutoff, the corresponding violation of locality does not manifest itself at large distances. The low energy expansion only concerns momenta which are small compared to the mass m_H of the Higgs particle and is not sensitive to the structure of the theory at distances which are small compared to $1/m_H$. Incidentally, the general low energy analysis also applies in three dimensions, where the Lagrangian (3.1) does specify a nontrivial local field theory.

The ground state of the model is characterized by a nonzero expectation value of the field. With a suitable choice of the field basis, $< 0 \mid \vec{\phi} \mid 0 >$ points along the first axis such that the perturbative expansion starts with

$$< 0 \mid \phi^0 \mid 0 >= v\{1 + O(\lambda)\}$$
$$< 0 \mid \phi^i \mid 0 >= 0 \quad i = 1, \ldots, n-1. \tag{3.2}$$

Visibly, the ground state is not symmetric under G, but breaks the symmetry down to the little group of the vector $< 0 \mid \vec{\phi} \mid 0 >$, i.e. to $H = O(n-1)$. The number of parameters needed to label the elements of $O(n)$ is $\frac{1}{2}n(n-1)$. The spectrum of the theory must therefore contain $n_G - n_H = n - 1$ Goldstone bosons.

In the present case, the occurrence of massless modes can directly be confirmed within perturbation theory. Setting $\vec{\phi} =< 0 \mid \vec{\phi} \mid 0 > + \vec{\varphi}$ and expanding the

Lagrangian in powers of $\vec{\varphi}$ to order $\vec{\varphi}^2$, one indeed finds that only one of the n fields, viz. φ^0, receives a mass term, $M^2 = 2\lambda v^2$, while the remaining $n-1$ fields are massless.

4 Quantum chromodynamics

As a second example, consider QCD

$$\mathcal{L} = -\frac{1}{4g^2}G^a_{\mu\nu}G^{\mu\nu a} + \bar{q}i\gamma^\mu D_\mu q - \bar{q}\mathcal{M}q \tag{4.1}$$

where $\mathcal{M}$ is the quark mass matrix,

$$\mathcal{M} = \begin{pmatrix} m_u & & & \\ & m_d & & \\ & & m_s & \\ & & & \ddots \end{pmatrix} \tag{4.2}$$

Since the quark masses are different from one another, the symmetry group of this theory is $G = U(1) \times U(1) \times \ldots$. There is one conserved charge for every one of the quark flavours; the corresponding conserved currents are $\bar{u}\gamma^\mu u$, $\bar{d}\gamma^\mu d, \ldots$

If the quark masses are set equal to zero, the symmetry group of the theory becomes larger. The Lagrangian does then not contain terms which connect the left- and right-handed components of the quark fields and remains invariant under a set of chiral transformations of the type (ii) which are characteristic for massless fermions (see section 1). For N massless quark flavours, the Lagrangian is invariant under independent rotations of the left- and right-handed quark fields,

$$q_L \rightarrow V_L q_L \qquad q_R \rightarrow V_R q_R \tag{4.3}$$

where q_L, e.g., stands for $\frac{1}{2}(1 - \gamma_5)q$ and V_L, V_R are unitary $N \times N$-matrices. One of the $2N^2$ currents whose charges generate these transformations is however anomalous: despite the symmetry of the Lagrangian, the Noether current $\bar{q}\gamma_\mu\gamma_5 q$ fails to be conserved, the divergence being proportional to the winding number density of the gluon field. The actual symmetry group of massless QCD consists of those pairs of elements $V_L, V_R \in U(N)$ which obey the constraint $\det(V_L V_R^{-1}) = 1$, i.e.

$$G = \mathrm{SU}(N)_L \times \mathrm{SU}(N)_R \times \mathrm{U}(1)_{L+R}. \tag{4.4}$$

One generally assumes that (if N is not too large) this symmetry is spontaneously broken, the vacuum being invariant only under the subgroup generated by the vector currents $\bar{q}\gamma_\mu \frac{1}{2}\lambda_i q$,

$$H = \mathrm{SU}(N)_{R+L} \times \mathrm{U}(1)_{R+L}. \tag{4.5}$$

Accordingly, the spectrum of massless QCD must contain $N^2 - 1$ Goldstone bosons where N is the number of quark flavours. Their quantum numbers can be read off from those of the states $Q_i^A \mid 0 >$ which result if the axial charge operators

$$Q_i^A = \int d^3x\, A_i^0(x) \quad ; \quad A_i^\mu = \bar{q}\gamma^\mu\gamma_5\frac{1}{2}\lambda_i q \tag{4.6}$$

are applied to the vacuum: $J^P = 0^-$.

The symmetry group in eq. (4.4) pertains to the theoretical limit where the quark masses are set equal to zero. In reality, the symmetry is broken by the quark mass term occurring in the Lagrangian. Indeed, the observed spectrum of particles does not contain massless hadrons. Remarkably, however, the eight lightest hadrons ($\pi^\pm, \pi^0, K^\pm, K^0, \bar{K}^0, \eta$) are pseudoscalars and three of them are particularly light. This pattern is to be expected if

(i) m_u, m_d, m_s happen to be small, such that the Lagrangian is approximately invariant under $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$. The spontaneous breakdown of this approximate symmetry to $\mathrm{SU}(3)_{L+R}$ generates $3^2 - 1 = 8$ approximately massless particles.

(ii) m_u, m_d are small compared to m_s, such that the group $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ is an almost exact symmetry of the Lagrangian. This is why $2^2 - 1 = 3$ of the pseudoscalar mesons are particularly light.

We will discuss the manner in which the masses of the Goldstone bosons are

affected by the symmetry breaking in detail below and show that the observed masses of the eight mesons listed above require $m_u : m_d : m_s \simeq 0.55 : 1 : 20$.

The masses of the other quarks occurring in the Standard Model, c, b, t, are not small. The lightest pseudoscalar bound state with the quantum numbers of $\bar{d}c$, e.g. sits at $M_{D^+} \simeq 1.87$ GeV. Even though there is a theoretical limit ($m_d = m_c = 0$) where this state is massless and plays the role of a Goldstone boson, this limit is not very useful, because the mass term $m_c\bar{c}c$ cannot be treated as a small perturbation. In the context of QCD, applications of chiral perturbation theory either concern the group $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ broken spontaneously to isospin symmetry or the group $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$, where the symmetry of the vacuum is the eightfold way. In either case, the Lagrangian can be decomposed into two pieces $\mathcal{L}_0 + \mathcal{L}_1$ where $\mathcal{L}_0$ is the symmetric part while $\mathcal{L}_1$ contains the symmetry breaking terms. If $\mathcal{L}_1$ is identified with $-(m_u\bar{u}u + m_d\bar{d}d)$, then $\mathcal{L}_0$ is symmetric under $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ and the perturbative expansion in powers of $\mathcal{L}_1$ amounts to a power series in m_u and m_d. If the mass term of the strange quarks is included in $\mathcal{L}_1$, the symmetry group of $\mathcal{L}_0$ is increased and now contains the factor $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$. The price to pay is that the expansion in powers of $\mathcal{L}_1$ now gives rise to powers of m_s - the chiral perturbation series converges less rapidly.

5 Effective Field Theory

In the context of the strong interactions, the main consequences of the hidden approximate symmetry were derived in the sixties from a direct analysis of the Ward identities, using current algebra and pion pole dominance. In the meantime, an alternative method has been developed which deals with the problem in a more systematic manner and which is considerably more efficient. I now turn to a discussion of this method [3] [4] [5].

For definiteness, consider QCD with two massless flavours, where three Goldstone bosons occur. In the Green functions of the theory, massless one-particle states manifest themselves as poles at $p^2 = 0$. The two-point function of the axial current, e.g., contains a pole term due to the exchange of a pion between the two currents. The residue of the pole is given by the square of the matrix element

$$< 0 \mid A_i^\mu \mid \pi^k(p) >= ip^\mu \delta_{ik} F. \qquad (5.1)$$

The pole occurring in the two-point function entails corresponding singularities in other Green functions. The Ward identities obeyed by the three-point function with two axial and one vector current, e.g., require that this quantity contains a simultaneous pole in the two axial legs. Picturing the pole factors as pions which propagate in space, the leading low energy contributions to the two- and three-point functions are shown in Figs. 1a, b. The Ward identities for the four-point function of the axial current relate this object to the three-point function just mentioned. They admit a solution only if the four-point function contains a simultaneous pole in all four legs as indicated in Fig. 1c. When analyzing these amplitudes in the framework of current algebra, one assumes that, at leading order in the low energy expansion,

(i) the only singularities which occur are poles originating in the progagation of pions and

(ii) the residues can be expanded in powers of the momenta.

Truncating the expansion at the first nontrivial order, the residues reduce to polynomials in the momenta. The coefficients occurring in these polynomials can then be worked out by explicitly solving the Ward identities.

The effective Lagrangian technique is based on the following heuristic considerations. The graphs shown in Fig. 1 may be viewed as tree graphs of a field theory which involves pion fields as basic variables. Vertices which exclusively emit pion lines - like the central vertex in Fig. 1c - represent interaction terms of the corresponding Lagrangian. If these vertices involve polynomials of the momenta, the corresponding interaction term contains derivatives of the pion field. The pion propagator originates in a kinetic term of the form $\frac{1}{2}\partial_\mu\vec{\pi}\partial^\mu\vec{\pi}$. The full Lagrangian is obtained by adding all interaction terms, involving aribtrarily many pion fields.

The vertex shown in Fig. 1a links the pion field to the axial current. In the

language of the pion field theory, the occurrence of this vertex implies that the axial current contains a term linear in the pion field,

$$A^i_\mu = -F\partial_\mu\pi^i + \ldots \qquad (5.2)$$

while a term of the form $\partial_\mu\pi\pi\pi$, e.g., corresponds to a vertex where the axial current emits three pion lines. The Ward identities require the vector and axial currents to be conserved. According to the Noether theorem, the pion field theory does give rise to conserved vector and axial currents if the corresponding action is invariant under $G = \mathrm{SU}(2)\times\mathrm{SU}(2)$. This indicates that the solutions of the Ward identities which obey the conditions (i) and (ii) are in one-to-one correspondence with the tree graphs of a pion field theory for which

(a) the Lagrangian admits an expansion in powers of the derivatives of the field,

(b) the pion field transforms according to a nontrivial representation of the group G,

(c) the action is invariant under these transformations.

The above arguments are plausibility considerations. To my knowledge, a rigorous proof of the claim that the requirements (i) and (ii) imply the existence of an effective field theory with the properties (a)-(c) is still lacking. If you are looking for a decent problem in mathematical physics: this is one. Note also that (i) and (ii) are assumptions, within both the current algebra approach and the effective Lagrangian technique. When you are through with a proof of the above proposition, could you please also demonstrate that the Goldstone bosons necessarily dominate the low energy behaviour of the Green functions in this specific sense? In the following, I assume that they do, take it for granted that the low energy structure can be worked out by means of an effective Lagrangian and proceed by analyzing its specific form. As a first step in this direction, we need to look more closely at the transformation properties of the pion field.

6 Transformation properties of the Goldstone bosons

The pion field transforms according to a representation of the group G, i.e. every $g \in G$ induces a mapping of the form $\vec{\pi}' = \vec{f}(g, \vec{\pi})$. For this mapping to be a representation, the function $\vec{f}$ must obey the composition law

$$\vec{f}(g_1, \vec{f}(g_2, \vec{\pi})) = \vec{f}(g_1 g_2, \vec{\pi}). \tag{6.1}$$

Remarkably, this property determines the function $\vec{f}(g, \vec{\pi})$ essentially uniquely [6]. To verify this claim, we first consider the image of the origin, $\vec{f}(g, 0)$. The composition law shows that the set of elements h which map the origin onto itself forms a subgroup $H \subset G$. Moreover, $\vec{f}(gh, 0)$ coincides with $\vec{f}(g, 0)$ for any $g \in G$, $h \in H$. Hence the function $\vec{f}(g, 0)$ lives on the space G/H which is obtained from G by identifying elements g, g' which only differ by right multiplication with a member of H, $g' = gh$. The function $\vec{f}(g, 0)$ thus maps the elements of G/H into the space of pion field variables. The mapping is invertible, because $\vec{f}(g_1, 0) = \vec{f}(g_2, 0)$ implies $g_1^{-1} g_2 \in H$. One can therefore identify the values of the pion field with the elements of G/H: the Goldstone bosons live in this space. Next, choose a representative element n in each one of the equivalence classes $\{gh, h \in H\}$, such that every group element can be decomposed as $g = nh$. The composition law (6.1) then shows that the image n' of the element n under the action of $g \in G$ is obtained by decomposing the product gn into $n'h$ - the standard action of G on the space G/H. This shows that the geometry fully fixes the transformation law of the pion field, except for the freedom in the choice of coordinates on the manifold G/H.

In the case of $G = \mathrm{SU}(2) \times \mathrm{SU}(2)$ and $H = \mathrm{SU}(2)$, the quotient G/H is the group SU(2). The pion field may therefore be represented as a 2×2 matrix field $U(x) \in \mathrm{SU}(2)$. Since one needs three coordinates to parametrize the elements of SU(2), the matrix field $U(x)$ is equivalent to a set of 3 ordinary fields $\pi^1(x), \pi^2(x), \pi^3(x)$. One may use, e.g., canonical coordinates,

$$U(x) = \exp i\alpha\pi(x) \quad ; \quad \pi(x) = \sum_{i=1}^{3} \pi^i(x)\tau_i \tag{6.2}$$

where τ_1, τ_2, τ_3 are the Pauli matrices. The constant α carries dimension $[\mathrm{mass}]^{-1}$; we will fix it later on. I recommend it as an exercise to verify that, in the present case, the action of G on the pion field is given by

$$U'(x) = V_R U(x) V_L^+. \tag{6.3}$$

The matrix $U(x)$ thus transforms linearly. Note, however, that the corresponding transformation law for the pion field $\vec{\pi}(x)$ is nonlinear. As indicated by the above general discussion, the occurrence of nonlinear realizations of the symmetry group is a characteristic feature of the effective Lagrangian technique.

7 Form of the effective Lagrangian

I now turn to the requirements (a) and (c) of section 5. Lorentz invariance implies that the leading terms in the expansion of the effective Lagrangian in powers of derivatives are of the form

$$\mathcal{L}_{eff} = f_1(U) + f_2(U) \times \Box U + f_3(U) \times \partial_\mu U \times \partial^\mu U + O(p^4)$$

where the symbol $O(p^4)$ indicates that the remainder contains four or more derivatives of the pion field. The crosses refer to the fact that the coefficients $f_2(U)$ and $f_3(U)$ carry indices which are contracted against those of the matrices $\Box U$ and $\partial_\mu U$. The first term does not contain derivatives. The corresponding action is invariant under $U \to V_R U V_L^+$ if and only if $f_1(U)$ is independent of U. Hence the first term is an irrelevant cosmological constant and can be dropped. Integrating by parts, the second term can be transformed into the third one, so that we can drop f_2, too. Without loss of generality we can then write the Lagrangian in the form $\tilde{f}_3(U) \times \Delta_\mu \times \Delta^\mu$ where Δ_μ stands for $-iU^{-1}\partial_\mu U$. The advantage of this manipulation is that Δ_μ is invariant under $U \to V_R U$, such that only $\tilde{f}_3(U)$ is affected by this operation. The requirement that the action must remain invariant therefore implies that $\tilde{f}_3\,(U)$ is independent of U. Finally, under the transformation $U \to UV_L^+$, the traceless quantity Δ_μ transforms according to the representation $D^{(1)}$. Since the product $D^{(1)} \times D^{(1)}$ contains the identity only once, there is a single invariant of order p^2,

$$\mathcal{L}_{\text{eff}} = g\,\text{tr}\Delta_\mu\Delta^\mu = g\,\text{tr}\,(\partial_\mu U \partial^\mu U^+). \tag{7.1}$$

This shows that the leading term in the derivative expansion of the effective Lagrangian contains only one free coupling constant, g.

Expanding the matrix $U(x) = \exp i\alpha\pi(x)$ in powers of the pion field, we obtain

$$\mathcal{L}_{\text{eff}} = 2g\alpha^2 \partial_\mu \vec{\pi} \partial^\mu \vec{\pi} + \frac{g\alpha^4}{12}\,\text{tr}\,([\partial_\mu \pi, \pi][\partial^\mu \pi, \pi]) + \ldots$$

where interactions involving six or more pion fields are omitted. In order for the first term to agree with the standard normalization of the kinetic energy, we fix the parameter α introduced in eq. (6.2) in terms of the coupling constant g by setting $\alpha^2 = 1/4g$. The Noether currents associated with the SU(2)×SU(2) symmetry of the Lagrangian (7.1) are

$$V_i^\mu = ig\,\text{tr}\,(\tau_i[\partial^\mu U, U^+])$$
$$A_i^\mu = ig\,\text{tr}\,(\tau_i\{\partial^\mu U, U^+\}). \tag{7.2}$$

Comparing the expression for the axial current with eq. (5.1) or (5.2), we see that the coupling constant g is related to the pion decay constant F by $g = F^2/4$. At leading order in the derivative expansion, the effective Lagrangian therefore only involves the pion decay constant,

$$\mathcal{L}_{\text{eff}} = \frac{F^2}{4}\,\text{tr}\,(\partial_\mu U \partial^\mu U^+)$$
$$U(x) = \exp\left\{\frac{i}{F}\sum_{k=1}^{3} \pi^k(x)\tau_k\right\}. \tag{7.3}$$

The field theory characterized by this Lagrangian is referred to as the nonlinear σ-model.

It is well-known that for $d > 2$ this model is not renormalizable - taken by itself, it is not a decent theory. Are we running after a cloud of dust here? Actually, in the above analysis only the tree graphs of the effective Lagrangian played a role. Renormalizability is not an issue which concerns the tree graphs. We will have occasion to discuss the significance of loop graphs later on, when we consider the low energy expansion beyond leading order. Clearly, the effective Lagrangian must then also be worked out beyond the leading term in the derivative expansion. In the framework of the effective Lagrangian, the nonlinear σ-model only represents one building block of the construction - it does not occur by itself. As we will see, the effective theory as a whole is a perfectly renormalizable scheme.

8 Universality

In the preceding section, we arrived at the conclusion that the leading term in the derivative expansion of the effective Lagrangian associated with the low energy behaviour of QCD is an explicitly known expression which only involves the pion decay constant. This result is remarkable in more than one respect. First, it rigorously establishes that all of the Ward identities for all of the vector and axial current Green functions admit a solution which obeys the two requirements (i) and (ii) formulated in section 5. The solution is given by the tree graphs of the nonlinear σ-model. If the plausibility arguments underlying the effective Lagrangian technique are taken for granted, the uniqueness of the effective Lagrangian also implies that, at leading order in the low energy expansion, the solution of the Ward identities is unique and only contains the pion decay constant. Since the analysis applies to any theory for which SU(2) × SU(2) is spontaneously broken to SU(2), the specific properties of the underlying theory are irrelevant - the low energy structure is universal.

The extension to three massless flavours is straightforward. In this case, the

Goldstone bosons live in $G/H = \mathrm{SU}(3)$. Accordingly, there are eight pion fields which may be identified with the canonical coordinates on SU(3),

$$U(x) = \exp \frac{i}{F} \sum_{k=1}^{8} \pi^k(x)\lambda_k \tag{8.1}$$

where $\lambda_1, \ldots, \lambda_8$ are the Gell-Mann matrices. There is again only one invariant at order p^2, the one given in eq. (7.3).

In the case of $G = O(n)$, $H = O(n-1)$, the quotient G/H is the $(n-1)$-dimensional sphere. Accordingly, the pion field is a unit vector $\vec{U}(x)$ with n components. The derivative expansion of the effective Lagrangian again starts with a term of order p^2 and involves a single coupling constant,

$$\mathcal{L}_{\text{eff}} = \frac{1}{2} F^2 \partial_\mu \vec{U} \partial^\mu \vec{U}. \tag{8.2}$$

In particular, if n is equal to four, the pion field lives in the three-dimensional sphere. This manifold can be mapped one-to-one onto the group SU(2), setting

$$U = U^0 1\!\!1 + i \sum_{k=1}^{3} U^k \tau_k. \tag{8.3}$$

One readily checks that the Lagrangians (7.3) and (8.2) are identical. This is not a great surprise, because the groups $G = O(4)$ and $H = O(3)$ occurring in the spontaneous breakdown of the Higgs model are locally isomorphic to $\mathrm{SU}(2)\times\mathrm{SU}(2)$ and to SU(2), respectively. The equivalence of the two effective theories implies that, at low energies, the Green functions of the Higgs model and of QCD with two massless flavours are the same, except for the magnitude of the pion decay constant. For QCD, $F \simeq 93$ MeV, while in the case of the Higgs model, $F \simeq 245$ GeV.

9 Geometry

Quite generally, the leading term in the low energy expansion of the effective Lagrangian is determined by the geometry of the manifold G/H [7] [5]. Denote

the coordinates on this manifold by $\omega^1, \ldots, \omega^r$ such that the Goldstone bosons are described by the fields $\omega^1(x), \ldots, \omega^r(x)$. Under the action of G, the coordinates transform in a nonlinear manner. There always exists a metric $g_{ab}(\omega)$ on G/H such that the line element

$$ds^2 = \sum_{a,b} g_{ab}(\omega) d\omega^a d\omega^b \tag{9.1}$$

is invariant under the action of G. In the examples considered above, this requirement fixes the metric uniquely up to a normalization constant. This is not always the case, however. In general, there are several independent quadratic forms on G/H which are invariant under G and the general G-invariant metric is given by a linear combination of these. Denote the general metric by $g_{ab}(\omega)$. The leading term in the derivative expansion of the effective Lagrangian is given by

$$\mathcal{L}_{\text{eff}} = \sum_{a,b} g_{ab}(\omega) \partial_\mu \omega^a \partial^\mu \omega^b. \tag{9.2}$$

The coefficients of the linear combination referred to above are the coupling constants of this Lagrangian.

An example where the effective Lagrangian involves more than one coupling constant at order p^2 is the following. Take $G = \mathrm{SU}(N)$ and identify H with the trivial subgroup $H = \{e\}$. The quotient G/H then coincides with G and the action of the symmetry group on the pion field $U(x) \in \mathrm{SU}(N)$ is of the form $U(x) \to VU(x)$. It is easy to see that in this case, there are several independent G-invariant quadratic forms. Consider two neighbouring elements U, $U + dU$ of G/H. The product $U^{-1}dU$ is invariant under G. Hence the linear forms Ω^a which occur if this product is expanded in the Gell-Mann basis

$$U^{-1} dU = i \sum_{a=1}^{N^2-1} \Omega^a \lambda_a. \tag{9.3}$$

are invariant under G. They can be used to construct the general G-invariant quadratic form as

$$ds^2 = \sum_{a,b} c_{ab}\Omega^a\Omega^b \tag{9.4}$$

where c_{ab} are arbitrary constants. In this case, the leading term in the derivative expansion of the effective Lagrangian thus involves $\frac{1}{2}N^2(N^2-1)$ coupling constants. Physically, this embarras de richesse reflects the fact that if the vacuum breaks the symmetry totally, then there is no symmetry left over which would require the pion matrix elements of the currents associated with G to be the same. Accordingly, the pole terms in the two-point functions of these currents need not have the same residue - several low energy constants are needed to specify the leading terms in the low energy expansion of the two-point functions. The Ward identities do not shed any light on the values of these constants - they merely require that Green functions involving more than two currents contain singularities which match those of the two-point functions.

10 Symmetry breaking

In the preceding sections, we have dropped the masses of the u- and d-quarks. In their presence, the Lagrangian of the theory is not invariant under $SU(2) \times SU(2)$, because the mass term

$$\mathcal{L} = \mathcal{L}_0 - \bar{q}\mathcal{M}q \tag{10.1}$$

connects the right- and left-handed components of the quark fields,

$$\bar{q}\mathcal{M}q = \bar{q}_R\mathcal{M}q_L + \text{h.c.}. \tag{10.2}$$

Note that the mass terms associated with s, c, b, t are included in $\mathcal{L}_0$; the quark field q only contains u and d and $\mathcal{M}$ is the matrix

$$\mathcal{M} = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix}. \tag{10.3}$$

It is instructive to compare the QCD Lagrangian with the Hamiltonian of a Heisenberg ferromagnet,

$$H = H_0 - \sum_a \mu\vec{s}_a \cdot \vec{H}. \tag{10.4}$$

Here, $\vec{s}_a$ is the spin associated with lattice site a, μ is the magnetic moment and $\vec{H}$ is an external magnetic field. The term H_0 is invariant under simultaneous $O(3)$-rotations of all spin variables, while the term which involves the external magnetic field breaks this symmetry. Clearly, the quark masses (m_u, m_d) play a role analogous to the external magnetic field and the quark condensate $< 0 \mid \bar{u}u \mid 0 >$, $< 0 \mid \bar{d}d \mid 0 >$ is analogous to the magnetization. In particular, spontaneous magnetization at zero external field corresponds to a nonzero value of the quark condensate in the chiral limit $m_u, m_d \to 0$.

In the case of the magnet, the symmetry breaking term transforms according to the spin 1 representation of $O(3)$. The decomposition of the quark mass term given in eq. (10.2) shows that this term transforms according to the representation $D^{(\frac{1}{2},\frac{1}{2})}$ of $SU(2)_L \times SU(2)_R$. Equivalently, we may say that the QCD Lagrangian is invariant under the transformation (4.3) of the quark fields, provided the mass matrix is transformed accordingly,

$$\mathcal{M} \to V_R\mathcal{M}V_L^+. \tag{10.5}$$

The occurrence of a mass term of course modifies the form of the effective Lagrangian,

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{eff}}(U, \partial U, \partial^2 U, \ldots, \mathcal{M}) \tag{10.6}$$

which now remains invariant under the transformation $U(x) \to V_R U(x) V_L^+$ of the pion field only if one simultaneously also transforms the quark mass matrix in the same manner. The modification of the Lagrangian generated by the quark masses can be analyzed by expanding in powers of $\mathcal{M}$. The first term in this expansion is the effective Lagrangian of the massless theory which we have considered in the preceding sections. The term linear in $\mathcal{M}$ is of the form

$$\mathcal{L}_{sb} = f(U, \partial U, \ldots) \times \mathcal{M}. \tag{10.6}$$

Next, we observe that derivatives of the pion field are suppressed by powers of the momenta. At leading order in an expansion in both, powers of $\mathcal{M}$ and powers of derivatives, the symmetry breaking term in the effective Lagrangian reduces to an expression of the form $f(U) \times \mathcal{M}$. Moreover, this expression must be invariant under simultaneous chiral transformations of the matrices U and $\mathcal{M}$. There are only two independent invariants: $\mathrm{tr}(\mathcal{M}U^+)$ and its complex conjugate. Hence the leading symmetry breaking contribution is of the form

$$\mathcal{L}_{sb} = \frac{F^2}{2}\{B\,\mathrm{tr}(\mathcal{M}U^+) + B^*\,\mathrm{tr}(\mathcal{M}^+U)\} \tag{10.7}$$

where I have extracted a factor F^2 for later convenience. The symmetry breaking involves a new low energy constant, B, which need not be real. Since only the product $B\mathcal{M}$ matters, the phase of B occurs together with the phase of the quark mass matrix and is related to the possible occurrence of a parity violating term of the form $\Theta\, G_{\mu\nu} \tilde{G}^{\mu\nu}$ in the Lagrangian of QCD. The fact that the neutron dipole moment is very small implies that the strong interactions conserve parity to a very high degree of occuracy. Let us therefore require that the effective Lagrangian is parity invariant. Using the standard basis where the quark mass matrix is diagonal and real, this requirement implies that B is real (the parity operation sends π into $-\pi$ and hence interchanges U with U^+),

$$\mathcal{L}_{sb} = \frac{F^2 B}{2}\mathrm{tr}\mathcal{M}(U + U^+). \tag{10.8}$$

Since U is an element of SU(2), the sum $U + U^+$ is proportional to the unit matrix. Accordingly, the leading contribution to the symmetry breaking part of the effective Lagrangian only involves the sum $m_u + m_d$ of the two quark masses and therefore conserves isospin - the breaking of isospin symmetry generated by the mass difference $m_u - m_d$ only shows up if the low energy expansion is carried

beyond leading order.

Expanding $U = \exp(i\vec{\pi}\cdot\vec{\tau}/F)$ in powers of the pion field $\vec{\pi}$, the Lagrangian (10.8) gives rise to the following contributions:

$$\mathcal{L}_{sb} = (m_u + m_d)B\left\{F^2 - \frac{1}{2}\vec{\pi}^2 + \frac{\vec{\pi}^4}{24F^2} + \ldots\right\}. \tag{10.9}$$

Up to a sign, the first term represents the vacuum energy generated by the symmetry breaking. The second is quadratic in the pion field and therefore amounts to a pion mass term. The remaining contributions show that the symmetry breaking necessarily also modifies the interaction among the Goldstone bosons.

The derivative of the QCD Hamiltonian with respect to m_u is the operator $\bar{u}u$. The corresponding derivative of the vacuum energy therefore represents the vacuum expectation value of $\bar{u}u$. Evaluating this derivative with the first term in eq. (10.9), we obtain

$$< 0 \mid \bar{u}u \mid 0 >=< 0 \mid \bar{d}d \mid 0 >= -F^2 B\{1 + O(\mathcal{M})\}. \tag{10.10}$$

The low energy constant B is therefore related to the value of the quark condensate. In analyzing the form of the effective Lagrangian, we have retained only terms linear in the quark masses. The curly bracket in eq. (10.10) indicates that, in this relation, the higher terms generate contributions of order $\mathcal{M}$.

According to eq. (10.9), the pion mass is given by

$$M_\pi^2 = (m_u + m_d)B\{1 + O(\mathcal{M})\}. \tag{10.11}$$

If the quark masses are set equal to zero, the pion mass vanishes, as it should - SU(2) $\times$ SU(2) is then an exact symmetry and the Goldstone bosons are strictly massless. As long as the symmetry breaking is small, the Goldstone bosons only pick up a small mass which is proportional to the quare root of the symmetry breaking parameter $m_u + m_d$. In accord with the remarks made above, isospin breaking does not manifest itself at this order of the expansion - the masses of π^+, π^0 and π^- are the same.

Eliminating the low energy constant B, the relations (10.10) and (10.11) lead to the well-known result of Gell-Mann, Oakes and Renner [8],

$$F^2 M_\pi^2 = -(m_u + m_d) < 0 \mid \bar{u}u \mid 0 > + O(\mathcal{M}^2). \qquad (10.12)$$

The extension to N quark flavours is straightforward - the above analysis goes through without any essential modifications and leads to an effective Lagrangian of the same form,

$$\mathcal{L}_{\rm eff} = \frac{F^2}{4} \operatorname{tr}\{\partial_\mu U \partial^\mu U^+ + 2B\mathcal{M}(U + U^+)\}. \qquad (10.13)$$

The field $U(x)$ now is an element of SU(N) and describes N^2-1 Goldstone bosons; $\mathcal{M}$ is the diagonal matrix formed with the N quark masses $m_u, m_d, m_s, \ldots$.

I recommend it as an exercise to work out the kinetic part of the Lagrangian (10.13) for the case of three flavours and to show that the meson masses are given by

$$\begin{aligned} M_{\pi^+}^2 &= (m_u + m_d)B + O(\mathcal{M}^2) + O(e^2) \\ M_{K^+}^2 &= (m_u + m_s)B + O(\mathcal{M}^2) + O(e^2) \\ M_{K^0}^2 &= (m_d + m_s)B + O(\mathcal{M}^2) + O(e^2) \end{aligned} \qquad (10.14)$$

The states π^0 and η mix with an angle of order $(m_u - m_d)/(m_s - \hat{m}) \ll 1$, such that the two levels repel. Show that this generates a mass difference between π^0 and π^+, given by

$$\begin{aligned} M_{\pi^0}^2 &\simeq M_{\pi^+}^2 - \frac{1}{4}\left(\frac{m_u - m_d}{m_s - \hat{m}}\right)^2 (M_K^2 - M_\pi^2) \\ \hat{m} &\equiv \frac{1}{2}(m_u + m_d). \end{aligned} \qquad (10.15)$$

Numerically, this effect is tiny - the observed mass difference mainly originates in the electromagnetic self-energy of the charged pion. Show that the squares of the masses obey the Gell-Mann-Okubo-formula.

The mass formulae (10.14) indicate that the Goldstone boson masses break SU(3) symmetry - the absence of isospin breaking at leading order observed in the case of SU(2) × SU(2) does not repeat itself here. In fact, none of the other multiplets shows SU(3) breaking effects comparable to those seen in the masses of the pseudoscalar octet. The reason is now evident: M_K^2 is 13 times larger than M_π^2, because m_s happens to be about 13 times larger than $m_u + m_d$ - the quark masses violently break SU(3) symmetry. Unlike the mass of say the nucleon which approaches a nonzero limit if the quark masses are turned off, the masses of the Goldstone bosons are exclusively due to symmetry breaking and are therefore just as asymmetric as the quark masses themselves.

Plenty of further exercises are available to familiarize yourself with the effective Lagrangian technique. You might, e.g., calculate the elastic $\pi\pi$ scattering amplitude by evaluating the terms of order π^4 in the effective SU(2) × SU(2) Lagrangian and derive the low energy theorems of Weinberg [9], e.g.

$$T_{\pi^+\pi^0 \to \pi^+\pi^0} = \frac{t - M_\pi^2}{F^2} + O(p^4, p^2\mathcal{M}, \mathcal{M}^2) \qquad (10.16)$$

where $t = (p'_+ - p_+)^2$ is the invariant momentum transfer. Another instructive example is the decay $\eta \to 3\pi$. Again, it suffices to work out the terms of order π^4, except that in this case one needs to use the Lagrangian for three flavours where the field $U(x)$ includes the η. Also, one must be careful to use a field basis where the kinetic part of the Lagrangian is diagonal (in a basis where terms proportional to $\pi^0\eta$ occur, the η field does not exclusively create η-particles). Show that, to first order in $m_u - m_d$, the transition amplitude is given by

$$T_{\eta \to \pi^+\pi^-\pi^0} = \frac{\sqrt{3}}{4} \, \frac{m_u - m_d}{m_s - \hat{m}} \, \frac{s - \frac{4}{3}M_\pi^2}{F^2} + \ldots \qquad (10.17)$$

11 Higher order terms

The expression (10.13) for the effective Lagrangian is the leading term in a double series: powers of derivatives and powers of the quark mass matrix. When working out pion matrix elements, one is considering momenta which obey $p^2 = (m_u + m_d)B$. It is therefore convenient to count the quark masses like two powers of momentum and to order the double series accordingly. Both of the terms occurring in eq. (10.13) are then quantities of order p^2. Denoting this contribution by $\mathcal{L}^{(2)}$, the full effective Lagrangian is a series of the form

$$\mathcal{L}_{eff} = \mathcal{L}^{(2)} + \mathcal{L}^{(4)} + \mathcal{L}^{(6)} + \ldots \qquad (11.1)$$

The term $\mathcal{L}^{(4)}$ contains contributions with four derivatives of the pion field, terms with two derivatives and one power of $\mathcal{M}$ and terms with two powers of $\mathcal{M}$. Note that, in four dimensions, Lorentz invariance implies that the number of derivatives is even, because the tensors $g_{\mu\nu}$ and $\epsilon_{\mu\nu\rho\sigma}$ available to form invariants are of even rank. At order p^4, the following invariants occur [10]:

$$\begin{aligned}
P_0 &= < \partial_\mu U \partial_\nu U^+ \partial^\mu U \partial^\nu U^+ > \\
P_1 &= < \partial_\mu U \partial^\mu U^+ >^2 \\
P_2 &= < \partial_\mu U \partial_\nu U^+ >< \partial^\mu U \partial^\nu U^+ > \\
P_3 &= < \partial_\mu U \partial^\mu U^+ \partial_\nu U \partial^\nu U^+ > \\
P_4 &= < \partial_\mu U \partial^\mu U^+ >< \chi U^+ + U \chi^+ > \\
P_5 &= < \partial_\mu U \partial^\mu U^+ (\chi U^+ + U\chi^+) > \\
P_6 &= < \chi U^+ + U \chi^+ >^2 \\
P_7 &= < \chi U^+ - U \chi^+ >^2 \\
P_8 &= < \chi U^+ \chi U^+ + U \chi^+ U \chi^+ >
\end{aligned} \qquad (11.2)$$

where $< A >$ stands for the trace of the matrix A and χ is proportional to the mass matrix:

$$\chi = 2B\mathcal{M}. \qquad (11.3)$$

If there are less than four flavours, then not all of these invariants are independent. For $N = 3$, P_0 can be dropped because it can be represented as a linear combination of P_1, P_2 and P_3; for $N = 2$, only P_1, P_2, P_4, P_6 and P_7 are independent. The list (11.2) is however not complete. There are three additional categories of terms:

(i) total derivatives, i.e. contributions of the form $\partial_\mu \omega^\mu$,

(ii) expressions which contain $\Box U$,

(iii) terms involving the ϵ-tensor,

$$\mathcal{L}_{WZ} = \epsilon^{\mu\nu\rho\sigma} f(U) \times \partial_\mu U \times \partial_\nu U \times \partial_\rho U \times \partial_\sigma U. \qquad (11.4)$$

The first category is irrelevant, because total derivatives do not affect the action. Terms involving the second derivative $\Box U$ differ from $P_0, \ldots, P_8$ only by a multiple of the classical equation of motion associated with $\mathcal{L}^{(2)}$,

$$\partial_\mu \{\partial^\mu U U^+\} = \frac{1}{2}(\chi U^+ - U\chi^+) - \frac{1}{2N} < \chi U^+ - U\chi^+ > . \qquad (11.5)$$

These terms can be removed from the effective Lagrangian with a suitable change of the field variables. Consider, e.g., the transformation

$$U' = U\{1 + \alpha(U^+ \Box U - \Box U^+ U) + O(p^4)\}, \qquad (11.6)$$

If α is real, the map preserves unitarity and also keeps the determinant equal to one. Inserted in the leading term $\mathcal{L}^{(2)}$, this change of variables generates a contribution of the form $< \Box U^+ \Box U >$, proportional to the arbitrary constant α. With a suitable choice of α, we can therefore bring the effective Lagrangian to a form where such a contribution does not occur. Here it is essential that, in the framework of the effective Lagrangian, the pion field plays the role of a mere auxiliary variable devoid of physical significance - in QCD, a pion field does not occur. In fact, the matrix elements of the operator $\vec{\pi}(x)$ are ambiguous. In particular, they depend on the conventions chosen when introducing coordinates on SU(N). The change of variables occurring in eq. (11.6) amounts to a coordinate transformation which not only involves the field itself, but also the derivatives thereof,

$$\vec{\pi}' = \vec{f}(\vec{\pi}, \partial\vec{\pi}, \partial^2\vec{\pi}, \ldots, \mathcal{M}). \tag{11.7}$$

Since the particular change of variables specified in eq. (11.6) respects the transformation properties of the matrix U under $\mathrm{SU}(N)_L \times \mathrm{SU}(N)_R$, there is no reason to prefer the field $\vec{\pi}(x)$ to $\vec{\pi}(x)'$. I recommend it as an exercise in the technology of effective Lagrangians to verify that all of the terms of category (ii) can be eliminated with a change of variables of this type.

In contrast to the first two categories, (iii) is physically interesting. As pointed out by Wess and Zumino [11], the occurrence, of anomalies in the Ward identities for the currents of $\mathrm{SU}(N)_L \times \mathrm{SU}(N)_R$ implies that the effective Lagrangian contains specific vertices of order p^4 which involve the ϵ-tensor (note that I am not referring to the anomalous divergence of the flavour singlet axial current here, but to the anomalies contained in the three- and four-point functions of the non-singlet-currents, e.g., in $< TA^i_\lambda V^k_\mu V^l_\nu >$.) In fact, the function $f(U)$ occurring in eq. (11.4) turns out to be fully determined by the anomalies, which in turn are governed by the short distance singularities of the underlying theory. For QCD, $f(U)$ is proportional to the number of colours while, if the underlying theory only involves scalar fields, then $f(U)$ vanishes. The geometric significance of the Wess-Zumino term was uncovered by Witten [12] who pointed out that it can be represented in compact form if the four-dimensional base manifold is treated as boundary of a five-dimensional domain. I do not dwell on the structure of this term further as this would necessitate a discussion of the anomalies mentioned above. Suffice it to say that the effective Lagrangian machine can be equipped with the appropriate wheels (external vector and axial vector fields) to provide a systematic low energy expansion also for Green functions involving currents and that this extension leads to a rather elegant and powerful vehicle. Quite apart from the Wess-Zumino term which belongs to the standard equipment of this vehicle, it also features a number of physically interesting extras. If this advertisement prompts you to try it out, you can find it in the garage [10]. No strings attached.

At order p^4, the general effective Lagrangian is a linear combination of the terms listed in eq. (11.2), supplemented by the Wess-Zumino term,

$$\mathcal{L}^{(4)} = \sum_{i=0}^{8} L_i P_i + \mathcal{L}_{WZ}. \tag{11.8}$$

While the leading term $\mathcal{L}^{(2)}$ only contains two low energy constants (F and B), this expression involves nine such constants: $L_0, \ldots, L_8$ (as mentioned above, L_0 can be dropped if $N = 3$). The virtue of using the quantity $\chi = 2B\mathcal{M}$ instead of the mass matrix itself is that all of these constants are dimensionless.

If the derivative expansion is carried further, an ever growing number of low energy constants occurs - the effective theory contains infinitely many coupling constants. In this respect, the situation is the same as for theories containing a mass gap, where the low energy expansion is given by the Taylor series and also involves an infinite sequence of low energy parameters (scattering length, effective range, ...). As mentioned above, the effective Lagrangian can be extended in such a way that it also yields the low energy expansion of the Green functions. The price to pay is that the effective Lagrangian then contains even more terms. At first nonleading order, altogether thirteen constants are needed to specify the scattering amplitude and the Green functions of the vector, axial, scalar and pseudoscalar currents if $N \geq 4$ (twelve for $N = 3$ and ten for $N = 2$).

The underlying theory contains much less parameters. In the case of QCD, e.g., F and B are multiples of Λ_{QCD} which in principle, are calculable and the coupling constants L_i also represent calculable numbers. Indeed, lattice calculations do provide rough determinations of F and B which will become more accurate in the future. At the present time, the main source of information about the coupling constants is low energy phenomenology which allows one to pin down most of the coupling constants occurring in $\mathcal{L}^{(4)}$ to within rather narrow limits [10]. Quite independently, a rough a priori estimate of these constants can be obtained on the basis of the following observation. The effective Lagrangian explicitly describes the poles and cuts generated by the Goldstone bosons, but replaces the low energy singularities due to the exchange of other particles (such as ρ-mesons) by the corresponding Taylor series. In fact, in the case of SU(2)×SU(2), the size of all of the coupling constants occurring in $\mathcal{L}^{(4)}$ can be understood in a semi-quantitative manner if one assumes that they are dominated by the contribution from ρ-exchange [10]. More recently [13], it was shown that the values of the coupling constants occurring in the effective Lagrangian for $N = 3$ can be predicted by evaluating the pole graphs due to the exchange of the lightest massive mesons (octets with $J^P = 1^-, 1^+, 0^+$ and singlets with $J^P = 0^-, 0^+$). The result agrees remarkably well with the outcome of the phenomenological analysis. I will illustrate the method with a specific example below.

12 Unitarity, loops, renormalizability

In the preceding sections we were dealing only with tree graphs, i.e., with classical field theory. It is not legitimate, however, to simply disregard graphs which contain loops, because the tree graph contributions to the scattering amplitudes do not satisfy unitarity. As discussed by Lehmann at the Schladming Winter School in 1973 [14], unitarity requires that the low energy expansion of the $\pi\pi$ scattering amplitude contains specific contributions of order p^4 which are not polynomials in the invariants s, t but exhibit logarithmic branch points. In the language of the Feynman diagrams associated with the Lagrangian $\mathcal{L}^{(2)}$, these contributions are represented by the one-loop graph shown in Fig. 2a and its crossed variants. At one-loop order, the tadpole graphs of Fig. 2b also need to be taken into account, which renormalize the pion mass and the pion decay constant. Graphs involving two or more loops of course also occur - only the sum of all Feynman diagrams of a local field theory leads to a unitary S-matrix. Chiral perturbation theory is based on the fact that not all of these graphs are of the same order of magnitude. While the tree graph contribution to the $\pi\pi$ scattering amplitude is of order p^2 [see eq. (10.16)], the one-loop graphs are of order p^4. More generally, graphs containing a different number of loops occur at different orders of the low energy expansion: in d dimensions, graphs with ℓ loops are suppressed compared to the tree graphs by

the power $[p^{d-2}]^\ell$. The rule is readily checked for individual graphs such as those shown in Figs. 2a or 2b. The loop integrals are homogeneous functions of the external momenta and of the pion mass occurring in the propagators. The degree of homogeneity is determined by the dimension of the integral which in turn is fixed by the overall power of the pion decay constant arising from the various vertices. A more thorough discussion of the issue can be found in ref. [3].

As discussed above, the Lagrangian $\mathcal{L}^{(2)}$ is not the full story - graphs involving vertices of $\mathcal{L}^{(4)}, \mathcal{L}^{(6)}, \ldots$ also need to be taken into account. In the case of the $\pi\pi$ scattering amplitude, graphs containing ℓ loops are of order $p^{2+\ell(d-2)}$ only if they exclusively involve vertices of $\mathcal{L}^{(2)}$. Graphs containing one vertex of $\mathcal{L}^{(4)}$ ($\mathcal{L}^{(6)}$) are smaller by one (two) powers of p^2. Hence, to evaluate the scattering amplitude in four dimensions to order p^4, we need to work out the tree and one-loop graphs of $\mathcal{L}^{(2)}$ shown in Fig. 2 and add the tree graphs of Fig. 3 which involve one vertex from $\mathcal{L}^{(4)}$. Higher orders in the derivative expansion of the effective Lagrangian and two-loop graphs only start contributing at order p^6.

Note that the graphs can be ordered by counting powers of the momentum only if $d > 2$. In two dimensions, the constant F is dimensionless and the degree of homogeneity is therefore independent of the number of loops. In $d = 2$, the Lagrangian $\mathcal{L}^{(2)}$ taken by itself specifies a decent, renormalizable theory, which moreover is asymptotically free and thus shares the qualitative properties of four-dimensional nonabelian gauge theories. In particular, the low energy structure of the theory cannot be analyzed perturbatively. [Incidentally, supplementing $\mathcal{L}^{(2)}$ by the Wess-Zumino term, one arrives at a two-dimensional field theory with very peculiar properties: the Wess-Zumino-Novikov-Witten model. In this model, the coupling constant F can be tuned in such a fashion that the β-function vanishes - the theory becomes conformally invariant.]

In $d = 4$, the Lagrangian $\mathcal{L}^{(2)}$ by itself is meaningless, but taken together with the infinite string of higher order terms $\mathcal{L}^{(4)}, \mathcal{L}^{(6)}, \ldots$ it does specify a renormalizable framework. It is convenient to regularize the loop integrals by means of dimensional regularization, because this method preserves the symmetries of the Lagrangian. The poles occurring at $d = 4$ then only require counter terms which are Lorentz invariant and symmetric under SU(N)×SU(N). By construction, the full effective Lagrangian contains all terms permitted by this symmetry. The divergences can therefore be absorbed in a renormalization of the coupling constants occurring in the Lagrangian. In particular, the divergences contained in the one-loop graphs are absorbed in a renormalization of the coupling constants $L_0, \ldots, L_8$ occurring in $\mathcal{L}^{(4)}$. Dimensional regularization also takes care of a technical complication connected with the fact that the effective Lagrangian contains derivative couplings. This property implies that the measure occurring in the functional integral does not coincide with the standard translation invariant measure on the space of the pion fields. In general, the measure generates additional contributions involving power divergences, such as $\delta(0) \sim \Lambda^4$. In dimensional regularization, however, power divergences do not occur (in particular, $\delta(0)$ vanishes) and the complications associated with the measure can simply be ignored [for a more detailed discussion and references to the literature, see, e.g. [10]].

13 Mass of the Goldstone bosons to one loop

As an illustration of the machinery, let us work out the pion mass to second order in the expansion in powers of the quark masses. For simplicity, I take the masses of the N quark flavours to be the same, $m_u = m_d = \ldots = m$, such that the spectrum of the theory consists of degenerate multiplets of SU(N). In particular, the $N^2 - 1$ Goldstone bosons then obtain the same mass M. The leading term in the expansion of M^2 in powers of m is given in eq. (10.11). Denote this term by M_1^2,

$$M_1^2 \equiv 2mB. \tag{13.1}$$

At order m^2, we need to evaluate the tree graphs of $\mathcal{L}^{(4)}$ and add the one-loop contributions generated by $\mathcal{L}^{(2)}$. The tree graph contributions to the mass are determined by the kinetic part of the Lagrangian, i.e. by the piece which is quadratic

in the pion field $\vec{\pi}(x)$. The invariants P_0, P_1, P_2, P_3 only involve vertices with four or more pion fields and do therefore not affect the mass, but the term P_4 does contain a quadratic piece,

$$P_4 = \frac{4N}{F^2} M_1^2 \partial_\mu \vec{\pi} \partial^\mu \vec{\pi} + O(\pi^4), \tag{13.2}$$

and P_5 is of a similar structure. In the case of P_6, the expansion in powers of $\vec{\pi}$ yields

$$P_6 = M_1^4 \left\{ 4N^2 - \frac{8N}{F^2} \vec{\pi}^2 + O(\pi^4) \right\}. \tag{13.3}$$

The term P_7 only contains vertices with six or more pion fields and P_8 is similar to P_6. Collecting the various contributions, the kinetic part of the effective Lagrangian becomes

$$\begin{aligned} \mathcal{L}^{(2)} + \mathcal{L}^{(4)} &= \frac{1}{2} \partial_\mu \vec{\pi} \partial^\mu \vec{\pi} \left\{ 1 + \frac{8M_1^2}{F^2}(NL_4 + L_5) \right\} \\ &- \frac{1}{2} M_1^2 \vec{\pi}^2 \left\{ 1 + \frac{16 M_1^2}{F^2}(NL_6 + L_8) \right\}. \end{aligned} \tag{13.4}$$

Graphically, this result corresponds to the tree graphs shown in Figs. 4a and 4b. The one-loop graph of Fig. 4c stems from the vertex $\sim \mathrm{tr}([\partial_\mu \pi, \pi][\partial^\mu \pi, \pi])$ contained in $\mathcal{L}^{(2)}$, two of the pion fields being contracted with

$$\begin{aligned} <0 \mid T\pi^a(x)\pi^b(y) \mid 0> &= \frac{1}{i} \delta^{ab} \Delta(x-y) \\ \Delta(z) &= \frac{1}{(2\pi)^d} \int d^d p \frac{e^{-ipz}}{M_1^2 - p^2 - i\epsilon}. \end{aligned} \tag{13.5}$$

Since the arguments of the pion fields coincide, the contractions are proportional to the propagator or its derivatives at the origin. The term $\partial_\mu \Delta(z)$ vanishes at $z = 0$, as it is antisymmetric under $z \to -z$. The second derivative can be evaluated with the differential equation

$$\Box \Delta(z) + M_1^2 \Delta(z) = \delta(z), \tag{13.6}$$

using the fact that, in dimensional regularization, $\delta(0)$ vanishes. The sum over the flavour of the meson circling around the loop leads to

$$\sum_a \lambda^a \lambda^a = \frac{2(N^2-1)}{N}; \quad \sum_a \lambda^a \lambda^b \lambda^a = -\frac{2}{N}\lambda^b. \tag{13.7}$$

The result for the mass of the Goldstone bosons then takes the form

$$M^2 = M_1^2 \left\{1 - \frac{8M_1^2}{F^2}(NL_4 + L_5 - 2NL_6 - 2L_8) - \frac{i}{NF^2}\Delta(0)\right\}. \tag{13.8}$$

Performing a Wick rotation, the propagator at the origin can be represented as

$$-i\Delta(0) = \frac{1}{(2\pi)^d}\int \frac{d^dk}{M_1^2+k^2} = \frac{1}{(2\pi)^d}\int_0^\infty d\lambda \int d^dk e^{-\lambda(M_1^2+k^2)}$$
$$= \frac{1}{(4\pi)^{d/2}}\Gamma(1-\frac{d}{2})M_1^{d-2}. \tag{13.9}$$

The Γ-function contains a pole at $d=4$, $\Gamma(1-d/2) = 2/(d-4)+\ldots$, generating a divergence in $\Delta(0)$ which is proportional to M_1^2. Eq. (13.8) shows that this divergence can indeed be absorbed with a suitable renormalization of the coupling constants $L_4, \ldots, L_8$, as claimed above. The singularity can be extracted by rewriting $\Delta(0)$ as

$$-i\Delta(0) = M_1^2\{c + \bar{c}(M_1)\} \tag{13.10}$$

where the divergent part c

$$c \equiv \frac{\Gamma(1-d/2)}{(4\pi)^{d/2}}\mu^{d-4} \tag{13.11}$$

is independent of M_1, but contains an arbitrary renormalization scale μ. The remainder

$$\bar{c}(M_1) = \frac{\Gamma(1-d/2)}{(4\pi)^{d/2}}(M_1^{d-4} - \mu^{d-4}) \tag{13.12}$$

approaches a finite limit as as $d \to 4$, given by

$$\bar{c}(M_1) = \frac{1}{8\pi^2}\ln\frac{M_1}{\mu}. \tag{13.3}$$

The Goldstone boson mass then becomes

$$M^2 = M_1^2\left\{1 - \frac{8M_1^2}{F^2}(NL_4^r + L_5^r - 2NL_6^r - 2L_8^r) + \frac{M_1^2}{8\pi^2 NF^2}\ln\frac{M_1}{\mu}\right\} + O(m^3) \tag{13.14}$$

where $L_i^r = L_i^r(\mu)$ is the renormalized coupling constant at running scale μ (note that the curly bracket is independent of this scale). The formula shows that the expansion in powers of the quark mass is not an ordinary Taylor series, but contains a nonanalytic piece $\sim m^2 \log m$, referred to as a "chiral logarithm". The occurrence of such contributions is characteristic of chiral perturbation theory.

14 Masses of the light quarks

One of the many applications of chiral perturbation theory concerns the relative magnitude of m_u, m_d and m_s. In the following, I briefly discuss this example which adequately illustrates both the strengths and the weaknesses of the method.

Let me start with the estimate of the light quark masses we obtained in 1975 [15]:

$$m_u \simeq 4 \text{ MeV}, \; m_d \simeq 6 \text{ MeV}, \; m_s \simeq 135 \text{ MeV}. \tag{14.1}$$

This pattern is remarkable in several respects. First of all, in accordance with the qualitative discussion of section 4, m_s is large compared to m_u, m_d. The quark mass term $m_u\bar{u}u + m_d\bar{d}d + m_s\bar{s}s$ is therefore dominated by the strange quark. This implies that the chiral $SU(3)_L\times SU(3)_R$ symmetry which results if all three quark masses are set equal to zero is on the same footing as the eightfold way which results if m_u, m_d and m_s are set equal. Since the eightfold way is known to represent a decent approximate symmetry of the strong interaction, the same should also be true of $SU(3)_L\times SU(3)_R$. Furthermore, the mass term of the strange quark does not break chiral $SU(2)_L \times SU(2)_R$. This group should therefore represent an almost exact symmetry of the strong interaction. In fact, the estimate (14.1) indicates that the difference $m_u - m_d$ is by no means small compared to m_u, m_d. The breaking of $SU(2)_L \times SU(2)_R$ therefore ought to be comparable to isospin breaking which is known to be very small. This conclusion leads to a puzzle: the pion mass is an $SU(2)_L \times SU(2)_R$ symmetry breaking effect. Why do the pions not show comparable breaking of isospin symmetry? We have already solved this puzzle, in section 10. As noted there, the leading term in the derivative expansion of the effective Lagrangian associated with chiral $SU(2)_L \times SU(2)_R$ is isospin symmetric - despite the fact that the underlying theory is not. In the effective Lagrangian for *three* quark flavours, the mass difference $m_u - m_d$ does show up at leading order, generating a splitting between K^+ and K^0 [compare eq. (10.14)]. The relative magnitude of this isospin breaking effect is however also small, because it is of order $(m_u - m_d)/m_s$ and m_s happens to be large compared to $m_u - m_d$. In the case of the nucleon, isospin breaking is even less significant: the mass difference $M_p - M_n$ is of the same order of magnitude as $m_u - m_d$, but the scale to compare this with is the nucleon mass. To summarize: the magnitude of the isospin breaking effects is small compared to $(m_u - m_d)/(m_u + m_d)$, because the relevant scale is not given by $m_u + m_d$. In the pseudoscalar octet, the scale is set by m_s while for the remaining states, the magnitude of isospin breaking effects is determined by the ratio $(m_u - m_d)/\Lambda$ where Λ is a typical hadronic scale such as the mass of the nucleon.

Since 1975, the literature offers a considerable number of papers dealing with the pattern of the light quark masses, both concerning the ratios $m_u : m_d : m_s$ and the absolute magnitude, say of m_s. Chiral perturbation theory does not shed much light on the value of m_s, because the quark mass matrix only enters the effective Lagrangian through the product $B\mathcal{M}$ and direct phenomenological information on the low energy constant B is not available. Chiral symmetry merely relates the magnitude of the quark masses to the size of the quark condensate [see eq. (10.12)] - some of the methods proposed to estimate the light quark masses are

based on this relation. I do not review the various estimates here, because chiral perturbation theory does not play an important role in this context. It does allow one, however, to determine the ratios $m_u : m_d : m_s$ to within rather narrow limits and I now turn to this topic.

The main source of information about the light quark masses is the mass spectrum of the pseudoscalar octet. Using the lowest order mass formulae (10.14) and ignoring the electromagnetic interaction, the experimental values of M_{π^+}, M_{K^+} and M_{K^0} imply $m_u : m_d : m_s = 0.66 : 1 : 20$. There are two categories of corrections to this result: electromagnetic contributions and higher order terms in the quark mass expansion.

A. Corrections of order e^2

Dashen's theorem [16] states that, in the limit $m_u = m_d = m_s = 0$, the electromagnetic self-energies of the pseudoscalar mesons are proportional to the square of their charge Q_P,

$$M_P^2 = (M_P^2)_{\text{QCD}} + e^2 Q_P^2 C + O(e^2 \mathcal{M}). \tag{14.2}$$

The constant C can be expressed as an integral over the difference between the vector and axial vector spectral functions [17] which can be evaluated on the basis of τ-decay data [18]. Alternatively, C can be determined from the mass difference between π^+ and π^0. As mentioned above, in the absence of the electromagnetic interaction, this difference is tiny, of order $(m_u - m_d)^2$. Hence the observed mass difference must almost entirely be due to the electromagnetic self-energy, i.e. $e^2 C$ must approximately be equal to $M_{\pi^+}^2 - M_{\pi^0}^2$. The result of the spectral function analysis is in good agreement with this conclusion. The relation (14.2) therefore implies

$$M_{K^0}^2 - M_{K^+}^2 = (M_{K^0}^2 - M_{K^+}^2)_{\text{QCD}} - (M_{\pi^+}^2 - M_{\pi^0}^2) + O(e^2 \mathcal{M}). \tag{14.3}$$

Numerically, this amounts to

$$M_{K^0} - M_{K^+} = 5.3\,\mathrm{MeV} - 1.3\,\mathrm{MeV}$$

where the first term is the contribution generated by $m_u - m_d$, while the second term represents the electromagnetic self-energy.

Correcting the masses M_{π^+} and M_{K^+} for electromagnetic contributions in this manner and neglecting the terms of order $\mathcal{M}^2$, the mass formulae (10.14) imply $m_u : m_d : m_s = 0.55 : 1 : 20.1$, the ratios advocated by Weinberg [19].

B. Corrections of order $\mathcal{M}^2$

We have evaluated the corrections of order $\mathcal{M}^2$ to the pseudoscalar masses in the preceding section, however only for the case of N quark flavours of equal mass. Since we are now considering $m_u \neq m_d \neq m_s$, the various mesons propagating in the loops carry different masses and the vertices generated by the symmetry breaking terms of $\mathcal{L}^{(2)}$ and $\mathcal{L}^{(4)}$ also distinguish between m_u, m_d and m_s. Otherwise, the calculation is the same. There are five different masses in the octet: $\pi^0, \pi^+, K^0, K^+, \eta$. Since we are using the observed difference between π^0 and π^+ to correct for the electromagnetic contributions, only four of these masses provide information about the mass spectrum in QCD. It is convenient to analyze this information in terms of the following three dimensionless ratios:

$$\begin{aligned} \Delta_{GMO} &\equiv (4M_K^2 - 3M_\eta^2 - M_\pi^2)/(M_\eta^2 - M_\pi^2) \\ Q_1 &\equiv M_K^2/M_\pi^2 \qquad (14.4) \\ Q_2 &\equiv (M_{K^0}^2 - M_{K^+}^2 + M_{\pi^+}^2 - M_{\pi^0}^2)/(M_K^2 - M_\pi^2). \end{aligned}$$

At leading order in the expansion in powers of $\mathcal{M}$, the squares of the meson masses obey the Gell-Mann-Okubo formula. Hence Δ_{GMO} is a quantity of order $m_s - \hat{m}$. The one-loop calculation leads to the following explicit expression [10]

$$\Delta_{GMO} = \frac{8}{F^2}(M_K^2 - M_\pi^2)\{L_5^r - 12L_7^r - 6L_8^r\} + \chi\log \qquad (14.5)$$

where the term "$\chi\log$" stands for chiral logarithms (compare eq. (13.14)). In the chiral expansion of Q_1 and Q_2, the leading term is different from zero - it is given by a ratio of quark masses. The contributions generated by the tree graphs of $\mathcal{L}^{(4)}$ and by the one-loop graphs of $\mathcal{L}^{(2)}$ amount to a correction of order $m_s - \hat{m}$:

$$\begin{aligned} Q_1 &= \frac{m_s + \hat{m}}{m_u + m_d}\left\{1 + \Delta_M + O(\mathcal{M}^2)\right\} \\ Q_2 &= \frac{m_d - m_u}{m_s - \hat{m}}\left\{1 + \Delta_M + O(\mathcal{M}^2)\right\}. \qquad (14.6) \end{aligned}$$

Remarkably, the correction is the same in the two cases [10]:

$$\Delta_M = \frac{8}{F^2}(M_K^2 - M_\pi^2)(2L_8^r - L_5^r) + \chi\log\,. \qquad (14.7)$$

The ratio $Q^2 \equiv Q_1/Q_2$ therefore agrees with the corresponding ratio of quark masses, up to corrections of second order

$$Q^2 = \frac{m_s^2 - \hat{m}^2}{m_d^2 - m_u^2}\left\{1 + O(\mathcal{M}^2)\right\}. \qquad (14.8)$$

Neglecting the terms of order $\mathcal{M}^2$, this implies that the quark mass ratios are constrained to an ellipse

$$\left(\frac{m_u}{m_d}\right)^2 + \frac{1}{Q^2}\left(\frac{m_s}{m_d}\right)^2 = 1 \qquad (14.9)$$

whose semi-axes are given by 1 and Q, respectively ($\hat{m}^2$ is negligibly small compared to m_s^2 - I have dropped this term). The observed values of the meson masses give $Q \simeq 24$. The corresponding ellipse is shown as a dashed line in Fig. 5, while the full lines represent an estimate of the uncertainties due to the higher order contributions occurring in eqs. (14.3) and (14.8) (for details, see ref. [20]). Note that the result (14.9) does not contain any of the coupling constants occurring in the effective Lagrangian. This relation neatly illustrates the fact that, although the chiral Lagrangian contains a considerable number of parameters which chiral symmetry leaves unspecified, chiral perturbation theory nevertheless leads to parameter free predictions.

The elliptic constraint (14.9) does not suffice to determine the two individual mass ratios $m_u : m_d$ and $m_s : m_d$. In particular, as emphasized by Kaplan and Manohar [21], this constraint does not exclude the possibility that m_u vanishes (the quark mass matrix could then be brought to real, diagonal form by means of a chiral $SU(N_f)_L \times SU(N_f)_R$ rotation and QCD would therefore necessarily be invariant under P, C and T; if the determinant of the quark mass matrix is different from zero, this property of the strong interactions can be accommodated, but is not explained). The recent literature contains several papers [22] which deal with this possibility, pointing out that the mass ratios advocated by Weinberg [19] are based on lowest order chiral perturbation theory and are therefore subject to corrections of order $m_s - \hat{m}$ which could be substantial (the fact that the ratio $F_K/F_\pi = 1.22$ differs from one also originates in a correction of this type).

In our ancient treatise on quark masses [23], we arrived at a rather accurate value for the ratio $R = (m_s - \hat{m})/(m_d - m_u)$ which, together with the relation (14.9) pins the quark mass ratios down to within small uncertainties. This value was obtained from a variety of sources (mass splittings in the baryon octet, $K^0 - K^+$, $\rho\omega$-mixing) which were shown to lead to consistent results provided the terms of order $\mathcal{M}^{3/2}$ in the chiral expansion of the baryon masses are accounted for. As we pointed out at that time, our analysis excludes the value $m_u = 0$ by many standard deviations. Conversely, if m_u were to vanish, that analysis would be entirely wrong, because the higher order terms in the chiral expansion could then not be treated as small corrections. [At leading order and for $m_u = 0$, the ratios Q_1, Q_2 are related by $1/Q_2 = Q_1 - 1$. Numerically, this amounts to $43 = 12$! Evidently, the chiral expansion would be in bad shape if m_u were zero.]

To determine the individual ratios $m_u : m_d$ and $m_s : m_d$ from the pseudoscalar mass spectrum, we need to analyze the correction Δ_M which occurs in the chiral expansion (14.6) of the ratio $Q_1 = M_K^2/M_\pi^2$. Here, the coupling constants don't do us the favour to drop out. We need an estimate of the magnitude of $2L_8 - L_5$. The value of L_5 can be determined on phenomenological grounds, exploiting the fact that the ratio F_K/F_π is also determined by this coupling constant,

$$\frac{F_K}{F_\pi} = 1 + 4\frac{(M_K^2 - M_\pi^2)}{F^2} L_5^r + \chi \log \qquad (14.10)$$

and using the experimental value $F_K/F_\pi = 1.22$. In the case of L_8, however, phenomenological information is not available because of a hidden symmetry first pointed out in ref. [21].

15 The hidden symmetry of Kaplan and Manohar

The symmetry originates in the fact that the effective Lagrangian exclusively incorporates chiral invariance. In particular, the only property of the quark mass matrix which the effective Lagrangian was told about is that it transforms according to $\mathcal{M} \rightarrow V_R \mathcal{M} V_L^+$. As pointed out by Kaplan and Manohar [21], the matrix

$$\mathcal{M}' = \alpha_1 \mathcal{M} + \alpha_2 (\mathcal{M}^+)^{-1} \det \mathcal{M} \qquad (15.1)$$

transforms in the same manner, α_1 and α_2 being arbitrary constants. For a real diagonal mass matrix, this amounts to

$$m_u' = \alpha_1 m_u + \alpha_2 m_d m_s \quad (\text{cycl. } u \rightarrow d \rightarrow s \rightarrow u). \qquad (15.2)$$

Symmetry alone does therefore not distinguish $\mathcal{M}'$ from $\mathcal{M}$. If $\mathcal{L}(U, \partial U, \ldots, \mathcal{M})$ is an effective Lagrangian consistent with chiral symmetry, so is $\mathcal{L}(U, \partial U, \ldots, \mathcal{M}')$. Since only the product $B\mathcal{M}$ enters the Lagrangian, α_1 merely changes the value of the constant B. The term proportional to α_2 is a correction of order $\mathcal{M}^2$ which, upon insertion in $\mathcal{L}^{(2)}$ generates a contribution to $\mathcal{L}^{(4)}$. The contribution can again be removed by changing some of the coupling constants:

$$B' = B/\alpha_1 \;\; ; \;\; L_6' = L_6 - \alpha$$
$$L_7' = L_7 - \alpha \;\; ; \;\; L_8' = L_8 + 2\alpha \qquad (15.3)$$

where $\alpha = \alpha_2 F^2/32\alpha_1 B$. The effective Lagrangian is therefore invariant under a simultaneous change of the quark mass matrix and of the coupling constants. Accordingly, the meson masses and scattering amplitudes which one calculates with the effective Lagrangian are also invariant under this operation. The quantity Δ_{GMO}, e.g., only involves the combination $2L_7 + L_8$ which is indeed invariant (compare eq. (14.5)). I recommend to verify that the chiral representations for Q_1 and Q_2 and the elliptic constraint (14.8) also pass the test (up to terms of order $(m_u - m_d)^2$ which were neglected in the derivation of these formulae). The chiral representation for the Green functions of the vector and axial currents are also invariant [20]. Since there is experimental information only about masses, scattering amplitudes and matrix elements of the electromagentic or weak currents and since the chiral representation for these does not distinguish $\mathcal{M}, B, L_i$ form $\mathcal{M}', B', L_i'$, phenomenology does not allow us to determine the magnitude of the constants B, L_6, L_7, L_8.

I emphasize that we are not dealing with a hidden symmetry of QCD here - this theory is not invariant under the change (15.2) of the quark masses. The quark masses can be determined, e.g., by measuring the current correlation functions very accurately at short distances. Also, when simulating QCD by means of Monte Carlo calculations on a lattice, a symmetry of this sort does not occur - the mass spectra associated with the quark mass matrices $\mathcal{M}$ and $\mathcal{M}'$ are different. The symmetry arises because we are not making use of the explicit form of the QCD Lagrangian, but are only exploiting its symmetry properties under chiral rotations. Attempts at elevating the above symmetry to a basic property of QCD are futile - the symmetry pertains to the method by means of which we are analyzing the theory, not to QCD itself. We do however face the problem that phenomenological information about the constant L_8 which occurs in Δ_M is not available and we therefore have to resort to theoretical arguments to estimate its magnitude.

16 Singularities generated by low lying states

Consider the nonet of pseudoscalar currents

$$P^A = \bar{q}\frac{\lambda^A}{2} i\gamma_5 q \ ; \ A = 0, \ldots, 8 \qquad (16.1)$$

with $\lambda^0 = \sqrt{2/3}$. In the chiral limit, the corresponding two-point functions are described by two independent spectral functions $\rho^8_{PP}(s)$ and $\rho^1_{PP}(s)$:

$$i \int dx e^{iqx} < 0 \mid TP^i(x)P^k(0) \mid 0 >= \delta^{ik} \int_0^\infty \frac{ds}{s - q^2 - i\epsilon}\, \rho^8_{PP}(s)$$
$$i \int dx e^{iqx} < 0 \mid TP^0(x)P^0(0) \mid 0 >= \int_0^\infty \frac{ds}{s - q^2 - i\epsilon}\, \rho^1_{PP}(s) \qquad (16.2)$$

The scalar and pseudoscalar currents represent the response of the QCD Lagrangian to an infinitesimal deformation of the quark mass matrix. The above two-point-functions can therefore be obtained by treating the quark mass matrix as an external field and taking the second derivative of the vacuum-to-vacuum amplitude with respect to this field. The same procedure can also be applied at the level of the effective theory, calculating the vacuum-to-vacuum amplitude with the effective Lagrangian specified in the preceding sections. Note however that chiral symmetry permits a term proportional to $\mathrm{tr}(\mathcal{M}\mathcal{M}^+)$. We did not include this term in eq. (11.8), because it does not affect the scattering amplitude or the meson masses, being independent of the pion field. In the present context, this term cannot be dropped; in the chiral representation for the above two-point-functions, it shows up as a contact contribution independent of q^2. It does drop out, however, in the difference between the singlet and the octet:

$$\frac{1}{B^2}\int_0^\infty \frac{ds}{s - q^2 - i\epsilon}\left\{\rho^1_{PP}(s) - \rho^8_{PP}(s)\right\} = \frac{F^2}{q^2} - 48L_7 + O(q^2). \qquad (16.3)$$

In the chiral limit, this representation is exact. The leading term of order $1/q^2$ originates in the one-pion intermediate state occurring in the two-point function of the octet while the contribution of order one is determined by the coupling constant

L_7. The spectral functions do not receive contributions from $\pi\pi$ intermediate states, on account of parity. The continuum starts with three pions and only manifests itself at the two-loop level (i.e. through terms of order q^2 in eq. (16.3)).

The formula (16.3) unambiguously specifies the physical significance of the coupling constant L_7. The hidden symmetry of the effective Lagrangian discussed in the preceding section (under which $L_7 \to L_7 - \alpha$) does not apply to the chiral representation of the Green functions involving scalar or pseudoscalar currents.

The operator expansion of the product $P^i(x)P^k(0)$ starts with the unit operator. Up to logarithmic factors which reflect the anomalous dimension of P^i, the coefficient of the unit operator is proportional to x^{-6}. Since P^0 and P^i belong to the same chiral multiplet, the coefficient of the unit operator in the expansion of $P^0(x)P^0(0)$ is the same. The short distance singularity associated with the unit operator therefore drops out in the difference we are considering here. Chirality conservation also implies that the coefficient of the operator $\bar{q}q$ is proportional to the quark mass and hence vanishes in the chiral limit. In the case of the operator $G^a_{\mu\nu}G^{\mu\nu a}$, the singularity again drops out because this operator is a chiral singlet. This shows that the difference of the two correlation functions is less singular than x^{-2}. The spectral functions therefore obey the sum rule

$$\int_0^\infty ds \left\{\rho^1_{PP}(s) - \rho^8_{PP}(s)\right\} = 0. \qquad (16.4)$$

The two relations (16.3) and (16.4) can be compared with the Weinberg sum rules obeyed by the spectral functions associated with the vector and axial currents [24]. In the chiral limit, these relations read

$$\int_0^\infty ds \left\{\rho^8_{VV}(s) - \rho^8_{AA}(s)\right\} = F^2$$
$$\int_0^\infty ds\, s \left\{\rho^8_{VV}(s) - \rho^8_{AA}(s)\right\} = 0. \qquad (16.5)$$

As it is well known, the contributions to the spectral functions generated by the lowest intermediate states (π, A_1, ρ) nearly saturate the Weinberg sum rules. In the case of the pseudoscalar spectral functions, the lowest states are π and η'.

Assuming that the corresponding contributions saturate the integrals occurring in eqs. (16.3) and (16.4), we obtain [10] [20]

$$< 0 \mid P^i \mid \pi^k > \simeq \delta^{ik} < 0 \mid P^0 \mid \eta' >$$
$$L_7 \simeq -\frac{F^2}{48 M^2_{\eta'}}. \qquad (16.6)$$

In the large N_c limit, both of these relations are exact. The η' then also plays the role of a Goldstone boson which can be included in the effective Lagrangian. A detailed discussion of the role of the η' in the context of chiral perturbation theory is given in ref. [10].

The estimate (16.6) expresses the coupling constant L_7 in terms of the mass of the lowest lying bound state with the appropriate quantum numbers. As mentioned in section 11, analogous estimates can be given for all of the coupling constants occurring in the effective Lagrangian [13]. Both the signs and the magnitudes of the coupling constants can thus be understood theoretically, on the basis of the observed spectrum of low lying bound states: the coupling constants are not free parameters but are determined by the low energy singularities which remain, once the Goldstone boson poles and cuts are removed.

As an illustration, consider the coupling constant L_5. The corresponding estimate reads [13]

$$L_5 \simeq \frac{F^2}{4M^2_S} \qquad (16.7)$$

where M_S is the mass of the lowest lying scalar multiplet. Ignoring the chiral logarithm associated with the $\pi\pi$ continuum underneath the resonance, the formula (14.10) thus predicts

$$\frac{F_K}{F_\pi} \simeq 1 + \frac{M^2_K - M^2_\pi}{M^2_S}.$$

This explains why SU(3) breaking effects only occur at the level of 20 or 30 %: the magnitude of these effects is determined by the ratio $(M^2_K - M^2_\pi)/M^2_S \simeq 0.25$.

Qualitatively, the chiral expansion in powers of m_u, m_d and m_s is a meaningful notion, because the corresponding eight Goldstone bosons are indeed the eight lightest hadrons. In chiral perturbation theory, this observation acquires quantitative meaning: the corrections of order $\mathcal{M}$ are given by M_P^2/M_S^2 or $M_P^2/M_{\eta'}^2$ where M_P stands for the mass of one of the Goldstone bosons while M_S or $M_{\eta'}$ are the masses of the lightest scalar or pseudoscalar non-Goldstone states. The two ratios are relatively small, since the masses $M_S \simeq M_{\eta'} \simeq 1$ GeV exceed the Goldstone boson masses by about a factor of two (for a more detailed discussion, see ref. [20]).

Let us now return to L_7. There is an independent phenomenological check on the estimate given in eq. (16.6). As pointed out in ref. [10], the coupling constant L_7 also determines the $\eta\eta'$ mixing angle:

$$\sin^2\Theta_{\eta\eta'} = -\frac{24(M_\eta^2 - M_\pi^2)^2}{F^2(M_{\eta'}^2 - M_\eta^2)} L_7. \qquad (16.8)$$

The phenomenology of the decays $\eta \to \gamma\gamma$ and $\eta' \to \gamma\gamma$ indicates that the mixing angle is in the range $20^0 < \Theta_{\eta\eta'} < 25^0$ while the estimate (16.6) corresponds to $\Theta_{\eta\eta'} \simeq 16^0$.

Once L_7 is pinned down, it is a straightforward matter to determine the value of the correction Δ_M which occurs in the mass formulae for the pseudoscalar mesons. Using the experimental values for Δ_{GMO} and for F_K/F_π, one arrives at the numbers shown in the upper right of Fig. 5. Clearly, the phenomenological information about the mixing angle indicates that Δ_M is small, consistent with zero. The corresponding quark mass ratios are obtained by intersecting the straight lines which represent a given value of Δ_M with the elliptic band discussed before. The figure also shows that the independent information about the ratio $R = (m_s - \hat{m})/(m_d - m_u)$ extracted from the baryon masses in ref. [23] is in perfect agreement with these results.

17 Final remarks

There are plenty of further applications of chiral perturbation theory. Although I did not make an attempt at a comprehensive survey in these lectures, I did discuss a few additional examples. In particular, I described the present status of our knowledge about the σ-term in πN scattering, a recurrent theme in the literature on chiral perturbation theory from the very beginning. This material is described in detail in ref. [25]. I also discussed the extension of chiral perturbation theory needed to analyze thermal expectation values. In particular, I reviewed some work concerning the melting of the quark condensate with rising temperature [26] and the kinetic properties of the hadronic phase [27]. Finally, some predictions of chiral perturbation theory concerning finite size effects were briefly discussed. The beautiful data on the distribution of the magnetization obtained from Monte Carlo simulations of the Heisenberg model in $d = 3$ and of the Higgs model in $d = 4$ [28] are in perfect agreement with the calculated shape of the constraint effective potential [29]. When the lattice simulations of QCD will reach realistic pion masses at sufficiently small lattice spacings, chiral perturbation theory should also prove to be useful there. The finite size effects in several correlation functions have been calculated explicitly [30] in view of these applications.

I did not discuss K-decays which are currently under active investigation within chiral perturbation theory [31]. The planned Dafne accelerator at Frascati will allow several of these processes to be investigated experimentally - the comparison with the theoretical analysis should lead to a considerable improvement in our understanding of the low energy structure of QCD. Chiral perturbation theory can also be used to analyze the low energy structure of the Higgs sector and of its analogue in extended versions of the Standard Model - this topic is discussed in detail in the lectures of Bagger [32].

In conclusion, let me underline the limitations of the method:

(i) The chiral expansion is useful only at low energies, small quark masses, low temperatures and large volumes

$$p, m_{quark}, T, \frac{1}{L} \ll \text{scale of theory}.$$

(ii) The effective Lagrangian involves many coupling constants. Although theoretical a priori estimates can be given, an accurate determination of these constants requires phenomenological input.

Despite these limitations, chiral perturbation theory is an unambiguous framework which leads to results of physical interest.

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Figure captions

Fig. 1: Pion pole term occurring in the 2-, 3- and 4-point functions of the vector and axial currents.

Fig. 2: Chiral perturbation theory graphs contributing to the low energy expansion of the $\pi\pi$ scattering amplitude to first nonleading order. The full circles represent vertices generated by the leading term in the derivative expansion of the effective Lagrangian, $\mathcal{L}^{(2)}$.

Fig. 3: Contributions to the $\pi\pi$ scattering amplitude generated by the first nonleading term in the derivative expansion of the effective Lagrangian: the full squares represent vertices of $\mathcal{L}^{(4)}$.

Fig. 4: Meson self energy graphs. At leading order in the low energy expansion only a tree graph generated by $\mathcal{L}^{(2)}$ contributes (a). At first nonleading order, a tree graph from $\mathcal{L}^{(4)}$ occurs (b) as well as a one-loop graph from $\mathcal{L}^{(2)}$ (c).

Fig. 5: The elliptic band indicates the range of quark mass ratios permitted by the low energy theorem (14.8). The cross-hatched area is the intersection of this band with the sector allowed by the phenomenology of $\eta\eta'$ mixing. The shaded wedge shows the constraint imposed on the ratio R by the mass splittings in the baryon octet, according to ref. [23].

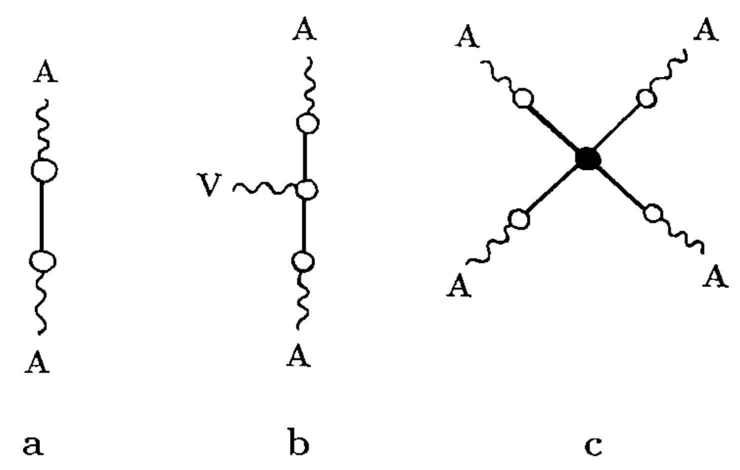

Fig.1

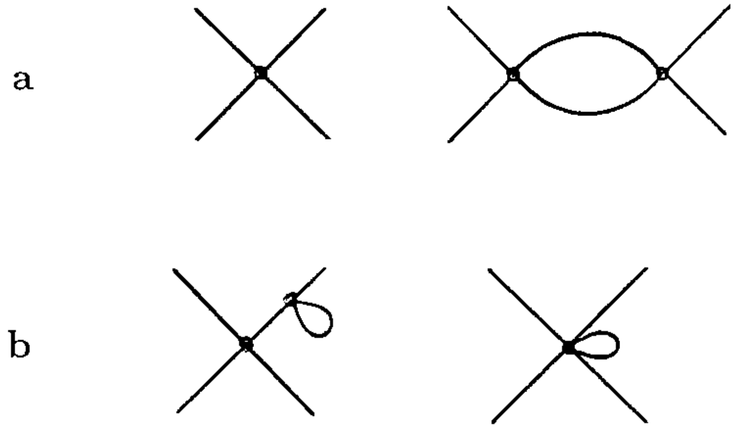

Fig.2

Fig.3

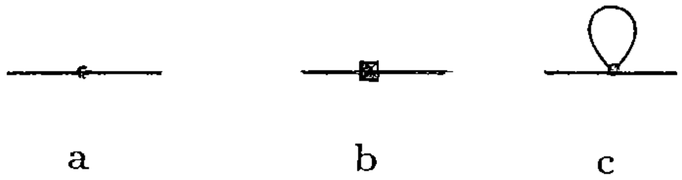

Fig.4

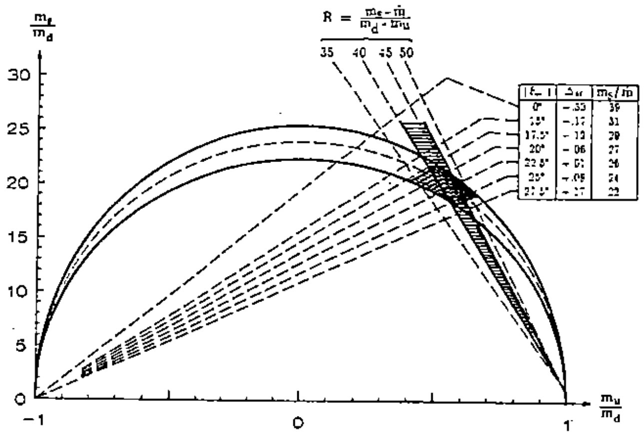

Fig.5